Hautus test for linear parabolic systems

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$$\Omega \subset \mathbb{R}^N, \quad Q_T = (0, T) \times \Omega, \quad \Sigma_T = (0, T) \times \partial\Omega,$$
$$\omega \subset \Omega, \quad \gamma \subset \partial\Omega.$$

• GOAL n° 1 : characterize the approximate controllability of linear parabolic systems of the form :

$$\begin{cases} \partial_t y = \Big(\Delta + G(x) \cdot \nabla + V(x)\Big) y + 1_\omega(x) B_1 f(t,x) \text{ in } Q_T \\ y = 1_\gamma(\sigma) B_2 g(t,\sigma) \text{ on } \Sigma_T \\ y(0) = y_0 \text{ in } \Omega \end{cases}$$

where the matrices D, G and V couple the equations and B_1, B_2 are the matrices through which the controls f and g operate.

• GOAL n° 2 : give usefull characterizations.

• IDEA : prove and use the Hautus test.

From now on, we will work on the adjoint system.

? ? ? What is the Hautus test ? ? ?

- In finite dimension, for matrices A and C, it is known that

Theorem (M.L.J. Hautus)

The pair (A, C) is observable (that is $Ce^{tA}x = 0$ for every $t > 0$ implies $x = 0$) if and only if

$$\mathrm{Ker}\,(sI - A) \cap \mathrm{Ker}\,(C) = \{0\}\,, \quad \forall s \in \mathbb{C}.$$

- We want to generalize formally this test to our framework.

The generalization of D.L. Russell and G. Weiss

There already exists an attempt of generalization of the Hautus test due to D.L. Russell and G. Weiss :

• Their generalization's condition is always necessary and sometimes sufficient.

But :

1. this has been made in view of exact observability
2. the most general results are for normal operators

See the works of D.L. Russell and G. Weiss, B. Jacob and H. Zwart, L. Miller, etc...

• On the same manner, the formal generalization of the Hautus test is always necessary and we will prove that in some cases it is sufficient.

The operator $-\Delta + V$

For

- $V \in \mathcal{M}_n(\mathbb{C})$
- $C : \mathcal{D}(C) \subset L^2(\Omega)^n \longrightarrow U^n$ (U Hilbert) Δ-bounded,

we have :

Theorem

The pair $(-\Delta + V, C)$ is observable in $L^2(\Omega)^n$ on $(0, +\infty)$ if and only if it satisfies the Hautus test

$$\mathrm{Ker}\,(sI - (-\Delta + V)) \cap \mathrm{Ker}\,(C) = \{0\}\,, \quad \forall s \in \mathbb{C}.$$

Interesting case :

- $C = BG$ with $B \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $G : \mathcal{D}(G) \subset L^2(\Omega) \longrightarrow U$.

Proposition

The pair $(-\Delta + V, BG)$ satisfies the Hautus test if and only if

1. $\mathrm{Ker}\,(\lambda I + \Delta) \cap \mathrm{Ker}\,(G) = \{0\}\,, \quad \forall \lambda \in \mathbb{C}.$
2. $\mathrm{Ker}\,(\theta I - V) \cap \mathrm{Ker}\,(B) = \{0\}\,, \quad \forall \theta \in \mathbb{C}.$
3. *For every distincts indices $l_1, \ldots, l_r$ and $i_1, \ldots, i_r$ such that*

$$\lambda_{l_1} + \theta_{i_1} = \ldots = \lambda_{l_r} + \theta_{i_r},$$

we, denoting s this common value,

$$\mathrm{Ker}\,(sI - (-\Delta + V)) \cap \mathrm{Ker}\,(BG) = \{0\}\,.$$

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Hautus test on the scalar differential part

First result and applications

Interesting case :

- $C = BG$ with $B \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $G : \mathcal{D}(G) \subset L^2(\Omega) \longrightarrow U$.

Proposition

The pair $(-\Delta + V, BG)$ *satisfies the Hautus test if and only if*

1. $\operatorname{Ker}(\lambda I + \Delta) \cap \operatorname{Ker}(G) = \{0\}, \quad \forall \lambda \in \mathbb{C}.$
2. $\operatorname{Ker}(\theta I - V) \cap \operatorname{Ker}(B) = \{0\}, \quad \forall \theta \in \mathbb{C}.$
3. *For every distincts indices* $l_1, \ldots, l_r$ *and* $i_1, \ldots, i_r$ *such that*

$$\lambda_{l_1} + \theta_{i_1} = \ldots = \lambda_{l_r} + \theta_{i_r},$$

we, denoting s this common value,

$$\operatorname{Ker}(sI - (-\Delta + V)) \cap \operatorname{Ker}(BG) = \{0\}.$$

Hautus test on the vectorial algebraic part

Interesting case :

- $C = BG$ with $B \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $G : \mathcal{D}(G) \subset L^2(\Omega) \longrightarrow U$.

Proposition

The pair $(-\Delta + V, BG)$ satisfies the Hautus test if and only if

1. $\mathrm{Ker}\,(\lambda I + \Delta) \cap \mathrm{Ker}\,(G) = \{0\}, \quad \forall \lambda \in \mathbb{C}.$
2. $\mathrm{Ker}\,(\theta I - V) \cap \mathrm{Ker}\,(B) = \{0\}, \quad \forall \theta \in \mathbb{C}.$
3. *For every distincts indices $l_1, \ldots, l_r$ and $i_1, \ldots, i_r$ such that*

$$\lambda_{l_1} + \theta_{i_1} = \ldots = \lambda_{l_r} + \theta_{i_r},$$

we, denoting s this common value,

$$\mathrm{Ker}\,(sI - (-\Delta + V)) \cap \mathrm{Ker}\,(BG) = \{0\}.$$

Hautus test on the "non natural" multiple eigenvalues of the system

Examples

• With this proposition we are able to recover some known results of F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. González-Burgos (2009) ; E. Fernández-Cara, M. González-Burgos, L. de Teresa (2010) and F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa (2011) .

• This characterizes the boundary controllability in any space dimension.

• We apply this characterization to discuss the boundary controllability on a rectangle.

Let now

- A_0 and B are Δ-bounded.

We form the following pair (A, C) :

$$A = \begin{pmatrix} -\Delta & A_0 \\ 0 & -\Delta \end{pmatrix}, \quad C = \begin{pmatrix} B & 0 \end{pmatrix}.$$

Theorem

If B is bounded in $L^2(\Omega)$ or $N = 1$, then the pair (A, C) is observable in $L^2(\Omega)^2$ on $(0, +\infty)$ if and only if

$$\mathrm{Ker}\,(sI - A) \cap \mathrm{Ker}\,(C) = \{0\}, \quad \forall s \in \mathbb{C}.$$

Examples

• We recover the results of O. Kavian, L. de Teresa (2010) and M. Gonzáles-Burgos (2012) .

• We provide some new examples of distributed and boundary controllability.

Summarizing

- We prove the Hautus test in two cases.
- We gave some reformulations of the Hautus, as best as we can.
- We can recover some known results.
- We provide new examples.

Hope to see you soon !