On the controllability of driftless swimmers

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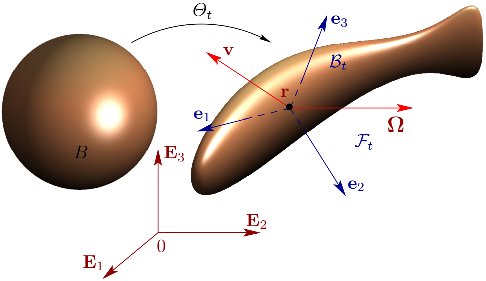
Modeling of Swimming

Kinematics



- $\mathfrak{E} := (0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ is Galilean and $\mathfrak{e} := (\mathbf{r}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is moving.
- The position of $\mathfrak{e}$ is described by $\mathbf{q} := (R, \mathbf{r}) \in \mathcal{Q} := \mathrm{SE}(3)$.
- $\mathbf{v}$ and $\boldsymbol{\Omega}$ are the linear and angular velocities of the fish in $\mathfrak{e}$.
- The shape of the fish is described with respect to $\mathfrak{e}$.
- $\mathcal{B}_t$: image of the unit ball B by a C^1 diffeomorphism Θ_t.
- The shape changes are prescribed: $t \in [0, T] \mapsto \Theta_t$.

Dynamics : General settings

Fluid's dynamics : Navier Stokes equations

$$\begin{aligned} \varrho_f\Big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{0} && \text{in } \mathcal{F}_t && t>0,\\ \operatorname{div}(\mathbf{u}) &= 0 && \text{in } \mathcal{F}_t && t>0,\\ \mathbf{u} &= \mathbf{w} && \text{on } \Sigma_t && t>0. \end{aligned}$$

Velocity of the swimmer

$$\mathbf{w} = R\boldsymbol{\Omega}\times(x-\mathbf{r}) + R\mathbf{v} + R\mathbf{w}^d \qquad \text{in } \mathcal{B}_t \qquad t>0.$$

Newton's laws

$$\begin{aligned} m\frac{d}{dt}(R\mathbf{v}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\,\mathrm{d}\sigma && t>0,\\ \frac{d}{dt}(\mathbb{J}R\boldsymbol{\Omega}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\times(x-\mathbf{r})\,\mathrm{d}\sigma && t>0,\\ \mathbb{T} &= -p\mathrm{Id} + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) && t>0. \end{aligned}$$

Dynamics : Driftless models

Fluid's dynamics : Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{0} && \text{in } \mathcal{F}_t && t > 0, \\ \operatorname{div}(\mathbf{u}) &= 0 && \text{in } \mathcal{F}_t && t > 0, \\ \mathbf{u} &= \mathbf{w} && \text{on } \Sigma_t && t > 0. \end{aligned}$$

Velocity of the swimmer

$$\mathbf{w} = R\boldsymbol{\Omega} \times (x - \mathbf{r}) + R\mathbf{v} + R\mathbf{w}^d \qquad \text{in } \mathcal{B}_t \qquad t > 0.$$

Newton's laws

$$\begin{aligned} \mathbf{0} &= -\int_{\Sigma_t} \mathbb{T}\mathbf{n}\, \mathrm{d}\sigma && t > 0, \\ \mathbf{0} &= -\int_{\Sigma_t} \mathbb{T}\mathbf{n} \times (x - \mathbf{r})\, \mathrm{d}\sigma && t > 0, \\ \mathbb{T} &= -p\mathrm{Id} + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) && t > 0. \end{aligned}$$

Dynamics : Driftless models

Fluid's dynamics : Euler equations

$$\begin{aligned} \varrho_f\Big(\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\Big) + \nabla p &= \mathbf{0} && \text{in } \mathcal{F}_t && t>0,\\ \operatorname{div}(\mathbf{u}) &= 0 && \text{in } \mathcal{F}_t && t>0,\\ \mathbf{u}\cdot\mathbf{n} &= \mathbf{w}\cdot\mathbf{n} && \text{on } \Sigma_t && t>0. \end{aligned}$$

Velocity of the swimmer

$$\mathbf{w} = R\boldsymbol{\Omega}\times(x-\mathbf{r}) + R\mathbf{v} + R\mathbf{w}^d \qquad \text{in } \mathcal{B}_t \qquad t>0.$$

Newton's laws

$$\begin{aligned} m\frac{d}{dt}(R\mathbf{v}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\,\mathrm{d}\sigma && t>0,\\ \frac{d}{dt}(\mathbb{J}R\boldsymbol{\Omega}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\times(x-\mathbf{r})\,\mathrm{d}\sigma && t>0,\\ \mathbb{T} &= -p\mathrm{Id} && t>0. \end{aligned}$$

Dynamics : Driftless models

Fluid's dynamics : Potential flow

$$\begin{aligned} -\Delta\phi_t &= 0 && \text{in } \mathcal{F}_t && t>0,\\ \mathbf{u} &= \nabla\phi_t && \text{in } \mathcal{F}_t && t>0,\\ \partial_n\phi_t &= \mathbf{w}\cdot\mathbf{n} && \text{on } \Sigma_t && t>0. \end{aligned}$$

Velocity of the swimmer

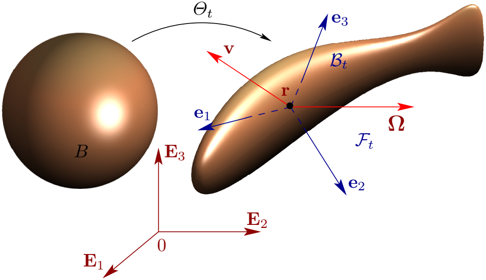
$$\mathbf{w} = R\boldsymbol{\Omega}\times(x-\mathbf{r}) + R\mathbf{v} + R\mathbf{w}^d \qquad \text{in } \mathcal{B}_t \qquad t>0.$$

Newton's laws

$$\begin{aligned} m\frac{d}{dt}(R\mathbf{v}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\,\mathrm{d}\sigma && t>0,\\ \frac{d}{dt}(\mathbb{J}R\boldsymbol{\Omega}) &= -\int_{\Sigma_t}\mathbb{T}\mathbf{n}\times(x-\mathbf{r})\,\mathrm{d}\sigma && t>0,\\ \mathbb{T} &= -p\mathrm{Id} && t>0,\\ p &= p_0 + \varrho_f\left(\partial_t\phi_t + |\nabla\phi_t|^2/2\right) && t>0. \end{aligned}$$

Modeling of Swimming

Statement of the problems



1. Direct problem: To determine the *rigid displacement* $t \mapsto (R(t), \mathbf{r}(t))$, the shape changes being given.

2. Control problem: Can the fish follow any given trajectory with appropriate shape changes?

More about the control function

Some function spaces

$C^1_0(\mathbb{R}^3) = \{\vartheta \in C^1(\mathbb{R}^3, \mathbb{R}^3) \text{ s. t. } \vartheta(x) \text{ and } \nabla\vartheta(x) \to 0 \text{ as } \|x\| \to +\infty\}.$

$D^1_0(\mathbb{R}^3) = \{\vartheta \in C^1_0(\mathbb{R}^3) \text{ s. t. } \mathrm{Id} + \vartheta \text{ is a diffeomorphism of } \mathbb{R}^3\}.$

We seek Θ_t the control function in the form:

$$\Theta_t = \mathrm{Id} + \vartheta_t$$

with $\vartheta \in W^{1,1}([0,T], D^1_0(\mathbb{R}^3))$.

Self-propelled constraints

Viscous case

A control function Θ is said to be allowable if for a. e. $t \in [0, T]$:

$$\int_\Sigma \Theta_t \, \mathrm{d}x = \mathbf{0} \qquad \text{and} \qquad \int_\Sigma \partial_t \Theta_t \times \Theta_t \, \mathrm{d}x = \mathbf{0}.$$

Potential flow

A pair (ϱ, Θ) is said to be allowable if for a. e. $t \in [0, T]$:

$$\int_B \varrho \, \Theta_t \, \mathrm{d}x = \mathbf{0} \qquad \text{and} \qquad \int_B \varrho \, \partial_t \Theta_t \times \Theta_t \, \mathrm{d}x = \mathbf{0}.$$

Kirchhoff's law

The decomposition of the swimmer's velocity:

$$\mathbf{w} = \underbrace{\sum_{i=1}^{3} \Omega_i (\mathbf{e}_i \times x) + v_i \mathbf{e}_i}_{\text{rigid motion}} + \underbrace{\partial_t \Theta_t ({\Theta_t}^{-1})}_{\text{deformation}},$$

leads to the decomposition of the fluid velocity:

- Potential flow:

$$\phi_t = \sum_{i=1}^{3} \Omega_i \psi_t^i + v_i \psi_t^{i+3} + \varphi_t,$$

- Viscous case:

$$\mathbf{u} = \sum_{i=1}^{3} \Omega_i \mathbf{u}^i + v_i \mathbf{u}^{i+3} + \mathbf{w}^d, \quad p = \sum_{i=1}^{3} \Omega_i p^i + v_i p^{i+3} + p^d.$$

Mass matrices

- Potential flow:

$$\mathbb{M}(t) := \begin{pmatrix} \mathbb{I}(t) & 0 \\ 0 & m\mathrm{Id} \end{pmatrix} + \varrho_f \left(\int_{\mathcal{F}_t} \nabla\psi_t^i \cdot \nabla\psi_t^j \,\mathrm{d}x \right)_{1\leq i,j\leq 6}$$

$$\mathbf{N}(t) := \varrho_f \left(\int_{\mathcal{F}_t} \nabla\psi_t^i \cdot \nabla\varphi_t \,\mathrm{d}x \right)_{1\leq i\leq 6}$$

- Viscous flow:

$$\mathbb{M}(t) := \nu \left(\int_{\mathcal{F}_t} \mathrm{D}\mathbf{u}^i : \mathrm{D}\mathbf{u}^j \,\mathrm{d}x \right)_{1\leq i,j\leq 6}$$

$$\mathbf{N}(t) := \nu \left(\int_{\mathcal{F}_t} \mathrm{D}\mathbf{u}^i : \mathrm{D}\mathbf{w}^d \,\mathrm{d}x \right)_{1\leq i\leq 6}$$

where

$$D\mathbf{u} := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right).$$

Dynamics of the fluid/swimmer system

Dynamics

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} R \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} R\widehat{\boldsymbol{\Omega}} \\ R\mathbf{v} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{\Omega} \\ \mathbf{v} \end{pmatrix} = -\mathbb{M}(t)^{-1}\mathbf{N}(t) \tag{1}$$

[Marsden et al. 2005] for potential fluid.

where

$$\widehat{\boldsymbol{\Omega}} := \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

- The control is $t \in [0, T] \mapsto \Theta_t$.
- The expressions of $\mathbb{M}(t)$ and $\mathbf{N}(t)$ require solving BVPs in $\mathcal{F}_t$.
- The control arises in the expressions of $\mathbb{I}(t)$, $\varphi_t \backslash \mathbf{u}_t$ and $\mathcal{F}_t$.

Main Controllability Result

Theorem (potential flow [T. Chambrion, A. M. 2010])

The following swimmer's data are given:

1. $\bar{\varrho} > 0$ *in* $C(\bar{B})$ *(the reference density) and* $\bar{\vartheta} \in W^{1,1}([0,T], D_0^1)$ *(the reference shape changes) such that the pair* $(\bar{\varrho}, \bar{\Theta})$ *be allowable;*
2. $(\bar{R}, \bar{\mathbf{r}}) \in C([0,T], \mathrm{SE}(3))$ *(the reference trajectory).*

Then, for any $\varepsilon > 0$*, there exists a function* $\varrho > 0$ *in* $C(\bar{B})$ *(the actual density of the swimmer) and a function* $\vartheta \in W^{1,1}([0,T], D_0^1)$ *such that the pair* (ϱ, Θ) *be allowable and*

- $\|\bar{\varrho} - \varrho\|_{C^0(\bar{B})} < \varepsilon$;
- $\sup_{t\in[0,T]} \left(\|\bar{\vartheta}_t - \vartheta_t\|_{C^1} + \|\bar{R}_t - R_t\|_{\mathrm{SO}(3)} + \|\bar{\mathbf{r}}_t - \mathbf{r}_t\|_{\mathbf{R}^3} \right) < \varepsilon$;

where $t \in [0,T] \mapsto (R_t, \mathbf{r}_t) \in \mathrm{SE}(3)$ *is the unique solution to System* (1) *with initial data* $(R_0, \mathbf{r}_0) = (\bar{R}_0, \bar{\mathbf{r}}_0)$ *and control* ϑ.

Main Controllability Result

Theorem (viscous case [J. Lohéac, A. M. 2011])

The following swimmer's data are given:

1. *An allowable shape function* $\bar{\vartheta} \in W^{1,1}([0,T], D_0^1)$ *(the reference shape changes);*
2. $(\bar{R}, \bar{\mathbf{r}}) \in C([0,T], \mathrm{SE}(3))$ *(the reference trajectory).*

Then, for any $\varepsilon > 0$*, there exists an allowable function* $\vartheta \in W^{1,1}([0,T], D_0^1)$ *such that*

$$\sup_{t\in[0,T]} \Big(\|\bar{\vartheta}_t - \vartheta_t\|_{C^1} + \|\bar{R}_t - R_t\|_{\mathrm{SO}(3)} + \|\bar{\mathbf{r}}_t - \mathbf{r}_t\|_{\mathbf{R}^3} \Big) < \varepsilon;$$

where $t \in [0,T] \mapsto (R_t, \mathbf{r}_t) \in \mathrm{SE}(3)$ *is the unique solution to System (1) with initial data* $(R_0, \mathbf{r}_0) = (\bar{R}_0, \bar{\mathbf{r}}_0)$ *and control* ϑ*.*

Remarks

- The regularity of the control ranges from $W^{1,1}([0,T],D^1_0)$ to analytic.
- One can probably choose $\varrho = \bar{\varrho}$ in the potential case;
- When no shape changes are prescribed, swimming is (generically) possible with a combination of any
  - 5 basic movements (potential case);
  - 4 basic movements (viscous case);

Taste of the proof

1. To identify a finite number of parameters characterizing a swimmer (and its way of swimming) : the swimmer's signature;
2. The set of all of the swimmer's signatures is an analytic connected infinite dimensional submanifold, embedded in a Banach space;
3. The property of being controllable is equivalent to the non vanishing of an analytic function in the signature;
4. To exhibit an example of controllable swimmer;
5. To conclude, invoking a classical property of analytic functions.

Some References

Swimming in a perfect fluid:

- Lamb (Hydrodynamics) 1879;
- V. V. Kozlov and D. A. Onishchenko (2003, Control);
- Kanso, Marsden, Rowley, Melli-Hubert 2005 (Dynamics of deformable bodies);
- A. M. 2008; A. M. 2009 (well posedness, modeling);
- A. M. and Pinçon 2010 (2D, numerics);
- Chambrion and A.M. 2010 (2D, control);

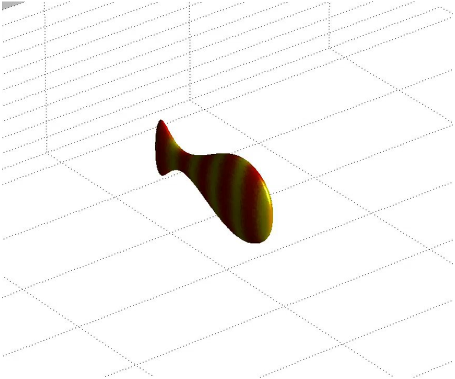
Swimming in a viscous fluid:

- G. Taylor, 1951;
- Purcell 1977;
- Shapere and Wilczek 1989;
- Alouges, Desimone, Lefebvre 2008, 2009, with Merlet 2010 (control);
- Loheac, Scheid, Tucsnak (optimal control) 2011;
- Kelly 2011 (Driftless models).

SOLEIL project (with B. pinçon and M. Fuentes)

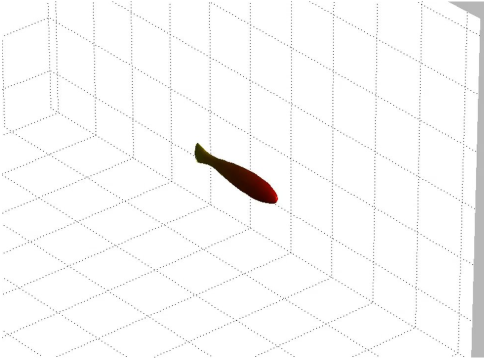
SOLEIL: (SOLveur d'Equations Integrales pour la Locomotion).

- The equations are restated in terms of integrals equations;
- Only the surface of the swimmer need to be meshed;
- Matlab interface.



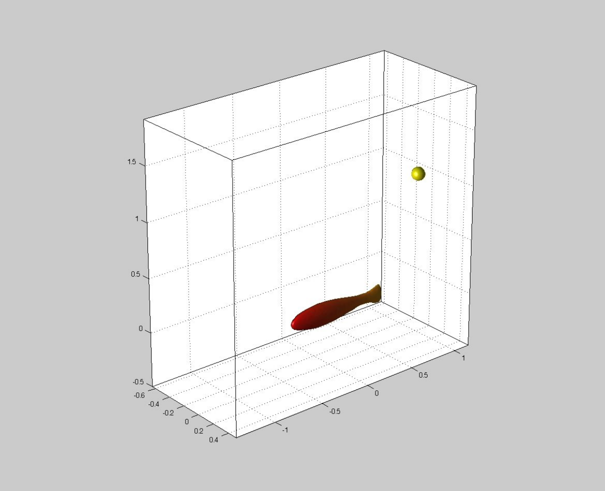
SOLEIL project

Matlab interface makes easy the modification of the fish's characteristics.



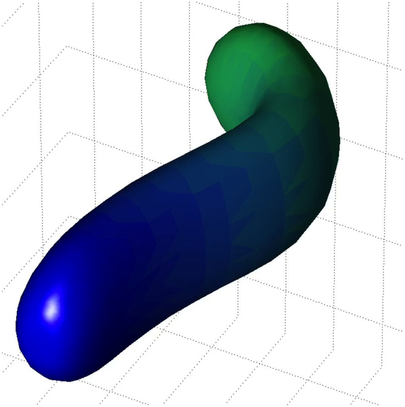
SOLEIL project

Closed loop control example.



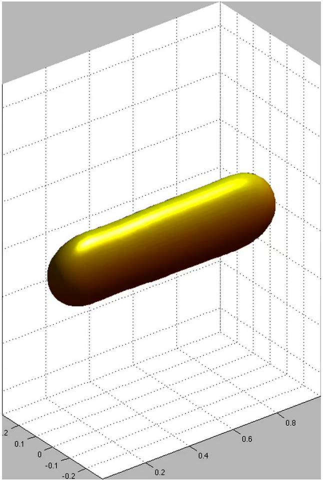
SOLEIL project

Shape optimization: Starting with a simple shape.



SOLEIL project

Use of the build-in function `fmincon` of the optimization toolbox.



SOLEIL project

Comparing both swimmers in a race.