Existence of a weak solution for a moving boundary fluid-structure interaction problem in blood flow

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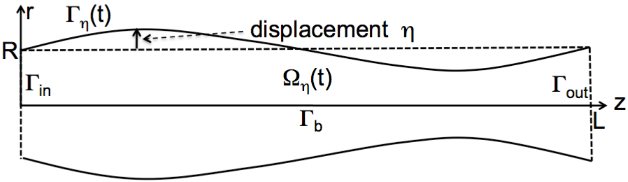
Description of the geometry



- η denote the vertical displacement of the deformable boundary.
- Fluid domain at time t:

$$\Omega_\eta(t) = \{(x, z) : 0 < z < L, 0 < r < R + \eta(t, z)\}.$$

- $\Gamma(t) = \{(z, R + \eta(t, z)) : 0 < z < L\}$ is deformable boundary.
- Longitudinal displacement is neglected.

Fluid and structure equations

- The fluid flow is governed by the incompressible Navier-Stokes equations:

$$\varrho_f(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = \nabla\cdot\sigma,\ \nabla\cdot\mathbf{u} = 0,\ \text{in } \Omega_\eta(t),\ t\in(0,T),$$

- ϱ_f is fluid density and $\sigma = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u})$ fluid stress tensor.
- The structure is modeled by a cylindrical linearly viscoelastic Koiter shell model:

$$\varrho_s h\partial_t^2\eta + C_0\eta - C_1\partial_x^2\eta + C_2\partial_x^4\eta +$$
$$+ D_0\partial_t\eta - D_1\partial_x^2\partial_t\eta + D_2\partial_x^4\partial_t\eta = f.$$

- It is physiologically reasonable structure model.
- In this presentation we present purely elastic case, i.e. $D_0 = D_1 = D_2 = 0$ (mathematically harder case).

Coupling conditions and boundary conditions

- The fluid and structure are coupled through the kinematic and dynamic coupling conditions, respectively:

$$\mathbf{u}(t,z,\eta(t,z)) = \partial_t \eta(t,z)\mathbf{e}_z,$$

$$f = -\sqrt{1+(\partial_z\eta)^2}\sigma\mathbf{n}\cdot\mathbf{e}_z,\ \text{on}\ (0,L),\ t\in(0,T).$$

- The flow is driven by a prescribed dynamic pressure drop at the inlet and outlet boundaries: $p+\frac{1}{2}|u|^2 = P_{in/out}(t),\ \mathbf{u}\times\mathbf{n}=0, \quad \text{on}\ \Gamma_{in/out}$.
- The structure is clamped: $\eta(0)=\partial_z\eta(0)=\eta(L)=\partial_z\eta(L)=0$.
- The system is supplemented with initial conditions: $\mathbf{u}(0,.)=\mathbf{u}_0,\ \eta(0,.)=\eta_0,\ \partial_t\eta(0,.)=v_0$.

Full FSI problem

Find $\mathbf{u} = (u_z(t,z,r), u_r(t,z,r)), p(t,z,r)$, and $\eta(t,z)$ such that

$$\left.\begin{array}{rcl} \rho_f\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) & = & \nabla\cdot\sigma \\ \nabla\cdot\mathbf{u} & = & 0 \end{array}\right\} \text{ in } \Omega_\eta(t),\ t\in(0,T),$$

$$\left.\begin{array}{rcl} \mathbf{u} & = & \partial_t\eta\mathbf{e}_r \\ \rho_s h\partial_t^2\eta + C_0\eta - C_1\partial_z^2\eta + C_2\partial_z^4\eta & = & -J\sigma\mathbf{n}\cdot\mathbf{e}_r \end{array}\right\} \text{ on } (0,T)\times(0,L),$$

$$\left.\begin{array}{rcl} u_r & = & 0, \\ \partial_r u_z & = & 0 \end{array}\right\} \text{ on } (0,T)\times\Gamma_b,$$

$$\left.\begin{array}{rcl} p + \frac{\rho_f}{2}|u|^2 & = & P_{in/out}(t), \\ u_r & = & 0, \end{array}\right\} \text{on } (0,T)\times\Gamma_{in/out},$$

$$\left.\begin{array}{rcl} \mathbf{u}(0,.) & = & \mathbf{u}_0, \\ \eta(0,.) & = & \eta_0, \\ \partial_t\eta(0,.) & = & v_0. \end{array}\right\} \text{at } t=0.$$

Description of the result

We prove existence of a weak solution of $2D - 1D$ FSI problem in blood flow.

- We use pressure inlet and outlet boundary conditions which introduce some technical difficulties.
- The most important novelty of this work is related to the method of proof. The proof is based on a semi-discrete, operator splitting Lie scheme, which was used by Guidoboni, Čanić et al. ('09) for a design of a stable, loosely coupled numerical scheme, called the kinematically coupled scheme. Therefore, in this work, we effectively prove that the kinematically coupled scheme converges to a weak solution of the underlying FSI problem.

History

Existence result for FSI problems in various settings - Conca, San Martín, Tucsnak ('99), Desjardins, Esteban ('99), San Martín, Starovoitov, Tucsnak ('02), Desjardins, Esteban, Grandmont, Le Tallec ('03), M. Boulakia ('03), Feireisl ('03), Takahashi ('03), Barbu, Grujić, Lasiecka, Tuffaha ('08), Houot, Munnier ('08), Galdi, Kyed ('09), Houot, San Martin, Tucsnak ('10), Guidoboni, Guidorzi, Padula ('12), ...

Existence results for moving boundary problem for Navier-Stokes coupled with elasticity:

- Strong solution: Beirão da Veiga ('04), Cheng and Shkoller ('10), Coutand and Shkoller. ('05, '06), Kukavica and Tuffaha ('12)
- Weak solutions : Chambolle, Desjardins, Esteban, and Grandmont ('05), Grandmont ('08)

Energy inequality

By formally taking solution $(\mathbf{u}, \partial_t \eta)$ as test function in the weak formulation we get following energy estimate:

$$\frac{d}{dt}E + D \leq C(P_{in/out}),$$

where

$$E = \frac{\varrho_f}{2}\|\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\varrho_s h}{2}\|\eta_t\|^2_{L^2(\Gamma)} + \frac{1}{2}\big(C_0\|\eta\|^2_{L^2} + C_1\|\partial_z\eta\|^2_{L^2} + C_2\|\partial_z^2\eta\|^2_{L^2}\big),$$

$$D = \mu\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}.$$

ALE formulation on reference domain

- We want to rewrite problem in the reference configuration $\Omega = (0, L) \times (0, 1)$.
- Since we consider control domain, we can not use Lagrangian coordinates.
- We use ALE mapping $A_\eta(t) : \Omega \to \Omega_\eta(t)$,

$$A_\eta(t)(\tilde{x}, \tilde{z}) = \begin{pmatrix} \tilde{x} \\ (1 + \eta(t, x))\tilde{z} \end{pmatrix}, \quad (\tilde{x}, \tilde{z}) \in \Omega.$$

- We have problem on a fixed domain, but with coefficients that depend on the solution.
- Test functions still depend on the solution (because of divergence-free condition).

Lie (Marchuk-Yanenko) operator splitting scheme

- We consider initial value problem $\frac{d}{dt}\phi + A(\phi) = 0$, $\phi(0) = \phi_0$.
- We suppose that $A = A_1 + A_2$.
- Let $\Delta T = T/N$ be time-dicretization step and $t_n = n\Delta t$. Then we define:
$$\frac{d}{dt}\phi_{n+\frac{i}{2}} + A_i(\phi_{n+\frac{i}{2}}) = 0 \quad \text{in } (t_n, t_{n+1}),$$
$$\phi_{n+\frac{i}{2}}(t_n) = \phi^{n+\frac{i-1}{2}},\ n = 0, \dots, N-1,\ i = 1, 2,$$
where $\phi^{n+\frac{i}{2}} = \phi_{n+\frac{i}{2}}(t_{n+1})$.
- To apply Lie scheme, we must rewrite original problem as first order problem. Therefore we introduce new unknown, structure velocity $v = \partial_t \eta$.

Semi-Discretization

- We use Semi-Discretization in time, i.e. we discretize only time variable t.
- Scheme is designed in such a way that we get semi-discrete energy inequalities (analogous to continuous case).
- That guarantees stability of the numerical scheme (which is not a case for classical loosely coupled schemes for FSI problems).
- In every subset scheme is implicit. It is crucial for stability!

Step 1 - Elastodynamics

Given $(\mathbf{u}^n, \eta^n, v^n)$ from previous step, find $(\mathbf{u}^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}})$ such that:

$$\begin{cases} \mathbf{u}^{n+\frac{1}{2}} = \mathbf{u}^n, \quad \text{in } \Omega \\ \dfrac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t} = v^{n+\frac{1}{2}} \quad \text{on } (0, L) \\ \rho_s h \dfrac{v^{n+\frac{1}{2}} - v^n}{\Delta t} + C_0 \eta^{n+\frac{1}{2}} - C_1 \partial_z^2 \eta^{n+\frac{1}{2}} + C_2 \partial_z^4 \eta^{n+\frac{1}{2}} = 0 \quad \text{on } (0, L), \\ \eta^{n+\frac{1}{2}}(0) = \partial_z \eta^{n+\frac{1}{2}}(0) = \eta^{n+\frac{1}{2}}(L) = \partial_z \eta^{n+\frac{1}{2}}(L) = 0. \end{cases} \tag{1}$$

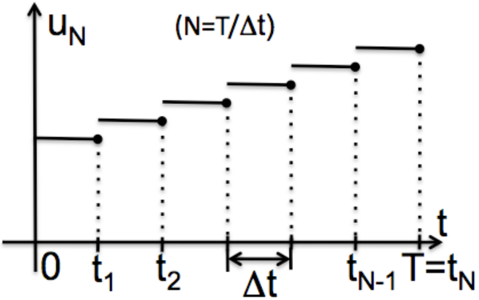
Step 2 - Fluid + Structure inertia

find $(\mathbf{u}^{n+1}, v^{n+1}, \eta^{n+1})$ such that:

$$\left\{\begin{array}{l}
\rho_f\Big(\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n+\frac{1}{2}}}{\Delta t}+((\mathbf{u}^n-\mathbf{w}^{\eta^{n+\frac{1}{2}}})\cdot\nabla^{\eta^n})\mathbf{u}^{n+1}\Big)=\nabla^{\eta^n}\cdot\sigma^{\eta^n}(\mathbf{u}^{n+1}),\\
\nabla^{\eta^n}\cdot\mathbf{u}^{n+1}=0 \quad \text{in } \Omega,\\
u_r^{n+1}=0,\ \partial_r u_z^{n+1}=0 \ \text{ on } \Gamma_b,\\
p^{n+1}+\frac{\rho_f}{2}|u^{n+1}|^2=P_{in/out}(t),\ u_r^{n+1}=0 \ \text{ on } \Gamma_{in/out},\\
\mathbf{u}^{n+1}=v^{n+1}\mathbf{e}_r \ \text{ on } (0,L),\\
\eta^{n+1}=\eta^{n+\frac{1}{2}} \quad \text{in } (0,L),\\
\rho_s h\dfrac{v^{n+1}-v^{n+\frac{1}{2}}}{\Delta t}=-J^{n+1}\sigma\mathbf{n}\cdot\mathbf{e}_r \ \text{ on } (0,L).
\end{array}\right. \tag{2}$$

Approximate solution

$$\mathbf{u}_N(t,.) = u_N^n,\ \eta_N(t,.) = \eta_N^n,\ v_N(t,.) = v_N^n,\ v_N^*(t,.) = v_N^{n-\frac{1}{2}},$$
$$t \in ((n-1)\Delta t, n\Delta t],\ n = 1 \dots N.$$



- Semi-discrete energy estimates imply that approximate solution are bounded in suitable norms uniformly in N.
- From weak compactness we get limit functions $\mathbf{u}$, v, v^* and η.
- $v = v^*$.

Passing to the limit

We can get following convergences (on a subsequence)

- $\eta_N \to \eta$ in $C([0,T]; H^s(0,L))$, $s < 2$,
- $\eta_N \rightharpoonup \eta$ weakly* in $L^\infty(0,T; H_0^2(0,L))$,
- $v_N \to v$ in $L^2((0,T) \times (0,L))$,
- $v_N \rightharpoonup v$ weakly* in $L^\infty(0,T; L^2(0,L)))$,
- $\mathbf{u}_N \to \mathbf{u}$ in $L^2((0,T) \times \Omega)$,
- $\mathbf{u}_N \rightharpoonup \mathbf{u}$ weakly in $L^2(0,T; H^1(\Omega))$.

To get strong convergence in L^2 we use:

$$\|\tau_{\Delta t}\mathbf{u}_N - \mathbf{u}_N\|^2_{L^2((0,T)\times\Omega)} + \|\tau_{\Delta t}v_N - v_N\|^2_{L^2((0,T)\times(0,L))} \leq C\Delta t.$$

This inequality follows from semi-discrete version of energy inequality.

Test function

- Test functions depends on N!
- Therefore one can not pass to the limit directly.
- We construct dense subset $\mathcal{X}$ of test functions on original domain such that every test function $\mathbf{q} \in \mathcal{X}$ can be transform to reference domain via ALE mapping A_{η_N}, for $q \geq N(\mathbf{q})$.
- We use fact that $\eta_N \to \eta$ in $C([0,T];C^1[0,L])$ and $\eta_N(x) \geq r_{min}$, $N \in \mathbb{N}$ (this can be ensured by taking T small enough).
- We have shown existence of local in time solution (in a sense that solution exists as long upper boundary does not touch lower boundary).

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Thank you for your attention!