Inverse problems for elliptic PDEs and Hardy classes of generalized analytic functions

Juliette Leblond

Sophia-Antipolis, France team APICS

From joint work with

L. Baratchart, Y. Fischer (INRIA Bordeaux), J.R. Partington (Univ. Leeds, UK)

Overview

- Planar conductivity PDE and (pseudo-) holomorphic functions

  (σ-harmonic, generalized analytic)

- Properties of associated Hardy spaces (review)

- Consequences for direct and inverse PDE problems

- Conclusion

Conductivity equation

Let $\Omega \subset \mathbb{R}^2$ with smooth boundary $\Gamma = \partial\Omega$ (Hölder smooth)

$\Omega \simeq$ disk $\mathbb{D}$, $\Gamma \simeq$ circle $\mathbb{T}$ or annulus: $\Omega \simeq \mathbb{A}$, $\Gamma \simeq \mathbb{T} \cup \varrho\mathbb{T}$

$\simeq$ conformally, simply or doubly connected (also with several holes) $0 < \varrho < 1$

Conductivity coefficient σ Lipschitz smooth function in Ω

σ known, $0 < c \leq \sigma \leq C$

Consider solutions u to (u):

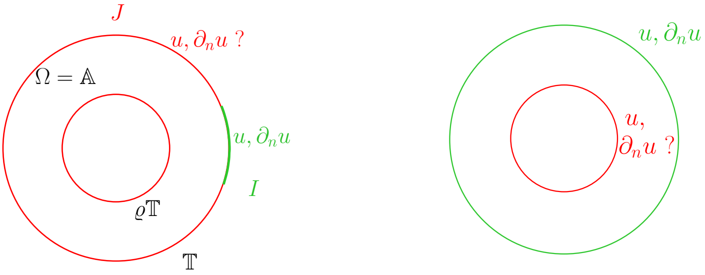
$$\boxed{\text{div}\,(\sigma\,\text{grad}\,u) = \nabla\,.\,(\sigma\,\nabla\,u) = 0} \text{ in } \Omega \qquad \text{(u)}$$

second order elliptic PDE, distributional sense $\nabla.(\sigma\nabla u) = \Delta u + \nabla(\log\sigma).\nabla u = 0$

$\sigma = 1$ or cst: $\Delta u = 0$, Laplace equation, u harmonic in Ω

Geometries

$\Omega = \mathbb{A}$, $I \subset \mathbb{T}$, $J = (\mathbb{T} \setminus I) \cup \varrho\mathbb{T}$ $\qquad$ $I = \mathbb{T}$, $J = \varrho\mathbb{T}$



or $\Omega = \mathbb{D}$, $I \subset \mathbb{T}$, $J = \mathbb{T} \setminus I$

or conformally equivalent domains (conformal invariance)

Boundary value problems

- Dirichlet (direct) problem:

Given measures of u on Γ, recover u in Ω (and $\sigma\partial_n u$, on Γ)

(n outer unit normal) well-posed for smooth Dirichlet data $\phi \rightsquigarrow \phi \in L^2_{\mathbb{R}}(\Gamma)$?

- Cauchy (inverse) problem: $|I|\,,\ |J| > 0$ partial overdetermined boundary data

Given measures u and $\sigma\,\partial_n u$ on $I \subset \Gamma$ of a solution u to (u),

(u): $\boxed{\nabla\,.\,(\sigma\,\nabla\,u) = 0}$ in Ω recover u, $\sigma\,\partial_n u$ on $J = \Gamma \setminus I$ (or $\partial_n u$)

1 pair of Dirichlet-Neumann data (ϕ_I, ψ_I) on I, $\phi_I \in L^2_{\mathbb{R}}(I)$, $\psi_I \in W^{-1,2}_{\mathbb{R}}(I)$? compatibility...

L^2 boundary data $\rightsquigarrow$ smooth conductivity σ, tradeoff

practically: pointwise corrupted boundary measurements

Other PDEs, other problems

(u): $\nabla \, . \, (\sigma \, \nabla \, u) = 0$ in $\Omega \subset \mathbb{R}^2$

Above boundary values problems also for

- Laplace: $\Delta u = 0$ from Cauchy data on $I \subset \Gamma$, recover $J = \Gamma \setminus I$ (unknown geometry), or Robin coefficient on J
- Schrödinger: $-\Delta w + q \, w = 0$ stationnary, for some potential q

Other related issues: unique continuation principles, density (Runge) properties, stability estimates, free boundary problems

σ-harmonic conjugation

Generalized Cauchy-Riemann equations: for $\Omega = \mathbb{D}$

u solution to (u): $\nabla . (\sigma \nabla u) = 0$ $\Leftrightarrow \exists v$ such that in Ω:

$$\begin{cases} \partial_x v = -\sigma \partial_y u \\ \partial_y v = \sigma \partial_x u \end{cases} \quad \text{and } \nabla . \left(\frac{1}{\sigma} \nabla v \right) = 0$$

Function v: σ-conjugated to u v unique up to additive constant

Want solution u to (u) and σ-conjugated v to have boundary values (traces) on Γ, whence:

$$\begin{cases} \partial_\theta v = \sigma \partial_n u \\ \partial_n v = -\sigma \partial_\theta u \end{cases} \quad \text{on } \Gamma$$

∂_θ tangential derivative for $\Omega = \mathbb{A}$: $\exists v$ if compatibility boundary condition

Generalized analytic functions

In $\Omega \simeq \mathbb{D} \subset \mathbb{R}^2 \simeq \mathbb{C}$ complex plane

$$X = (x, y) \simeq z = x + iy \,, \qquad \partial = \partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)\,,\ \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\,\partial_y)$$

u solution to (u): $\nabla\,.\,(\sigma\,\nabla\,u) = 0$ $\qquad \nabla \simeq \bar{\partial},\ \nabla\,. \simeq \mathrm{Re}\,\partial$

$\Leftrightarrow$ $f = u + i\,v$ satisfies Beltrami equation $\qquad$ conjugated B., pseudoholomorphic

$$\boxed{\bar{\partial} f = \nu \overline{\partial f}} \qquad \text{(f)}$$

$$\text{for } \nu = \frac{1-\sigma}{1+\sigma} \in W^{1,\infty}(\Omega)\,,\ |\nu| \leq \kappa < 1 \text{ in } \Omega$$

$$f \text{ solution to (f)} \iff u = \mathrm{Re}\, f \text{ solution to (u)}$$

(f) conformally invariant $\qquad$ ($\mathbb{R}$-linear, first order)

$\neq$ $\mathbb{C}$-linear Beltrami equation: $\bar{\partial} g = \nu \partial g$, quasi-conformal map. $\qquad$ $f(z, \bar{z})$, $u(x, y)$, $v(x, y)$

in $\Omega \simeq \mathbb{A}$, compatibility condition needed for $\Leftarrow$

Harmonic and analytic functions

Generalization of homogeneous situations $\sigma = \text{cst} \rightsquigarrow \sigma = 1,\ \nu = 0$

Holomorphic / complex analytic functions $\bar{\partial} F = 0$ in $\mathbb{D} \subset \mathbb{C}$:

$\Omega = \mathbb{D}$ unit disc or $\Omega \simeq \mathbb{D}$ conformally equivalent

$$X = (x, y) \simeq z = x + iy\,, \qquad \partial = \partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)\,,\ \ \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\,\partial_y)$$

Laplace operator $\Delta = 4\,\bar{\partial}\,\partial = 4\,\partial\,\bar{\partial} = \partial_x^2 + \partial_y^2$

$$F(z) = \sum_{k \geq 0} \hat{F}_k\, z^k = \sum_{k \geq 0} \hat{F}_k\, r^k\, e^{ik\theta}\,,\ \ z = r\, e^{i\theta} \in \mathbb{D}\,,\ \ r < 1$$

(Fourier series, coefficients $\hat{F}_k$) $\bar{\partial}\, F = 0$ (F holomorphic) $\Leftrightarrow F = u + iv$

with harmonic u and conjugate function v satisfying Cauchy-Riemann equations in $\mathbb{D}$:

$$\begin{cases} \partial_x v = -\partial_y u \\ \partial_y v = \partial_x u \end{cases}$$

Hardy spaces H^2 of analytic functions in $\mathbb{D}$

$$\sum_{k\geq 0} |\hat{F}_k|^2 = \|F\|_2^2 = \operatorname*{ess\,sup}_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty$$

$\Rightarrow F \in H^2(\mathbb{D})$: solutions to $\bar{\partial} F = 0$ in $\mathbb{D}$, $\|F\|_2 < \infty$

Hilbert space $\subset L^2(\mathbb{D})$ Parseval $p = 2$, also $\Omega = \mathbb{A}$ and Banach H^p

$\rightsquigarrow$ L^2 boundary values on $\mathbb{T}$: tr $H^2(\mathbb{D}) \subset L^2(\mathbb{T})$ closed (traces, non tg lim)

$\rightsquigarrow$ equivalent boundary $L^2(\mathbb{T})$ norm: $\|F\|_2 = \|\text{tr } F\|_{L^2(\mathbb{T})}$

$\rightsquigarrow$ Cauchy-Riemann equation in $\overline{\mathbb{D}}$, up to boundary $\mathbb{T}$:
tr $F = u + iv$ on $\mathbb{T}$, where $\partial_\theta v = \partial_n u$, $\partial_n v = -\partial_\theta u$

$\rightsquigarrow$ uniqueness and density results (stability)

also Poisson, Cauchy integral formulas, Hilbert-Riesz operator, $\perp$ decomposition + further properties

$\rightsquigarrow$ results for $\sigma = 1$, $\nu = 0$, Laplace equations (also in $\mathbb{A}$, in $\mathbb{R}^3$)

Generalized Hardy space H^2_ν

Hilbert space $H^2_\nu = H^2_\nu(\mathbb{D})$:

also $\Omega \simeq \mathbb{D}$ or $\simeq \mathbb{A}$ conformally; Banach spaces H^p_ν, $1 < p < \infty$

- solutions f to (f) $\boxed{\bar{\partial} f = \nu \overline{\partial f}}$ in $\mathbb{D}$
- bounded in Hardy norm in $\mathbb{D}$ $\|f\|_2 < \infty$

$$\|f\|_2^2 = \operatorname*{ess\,sup}_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty$$

(sup of L^2 norms on circles in $\mathbb{D}$)

H^2_ν shares many properties of $H^2 = H^2_0$

[Baratchart-L.-Rigat-Russ, 2010], [Fischer, 2011], [F.-L.-Partington-Sincich, 2011], [BFL, 2012]

Properties of H^2_ν

Generalize those of H^2 $\quad\Omega = \mathbb{D}$

Theorem [BLRR] $\quad f \in H^2_\nu(\mathbb{D}) \quad \bar\partial f = \nu \overline{\partial f}$, $\|f\|_2 < \infty$

- f admits a non tangential limit tr $f \in L^2(\mathbb{T})$ on $\mathbb{T}$ $\quad(\Gamma = \mathbb{T})$
- tr $f = 0$ a.e. on $I \subset \mathbb{T}$, $|I| > 0$ implies that $f \equiv 0$; if $f \not\equiv 0$, then its zeroes are isolated in $\mathbb{D}$ $\quad$ + Blaschke condition and $\log |\text{tr } f| \in L^1(\mathbb{T})$
- $\|\text{tr } f\|_{L^2(\mathbb{T})}$ is equivalent to $\|f\|_2$ on $H^2_\nu(\mathbb{D})$ $\quad$ Hardy norm
- Closedness of traces: $\quad$ tr $H^2_\nu(\mathbb{D})$ is closed in $L^2(\mathbb{T})$
- Re tr $f = 0$ a.e. on $\mathbb{T}$ implies that $f \equiv 0$ in $\mathbb{D}$ $\quad$ up to constant,

or whenever normalization on $\mathbb{T}$, $f \in H^{2,0}_\nu(\mathbb{D}) = \{f \in H^2_\nu(\mathbb{D})\,,\ \int_{\mathbb{T}} \text{Im tr } f = 0\}$

+ maximum principle in modulus

also in $\Omega = \mathbb{A}$, $\Gamma = \mathbb{T} \cup \varrho\mathbb{T}$ [BFL,F], and in H^p_ν

Properties of tr $H^2_\nu(\mathbb{D})$

Corollary [BLRR] — Dirichlet in $H^2_\nu(\mathbb{D})$, density

- $\forall \phi \in L^2_{\mathbb{R}}(\mathbb{T})$, $\exists !\ f \in H^{2,0}_\nu(\mathbb{D})$ such that $\mathrm{Re\,tr}\ f = \phi$

  moreover, $\|\mathrm{tr}\ f\|_{L^2(\mathbb{T})} \leq c_\nu \|\phi\|_{L^2(\mathbb{T})}$

- conjugation operator $\mathcal{H}_\nu$ bounded on $L^2_{\mathbb{R}}(\mathbb{T})$ — Hilbert-Riesz transform, $L^2(\mathbb{T})$

$$\mathrm{tr}\ u = \mathrm{Re\,tr}\ f = \phi \overset{\mathcal{H}_\nu}{\longmapsto} \mathrm{tr}\ v = \mathrm{Im\,tr}\ f = \mathcal{H}_\nu \phi$$

$$f \in H^{2,0}_\nu(\mathbb{D}) \Longleftrightarrow \mathrm{tr}\ f = (I + i\mathcal{H}_\nu)\phi,\ \phi \in L^2_{\mathbb{R}}(\mathbb{T})$$

- density (Runge): let $I \subset \mathbb{T}$, $J = \mathbb{T} \setminus I$ such that $|J| > 0$

then, restrictions to I of functions in tr $H^2_\nu(\mathbb{D})$ dense in $L^2(I)$

however, if $g_n \in \mathrm{tr}\ H^2_\nu$ converges to tr f in $L^2(I)$ while $\mathrm{tr}\ f \notin \mathrm{tr}\ H^2_\nu{}_{|_I}$, then $\|g_n\|_{L^2(J)} \to \infty$ — unstable

also $H^2_\nu(\mathbb{D})_{|_{\varrho\mathbb{D}}}$ dense in $H^2_\nu(\varrho\mathbb{D})$ — also in H^p_ν, $1 < p < \infty$

Best constrained approximation in H^2_ν

Regularization: bounded extremal problems (BEP)

Let $I \subset \Gamma$, $|I|$, $|J| > 0$, $\varepsilon > 0$ $\qquad \Omega = \mathbb{D}$, $\Gamma = \mathbb{T}$, $J = \mathbb{T} \setminus I$

$$\mathcal{B} = \left\{ f \in \text{tr } H^2_\nu \, , \ \|\text{Re } f\|_{L^2(J)} \le \varepsilon \right\} |_I \subset L^2(I) \, .$$

Theorem [BFL, FLPS] $\qquad$ (BEP) well-posed $\qquad$ $\nu = 0$: [BLP]

$\forall$ function $\Phi \in L^2(I)$, $\exists$ unique $f_* \in \mathcal{B}$ such that

$$\|\Phi - f_*\|_{L^2(I)} = \min_{f \in \mathcal{B}} \|\Phi - f\|_{L^2(I)}$$

Moreover, if $\Phi \notin \mathcal{B}$, then $\|\text{Re } f_*\|_{L^2(J)} = \varepsilon$

Proof: bounded conjugation, density result $\qquad$ also in $\Omega \simeq \mathbb{A}$, with $I \subset \mathbb{T}$, $J = (\mathbb{T} \setminus I) \cup \varrho\mathbb{T}$

also in H^p_ν, for $L^p(I)$ data, or with other norm constraints

Constructive issues in H^2_ν

Computation algorithm, from $\Phi \in L^2(I)$ $\Omega = \mathbb{D}, \mathbb{A}$ ($I \subseteq \mathbb{T}$) [AP,BFL,FLPS]

$\perp$ projection operator $L^2(\Gamma) \to \mathrm{tr}\ H^{2,0}_\nu$: $$P_\nu \phi = \frac{1}{2}(\phi + i\mathcal{H}_\nu \phi)$$

vanishing mean on $\mathbb{T}$

Solution to (BEP): given $\Phi \in L^2(I)$, $M > 0$ Toeplitz-Hankel operators on H^2_ν

$$P_\nu(\chi_I f_*) - \gamma P_\nu(\chi_J \mathrm{Re} f_*) = P_\nu(\Phi \vee 0)$$

for ! Lagrange parameter $\gamma < 0$ s.t. $\|f_*\|_{L^2(J)} = M$

$\min_{f \in \mathrm{tr}\ H^2_\nu} \|\Phi - f\|_{L^2(I)} + \gamma \|\mathrm{Re}\, f\|_{L^2(J)}$ γ % M smoothly decreasing

Complete families of solutions, for computations in $H^2_\nu(\Omega)$ and $L^2(\Gamma)$

$\rightsquigarrow$ Bessel/exponentials, toroidal harmonics [F] (w.r.t. σ or ν, and Ω) polynomials?

$\nu = 0$: Fourier basis, polynomials

For related conductivity PDE

u solution to (u) in Ω: $\quad \Omega \simeq \mathbb{D}$ or $\mathbb{A}$

$\nabla . (\sigma \nabla u) = 0 \Leftarrow u = \operatorname{Re} f$ with f solution to (f) in Ω $\quad \Leftrightarrow$ if $\Omega \simeq \mathbb{D}$

- Dirichlet boundary value problems:

from prescribed boundary data $\phi \in L^2_{\mathbb{R}}(\Gamma)$,

recover u in Ω solution to (u) such that tr $u = \phi$ on Γ

From Dirichlet theorem in $H^{2,0}_\nu(\Omega)$: $\quad \Omega \simeq \mathbb{A}$: + hyp. on ϕ

$\exists!$ u in $L^2_{\mathbb{R}}(\Omega)$ solution to (u) such that tr $u = \phi$

$$\mathrm{tr}\ f = \phi + i \int_\Gamma \sigma \partial_n u = \phi + i \mathcal{H}_\nu \phi\,,\ \|u\|_2 = \|\mathrm{tr}\ u\|_{L^2(\Gamma)} = \|\phi\|_{L^2(\Gamma)}$$

- Unique continuation properties $\quad$ isolated critical points of u in Ω

Bounded conjugation operator $\rightsquigarrow$ stability properties for (u)... Dirichlet-Neumann map: $\Lambda \phi = \partial_\theta\, \mathcal{H}_\nu \phi$

For related conductivity PDE

- Cauchy inverse problems, $I \subset \mathbb{T}$ $\Omega = \mathbb{D}$ or $\mathbb{A}$

Given ϕ_I and ψ_I in $L^2_{\mathbb{R}}(I)$, from Cauchy data, $J = \partial\Omega \setminus I$

recover u solution to (u) in Ω such that tr $u = \phi_I$, $\sigma \partial_n u = \psi_I$ on I

Let $\Phi = \phi_I + i \int_I \psi_I \in L^2(I) \setminus (\text{tr } H^2_\nu)_{|_I}$; from density results:

$\exists u_k = \text{Re } f_k$ solution to (u) in Ω $\|\text{tr } u_k - \phi_I\|_{L^2(I)} \to 0$

$\|\partial_\theta \mathcal{H}_\nu u_k - \psi_I\|_{L^2(I)} \to 0$ but $\|\text{tr } u_k\|_{L^2(J)} \to \infty$

$\rightsquigarrow$ ill-posed for non compatible boundary data ϕ_I, ψ_I on I

$\rightsquigarrow$ look for tr $u \simeq \phi_I$, $\sigma \partial_n u \simeq \psi_I$ on I with tr u bounded on J...

$\rightsquigarrow$ solve bounded extremal problems (BEP) for Φ in tr H^2_ν

interpolation (ill-posed) $\rightsquigarrow$ best constrained approximation, regularization (Lavrentiev, Tikhonov)

Plasma equilibrium model in a tokamak

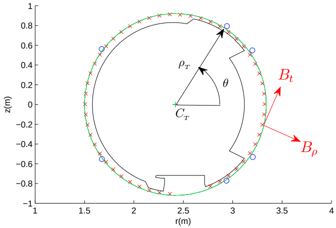
In 2D poloidal sections, poloidal magnetic flux u:

$$\nabla \,.\, \left(\frac{1}{x}\nabla u\right) = \nabla \,.\, (\sigma \nabla u) = 0 \text{ in the vacuum } \Omega \,, \text{ conductivity } \sigma = \frac{1}{x}$$

Maxwell equations, axisymmetric assumption, cylindrical coordinates $(x, y) = (R, Z)$ $(\varphi = \text{cte})$

$\Omega \simeq \mathbb{A}_0 \subset \mathbb{R}^2$ annular domain between plasma and chamber

$\Gamma = \Gamma_e \cup \Gamma_p$ limitor $\Gamma_l \subset \Omega$ inside plasma, Grad-Shafranov equation, control



From pointwise magnetic data
on outer boundary Γ_e (poloidal magnetic field)

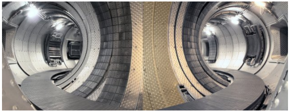
$$u \,,\; B_\rho = -\frac{1}{x}\partial_t u \,,\; B_t = \frac{1}{x}\partial_n u$$

recover plasma boundary Γ_p

outermost level line tangent to Γ_l

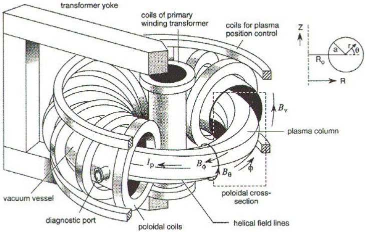
free boundary problem

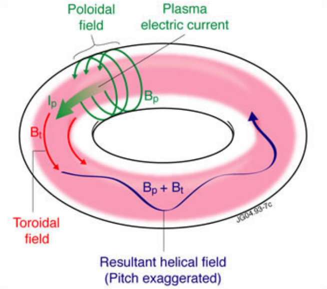
Application to plasma shaping in a tokamak



Tore Supra
(CEA-IRFM Cadarache)

magnetic field B, flux u





Plasma in tokamak

From measurements of u, $\sigma \partial_n u$ on outer boundary Γ_e, find level line Γ_p of associated solution u to (u), tangent to limitor Γ_l

Take a first such $\Gamma_{p,0}$ expand u on Γ_e (toroidal harmonics), compute $\max u = c_0$ on Γ_l

Data transmission $\Gamma_e \rightsquigarrow \Gamma_{p,0}$:

u, $\sigma \partial_n u$ on $I = \Gamma_e \rightsquigarrow u$, $\sigma \partial_n u$ on $J_0 = \Gamma_{p,0}$, u in Ω_0

Cauchy boundary inverse problem in Ω_0 solve (BEP)

$u\,,\ B_\rho \rightsquigarrow \phi_I,\ B_t = \partial_\theta v = \sigma \partial_n u \rightsquigarrow \psi_I \rightsquigarrow$ Cauchy data Φ on Γ_e

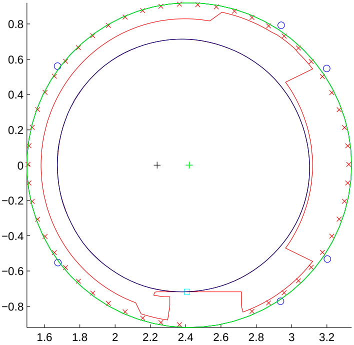
constraint $\|\mathrm{Re}\, f_* - c_0\|_{L^2(J_0)} \leq M$ small, c constant

Free boundary problem Γ_p: u, $\partial_n u$ on $\Gamma_l \rightsquigarrow \Gamma_{p,1} : \{u = \max_{\Gamma_l} u = c_1\}$

iterate 1st step $\rightsquigarrow \Gamma_p$, last closed level line tangent to Γ_l with shape optimization [BFP]

Plasma boundary recovery

Poloidal section of tokamak Tore Supra



Reconstruction of plasma boundary Γ_p from measurements ○ of poloidal flux u and × of poloidal magnetic field on Γ_e

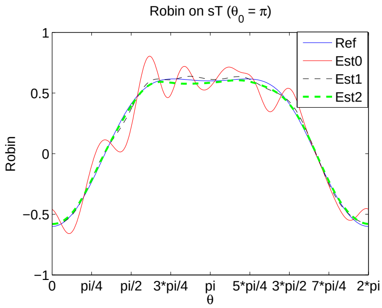
with series of toroidal harmonics (18 terms)

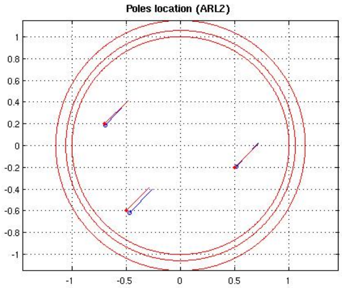
Other applications

Inverse problems for Laplace equations $\Delta u = 0$ in planar domains, from given Cauchy data on $I \subset \Gamma = \partial\Omega$ $\qquad \Omega = \mathbb{A}$

Robin coefficient recovery
on inner boundary J of $\mathbb{A}$
($I = \mathbb{T}$ outer boundary)

data transmission
on circles
(for sources localisation)



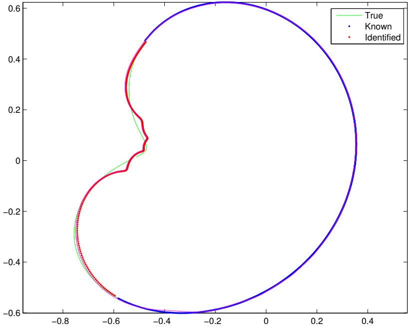


$\partial_n u + R\, u = 0$ on J

Other applications

Geometry recovery, $J \subset \Gamma$ (using conformal maps)

Given Cauchy data on $I \subset \Gamma$, recover J such that $\partial_n u = 0$ on J



Conclusion

Work in progress: ... or to be done?

- More about generalized Hardy classes H^p_ν

  factorization, operators, density of traces for $\Omega = \mathbb{A}$ reproducing kernel in H^2_ν? H^p_ν, $p = 1, \infty$?

- extremal problems minimize also w.r.t. c in constraint $\|\mathrm{Re}\, f - c\|_{L^2(J)}$
- solutions $w = e^s F$ to related $\bar{\partial} w = \alpha \bar{w}$ $\alpha = \bar{\partial} \log \sigma^{1/2}$

- Other elliptic operators (and relat-ed/-ing PDEs) + time t?

  Schrödinger $\Delta w \simeq |\alpha|^2 w + (\partial \alpha) \overline{w}$ 3D Laplace + symmetry properties $\rightsquigarrow$ 2D conductivity (u)

- Unique continuation principles for (u) and (w)

  stability, energy estimates Runge density properties $\rightsquigarrow$ EIT issues?

- Non smooth boundary geometry Γ other tokamaks, Jet, ITER: X point

  with geometrical issues: Bernoulli type (free boundary) problems

- Other classes of conductivities σ (or coefficients ν, α)?

  non smooth? (up to now, $\mathbb{R}$-valued Hölder smooth σ, $r > 2$, in $H^p_\nu(\Omega)$, $p > r/(r-1)$)

  anisotropic (matrix-valued)? in $\mathbb{R}^3$?

Main references

[AP] Astala, Päivärinta, Calderón inverse conductivity problem (2006)

[BLF] Baratchart, Fischer, Leblond, Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation (subm.)

[BLP] Baratchart, Leblond, Partington, Hardy approximation on subsets of the circle (1996-2000)

[BLRR] Baratchart, Leblond, Rigat, Russ, Hardy spaces of the conjugate Beltrami equation (2010)

[BN] Bers, Nirenberg, On a representation theorem for linear elliptic systems with discontinuous coeff. and appl. (1954)

[FLPS] Fischer, Leblond, Partington, Sincich, BEP in Hardy spaces for the conjugate Beltrami equation in simply conn. dom. (2011)

+ Vekua (1962), Kohn-Vogelius (1984), Sylvester-Uhlmann (1987), ...

... Astala-Iwaniec-Martin (2008), Kravchenko (2009), ...