Inverse problem for a parabolic equation with periodic conditions

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Juin/2012

Outline of the talk

Introduction

- **Direct problem** : the coefficients are known.
- $\rightarrow$ we determine solution.
- **Inverse problem** : the solution is known (partially).
- $\rightarrow$ we determine one of several coefficients.
- In general the inverse problems are ill posed in the sense of Hadamard :
  1. The solution can "not exist".
  2. The solution can "not be unique".
  3. The solution can "not be stable".

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We prove an inequality of stability by using a Carleman estimate for the following parabolic periodic problem :

$$\begin{cases} \partial_t u = \Delta u + \mu(x)u - \nu(x)u^2, & 0 < t < T, x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ u\big|_{\Gamma_j^0} = u\big|_{\Gamma_j^1}, \quad \frac{\partial u}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial u}{\partial x_j}\Big|_{\Gamma_j^1}, & 1 \le j \le n \end{cases} \tag{1}$$

- $\mu(x)$, $\nu(x)$ and $u_0(x)$ are L-periodic functions ($L = (L_j)_j$).
- $\mu(x)$ unknown, $\nu(x)$ known.
- where $\Omega = \Pi_{j=1}^{n}(0, L_j)$, $Q = (0,T) \times \Omega$, $Q_0 = (t_0, T) \times \Omega$
  $\Sigma = (0,T) \times \partial\Omega$,
  $\partial\Omega$ is the boundary of Ω,
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In biology :

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Objectives

Our main goal is to reconstruct the coefficient $\mu(x)$ from partial measurements of the solution u :

- We improve a Carleman inequality established by J.Choi [1] for a linear problem, by eliminating the constraints of the set of observation.
- We apply this inequality to a nonlinear problem.
- To end we prove the stability of the potential $\mu(x)$ (growth rate) with regard to the observations by using this Carleman inequality.

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Objectives

- Our goal is to prove the following theorem :

Theorem

Let $\omega \subset \Omega$ an open set, $\theta \in (0, T)$ fixed, then there exists a constant $C > 0$, $C = C(\omega, \Omega, T, \theta)$, such that :

$$||\mu - \tilde{\mu}||_{L^2(\Omega)} \leq C(||u - \tilde{u}||_{H^1(0,T;L^2(\omega))} + ||(u - \tilde{u})(\theta, .)||_{H^2(\Omega)})$$

Existence and uniqueness

Nonlinear case :

Theorem of existence and uniqueness (Berestycki-Hamel-Roques 2005)

We assume that $\mu, \nu \in \mathcal{C}^m(\mathbb{R}^n)$, $m \in (0, 1)$, $\mu(x)$, $\nu(x)$ et $u_0(x)$ are periodic functions, $\nu(x) > 0$. Then the problem (1) have an unique solution in $\mathcal{C}^{1+\frac{m}{2}, 2+m}((0, \infty) \times \mathbb{R}^n) \cap \mathcal{C}([0, \infty) \times \mathbb{R}^n)$.

Lemma (Carleman inequality)

Let $\omega \subset \Omega$, $\omega \neq \Omega$, there exists s_0, there exists constant $C > 0$, $C = C(\omega, \Omega, \theta, T)$, such that $\forall s \geq s_0$ the solution of the problem $Lz = g$ with periodic conditions boundary verifies the following inequality :

$$\int_{Q_{t_0}} \left(\frac{1}{s\varphi} \left(\left| \frac{\partial z}{\partial t} \right|^2 + |\Delta z|^2 \right) + s\varphi |\nabla z|^2 + s^3 \varphi^3 |z|^2 \right) e^{2s\alpha} dx\, dt$$

$$\leq C(\int_{Q_{t_0}} |g|^2 e^{2s\alpha} dxdt + \int_{]t_0, T[\times\omega} s^3 \varphi^3 |z|^2 e^{2s\alpha} dxdt)$$

with

$$Lz = \frac{\partial z}{\partial t} - \Delta z = g(t, x) \quad \text{in} \quad Q$$

Lemma (Weight function)

Let $\omega \subset \overline{\Omega}$, $\omega \neq \Omega$. Then there exists a periodic function $\psi \in \mathcal{C}^2(\overline{\Omega})$, such that :

$$\psi(x) > 0, \forall x \in \Omega, \quad et \quad \nabla\psi(x) \neq 0, \quad si \quad x \notin \omega$$

$$\psi\big|_{\Gamma_j^0} = \psi\big|_{\Gamma_j^1}, \quad \frac{\partial\psi}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial\psi}{\partial x_j}\Big|_{\Gamma_j^1}, \quad pour \quad 1 \leq j \leq n.$$

$$\varphi(t,x) = \frac{e^{\lambda\psi(x)}}{(t-t_0)(T-t)},$$

and

$$\alpha(t,x) = \frac{e^{\lambda\psi} - e^{2\lambda||\psi||_{C(\overline{\Omega})}}}{(t-t_0)(T-t)} < 0.$$

Remark

We use this periodic weight function to eliminate the boundary terms. Because of the boundary terms, Choi had to :

1. consider two weight functions [2].
2. assume that the open set ω (on which the gradient of the weight function can vanish) contains corners (in $\mathbb{R}^2$).

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Inverse problem

We have :

$$\begin{cases} \partial_t u(t,x) & = & \Delta u(t,x) - p(x,u)u(t,x) & t_0 < t < T, x \in \Omega \\ u(t_0,x) & = & u_0(x) & \\ u\big|_{\Gamma_j^0} & = & u\big|_{\Gamma_j^1}, \quad \frac{\partial u}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial u}{\partial x_j}\Big|_{\Gamma_j^1}, & 1 \le j \le n \end{cases} \qquad (P_{\mu,u_0})$$

with $p(x,u) = \mu(x) - \nu(x)u$.

Inverse problem

and

$$\left\{\begin{array}{lcl} \partial_t \tilde{u}(t,x) & = & \Delta \tilde{u}(t,x) - \tilde{p}(x,u)\tilde{u}(t,x) \qquad t_0 < t < T, x \in \Omega \\ \tilde{u}(t_0,x) & = & \tilde{u}_0(x) \\ \tilde{u}\big|_{\Gamma_j^0} & = & \tilde{u}\big|_{\Gamma_j^1}, \quad \frac{\partial \tilde{u}}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial \tilde{u}}{\partial x_j}\Big|_{\Gamma_j^1}, \qquad 1 \leq j \leq n \end{array}\right. \qquad (P_{\tilde{\mu},\tilde{u}_0})$$

with $\tilde{p}(x,\tilde{u}) = \tilde{\mu}(x) - \nu(x)\tilde{u}$.

Inverse problem

We formulate the inverse problem. Let u, (resp. $\tilde{u}$) solution of the problem (P_{μ,u_0}), (resp. $P_{\tilde{\mu},\tilde{u}_0}$).
Let $w = u - \tilde{u}$, $f = \mu - \tilde{\mu}$, $a(.) = u(\theta,.) - \tilde{u}(\theta,.)$
We obtain :

$$\begin{cases} w_t &= \Delta w + \mu w + \tilde{u} f - \nu w(u+\tilde{u}), & 0 \leq t \leq T, x \in \Omega \\ w(0,x) &= u_0(x) - \tilde{u}_0(x), & x \in \Omega \\ w\big|_{\Gamma_j^0} &= w\big|_{\Gamma_j^1}, \quad \frac{\partial w}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial w}{\partial x_j}\Big|_{\Gamma_j^1}, & 0 \leq t \leq T \end{cases}$$

Inverse problem

We set : $z(t,x) = w(t,x)/\tilde{u}(t,x)$

$$\begin{cases} z_t = \Delta z + P_1(z) + f, & t_0 \le t \le T, x \in \Omega \\ z(\theta, x) = a(x)/\tilde{u}(\theta, x), & x \in \Omega \\ z\big|_{\Gamma_j^0} = z\big|_{\Gamma_j^1}, \quad \frac{\partial z}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial z}{\partial x_j}\Big|_{\Gamma_j^1}, & t_0 \le t \le T \end{cases}$$

P_1 : first order operator.

$$P_1(z) = a_1(t,x)z + A_1(t,x)\nabla z$$

Inverse problem

We set $y = z_t$

$$\begin{cases} y_t = \Delta y + T_1(y) + Q_1(z) & t_0 \leq t \leq T, x \in \Omega, \\ y\big|_{\Gamma_j^0} = y\big|_{\Gamma_j^1}, \quad \frac{\partial y}{\partial x_j}\Big|_{\Gamma_j^0} = \frac{\partial y}{\partial x_j}\Big|_{\Gamma_j^1}, & t_0 \leq t \leq T \end{cases}$$

T_1, Q_1 : first order operator.

Inverse problem

By applying Carleman inequality to y, we obtain :

$$\int_{Q_{t_0}} \left(\frac{1}{s\varphi} \left(\left| \frac{\partial y}{\partial t} \right|^2 + |\Delta y|^2 \right) + s\varphi |\nabla y|^2 + s^3 \varphi^3 |y|^2 \right) e^{2s\alpha} dx\, dt$$

$$\leq C(\int_{Q_{t_0}} ((Q_1(z) + T_1(y))^2 e^{2s\alpha} dxdt + \int_{]t_0,T[\times\omega} s^3 \varphi^3 |y|^2 e^{2s\alpha} dxdt)$$

Inverse problem

To summarise, from :

$$f = \mu - \tilde{\mu},$$

$$f = z_t - \Delta z - P_1(z)$$

and

$$y = z_t$$

The Carleman estimate gives the following inequality of stability :

$$||\mu - \tilde{\mu}||_{L^2(\Omega)} \leq C(||u - \tilde{u}||_{H^1(0,T;L^2(\omega))} + ||(u - \tilde{u})(\theta, .)||_{H^2(\Omega)})$$

Work in progress

1. We study the same problem, but with non-smooth coefficient $\mu(x)$.
2. We use another idea for the reconstruction of the potential $\mu(x)$, we work in $\mathbb{R}^n$ with the same problem but we use differents data.

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Thank you for your attention.