Optimal Dirichlet boundary control for the Navier–Stokes equations

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Control of Fluid–Structure Systems and Inverse Problems
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Outline

Optimal Dirichlet boundary control

- Optimality system
- Discretization and stabilized FEM
- Numerical examples

Application to arterial blood flow

- Numerical examples

Conclusion and Outlook

Optimal control problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded Lipschitz domain, with boundary $\Gamma = \partial\Omega$.

- $\overline{\Gamma} = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}}$
- $\overline{\Gamma}_{\mathrm{D}} \cap \overline{\Gamma}_{\mathrm{c}} \neq \emptyset$ and $\overline{\Gamma}_{\mathrm{N}} \cap \overline{\Gamma}_{\mathrm{c}} = \emptyset$

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Problem

$$\min \; \mathcal{J}(\underline{u}, \underline{z}) = \frac{1}{2} \|\underline{u} - \overline{\underline{u}}\|^2_{L_2(\Omega)} + \frac{1}{2} \varrho \, |\underline{z}|^2_{H^{1/2}(\Gamma_\mathrm{c})}$$

under the constraint

$$\begin{aligned} -\nu \Delta \underline{u} + (\underline{u} \cdot \nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla \cdot \underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla \underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}, \end{aligned}$$

viscosity constant $\nu > 0$, cost coefficient $\varrho > 0$.

Realization of the $H^{1/2}$ semi–norm

Auxiliary problem

$$\begin{aligned} -\Delta \underline{u}_z &= \underline{0} && \text{in } \Omega, \\ \underline{u}_z &= \underline{0} && \text{on } \Gamma_{\mathrm{D}}, \\ (\nabla \underline{u}_z)\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}}, \\ \underline{u}_z &= \underline{z} && \text{on } \Gamma_{\mathrm{c}}. \end{aligned}$$

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Steklov–Poincaré operator

$$S : [\widetilde{H}^{1/2}(\Gamma_{\mathrm{c}})]^n \to [H^{-1/2}(\Gamma_{\mathrm{c}})]^n$$

with

$$[\widetilde{H}^{1/2}(\Gamma_{\mathrm{c}})]^n := \left\{ \underline{v} = \underline{\tilde{v}}|_{\Gamma_{\mathrm{c}}} : \underline{\tilde{v}} \in [H^{1/2}(\Gamma)]^n,\ \mathrm{supp}(\underline{\tilde{v}}) \subseteq \overline{\Gamma}_{\mathrm{c}} \right\}$$

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satisfying

$$|\underline{z}|^2_{H^{1/2}(\Gamma_\mathrm{c})} = \langle S\underline{z}, \underline{z} \rangle_{\Gamma_\mathrm{c}} = \langle (\nabla \underline{u}_z)\underline{n}, \underline{z} \rangle_{\Gamma_\mathrm{c}}.$$

for all $\underline{z} \in [H^{1/2}(\Gamma_\mathrm{c})]^n$. [Of, Than, Steinbach, 2009]

Optimality system

Primal problem

$$\begin{aligned} -\nu\Delta \underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_{\mathrm{D}}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_{\mathrm{c}}, \end{aligned}$$

Adjoint problem

$$\begin{aligned} -\nu\Delta \underline{w} - (\nabla\underline{w})\underline{u} - (\nabla\underline{w})^{\top}\underline{u} - \nabla r &= \underline{u} - \underline{\overline{u}} && \text{in } \Omega, \\ \nabla\cdot\underline{w} &= 0 && \text{in } \Omega, \\ \underline{w} &= \underline{0} && \text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}}, \\ \nu(\nabla\underline{w})\underline{n} + (\underline{u}\cdot\underline{w})\underline{n} + (\underline{u}\cdot\underline{n})\underline{w} + r\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}}, \end{aligned}$$

Optimality condition

$$-\nu(\nabla\underline{w})\underline{n} - (\underline{u}\cdot\underline{w})\underline{n} - (\underline{u}\cdot\underline{n})\underline{w} - r\underline{n} + \varrho S\underline{z} = \underline{0} \qquad \text{on } \Gamma_{\mathrm{c}}.$$

Discretization and stabilized FEM

Finite dimensional subspaces

$$V_h \subset [H_0^1(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n, \qquad Q_h \subset L_2(\Omega).$$

Discretization and stabilized FEM

Finite dimensional subspaces

$$V_h \subset [H^1_0(\Omega, \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}})]^n, \qquad Q_h \subset L_2(\Omega).$$

- $\mathcal{P}_1$–$\mathcal{P}_1$ element with Dohrmann–Bochev stabilization

$$c(q_h, p_h) := \frac{1}{\nu} \int_\Omega (p_h - \Pi_h p_h)(q_h - \Pi_h q_h)\, dx$$

with $L_2(\Omega)$–projection $\Pi_h : L_2(\Omega) \to Q_h^0$.

[Dohrmann, Bochev 2004]

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[Dohrmann, Bochev 2004]

Realization of the Steklov–Poincaré operator

Linear system for auxiliary problem

$$\begin{pmatrix} A_{II} & A_{IC} \\ A_{CI} & A_{CC} \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{z}_C \end{pmatrix} = \begin{pmatrix} \underline{0} \\ S_h \underline{z}_C \end{pmatrix}$$

For the Galerkin matrix S_h:

$$\Rightarrow S_h \underline{z}_C = (A_{CC} - A_{CI} A_{II}^{-1} A_{IC}) \underline{z}_C$$

Numerical example

- $\Omega = (0,1)^2$ with $\Gamma = \Gamma_c$
- $\nu = 1,\ \varrho = 1$
- $\underline{\overline{u}} = (x_2(x_2-1)+1, x_1(x_1-1)+1)^\top$
- $\underline{f} = \underline{1}$

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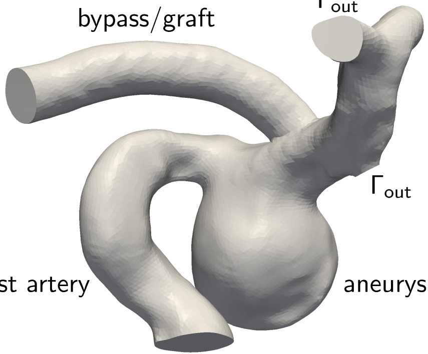
	$L_2(\Gamma)$ control		$H^{1/2}(\Gamma)$ control	
L	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{L_2(\Gamma)}$	eoc	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{L_2(\Gamma)}$	eoc
0	$3.55102\ e-01$	–	$9.64011\ e-03$	–
1	$2.43439\ e-01$	0.54	$3.47652\ e-03$	1.47
2	$1.70947\ e-01$	0.51	$1.72964\ e-03$	1.01
3	$1.19022\ e-01$	0.52	$5.30675\ e-04$	1.70
4	$8.29968\ e-02$	0.52	$1.77762\ e-04$	1.58
5	$5.75723\ e-02$	0.53	$7.22122\ e-05$	1.30
6	$3.92801\ e-02$	0.55	$3.21299\ e-05$	1.17
7	$2.57017\ e-02$	0.61	$1.36679\ e-05$	1.23
8	$1.48376\ e-02$	0.79	$4.55947\ e-06$	1.58
		0.6		1.3

Table: Errors and eoc for the control $\underline{z}$.

Application to arterial blood flow

Optimal control of the inflow in a bypass



- Parabolic inflow $\underline{g}$ on Γ_{in}
- $\underline{f} = \underline{0}$
- Kinematic viscosity $\nu = 0.04$
- $Re \approx 100$

Application to arterial blood flow

Velocity $\underline{u}$

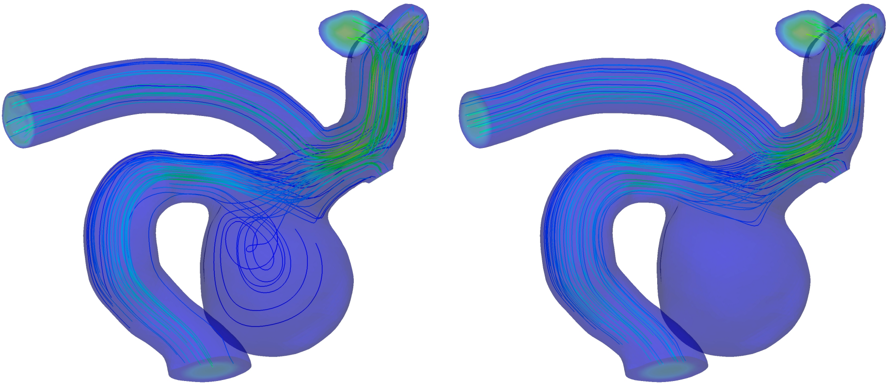
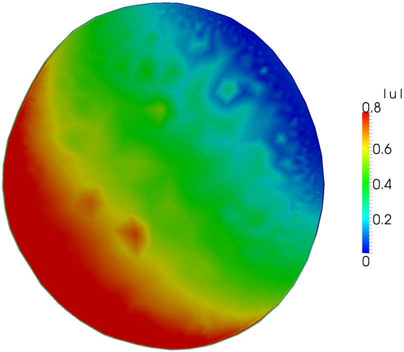


Figure: $L_2(\Gamma_c)$ control (left), $H^{1/2}(\Gamma_c)$ control (right)

Applications to arterial blood flow

Control $\underline{z}$



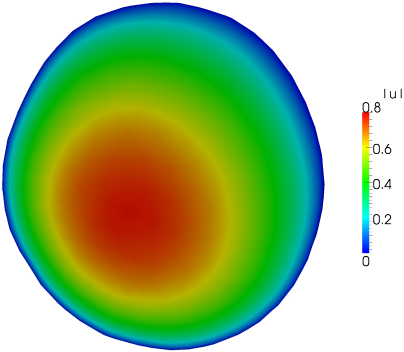


Figure: $L_2(\Gamma_c)$ control (left), $H^{1/2}(\Gamma_c)$ control (right)

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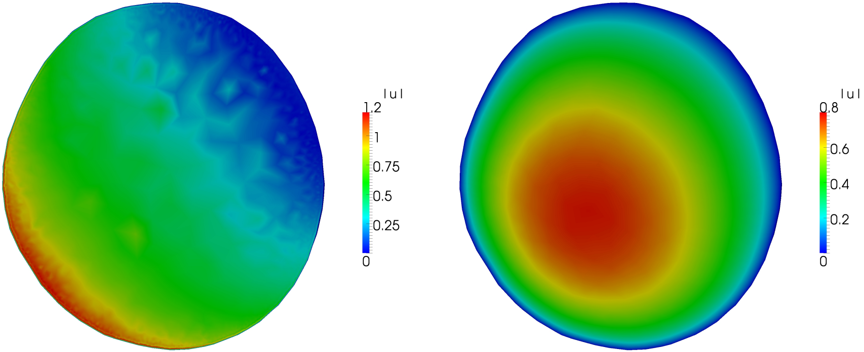

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Conclusion

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Outlook

- Optimal control of wall shear stresses
- Instationary problem
  - Solvers
  - Periodic bc's for pulsative flow
- Non–Newtonian fluids
- Optimal control of FSI

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