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Works with O. Glass (P9), Work O.Kavian (UVSQ), and in progress with G. Legendre (Paris 9

Lagrangian controllability of fluid models.

—Motivation—

- Assume that you have a fluid (modelled by some pde).
- Can one prescribe the motion of (some) fluid particles ? and if too difficult $--->$
- Can one prescribe the motion of a set of fluid particles ? and if too difficult $--->$
- Strive to prescribe as best as possible the motion of this set.

—Possible applications—

- Treatment of pollution: when a pollutant can be considered as a fluid and to simplify of the same type as the ambiant fluid.

- Flottation and application to discriminate between spoilt fluids.

- Displacement of species (animals, plant, alga,jellys) $--->$ related to fluid-structure interaction. We hope to apply our method to fluid-structure controllability.

- Partly related to the work of Khapolov and also the work of Caponigro-Agrachev.

- Ω regular bounded domain of $\mathbb{R}^{N=2,3}$, $T > 0$.
- Γ a part of $\partial\Omega$. n the normal to Ω.

$$(\text{Euler}) \quad \begin{cases} \partial_t u + (u.\nabla)u + \nabla p = 0 \text{ in } (0,T) \times \Omega, \\ \text{div}\,(u) = 0 \text{ in } (0,T) \times \Omega, \\ u(t=0) = u_0, \\ u.n = 0 \text{ on } (0,T) \times (\partial\Omega \setminus \Gamma), \\ \text{ on } (0,T) \times \Gamma \text{ nothing is said apart balance.} \end{cases}$$

- $\gamma_{0,1} \subset \Omega$ two smooth Jordan curves ($N = 2$), Jordan surfaces ($N = 3$).

- φ^X the (if well-defined) flow of a vector field X,

$$(\text{Flow}) \qquad \begin{cases} \varphi^X : [0,T] \times [0,T] \times \Omega \to \Omega, \\ \dfrac{\partial}{\partial t}\varphi^X(s,t,x) = X(t, \varphi^X(s,t,x)), \\ \varphi^X(s,s,x) = x. \end{cases}$$

–Lagrangian controllability (L.C.)—

- Def: There is exact L.C. between γ_0 and γ_1 in time T iff $\exists$ a solution of (Euler) (u, p), s.t. φ^u the solution of (Flow). satisfies

$$\varphi^u(0, T, \gamma_0) = \gamma_1, \text{ \underline{and}}$$

$$\varphi^u(0, [0, T], \gamma_0) \subset \Omega.$$

- Def: Approximate L.C. bet. γ_0 and γ_1 in time T and in norm $C^{k,\alpha}$ iff $\exists$ parameterization of γ_0 and γ_1 by $\mathbb{S}^{1,2}$ s.t. $\forall \varepsilon > 0$ there exists a solution of (Euler) (u, p), such that φ^u satisfies

$$\|\varphi^u(0, T, \gamma_0) - \gamma_1\|_{C^{k,\alpha}(\mathbb{S}^{1,2})} \leqslant \varepsilon,$$

$$\varphi^u(0, [0, T], \gamma_0) \subset \Omega.$$

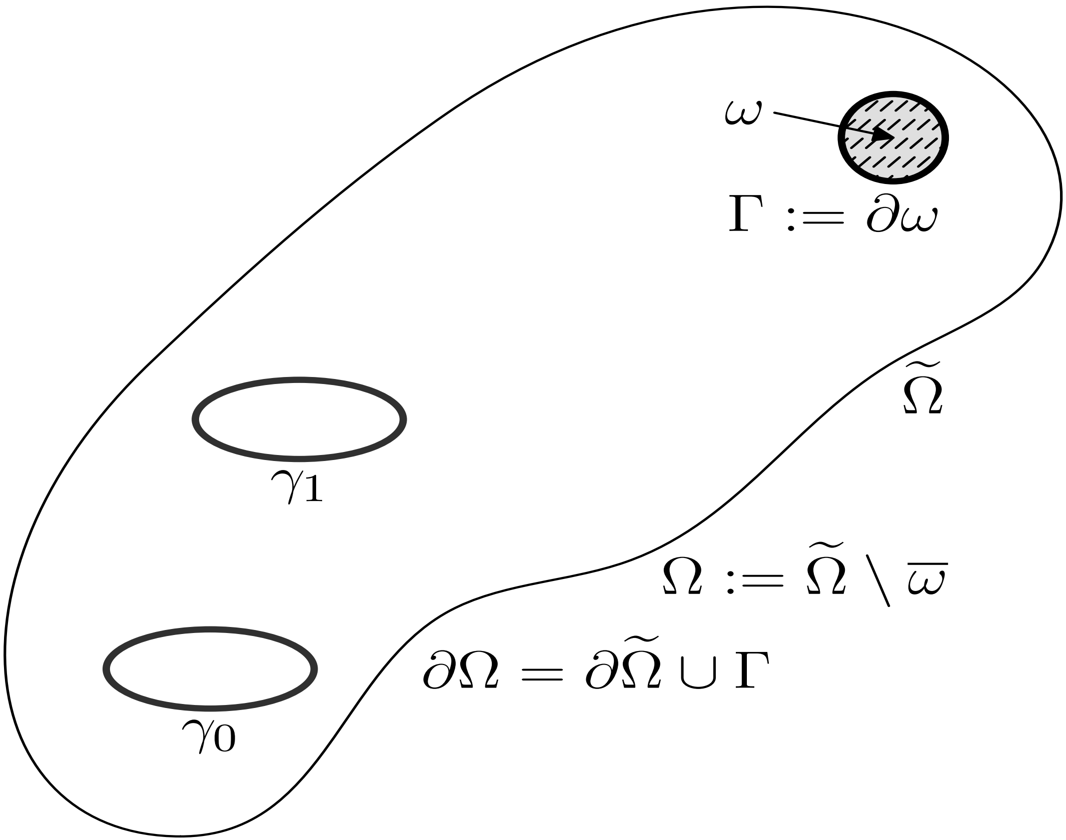
—Remarks—

- Possible problems in defining the flow.
- Incompressibility. if exact L.C occurs between γ_0 and γ_1, then

$$|\mathrm{int}(\gamma_0)| = |\mathrm{int}(\gamma_1)|,$$

  thus we will assume it.
- If one relaxes the condition $\varphi^u(0, [0, T], \gamma_0) \subset \Omega$, and if $|\mathrm{int}(\gamma_0)| = |\mathrm{int}(\gamma_1)|$, then exact L.C. occurs (methods due to Coron for $N = 2$, Glass for $N = 3$, provided Γ intersects each connected component of $\partial\Omega$).

—A Picture—


ω
$\Gamma := \partial\omega$
$\widetilde{\Omega}$
γ_1
$\Omega := \widetilde{\Omega} \setminus \overline{\omega}$
$\partial\Omega = \partial\widetilde{\Omega} \cup \Gamma$
γ_0

—Results—

- $N = 2$. Theorem. Let γ_0 and γ_1 two C^∞ Jordan curves included in $\Omega \subset \mathbb{R}^2$. If γ_0 and γ_1 are homotopic in Ω and surrounds domains of same area, for all $\varepsilon > 0$, for all $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, for all $u_0 \in C^\infty(\bar{\Omega})$ such that

$$\operatorname{div}(u_0) = 0, \; u_0.n = 0 \text{ on } \partial\Omega \setminus \Gamma,$$

there exists (u, p) solution of (Euler) such that

$$\begin{cases} \forall t \in [0, T], \; \varphi^u(0, t, \gamma_0) \subset \Omega, \\ \|\varphi^u(0, T, \gamma_0) - \gamma_1\|_{C^{k,\alpha}(\mathbb{S}^1)} < \varepsilon. \end{cases}$$

- $N = 3$. Theorem. Let γ_0 and γ_1 two C^∞ embedded spheres in Ω which are contractile in Ω, surrounding domains of same area, then for all $\varepsilon > 0$, any $u_0 \in C^\infty(\bar{\Omega})$ s.t. $\operatorname{div}(u_0) = 0$, $u_0.n = 0$ on $\partial\Omega \setminus \Gamma$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$ there exist $T > 0$, and (u,p) solution of (Euler) such that

$$\begin{cases} \forall t \in [0,T],\ \varphi^u(0,t,\gamma_0) \subset \Omega, \\ \|\varphi^u(0,T,\gamma_0) - \gamma_1\|_{C^{k,\alpha}(\mathbb{S}^2)} < \varepsilon. \end{cases}$$

—Remarks—

- 2D vs 3D: In 3D, blow-up may occur, we have to act before blow-up occurs.
- In any case, our solution will be C^∞ in space and time, φ^u is well-defined.
- We construct and mimic a volume preserving motion between γ_0 and γ_1 which is technically quite complicated to construct even in 2D.
- Implicit formulation of the control in (Euler), but one can take

$$u.n \text{ on } \Gamma, \ \text{curl}\, u \text{ on } \{u.n < 0\} \text{ in 2D}, \ \text{[Yudovich]}$$

$$u.n \text{ on } \Gamma, \text{curl}\, u \wedge n \text{ on } \{u.n < 0\} \text{ in 3D}, \ \text{[Khazikhov]}.$$

—Remarks again—

In both 2D and 3D, we have counterexamples to E.L.C

- For example $N = 2$ (similar if $N = 3$): Assume $u_0 = 0$ in a neighborhood of $\overline{\text{int}(\gamma_0)}$.
- Take $\omega := \operatorname{curl} u$, it satisfies $\omega_t + (u.\nabla)\omega = 0$. Thus on a neighborhood of $\varphi^u(0, t, \gamma_0)$, $\operatorname{curl} u(t, .) = 0$ for any $t \in [0, T]$, and thus u is harmonic —>
- if γ_0 is analytic so is $\varphi^u(0, t, \gamma_0)$ for any $t \in [0, T]$. Thus if γ_1 is not analytic there cannot be exact L.C.
- The vorticity equation shows also that we cannot prescribe the flow of γ_0 and $u(T, \cdot)$.

—Other Models—

- Burgers with dispersion

$$\partial_t u + u\partial_x u - \partial_{xx} u = 0 \text{ on } (0,1)$$
$$u(t, x=0) = 0, \quad u(t, x=1) = \text{ the control}$$

-local- T. Horsin (Ann. IHP-2008).

- For Stokes quasi-static (∼ N-S with low Reynolds numbers.)

$$-\Delta u(t) + \nabla p(t) = 0, \quad \forall t \in [0,T], \quad \text{ in } \Omega$$
$$\text{div}\,(u(t)) = 0, \quad \forall t \in [0,T], \quad \text{ in } \Omega$$

In dimension 2 and 3, A.L.C provided the same condition on γ_0 and γ_1 respectively (same method as the work with O. Glass for

Euler). Between low and high Reynolds ???, see the talk of A. Munier and the results of F. Alouges and al.

- As far as $u_0 = 0$, the work can be done for N-S equations but with potential flows.

—sketches of proofs for Euler—

- We want to move a set of particles —> it suffices to control the normal velocity of the set of particles.

- We use the return method: first assume that $u_0 = 0$ and $T = 1$. Construct a solution $(\overline{u}, \overline{p})$ which does the job —> look for potential flows, that is, take

$$\bar{u}(t,.) := \nabla\psi(t,.) \text{ in } \Omega,\ \bar{p} := -\frac{\partial \bar{u}}{\partial t} - |\nabla\psi|^2, \text{ with}$$

$$\Delta\psi(t,.) = 0 \text{ in } \Omega,\ \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

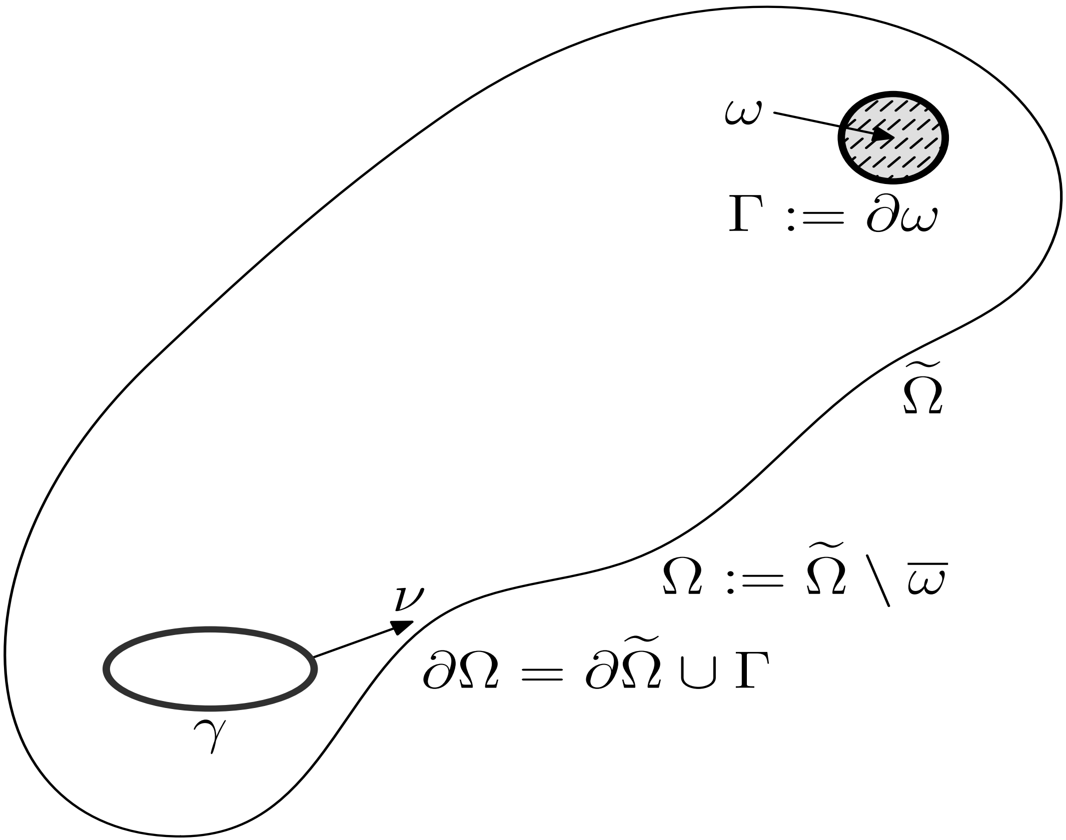
- <u>But</u> before, we have to determine a possible motion inside Ω from γ_0 to γ_1, and by keeping the incompressibility condition.

—sketches of proofs—

- Theorem. $N = 2, 3$. With the assumptions made on γ_0 and γ_1, there exists $X \in C_0^\infty([0,1], C_0^\infty(\Omega))$ such that div $(X(t,.)) = 0$ in Ω, and such that $\varphi^X(0,1,\gamma_0) = \gamma_1$.

- Either in 2D and 3D, you can construct X almost explicitally, but numerically difficult to exploit.

—Another picture—



—Sketches of proofs—

- Once X is given, we want to solve, ν denoting the exterior normal to $\varphi^X(0, t, \gamma_0)$,

$$
\begin{cases}
\Delta\psi(t, 0) = 0 \text{ in } \Omega, \\
\dfrac{\partial\psi(t, .)}{\partial\nu} = X(t, .) \cdot \nu \text{ on } \varphi^X(0, t, \gamma_0), \\
\dfrac{\partial\psi(t, .)}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma, \\
\displaystyle\int_\Gamma \dfrac{\partial\psi(t, .)}{\partial n} d\sigma = 0.
\end{cases}
$$

- Generally an ill-posed problem, but

- it can be approximately solved:

$$
(\mathrm{Approx}) \quad \begin{cases} \Delta\psi(t,0) = 0 \text{ in } \Omega \\ \left\| \dfrac{\partial\psi(t,.)}{\partial\nu} - X(t,.)\cdot\nu \right\|_{C^{k,\alpha}} \leqslant \varepsilon \text{ on } \varphi^X(0,t,\gamma_0), \\ \dfrac{\partial\psi(t,.)}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma, \\ \displaystyle\int_\Gamma \dfrac{\partial\psi(t,.)}{\partial n} d\sigma = 0, \end{cases}
$$

- The job is not complete, and we need some outside approximation of γ_0 (and consequently approximating X) with analytical Jordan domains to be able to apply some Gronwall results and prove the results in 2D and 3D. These approximations are done by the use of Whitney's results.

- Then we perform the return method, for $u_0 \neq 0$.

- Heuristically, in 2D, during $[0, T-\delta]$ we do not control and during $[T-\delta, T]$, we flow the rescaled potential solution:

$$\frac{1}{\delta}\bar{u}(\frac{t-(T-\delta)}{\delta}, x), \frac{1}{\delta^2}\bar{p}(\frac{t-(T-\delta)}{\delta}, x),$$

and use the vorticity formulation of the Euler equations (as in [Kato], later [Coron]).

- In 3D, we rescale u_0 such that the uncontrolled Euler equations do not blow-up during $[0, 1 - \delta]$ and then we flow quickly the potential solution again. We rescale back to get our initial data which rescales also the time during which we control: T depends on u_0 and use again the vorticity formulation (as Glass did).

—Numerical—with O. Glass, O.Kavian and G. Legendre

- Our construction of X is explicit but very difficult to exploit numerically.

- Once X is given, construct ψ that satisfies (Approx) . We have explicit construction of Runge's approximation which allow to treat the time dependance, but analytic expansions do not behave well numerically.

- Another way is to find a functional approach for finding a solution of (Approx) .

- For example, take the solution of (Approx) minimizing

$$\|\frac{\partial\Psi}{\partial n}\|_{H^{-1/2}(\partial\Omega)}.$$

- But we still have problems due to the very weak coercivity of the functionnal that is used to obtain the control by duality:

- Let us define $H_m^{-1/2}(\Gamma) := \{\nu \in H_m^{-1/2}(\partial\Omega), \nu = 0 \text{ on } \partial\Omega \setminus \Gamma,\ \langle\nu|1\rangle = 0\}$ (Γ is assumed to be closed with).

- In the case of the pictures $H_m^{-1/2}(\Gamma) = (H_m^{1/2}(\Gamma))'$ where $H_m^{1/2}(\Gamma) = \{\varphi \in H^{1/2}(\Gamma), \int_\Gamma \varphi\, d\sigma = 0\}$.

—Numerical—with O. Glass, O.Kavian and G. Legendre

We define a mapping

$$\Lambda : H_m^{-1/2}(\Gamma) \to H_m^{-1/2}(\gamma)$$
$$\nu \mapsto \frac{\partial \psi}{\partial \nu} \text{ on } \gamma$$
$$\text{where } \Delta\psi = 0 \text{ in } \Omega \quad \frac{\partial \psi}{\partial n} = \nu \text{ on } \partial\Omega$$

- If we want to obtain a target $g \in H_m^{-1/2}(\gamma)$ at order ε by minimizing ν, one has to minimize

$$\frac{1}{4}\|\Lambda^*(\varphi)\|^2 + \varepsilon\|\varphi\| - \langle g|\varphi\rangle$$

which is coercive but if $\varepsilon = 0$ it is no more coercive. Many numerical difficulties.