Indirect stabilization of weakly coupled systems

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Outline

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a weakly coupled system

$\Omega \subset \mathbb{R}^n$ bounded. Coupling a conservative and a damped wave equation through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R} \tag{1}$$

$$B.C. \qquad u = 0 = v \quad \text{on} \quad \partial\Omega \times \mathbb{R}$$

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Rewrite the system (1) as an evolution equation in
$\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

L_1, L_2 generators of $\mathcal{C}_0$-semigroups on $H_0^1(\Omega) \times L^2(\Omega)$

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lack of exponential stability (ctnd)

$\omega_0(\mathcal{L})$ type of the semigroup generated by $\mathcal{L}$
(blind to compact perturbation)

$\omega_0(\mathcal{L}) \geq 0 \quad \Rightarrow \quad$ system cannot be exponentially stable!
Nevertheless, the total energy of the system is non-increasing, since it satisfies a dissipation relation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback!

Then, we look for weaker decay rates—of polynomial type.

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a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^* , \qquad \langle A_i u, u\rangle \geq \omega_i |u|^2 \qquad (\omega_1, \omega_2 > 0)$$

(H2) B is a bounded linear operator on H such that

$$B = B^* , \qquad \langle Bu, u\rangle \geq \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$

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energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right)$$

total energy of the system $U = (u,p,v,q)$

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u,v\rangle$$

assumptions yield

$$\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big], \quad \nu(\alpha) > 0\,.$$

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reduction to a first order system

$$\begin{aligned} \mathcal{H} &= D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H \\ (U|\widehat{U}) &= \langle A_1^{1/2} u, A_1^{1/2} \widehat{u} \rangle + \langle p, \widehat{p} \rangle \\ &\quad + \langle A_2^{1/2} v, A_2^{1/2} \widehat{v} \rangle + \langle q, \widehat{q} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle \end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A} U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1) . \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A} U = (p, -A_1 u - Bp - \alpha v, q, -A_2 v - \alpha u) \end{cases}$$

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stability result for standard boundary conditions

Theorem (Alabau Boussouira–Cannarsa–Komornik)

Assume, for some integer $j \geq 2$,

$$|A_1 u| \leq c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad (ACK)$$

- $U_0 \in D(\mathcal{A}^n)$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty\,.$

Observe

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1 A_2^{-j/2} u| \leq c|u|$$

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example 1: Dirichlet boundary conditions

$\Omega \subset \mathbb{R}^n$ bounded $\Gamma = \partial\Omega$

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0$$

in this example $A_1 = A = A_2$ with

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = -\Delta u$$

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conclusion: for $0 < |\alpha| < C_\Omega$, for every $t > 0$,

$$\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx \leq \frac{c}{t} \left(\|u^0\|^2_{2,\Omega} + \|u^1\|^2_{1,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega} \right)$$

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more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for $j = 2$;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for $j = 2$;
- $A_2 = A_1^2$ with $j = 2$;
- $A_1 = A_2^2$ with $j = 4$.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$$

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- $(A_1, D(A_1)) = (A_2, D(A_2))$ for $j = 2$;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for $j = 2$;
- $A_2 = A_1^2$ with $j = 2$;
- $A_1 = A_2^2$ with $j = 4$.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$$

for every $t > 0$, for $U_0 \in D(\mathcal{A}^n)$.

example 2: hybrid boundary conditions

Consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty)$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0$$

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second stability result

Theorem (Alabau Boussouira–Cannarsa–G.)

Assume

$$D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2 u| \quad \forall u \in D(A_2) \quad (ACG)$$

Then

- $U_0 \in D(\mathcal{A}^n)$ (*some* $n \geq 1$) $\Rightarrow \mathcal{E}(U(t)) \leq \dfrac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0))$
- $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad$ *as* $\quad t \to \infty$.

Observe

$$(ACG) \quad \Longleftrightarrow \quad |\langle A_1 u, v\rangle| \leq c|A_2 v|\langle A_1 u, u\rangle^{1/2}$$

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main tools

proof uses

- energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- an abstract decay lemma
- interpolation techniques

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the energy of the solution to the boundary-value problem

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satisfies, for $0 < |\alpha| < C_\Omega$, for every $t > 0$,

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \leq \frac{c}{t^{1/4}} \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2 \right)$$

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some remarks

- The compatibility conditions (ACG) also covers systems with operators of different orders
- We can consider different coupling coefficients in the two equations

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0 \, . \end{cases}$$

- we show that stabilization does not occur when the coupling acts only in one components

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention
Merci bien!

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