Indirect stabilization of weakly coupled systems

Roberto Guglielmi

Universitá Tor Vergata (Italy) Université de Lorraine (France)

Control of Fluid-Structure Systems and Inverse Problems Toulouse, France June 25, 2012



<ロ>

-Outline

Outline

Indirect stabilization of weakly coupled systems

Abstract set-up

Stabilization results and compatibility conditions

Concluding remarks



・ロト ・四ト ・ヨト ・ヨト

-Outline

Outline

Indirect stabilization of weakly coupled systems

Abstract set-up

Stabilization results and compatibility conditions

Concluding remarks



ヘロト ヘヨト ヘヨト ヘヨト

-Outline

Outline

Indirect stabilization of weakly coupled systems

Abstract set-up

Stabilization results and compatibility conditions

Concluding remarks



-Outline

Outline

Indirect stabilization of weakly coupled systems

Abstract set-up

Stabilization results and compatibility conditions

Concluding remarks



- indirect stabilization

a weakly coupled system

 $\Omega \subset \mathbb{R}^n$ bounded. Coupling a conservative and a damped wave equation through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$
(1)

B.C.
$$u = 0 = v$$
 on $\partial \Omega \times \mathbb{R}$

any kind of stability for $\alpha \neq 0$?

a weakly coupled system

 $\Omega \subset \mathbb{R}^n$ bounded. Coupling a conservative and a damped wave equation through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$
(1)

B.C.
$$u = 0 = v$$
 on $\partial \Omega \times \mathbb{R}$

any kind of stability for $\alpha \neq 0$?



・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

-indirect stabilization

lack of exponential stability

Rewrite the system (1) as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

 $L_1, L_2 \text{ generators of } C_0 \text{-semigroups on } H_0^1(\Omega) \times L^2(\Omega)$ $K\begin{pmatrix} u\\ u' \end{pmatrix} = \begin{pmatrix} 0\\ \alpha u \end{pmatrix} \text{ compact operator in } H_0^1(\Omega) \times L^2(\Omega),$



-indirect stabilization

lack of exponential stability

Rewrite the system (1) as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix},$$

 $\begin{array}{l} L_1, L_2 \text{ generators of } \mathcal{C}_0 \text{-semigroups on } H_0^1(\Omega) \times L^2(\Omega) \\ \mathcal{K} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha u \end{pmatrix} \quad \text{compact operator in } H_0^1(\Omega) \times L^2(\Omega), \end{array}$



lack of exponential stability (ctnd)

 $\omega_0(\mathcal{L})$ type of the semigroup generated by \mathcal{L} (blind to compact perturbation)

 $\omega_0(\mathcal{L}) \ge 0 \implies$ system cannot be exponentially stable! Nevertheless, the total energy of the system is non-increasing, since it satisfies a dissipation relation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback





・ ロ ト ・ 雪 ト ・ 目 ト

lack of exponential stability (ctnd)

 $\omega_0(\mathcal{L})$ type of the semigroup generated by \mathcal{L} (blind to compact perturbation) $\omega_0(\mathcal{L}) \ge 0 \implies$ system cannot be exponentially stable! Nevertheless, the total energy of the system is non-increasing, since it satisfies a dissipation relation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback





・ 日 ・ ・ 雪 ・ ・ 日 ・ ・ 日 ・

lack of exponential stability (ctnd)

 $\omega_0(\mathcal{L})$ type of the semigroup generated by \mathcal{L} (blind to compact perturbation) $\omega_0(\mathcal{L}) \ge 0 \implies$ system cannot be exponentially stable! Nevertheless, the total energy of the system is non-increasing, since it satisfies a dissipation relation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback





・ロン ・雪 と ・ ヨ と ・ ヨ と

lack of exponential stability (ctnd)

 $\begin{array}{l} \omega_0(\mathcal{L}) \text{ type of the semigroup generated by } \mathcal{L} \\ (\text{blind to compact perturbation}) \\ \omega_0(\mathcal{L}) \geq 0 \quad \Rightarrow \quad \text{system cannot be exponentially stable!} \\ \text{Nevertheless, the total energy of the system is non-increasing,} \\ \text{since it satisfies a dissipation relation} \end{array}$

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback





lack of exponential stability (ctnd)

 $\begin{array}{l} \omega_0(\mathcal{L}) \text{ type of the semigroup generated by } \mathcal{L} \\ (\text{blind to compact perturbation}) \\ \omega_0(\mathcal{L}) \geq 0 \quad \Rightarrow \quad \text{system cannot be exponentially stable!} \\ \text{Nevertheless, the total energy of the system is non-increasing,} \\ \text{since it satisfies a dissipation relation} \end{array}$

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback





lack of exponential stability (ctnd)

 $\begin{array}{l} \omega_0(\mathcal{L}) \text{ type of the semigroup generated by } \mathcal{L} \\ (\text{blind to compact perturbation}) \\ \omega_0(\mathcal{L}) \geq 0 \quad \Rightarrow \quad \text{system cannot be exponentially stable!} \\ \text{Nevertheless, the total energy of the system is non-increasing,} \\ \text{since it satisfies a dissipation relation} \end{array}$

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback!



Then, we look for weaker decay rates—of polynomial type.



・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

lack of exponential stability (ctnd)

 $\begin{array}{l} \omega_0(\mathcal{L}) \text{ type of the semigroup generated by } \mathcal{L} \\ (\text{blind to compact perturbation}) \\ \omega_0(\mathcal{L}) \geq 0 \quad \Rightarrow \quad \text{system cannot be exponentially stable!} \\ \text{Nevertheless, the total energy of the system is non-increasing,} \\ \text{since it satisfies a dissipation relation} \end{array}$

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_{\Omega} |u'(t)|^2 dx$$

Hope to stabilize the full system by a single feedback!





a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0\\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \rightarrow H \ (i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^*$$
, $\langle A_i u, u \rangle \ge \omega_i |u|^2$ $(\omega_1, \omega_2 > 0)$

(H2) B is a bounded linear operator on H such that

$$B = B^*, \qquad \langle Bu, u \rangle \ge \beta |u|^2 \qquad (\beta > 0)$$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$



a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0\\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \rightarrow H$ (*i* = 1, 2) are densely defined closed linear operators such that

$$A_i = A_i^*$$
, $\langle A_i u, u \rangle \ge \omega_i |u|^2$ $(\omega_1, \omega_2 > 0)$

(H2) B is a bounded linear operator on H such that

 $B = B^*$, $\langle Bu, u \rangle \ge \beta |u|^2$ $(\beta > 0)$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$



a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0\\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \rightarrow H$ (*i* = 1, 2) are densely defined closed linear operators such that

$$A_i = A_i^*$$
, $\langle A_i u, u \rangle \ge \omega_i |u|^2$ $(\omega_1, \omega_2 > 0)$

(H2) B is a bounded linear operator on H such that

$$B = B^*$$
, $\langle Bu, u \rangle \ge \beta |u|^2$ $(\beta > 0)$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$



a system of second order evolutions equations

in a separable Hilbert space H

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0\\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

(H1) $A_i : D(A_i) \subset H \rightarrow H$ (*i* = 1, 2) are densely defined closed linear operators such that

$$A_i = A_i^*$$
, $\langle A_i u, u \rangle \ge \omega_i |u|^2$ $(\omega_1, \omega_2 > 0)$

(H2) B is a bounded linear operator on H such that

$$B = B^*$$
, $\langle Bu, u \rangle \ge \beta |u|^2$ $(\beta > 0)$

(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$



-abstract set-up

energies

energies associated to A_1, A_2

$$E_i(u,p) = rac{1}{2} \left(|A_i^{1/2}u|^2 + |p|^2
ight)$$

total energy of the system U = (u, p, v, q)

 $\mathcal{E}(U) := E_1(u, p) + E_2(v, q) + \alpha \langle u, v \rangle$

assumptions yield

 $\mathcal{E}(U) \geq \nu(\alpha) \Big[E_1(u,p) + E_2(v,q) \Big], \quad \nu(\alpha) > 0.$



-abstract set-up

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2} \left(|A_i^{1/2}u|^2 + |p|^2 \right)$$

total energy of the system U = (u, p, v, q)

$$\mathcal{E}(U) := E_1(u, p) + E_2(v, q) + \alpha \langle u, v \rangle$$

assumptions yield

 $\mathcal{E}(U) \ge \nu(\alpha) \Big[E_1(u, p) + E_2(v, q) \Big], \quad \nu(\alpha) > 0.$



-abstract set-up

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2} \left(|A_i^{1/2}u|^2 + |p|^2 \right)$$

total energy of the system U = (u, p, v, q)

$$\mathcal{E}(U) := E_1(u, p) + E_2(v, q) + \alpha \langle u, v \rangle$$

assumptions yield

$$\mathcal{E}(U) \geq \nu(\alpha) \Big[E_1(u, p) + E_2(v, q) \Big], \quad \nu(\alpha) > 0.$$



・ロト ・ 四ト ・ ヨト ・ ヨト ・ ヨ

-abstract set-up

reduction to a first order system

$$\begin{aligned} \mathcal{H} &= D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H \\ (U|\widehat{U}) &= \langle A_1^{1/2}u, A_1^{1/2}\widehat{u} \rangle + \langle p, \widehat{p} \rangle \\ &+ \langle A_2^{1/2}v, A_2^{1/2}\widehat{v} \rangle + \langle q, \widehat{q} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle \end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1). \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by $\begin{cases}
D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_1^{1/2}) \times D(\mathcal{A}_2) \times D(\mathcal{A}_2^{1/2}) \\
\mathcal{A}U = (p, -\mathcal{A}_1u - Bp - \alpha v, q, -\mathcal{A}_2v - \alpha u)
\end{cases}$



-abstract set-up

reduction to a first order system

$$\begin{aligned} \mathcal{H} &= D(\mathcal{A}_1^{1/2}) \times \mathcal{H} \times D(\mathcal{A}_2^{1/2}) \times \mathcal{H} \\ (\mathcal{U}|\widehat{\mathcal{U}}) &= \langle \mathcal{A}_1^{1/2} u, \mathcal{A}_1^{1/2} \widehat{u} \rangle + \langle \mathcal{p}, \widehat{\mathcal{p}} \rangle \\ &+ \langle \mathcal{A}_2^{1/2} v, \mathcal{A}_2^{1/2} \widehat{v} \rangle + \langle \mathcal{q}, \widehat{\mathcal{q}} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle \end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1). \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by $\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2v - \alpha u) \end{cases}$



-abstract set-up

reduction to a first order system

$$\begin{aligned} \mathcal{H} &= D(\mathcal{A}_1^{1/2}) \times \mathcal{H} \times D(\mathcal{A}_2^{1/2}) \times \mathcal{H} \\ (\mathcal{U}|\widehat{\mathcal{U}}) &= \langle \mathcal{A}_1^{1/2} u, \mathcal{A}_1^{1/2} \widehat{u} \rangle + \langle \mathcal{p}, \widehat{\mathcal{p}} \rangle \\ &+ \langle \mathcal{A}_2^{1/2} v, \mathcal{A}_2^{1/2} \widehat{v} \rangle + \langle \mathcal{q}, \widehat{\mathcal{q}} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle \end{aligned}$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1). \end{cases}$$

with
$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$
 defined by

$$\begin{cases} D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_1^{1/2}) \times D(\mathcal{A}_2) \times D(\mathcal{A}_2^{1/2}) \\ \mathcal{A}U = (p, -\mathcal{A}_1u - \mathcal{B}p - \alpha v, q, -\mathcal{A}_2v - \alpha u) \end{cases}$$



・ロト ・聞 ト ・ ヨト ・ ヨト ・ ヨ

stability result for standard boundary conditions

Theorem (Alabau Boussouira–Cannarsa–Komornik) Assume, for some integer $j \ge 2$,

$$|A_1 u| \le c |A_2^{j/2} u| \quad \forall u \in D(A_2^{j/2}) \quad (ACK)$$

$$\bullet \ U_0 \in D(\mathcal{A}^n) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

$$\bullet \ \forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Observe

 $(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1A_2^{-j/2}u| \le c|u|$



stability result for standard boundary conditions

Theorem (Alabau Boussouira–Cannarsa–Komornik) Assume, for some integer $j \ge 2$,

$$\begin{aligned} |A_1 u| &\leq c |A_2^{j/2} u| \quad \forall u \in D(A_2^{j/2}) \quad (ACK) \\ &\blacktriangleright U_0 \in D(\mathcal{A}^n) \text{ (some } n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0)) \\ &\blacktriangleright \forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

Observe

 $(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1A_2^{-j/2}u| \le c|u|$



stability result for standard boundary conditions

Theorem (Alabau Boussouira-Cannarsa-Komornik) Assume, for some integer j > 2,

$$|A_1 u| \le c |A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) \qquad (ACK)$$

$$\bullet \ U_0 \in D(\mathcal{A}^n) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

$$\bullet \ \forall U_0 \in \mathcal{H} \qquad \mathcal{E}(U(t)) \Rightarrow 0 \quad \text{as } t \Rightarrow \infty$$



stability result for standard boundary conditions

Theorem (Alabau Boussouira–Cannarsa–Komornik) Assume, for some integer $j \ge 2$,

$$|A_1 u| \le c |A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2})$$
(ACK)

$$U_0 \in D(\mathcal{A}^n) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

$$\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Observe

 $(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1A_2^{-j/2}u| \le c|u|$



stability result for standard boundary conditions

Theorem (Alabau Boussouira–Cannarsa–Komornik) Assume, for some integer $j \ge 2$,

$$\begin{split} |A_1 u| &\leq c |A_2^{j/2} u| \quad \forall u \in D(A_2^{j/2}) \quad (ACK) \\ \blacktriangleright \ U_0 \in D(\mathcal{A}^n) \; (some \; n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0)) \\ \blacktriangleright \ \forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty \, . \end{split}$$

$$(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \& |A_1A_2^{-j/2}u| \leq c|u|$$



example 1: Dirichlet boundary conditions

$$\Omega \subset \mathbb{R}^n \quad \text{bounded} \quad \Gamma = \partial \Omega$$

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t)$$
 on $\Gamma \quad \forall t > 0$

in this example $A_1 = A = A_2$ with

 $D(A) = H^2(\Omega) \cap H^1_0(\Omega), \qquad Au = -\Delta u$

so that (ACK) : $|A_1u| \le c |A_2^{j/2}u|$ holds with j = 2



example 1: Dirichlet boundary conditions

$$\begin{split} \Omega \subset \mathbb{R}^n & \text{bounded} \quad \Gamma = \partial \Omega \\ & \left\{ \begin{array}{l} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{array} \right. & \text{in} \quad \Omega \times (0, +\infty) \end{split}$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t)$$
 on $\Gamma \quad \forall t > 0$

in this example $A_1 = A = A_2$ with

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega), \qquad Au = -\Delta u$$

so that (ACK) : $|A_1u| \le c|A_2^{j/2}u|$ holds with j = 2



-Stabilization results

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times (0, +\infty) \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times (0, +\infty) \end{cases}$$
$$u(\cdot, t) = 0 = v(\cdot, t) & \text{on} \quad \Gamma \quad \forall t > 0 \\ \begin{cases} u(x, 0) = u^0(x), & u'(x, 0) = u^1(x) \\ v(x, 0) = v^0(x), & v'(x, 0) = v^1(x) \end{cases} \quad x \in \Omega \end{cases}$$
$$\text{nclusion: for} \quad 0 < |\alpha| < C_\Omega, \text{ for every } t > 0, \end{cases}$$
$$\int_{\Omega} \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx$$
$$\leq \frac{C}{t} \left(||u^0||_{2,\Omega}^2 + ||u^1||_{1,\Omega}^2 + ||v^0||_{2,\Omega}^2 + ||v^1||_{1,\Omega}^2 \right)$$



◆□▶ ◆□▶ ◆三▶ ◆三▶ 三三 のへで

-Stabilization results

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty) \\ u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0 \\ \begin{cases} u(x, 0) = u^0(x), & u'(x, 0) = u^1(x) \\ v(x, 0) = v^0(x), & v'(x, 0) = v^1(x) \end{cases} \quad x \in \Omega \end{cases}$$

conclusion: for $0 < |\alpha| < C_{\Omega}$, for every t > 0,

$$\int_{\Omega} \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx$$

$$\leq \frac{c}{t} \left(\|u^0\|_{2,\Omega}^2 + \|u^1\|_{1,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$



-Stabilization results

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times (0, +\infty) \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times (0, +\infty) \end{cases}$$
$$u(\cdot, t) = 0 = v(\cdot, t) & \text{on} \quad \Gamma \quad \forall t > 0 \\ \begin{cases} u(x, 0) = u^0(x), & u'(x, 0) = u^1(x) \\ v(x, 0) = v^0(x), & v'(x, 0) = v^1(x) \end{cases} \quad x \in \Omega \end{cases}$$
$$\text{conclusion: for} \quad 0 < |\alpha| < C_{\Omega}, \text{ for every } t > 0, \end{cases}$$

$$\int_{\Omega} \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx \\ \leq \frac{c}{t} \left(\|u^0\|_{2,\Omega}^2 + \|u^1\|_{1,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

・ロト ・ 四ト ・ ヨト ・ ヨト ・ ヨ

-Stabilization results

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;
- $A_2 = A_1^2$ with j = 2;
- $A_1 = A_2^2$ with j = 4.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

for every t > 0, for $U_0 \in D(\mathcal{A}^n)$

ъ

・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

-Stabilization results

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- ▶ $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;
- $A_2 = A_1^2$ with j = 2;
- $A_1 = A_2^2$ with j = 4.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

-Stabilization results

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;
- $A_2 = A_1^2$ with j = 2;
- $A_1 = A_2^2$ with j = 4.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

-Stabilization results

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;

•
$$A_2 = A_1^2$$
 with $j = 2$;

• $A_1 = A_2^2$ with j = 4.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

-Stabilization results

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;

•
$$A_2 = A_1^2$$
 with $j = 2$;

• $A_1 = A_2^2$ with j = 4.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

more classes of examples

Indeed, (ACK) applies to larger classes of examples:

- $(A_1, D(A_1)) = (A_2, D(A_2))$ for j = 2;
- $D(A_1) = D(A_2)$ (isomorphic as Banach spaces) for j = 2;

•
$$A_2 = A_1^2$$
 with $j = 2$;

•
$$A_1 = A_2^2$$
 with $j = 4$.

conclusion: for all the previous situations, we deduce that

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/j}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

U

example 2: hybrid boundary conditions

Consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

with boundary conditions

$$\begin{pmatrix} \frac{\partial u}{\partial \nu} + u \end{pmatrix} (\cdot, t) = 0 \text{ on } \Gamma \\ v(\cdot, t) = 0 \text{ on } \Gamma$$

The compatibility condition (ACK) does NOT apply!



example 2: hybrid boundary conditions

Consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty) \end{cases}$$

with boundary conditions

$$\begin{pmatrix} \frac{\partial u}{\partial \nu} + u \end{pmatrix} (\cdot, t) = 0 \text{ on } \Gamma \\ v(\cdot, t) = 0 \text{ on } \Gamma$$

The compatibility condition (ACK) does NOT apply!



example 2: hybrid boundary conditions

Consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty) \end{cases}$$

with boundary conditions

$$egin{pmatrix} \displaystyle \left(rac{\partial u}{\partial
u} + u
ight) (\cdot, t) &= 0 ext{ on } \Gamma \ v(\cdot, t) &= 0 ext{ on } \Gamma \ \end{split} rac{\partial t}{\partial
u} rac{\partial t}{\partial
u} > 0$$

The compatibility condition (ACK) does NOT apply!



・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

-Stabilization results

second stability result

Theorem (Alabau Boussouira–Cannarsa–G.) A*ssume*

 $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

► $U_0 \in D(\mathcal{A}^n)$ (some $n \ge 1$) $\Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$

 $\blacktriangleright \forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty.$

Observe

 $(ACG) \iff |\langle A_1 u, v \rangle| \leq c |A_2 v| \langle A_1 u, u \rangle^{1/2}$



-Stabilization results

second stability result

Theorem (Alabau Boussouira–Cannarsa–G.) *Assume*

$$D(A_2) \subset D(A_1^{1/2})$$
 & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

► $U_0 \in D(\mathcal{A}^n)$ (some $n \ge 1$) $\Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$

▶ $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty.$

Observe

 $(ACG) \iff |\langle A_1 u, v \rangle| \leq c |A_2 v| \langle A_1 u, u \rangle^{1/2}$



-Stabilization results

second stability result

Theorem (Alabau Boussouira–Cannarsa–G.) *Assume*

$$D(A_2) \subset D(A_1^{1/2})$$
 & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

•
$$U_0 \in D(\mathcal{A}^n) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

▶ $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty.$

$$(ACG) \iff |\langle A_1u,v\rangle| \leq c|A_2v|\langle A_1u,u\rangle^{1/2}$$



-Stabilization results

second stability result

Theorem (Alabau Boussouira–Cannarsa–G.) *Assume*

$$D(A_2) \subset D(A_1^{1/2})$$
 & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

•
$$U_0 \in D(\mathcal{A}^n) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$$

► $\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty.$

$$(ACG) \iff |\langle A_1u,v\rangle| \leq c|A_2v|\langle A_1u,u\rangle^{1/2}$$



-Stabilization results

second stability result

Theorem (Alabau Boussouira–Cannarsa–G.) *Assume*

$$D(A_2) \subset D(A_1^{1/2})$$
 & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

►
$$U_0 \in D(\mathcal{A}^n)$$
 (some $n \ge 1$) $\Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$

►
$$\forall U_0 \in \mathcal{H}, \quad \mathcal{E}(U(t)) \to 0 \quad as \quad t \to \infty.$$

$$(ACG) \iff |\langle A_1 u, v \rangle| \le c |A_2 v| \langle A_1 u, u \rangle^{1/2}$$



-Stabilization results

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- an abstract decay lemma
- interpolation techniques



э

・ロ・・ (日・・ ヨ・・

-Stabilization results

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- an abstract decay lemma
- interpolation techniques



ъ

・ ロ ト ・ 雪 ト ・ 国 ト ・ 日 ト

-Stabilization results

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- an abstract decay lemma
- interpolation techniques



Э

・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

-Stabilization results

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- an abstract decay lemma
- interpolation techniques



-Stabilization results

example 2: (ACG) applies

the energy of the solution to the boundary-value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\begin{pmatrix} \frac{\partial u}{\partial \nu} + u \end{pmatrix} (\cdot, t) = 0 \quad \text{on } \Gamma \\ v(\cdot, t) = 0 \quad \text{on } \Gamma \end{cases} \quad \forall t > 0$$

satisfies, for $0 < |\alpha| < C_{\Omega}$, for every t > 0,

 $E_{1}(u(t), u'(t)) + E_{2}(v(t), v'(t))$ $\leq \frac{c}{t^{1/4}} \left(|A_{1}u^{0}|^{2} + |A_{1}^{1/2}u^{1}|^{2} + |A_{2}v^{0}|^{2} + |A_{2}^{1/2}v^{1}|^{2} \right)$



(日)

-Stabilization results

example 2: (ACG) applies

the energy of the solution to the boundary-value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$egin{pmatrix} \left(rac{\partial u}{\partial
u}+u
ight)(\cdot,t)&=0\quad ext{on }\Gamma \ v(\cdot,t)&=0\quad ext{on }\Gamma \ \end{pmatrix} rac{dt}{dt} ext{ } ext{ } ext{on }
on \ ec{r}$$

satisfies, for $0 < |\alpha| < C_{\Omega}$, for every t > 0,

$$E_{1}(u(t), u'(t)) + E_{2}(v(t), v'(t))$$

$$\leq \frac{c}{t^{1/4}} \left(|A_{1}u^{0}|^{2} + |A_{1}^{1/2}u^{1}|^{2} + |A_{2}v^{0}|^{2} + |A_{2}^{1/2}v^{1}|^{2} \right)$$

<□> <圖> < □> < □> < □> < □> = □

-Final remarks and open problems

some remarks

- The compatibility conditions (ACG) also covers systems with operators of different orders
- We can consider different coupling coefficients in the two equations

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0. \end{cases}$$

we show that stabilization does not occur when the coupling acts only in one components



・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

-Final remarks and open problems

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention Merci bien!



・ ロ ト ・ 雪 ト ・ 国 ト ・ 日 ト

-Final remarks and open problems

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention Merci bien!



・ 日 ト ・ 雪 ト ・ 目 ト