Two uniqueness results for weak solutions of fluid-solid models

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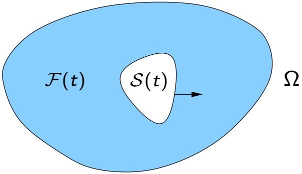
I. Introduction. Two models of a solid embedded in an incompressible fluid

- We consider the motion of a solid body embedded in an incompressible fluid in a smooth bounded open domain $\Omega \subset \mathbb{R}^2$.



- $\mathcal{S}(t)$ is the solid domain at time t, $\mathcal{F}(t)$ is the fluid domain

$$\Omega = \mathcal{S}(t) \cup \mathcal{F}(t).$$

  We take the convention that $\mathcal{S}(t)$ is closed and $\mathcal{F}(t)$ open.
- We will only examine the situation where $\text{dist}(\mathcal{S}(t), \partial\Omega) > 0$. In particular $\text{dist}(\mathcal{S}(0), \partial\Omega) > 0$.
- We will consider the two cases of a viscous and an inviscid fluid.

Solid motions

- The solid $\mathcal{S}(t)$ is obtained by a rigid motion with respect to its initial position :

$$\mathcal{S}(t) = \tau(t)\mathcal{S}_0,$$

  where $\tau(t) \in SE(2)$, the special Euclidean group :

$$\tau(t) \cdot x = h(t) + Q(t)(x - h(0)),$$

  where $h(t)$ the position of the center of mass of $\mathcal{S}$ at time t and

$$Q(t) := \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

  so that the angle $\theta(t)$ measures the rotation between $\mathcal{S}(t)$ and $\mathcal{S}_0$.

- We denote

$$\ell(t) := h'(t) \text{ and } r(t) := \theta'(t),$$

  the velocity of the center of mass and the angular velocity of the body, respectively.

The first model : a solid in a perfect incompressible fluid

- We now turn to our first model where the fluid is supposed to be inviscid (in addition to being incompressible).
- Hence it satisfies the incompressible Euler equation in the fluid domain :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0 \quad \text{for} \quad x \in \mathcal{F}(t)$$
$$\text{div } u = 0 \quad \text{for} \quad x \in \mathcal{F}(t).$$

- Here $u(t, \cdot) : \mathcal{F}(t) \to \mathbb{R}^2$ is the velocity field and $p(t, \cdot) : \mathcal{F}(t) \to \mathbb{R}$ is the pressure field in the fluid.

A solid in a perfect incompressible fluid, 2

- On the outer boundary and on the solid boundary, the fluid satisfies the non-penetration condition :

$$u \cdot n = u_{\mathcal{S}} \cdot n \text{ on } \partial\mathcal{S}(t) \text{ and } u \cdot n = 0 \text{ on } \partial\Omega,$$

where $u_{\mathcal{S}}$ is the solid velocity :

$$u_{\mathcal{S}}(t,x) := \ell + r(x - h(t))^{\perp}.$$

We used the notation $(x_1, x_2)^{\perp} := (-x_2, x_1)$.

- The solid motion is given by Newton's law. It evolves under the influence of the fluid pressure on its surface :

$$mh''(t) = \int_{\partial\mathcal{S}(t)} p\, n\, d\sigma,$$

$$\mathcal{J}\theta''(t) = \int_{\partial\mathcal{S}(t)} p\,(x - h(t))^{\perp} \cdot n\, d\sigma.$$

Here $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the inertia of the body.

The second model : a solid in a viscous incompressible fluid

- We now turn to our first model where the fluid is supposed to be Newtonionian and viscous :
- Hence it satisfies the incompressible Navier-Stokes equation in the fluid domain :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \Delta u + \nabla p = 0 \quad \text{for} \quad x \in \mathcal{F}(t)$$
$$\text{div } u = 0 \quad \text{for} \quad x \in \mathcal{F}(t).$$

- On the outer boundary and on the solid boundary, the fluid satisfies the no-slip condition :

$$u = u_{\mathcal{S}} = \ell + r(x - h(t))^{\perp} \quad \text{on } \partial\mathcal{S}(t) \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

A solid in a viscous incompressible fluid, 2

- The solid motion is given by Newton's law. It evolves under the influence of the whole Cauchy stress tensor :

$$mh''(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T} n \, d\sigma,$$

$$\mathcal{J}\theta''(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T} n \cdot (x - h(t))^{\perp} \, d\sigma,$$

where

$$\mathbb{T}(u,p) := -p\mathrm{Id} + 2Du \quad \text{with } Du := \frac{1}{2}(\nabla u + \nabla u^T).$$

II. The inviscid model

- Recall the model

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = 0 \ \text{ for } \ x \in \mathcal{F}(t),$$
$$\text{div } u = 0 \ \text{ for } \ x \in \mathcal{F}(t)$$
$$u\cdot n = u_{\mathcal{S}}\cdot n = [\ell + r(x-h(t))^{\perp}]\cdot n \ \text{ on } \partial\mathcal{S}(t) \ \text{ and } \ u\cdot n = 0 \ \text{ on } \partial\Omega,$$
$$mh''(t) = \int_{\partial\mathcal{S}(t)} p\, n\, d\sigma,$$
$$\mathcal{J}\theta''(t) = \int_{\partial\mathcal{S}(t)} p\,(x-h(t))^{\perp}\cdot n\, d\sigma.$$

- We add initial conditions :

$$h(0) = h_0,\ h'(0) = \ell_0,\ \theta(0) = 0,\ r(0) = r_0,\ u(0) = u_0 \text{ in } \mathcal{F}_0,$$

with

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}_0,\ u_0\cdot n = (\ell_0 + r_0(x-h_0)^{\perp})\cdot n \text{ on } \partial\mathcal{S}_0,\ u_0\cdot n = 0 \text{ on } \partial\Omega.$$

References for the Cauchy problem

- Ortega-Rosier-Takahashi (2005, 2007) : case of a single solid in the whole plane ($\Omega = \mathbb{R}^2$). Existence and uniqueness of classical solutions ($C^{1,\alpha}$) .
- Rosier-Rosier (2009), classical solutions for a ball in the whole space $\mathbb{R}^3$.
- Houot-San Martin-Tucsnak (2010) classical solutions (in Sobolev spaces) in a bounded domain of $\mathbb{R}^3$.
- G.-Lacave-Sueur (2011) weak solutions for a single solid in the whole plane :

  $$\omega := \operatorname{curl} u \in L^p_c(\Omega), \quad p > 2.$$

  This corresponds to solutions à la Yudovich when $p = +\infty$ and to solutions à la Di Perna-Majda when $p < +\infty$. Moreover, one has uniqueness when $p = +\infty$.
- Xin-Wang (2012) : in the whole plane, finite-energy weak solutions for $\omega \in L^1 \cap L^p$, $p > 4/3$ and G.-Sueur (2012) for $\omega \in L^p_c$, $p > 1$.
- Sueur (2012) : in the whole plane, finite-energy weak solutions for ω bounded Radon measure with symmetry.

Functional spaces

- Given functional space X, the notation $L^\infty(0, T; X(\mathcal{F}(t)))$ or $C([0, T]; X(\mathcal{F}(t)))$ stands for the space of functions :
  - defined for each t in the fluid domain $\mathcal{F}(t)$,
  - that can be extended to functions in $L^\infty(0, T; X(\mathbb{R}^2))$ or $C([0, T]; X(\mathbb{R}^2))$.
- Here $\mathcal{LL}(\mathcal{F}(t))$ stands for the space of log-Lipschitz functions on $\mathcal{F}(t)$, that is the set of functions $f \in L^\infty(\mathcal{F}(t))$ such that

$$\|f\|_{\mathcal{LL}(\mathcal{F}(t))} := \|f\|_{L^\infty(\mathcal{F}(t))} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|(1 + \ln^- |x-y|)} < +\infty. \tag{1}$$

Main result in the inviscid case

- We have the following counterpart of Yudovich's theorem (1963) :

Theorem

For any $u_0 \in C^0(\overline{\mathcal{F}_0}; \mathbb{R}^2)$, $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, *satisfying the above compatibility conditions and*

$$\omega_0 := \operatorname{curl} u_0 \in L^\infty(\mathcal{F}_0),$$

there exists $T > 0$ *and a unique solution*

$$(\ell, r, u) \in C^1([0,T]; \mathbb{R}^2 \times \mathbb{R}) \times [L^\infty(0,T; \mathcal{LL}(\mathcal{F}(t))) \cap C^0([0,T]; W^{1,q}(\mathcal{F}(t)))],$$

for all $q \in [1, +\infty)$, *of the system. Moreover, if* $T < +\infty$ *is maximal, then*

$$\operatorname{dist}(\mathcal{S}(t), \partial\Omega) \to 0 \text{ as } t \to T^-.$$

III. The viscous model

- Recall the model

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \quad \text{for} \quad x \in \mathcal{F}(t)$$

$$\text{div } u = 0 \quad \text{for} \quad x \in \mathcal{F}(t),$$

$$u = u_{\mathcal{S}} = \ell + r(x - h(t))^{\perp} \quad \text{on } \partial\mathcal{S}(t) \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega,$$

$$mh''(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T} n \, d\sigma,$$

$$\mathcal{J}\theta''(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T} n \cdot (x - h(t))^{\perp} \, d\sigma,$$

$$\mathbb{T}(u, p) := -p\mathrm{Id} + 2Du \quad \text{with } Du := \frac{1}{2}(\nabla u + \nabla u^T).$$

- We add initial conditions :

$$h(0) = h_0, \; h'(0) = \ell_0, \; \theta(0) = 0, \; r(0) = r_0, \; u(0) = u_0 \text{ in } \mathcal{F}_0,$$

with the compatibility conditions (for regular solutions) :

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}_0, \; u_0 = (\ell_0 + r_0(x - h_0)^{\perp}) \text{ on } \partial\mathcal{S}_0, \; u_0 = 0 \text{ on } \partial\Omega.$$

References for the Cauchy problem

There are many references concerning the Cauchy problem for this system :

- Weinberger (1973),
- Judakov (1974),
- Serre (1987),
- Galdi (1998, 1999, 2002)
- Hoffmann-Starovoitov (1999, 2000)
- Desjardins-Esteban (1999, 2000)
- Conca-San Martín-Tucsnak (2000)
- Grandmont-Maday (2000)
- Gunzburger-Lee-Seregin (2000)
- Feireisl (2001, 2002, 2003)
- Galdi-Silvestre (2002, 2005, 2006)
- San Martín-Starovoitov-Tucsnak (2002)
- Takahashi (2003)
- Takahashi-Tucsnsak (2004)
- Cumsille-Takahashi (2008)
- Geissert-Götze-Hieber (2012)
- ...

(Not to mention compressible/non-Newtonian fluids, flexible structures, etc.)

Extended velocity and density

We define

- an initial density globally on Ω by setting

$$\rho_0(x) = \rho_{\mathcal{S}_0}(x) \text{ in } \mathcal{S}_0 \ \text{ and } \ \rho_0(x) = 1 \text{ in } \mathcal{F}_0.$$

- the solid density at time t by

$$\rho_{\mathcal{S}}(t,x) = \rho_{\mathcal{S}_0}((\tau(t))^{-1}(x)) \text{ in } \mathcal{S}(t) \ \text{ and } \ \rho_{\mathcal{S}}(t,x) = 0 \text{ in } \mathcal{F}(t),$$

  with τ, $\mathcal{S}(t) := \tau(t)(\mathcal{S}_0)$ and $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$ determined by (ℓ, r).
- a density at time t globally on Ω by setting

$$\rho(t,x) = \rho_{\mathcal{S}}(t,x) \text{ in } \mathcal{S}(t) \ \text{ and } \ \rho(x) = 1 \text{ in } \mathcal{F}(t).$$

- We will say that $\overline{u}$ in $L^2(0,T;H^1(\Omega))$ is compatible with (ℓ, r) when

$$\overline{u}(t,x) = u_{\mathcal{S}}(t,x) = \ell(t) + r(t)(x - h(t))^{\perp} \ \text{ for } \ x \in \mathcal{S}(t),$$

  for almost every t.

Weak solutions "à la Leray"

Definition

Let $u_0 \in L^2(\mathcal{F}_0; \mathbb{R}^2)$ and $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$ satisfying :

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}_0, \; u_0 \cdot n = (\ell_0 + r_0(x - h_0)^\perp) \cdot n \text{ on } \partial\mathcal{S}_0, \; u_0 \cdot n = 0 \text{ on } \partial\Omega.$$

We say that

$$(\ell, r, \overline{u}) \in C^0([0, T]; \mathbb{R}^2 \times \mathbb{R}) \times [C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))]$$

is a weak solution of the system with the initial data (ℓ_0, r_0, u_0) if

- $\overline{u}$ is divergence free,
- $\overline{u}$ is compatible with (ℓ, r),
- and for any divergence free vector field $\phi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^2)$ such that $D\phi(t, x) = 0$ when $t \in [0, T]$ and $x \in \mathcal{S}(t)$, there holds :

$$\int_\Omega \rho_0 \overline{u}_0 \cdot \phi|_{t=0} - \int_\Omega (\rho \overline{u} \cdot \phi)|_{t=T} + \int_{(0,T)\times\Omega} \rho \overline{u} \cdot \frac{\partial \phi}{\partial t} + (\overline{u} \otimes \overline{u} - 2D\overline{u}) : D\phi = 0.$$

Existence of weak solutions

Theorem (Gunzburger-Lee-Seregin, Desjardins-Esteban, Feireisl, San Martin-Starovoitov-Tucsnak)

For any $u_0 \in L^2(\mathcal{F}_0; \mathbb{R}^2)$ *and* $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$ *compatible, for any* $T > 0$, *there exists a corresponding weak solution*

$$(\ell, r, \overline{u}) \in C^0([0,T]; \mathbb{R}^2 \times \mathbb{R}) \times [C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))].$$

Moreover, for any $t \in [0, T]$,

$$\frac{1}{2}\int_\Omega \rho(t,\cdot)|\overline{u}(t,\cdot)|^2\,dx + 2\int_{(0,t)\times\Omega} \rho(s,x)\,D\overline{u}(s,x) : D\overline{u}(s,x)\,dx\,ds = \frac{1}{2}\int_\Omega \rho_0(x)|\overline{u}_0(x)|^2\,dx.$$

Uniqueness of weak solutions

Our second main result states that the solution given by the previous theorem is unique as long as there is no collision.

Theorem

Let $T > 0$ and (ℓ, r, u) be as in the the previous theorem. Assume that for any $t \in [0, T]$, $\operatorname{dist}(\mathcal{S}(t), \partial\Omega)) > 0$. Let $(\tilde{\ell}, \tilde{r}, \tilde{u})$ be another weak solution on $[0, T]$ with the same initial data. Then $(\tilde{\ell}, \tilde{r}, \tilde{u}) = (\ell, r, u)$.

- This extends a result by Takahashi (2003) where u_0 is assumed to be in $H^1(\mathcal{F}_0; \mathbb{R}^2)$.
- The possibility of a collision is excluded in some particular cases, see Hesla (2005), Hillairet (2007), Gérard-Varet and Hillairet (2010).
- Such a weak solution cannot be unique if a collision occurs, see Hoffmann and Starovoitov (1999), Starovoitov (2005).

IV. Ideas of proof

A basic lemma

- Given $A \subset \mathbb{R}^2$ and $\delta > 0$, we denote

$$\mathcal{V}_\delta(A) := \Big\{ x \in \mathbb{R}^2 \Big/ \operatorname{dist}(x, A) \leq \delta \Big\}.$$

- Let $\operatorname{Diff}(\overline{\Omega})$ denote the set of C^∞-diffeomorphisms of $\overline{\Omega}$.

Proposition (Inoue-Wakimoto)

Let Ω and $\mathcal{S}_0$ be fixed as previously. There exist a compact neighborhood U of Id in $SE(2)$, $\delta > 0$ and $\Psi \in C^\infty(U; \mathit{Diff}(\overline{\Omega}))$ such that $\Psi[\mathrm{Id}] = \mathrm{Id}$ and that for all $\tau \in U$,

$$\Psi[\tau] \text{ is volume-preserving},$$

$$\Psi[\tau](x) = \tau(x) \text{ on } \mathcal{V}_\delta(\mathcal{S}_0) \quad \text{and} \quad \Psi[\tau](x) = x \text{ on } \mathcal{V}_\delta(\partial\Omega) \cap \overline{\Omega}.$$

A corollary

- We consider $SE(2) \subset \mathbb{R}^3$ so that we can use the $\mathbb{R}^3$ norm on the elements of $SE(2)$.
- When we consider a time-dependent family of rigid motions $(\tau(t))_{t\in[0,T]}$, we will write $\tau_t := \tau(t,\cdot)$.
- $\{\Psi[\tilde{\tau}_t]\}^{-1}$ denotes the inverse of $\Psi[\tilde{\tau}_t]$ with respect to the variable x.

Corollary

Reducing U if necessary one has for some $C > 0$:,

$$\forall \tau, \tilde{\tau} \in U, \quad \left\| \Psi[\tau] \circ \{\Psi[\tilde{\tau}]\}^{-1} - \mathrm{Id} \right\|_{C^2(\overline{\Omega})} \leq C \|\tau - \tilde{\tau}\|_{\mathbb{R}^3},$$

and if $\tau_t, \tilde{\tau}_t \in C^1([0,T]; SE(2))$, then for all $t_0 \in [0,T]$,

$$\left\| \left[\frac{d}{dt} \left(\Psi[\tau_t] \circ \{\Psi[\tilde{\tau}_t]\}^{-1} \right) \right]_{t=t_0} \right\|_{C^1(\overline{\Omega})} \leq C \Big(\|\tilde{\tau}'_{t_0}\|_{\mathbb{R}^3} \|\tau_{t_0} - \tilde{\tau}_{t_0}\|_{\mathbb{R}^3} + \|\tau'_{t_0} - \tilde{\tau}'_{t_0}\|_{\mathbb{R}^3} \Big).$$

Existence of solutions à la Yudovich

The structure of the proof is as follows.

- We first consider the case where the solid movement is prescribed. In this case the existence of solutions à la Yudovich is proved by Schauder's fixed point theorem. Uniqueness follows from Yudovich's argument for the case of a fixed boundary.
- We prove that these solutions depend continuously in C^0 on the solid movement.
- Then the existence of solutions à la Yudovich is obtained by a second Schauder's fixed point argument on (ℓ, r) relying on an added mass argument.

Added mass, 1

We use the decomposition of the pressure ∇p :

$$\nabla p = \nabla \mu - \nabla \left((\Phi_i)_{i=1,2,3} \cdot \begin{bmatrix} \ell \\ r \end{bmatrix}' \right),$$

where the functions $\Phi_i = \Phi_i(t,x)$ (the Kirchhoff potentials) and the function $\mu = \mu(t,x)$ are :

$$\begin{cases} -\Delta \Phi_i = 0 \quad \text{for} \quad x \in \mathcal{F}(t), \\ \dfrac{\partial \Phi_i}{\partial n} = K_i \quad \text{for} \quad x \in \partial \mathcal{S}(t), \\ \dfrac{\partial \Phi_i}{\partial n} = 0 \quad \text{for} \quad x \in \partial \Omega, \end{cases} \quad \text{where } K_i := \begin{cases} n_i & \text{if } i = 1,2, \\ (x - h(t))^{\perp} \cdot n & \text{if } \ i = 3, \end{cases}$$

and, defining ρ as the signed distance to the boundary,

$$\begin{cases} -\Delta \mu = \operatorname{trace}(\nabla u \cdot \nabla u) \quad \text{for} \quad x \in \mathcal{F}(t), \\ \frac{\partial \mu}{\partial n} = \nabla^2 \rho \left\{ u - u_{\mathcal{S}}, u - u_{\mathcal{S}} \right\} - n \cdot \left(r \left(2u - u_{\mathcal{S}} - \ell \right)^{\perp} \right) \quad \text{for} \quad x \in \partial \mathcal{S}(t), \\ \frac{\partial \mu}{\partial n} = -\nabla^2 \rho(u,u) \quad \text{for} \quad x \in \partial \Omega. \end{cases}$$

Added mass, 2

Using Green's theorem, the equations for the solid can be recast as :

$$\mathcal{M} \begin{bmatrix} \ell \\ r \end{bmatrix}' = \left[\int_{\mathcal{F}(t)} \nabla \mu \cdot \nabla \Phi_i \, dx \right]_{i \in \{1,2,3\}},$$

$$\mathcal{M} := \mathcal{M}_1 + \mathcal{M}_2,$$

$$\mathcal{M}_1 := \begin{bmatrix} m \, \mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} \quad \text{and} \quad \mathcal{M}_2 := \left[\int_{\mathcal{F}(t)} \nabla \Phi_i \cdot \nabla \Phi_j \; dx \right]_{i,j \in \{1,2,3\}}.$$

A priori estimates estimates for solutions à la Yudovich

- These solutions satisfy the following a priori estimates : for any t,

$$\forall q \in [2,\infty], \quad \| \operatorname{curl} u(t,\cdot)\|_{L^q(\mathcal{F}(t))} = \| \operatorname{curl} u_0\|_{L^q(\mathcal{F}_0)},$$
$$\|u(t,\cdot)\|^2_{L^2(\mathcal{F}(t))} + m|\ell(t)|^2 + \mathcal{J}|r(t)|^2 = \|u_0\|^2_{L^2(\mathcal{F}_0)} + m|\ell_0|^2 + \mathcal{J}|r_0|^2.$$

- Moreover, one has for all $t \in [0, T]$ (before collision) and $q \in [2,\infty)$,

$$\|u(t,\cdot)\|_{W^{1,q}(\mathcal{F}(t))} \leq Cq\big(\|\omega_0\|_{L^q(\mathcal{F}_0)} + |\ell_0| + |r_0| + \gamma\big).$$

  (γ takes into account the circulations around the connected components of the boundary, which are conserved by Kelvin's theorem).

- Using again the added mass effect, we infer that uniformly in $[0, T]$:

$$\|u(t)\|_{H^1(\mathcal{F}(t))} + \|\partial_t u\|_{L^2(\mathcal{F}(t))} + \|\nabla p\|_{L^2(\mathcal{F}(t))} \leq C.$$

Proof of uniqueness

- Consider (ℓ_1, r_1, u_1) and (ℓ_2, r_2, u_2) two solutions defined on $[0, T]$.
- It is sufficient to prove the uniqueness for $T > 0$ small.
- We let τ_1 and τ_2 in $C^2([0, T]; SE(2))$ the corresponding rigid movements associated to these solutions. For each $t \in [0, T]$ we introduce φ_t and ψ_t in $\mathrm{Diff}(\overline{\Omega})$ by

$$\varphi_t := \Psi[\tau_2(t)] \circ \{\Psi[\tau_1(t)]\}^{-1}, \quad \psi_t := \varphi_t^{-1}.$$

  φ_t is volume preserving and sends $\mathcal{F}_1(t)$ into $\mathcal{F}_2(t)$.
- Now we define

$$\tilde{u}_2(t, x) := [d\varphi_t(x)]^{-1} \cdot u_2(t, \varphi_t(x)), \ x \in \mathcal{F}_1(t),$$

  the pullback of u_2 by φ_t, which is a solenoidal vector field on $\mathcal{F}_1(t)$.
- We also define

$\tilde{p}_2(t, x) := p_2(t, \varphi_t(x)), \ x \in \mathcal{F}_1(t)$, and $\tilde{\ell}_2 := d(\tau_1 \circ \tau_2^{-1}) \cdot \ell_2 = Q_1 \cdot Q_2^{-1} \cdot \ell_2$.

Proof of uniqueness of solutions à la Yudovich

Now we define

$$\hat{u}(t,x) := u_1(t,x) - \tilde{u}_2(t,x) \text{ and } \hat{p}(t,x) := p_1(t,x) - \tilde{p}_2(t,x) \text{ in } \mathcal{F}_1(t),$$
$$\hat{h} := h_1 - h_2,\ \hat{\theta} := \theta_1 - \theta_2,\ \hat{\ell} := \ell_1 - \tilde{\ell}_2 \text{ and } \hat{r} := r_1 - r_2.$$

We deduce that

$$\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u} + (\hat{u} \cdot \nabla)\tilde{u}_2 + \nabla \hat{p} = \tilde{f} \text{ in } \mathcal{F}_1(t),$$

with

$$\begin{aligned} \tilde{f}^i = {} & (\partial_k \varphi^i - \delta_{ik})\partial_t \tilde{u}_2^k + \partial_k \varphi^i\, \partial_l \tilde{u}_2^k\, (\partial_t \psi^l) + (\partial_k \partial_t \varphi^i)\tilde{u}_2^k + (\partial^2_{kl}\varphi^i)\,(\partial_t \psi^l)\,\tilde{u}_2^k \\ & + \tilde{u}_2^l\, \partial_l \tilde{u}_2^k (\partial_k \varphi^i - \delta_{ik}) + (\partial^2_{lk}\varphi^i)\,\tilde{u}^l\,\tilde{u}^k + \partial_k \tilde{p}_2\,(\partial_i \psi^k - \delta_{ik}). \end{aligned}$$

In the above equation, all the factors between parentheses are small (in L^∞ norm) whenever $\|\varphi_t - \mathrm{Id}\|_{C^2(\overline{\Omega})} + \|\partial_t \varphi_t\|_{C^1(\overline{\Omega})}$ is small.

Energy estimate

Multiplying the previous equation by $\hat{u}$ and integrating over $\mathcal{F}_1(t)$, we deduce

$$\int_{\mathcal{F}_1(t)} (\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u}) \cdot \hat{u}\, dx + \int_{\mathcal{F}_1(t)} \hat{u} \cdot (\hat{u} \cdot \nabla)\tilde{u}_2\, dx + \int_{\mathcal{F}_1(t)} \hat{u} \cdot \nabla \hat{p}\, dx = \int_{\mathcal{F}_1(t)} \hat{u} \cdot \tilde{f}\, dx.$$

But

$$\int_{\mathcal{F}_1(t)} (\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u}) \cdot \hat{u}\, dx = \frac{d}{dt} \int_{\mathcal{F}_1(t)} \frac{|\hat{u}|^2}{2}\, dx,$$

$$\left| \int_{\mathcal{F}_1(t)} \hat{u} \cdot (\hat{u} \cdot \nabla)\tilde{u}_2\, dx \right| \leq \|\nabla \tilde{u}_2\|_{L^q} \|\hat{u}^2\|_{L^{q'}} \leq C_0 q\, \|\hat{u}^2\|_{L^2}^{\frac{2}{q'}},$$

$$\int_{\mathcal{F}_1(t)} \hat{u} \cdot \nabla \hat{p}\, dx = \frac{1}{2} \frac{d}{dt} \left(m|\hat{\ell}|^2 + \mathcal{J}|\hat{r}|^2 \right) - m\hat{r}\hat{\ell} \cdot \tilde{\ell}_2^{\perp}.$$

Right hand side

Concerning the right hand side, we see that

$$\left| \int_{\mathcal{F}_1(t)} \hat{u} \cdot \tilde{f} \, dx \right| \leq C \|\hat{u}(t)\|_{L^2(\mathcal{F}_1(t))} \left[\|\varphi_t - \mathrm{Id}\,\|_{C^2(\overline{\Omega})} + \|\partial_t \varphi_t\|_{C^1(\overline{\Omega})} \right]$$
$$\times \Big(1 + \|\partial_t \tilde{u}_2(t)\|_{L^2(\mathcal{F}_1(t))} + \|\tilde{u}_2(t)\|^2_{H^1(\mathcal{F}_1(t))} + \|\nabla \tilde{p}_2(t)\|_{L^2(\mathcal{F}_1(t))} \Big).$$
$$\leq C(\Psi, \ell_0, r_0, u_0) \, \|\hat{u}(t)\|_{L^2(\mathcal{F}_1(t))} \left(\|(\hat{h}, \hat{\theta})(t)\|_{\mathbb{R}^3} + \|(\hat{\ell}, \hat{r})(t)\|_{\mathbb{R}^3} \right).$$

Summing up, we obtain that for any $q \in [2, \infty)$:

$$\frac{d}{dt} \left(\|\hat{u}\|^2_{L^2(\mathcal{F}_1(t))} + |\hat{\ell}|^2 + |\hat{r}|^2 \right)$$
$$\leq C_0 \left(q \|\hat{u}\|^{\frac{2}{q'}}_{L^2(\mathcal{F}_1(t))} + \|\hat{u}\|^2_{L^2(\mathcal{F}_1(t))} + |\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2 \right).$$

Proof of uniqueness

- Concerning the solid movement, we have

$$|\hat{h}'| = |\ell_1 - \ell_2| \leq |\ell_1 - \tilde{\ell}_2| + |\ell_2 - \tilde{\ell}_2| \leq C(|\hat{\ell}| + |\hat{\theta}|),$$

so

$$\frac{d}{dt}\left(|\hat{h}|^2 + |\hat{\theta}|^2\right) \leq C\left(|\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2\right).$$

- Hence we obtain that

$$\begin{aligned}\frac{d}{dt}&\left(\|\hat{u}\|^2_{L^2(\mathcal{F}_1(t))} + |\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2\right)\\ &\leq C_1\left(q\|\hat{u}\|^{\frac{2}{q'}}_{L^2(\mathcal{F}_1(t))} + |\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2\right)\\ &\leq C_1 q\left(\|\hat{u}\|^2_{L^2(\mathcal{F}_1(t))} + |\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2\right)^{\frac{1}{q'}},\end{aligned}$$

by considering T sufficiently small.

- a comparison argument proves that

$$\|\hat{u}\|^2_{L^2} + |\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2 \leq (C_1 t)^q,$$

and we conclude that $\hat{h} = 0$, $\hat{\theta} = 0$ and $\hat{u} = 0$ for $t < 1/C_1$ by letting $q \to +\infty$.

A priori estimates of the solutions à la Leray

We begin by giving a priori estimates on a solution given by the above existence theorem.
We will also use, for $T > 0$, the notation

$$\mathcal{F}_T := \cup_{t \in (0,T)} \{t\} \times \mathcal{F}(t).$$

Moreover we assume that $\mathrm{dist}(\mathcal{S}(t), \partial\Omega) > 0$ on $[0, T]$. There holds

$$u, \ (u \cdot \nabla) u \in L^{\frac{4}{3}}(\mathcal{F}_T, \mathbb{R}^4).$$

Proposition

There holds

$$tu \in L^{\frac{4}{3}}(0, T; W^{2,\frac{4}{3}}(\mathcal{F}(t))), \quad (t\partial_t u, t\nabla p) \in L^{\frac{4}{3}}(\mathcal{F}_T; \mathbb{R}^4).$$

Sketch of proof of the proposition

The proof relies in a crucial way on the following auxiliary system with unknown $(\mathfrak{l}, \mathfrak{r}, v)$:

$$\frac{\partial v}{\partial t} - \Delta v + \nabla q = g \quad \text{for} \quad x \in \mathcal{F}(t),$$
$$\text{div } v = 0 \quad \text{for} \quad x \in \mathcal{F}(t),$$
$$v = v_{\mathcal{S}} \quad \text{for} \quad x \in \partial\mathcal{S}(t), \text{ and } v = 0 \quad \text{for} \quad x \in \partial\Omega,$$
$$m\mathfrak{l}'(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T}(v,p)n\, d\sigma + mg_1,$$
$$\mathcal{J}\mathfrak{r}'(t) = -\int_{\partial\mathcal{S}(t)} \mathbb{T}(v,p)n \cdot (x - h(t))^{\perp}\, d\sigma + \mathcal{J}g_2,$$
$$v_{\mathcal{S}}(t,x) := \mathfrak{l} + \mathfrak{r}(x - h(t))^{\perp},$$

where

- g, g_1 and g_2 are some source terms,
- $\mathcal{F}(t)$ and $\mathcal{S}(t)$ are prescribed as associated to the solution (ℓ, r, u) above,
- $h(t) = \int_0^t \ell$.

Sketch of proof of the proposition

Let us now explain how this system enters into the game. We define

$$v := tu, \quad q := tp, \quad \mathfrak{l} := t\ell, \quad \text{and } \mathfrak{r} := tr.$$

From the original equations we infer that $(\mathfrak{l}, \mathfrak{r}, v)$ is a weak solution of the previous system, with vanishing initial data and with, as source terms,

$$g := u - t(u \cdot \nabla)u \in L^{\frac{4}{3}}(\mathcal{F}_T; \mathbb{R}^2),$$

$$(g_1, g_2) := (\ell, r) \in L^{\frac{4}{3}}(0, T; \mathbb{R}^2 \times \mathbb{R}).$$

Regular solutions for the auxiliary system

Then we have the following result about the existence of regular solutions to the auxiliary system, see Geissert, Götze and Hieber (2012).

Theorem

There exists a unique solution of the auxiliary system on $[0, T]$ *with vanishing initial data, and this solution satisfies*

$$v \in L^{\frac{4}{3}}(0, T; W^{2,\frac{4}{3}}(\mathcal{F}(t))), \quad (\partial_t v, \nabla q) \in L^{\frac{4}{3}}(\mathcal{F}_T; \mathbb{R}^4),$$
$$(\mathfrak{l}, \mathfrak{r}) \in W^{1,\frac{4}{3}}((0, T); \mathbb{R}^3).$$

This result is obtained by using :

- the same type of change of variable as before,
- maximal regularity theory for the Stokes equation,
- an argument of added mass to deal with the non-homogeneous boundary term (see also the monograph of Galdi (2002)).

Proof of the uniqueness of solutions à la Leray

- We consider (ℓ_1, r_1, u_1) and (ℓ_2, r_2, u_2) two solutions in $[0, T]$ in the sense of the above existence theorem.
- It is sufficient to prove uniqueness for $T > 0$ small enough.
- We perform the same change of variable than in the proof of uniqueness of solutions à la Yudovich :

$$\begin{gathered}\tilde{u}_2(t,x) := [d\varphi_t(x)]^{-1} \cdot u_2(t, \varphi_t(x)),\ x \in \mathcal{F}_1(t),\\ \tilde{p}_2(t,x) := p_2(t, \varphi_t(x)),\ x \in \mathcal{F}_1(t),\\ \tilde{\ell}_2 := d(\tau_1 \circ \tau_2^{-1}) \cdot \ell_2 = Q_1 \cdot Q_2^{-1} \cdot \ell_2.\end{gathered}$$

  with $\varphi_t := \Psi[\tau_2(t)] \circ \{\Psi[\tau_1(t)]\}^{-1}, \ \ \psi_t := \varphi_t^{-1}$.
- We define as well

$$\begin{gathered}\hat{u}(t,x) := u_1(t,x) - \tilde{u}_2(t,x),\\ \hat{p}(t,x) := p_1(t,x) - \tilde{p}_2(t,x) \text{ in } \ \mathcal{F}_1(t),\\ \hat{h} := h_1 - h_2,\ \hat{\theta} := \theta_1 - \theta_2,\ \hat{\ell} := \ell_1 - \tilde{\ell}_2 \ \text{ and } \ \hat{r} := r_1 - r_2.\end{gathered}$$

Proof of the uniqueness of solutions à la Leray

We obtain the following equations :

$$\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u} + (\hat{u} \cdot \nabla)\tilde{u}_2 + \nabla \hat{p} - \Delta \hat{u} = \tilde{f} \quad \text{in} \quad \mathcal{F}_1(t),$$

$$\hat{u} = \hat{\ell}(t) + \hat{r}(t)(x - h_1(t))^{\perp} \quad \text{for} \quad x \in \partial \mathcal{S}_1(t), \;\; \hat{u} = 0 \quad \text{for} \quad x \in \partial\Omega,$$

$$m\hat{\ell}' = -\int_{\partial \mathcal{S}_1(t)} \mathbb{T}(\hat{u}, \hat{p}) n_1 \, d\sigma + m\hat{r}\tilde{\ell}_2^{\perp},$$

$$\mathcal{J}\hat{r}'(t) = -\int_{\partial \mathcal{S}_1(t)} \mathbb{T}(\hat{u}, \hat{p}) n_1 \cdot (x - h_1(t))^{\perp} \, d\sigma.$$

where

$$\begin{aligned}
\tilde{f}^i =& (\partial_k \varphi^i - \delta_{ik})\partial_t \tilde{u}_2^k + \partial_k \varphi^i \, \partial_l \tilde{u}_2^k \, (\partial_t \psi^l) + (\partial_k \partial_t \varphi^i)\tilde{u}_2^k \\
&+ (\partial_{kl}^2 \varphi^i) \, (\partial_t \psi^l) \, \tilde{u}_2^k + \tilde{u}_2^l \, \partial_l \tilde{u}_2^k (\partial_k \varphi^i - \delta_{ik}) + (\partial_{lk}^2 \varphi^i) \, \tilde{u}_2^l \, \tilde{u}_2^k \\
&+ \partial_k \tilde{p}_2 \, (\partial_i \psi^k - \delta_{ik}) - \partial_j \psi^m (\partial_{mk}^2 \varphi^i) \, \partial_l \tilde{u}_2^k \, \partial_j \psi^l \\
&- (\partial_k \varphi^i \partial_j \psi^m \partial_j \psi^l - \delta_{ik}\delta_{jm}\delta_{jl})\partial_{ml}^2 \tilde{u}_2^k - \partial_k \varphi^i \, \partial_l \tilde{u}_2^k (\partial_j^2 \psi^l) \\
&- \partial_j \psi^m (\partial_{mlk}^3 \varphi^i) \, \partial_j \psi^l \, \tilde{u}_2^k - (\partial_{lk}^2 \varphi^i) \, \partial_{jj}^2 \psi^l \, \tilde{u}_2^k - (\partial_{lk}^2 \varphi^i) \, \partial_j \psi^l \, \partial_j \psi^m \, \partial_m \tilde{u}_2^k.
\end{aligned}$$

Proof of the uniqueness of solutions à la Leray

Multiplying the previous equation by $\hat{u}$ and integrating over $\mathcal{F}_1(t)$, we deduce that for a.e. $t > 0$

$$\int_{\mathcal{F}_1(t)} (\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u}) \cdot \hat{u}\, dx + \int_{\mathcal{F}_1(t)} \hat{u} \cdot (\hat{u} \cdot \nabla)\tilde{u}_2\, dx + \int_{\mathcal{F}_1(t)} \hat{u} \cdot \nabla \hat{p}\, dx - \int_{\mathcal{F}_1(t)} \hat{u} \cdot \Delta \hat{u}\, dx = \int_{\mathcal{F}_1(t)} \hat{u} \cdot \tilde{f}\, dx.$$

Proceeding as in the proof for Euler, we have

$$\begin{aligned}
\int_{\mathcal{F}_1(t)} (\partial_t \hat{u} + (u_1 \cdot \nabla)\hat{u}) \cdot \hat{u}\, dx &= \frac{d}{dt} \int_{\mathcal{F}_1(t)} \frac{|\hat{u}|^2}{2}\, dx, \\
\left| \int_{\mathcal{F}_1(t)} \hat{u} \cdot (\hat{u} \cdot \nabla)\tilde{u}_2\, dx \right| &\leq C \|\nabla \tilde{u}_2\|_{L^2}^2 \|\hat{u}\|_{L^2}^2 + \frac{1}{4} \|\nabla \hat{u}\|_{L^2}^2, \\
\int_{\mathcal{F}_1(t)} \hat{u} \cdot \nabla \hat{p}\, dx - \int_{\mathcal{F}_1(t)} \hat{u} \cdot \Delta \hat{u}\, dx &= 2 \int_{\mathcal{F}_1(t)} D\hat{u} : D\hat{u}\, dx \\
&\quad + \frac{1}{2} \frac{d}{dt} \left(m|\hat{\ell}|^2 + \mathcal{J}|\hat{r}|^2 \right) - m\hat{r}\hat{\ell} \cdot \tilde{\ell}_2^{\perp}.
\end{aligned}$$

Proof of the uniqueness of solutions à la Leray

Regarding the right hand side of the energy identity, we obtain that for some constant $C > 0$ depending on the geometry only and defining the function $\mathcal{B} \in L^1(0, T)$ by

$$
\begin{aligned}
\mathcal{B}(t) \quad := \quad & \|\tilde{u}_2\|_{L^\infty(0,T;L^2(\mathcal{F}_1(t)))}(1 + \|\nabla \tilde{u}_2(t,\cdot)\|_{L^2(\mathcal{F}_1(t))}) \\
& + \|\tilde{u}_2\|^{1/2}_{L^\infty(0,T;L^2(\mathcal{F}_1(t)))} \|\nabla \tilde{u}_2(t)\|^{1/2}_{L^2(\mathcal{F}_1(t))} \|t \nabla \tilde{u}_2(t)\|_{L^4(\mathcal{F}_1(t))} \\
& + \big(\|t \partial_t \tilde{u}_2\|_{L^{4/3}(\mathcal{F}_1(t))} + \|t \tilde{u}_2\|_{W^{2,4/3}(\mathcal{F}_1(t))} + \|t \nabla \tilde{p}_2\|_{L^{4/3}(\mathcal{F}_1(t))}\big)^{4/3},
\end{aligned}
$$

one has the following estimate on the right hand side :

$$
\left| \int_0^T \int_{\mathcal{F}_1(t)} \hat{u} \cdot \tilde{f} \, dx \, dt \right| \leq \frac{1}{4} \int_0^T \int_{\mathcal{F}_1(t)} |\nabla \hat{u}|^2 \, dx \, dt
$$

$$
+ C \int_0^T \mathcal{B}(t) \Big[\max_{\tau \in [0,t]} \|\hat{u}(\tau, \cdot)\|^2_{L^2(\mathcal{F}_1(t))} + \max_{[0,t]} |(\hat{h}, \hat{\theta}, \hat{\ell}, \hat{r})|^2 \Big] \, dt.
$$

Proof of the uniqueness of solutions à la Leray

Now we use that

$$\int_{\mathcal{F}_1(t)} |\nabla \hat{u}|^2 \, dx \leq 2 \int_{\mathcal{F}_1(t)} |D\hat{u}|^2 \, dx.$$

and we take into account the vanishing initial condition for $(\hat{\ell}, \hat{r}, \hat{u})$ to deduce that for any $T > 0$ sufficiently small,

$$m|\hat{\ell}(T)|^2 + \mathcal{J}|\hat{r}(T)|^2 + \|\hat{u}(T)\|^2_{L^2(\mathcal{F}_1(T))}$$
$$\leq C \int_0^T \mathcal{B}(t) \Big[\max_{\tau \in [0,t]} \|\hat{u}(\tau, \cdot)\|^2_{L^2(\mathcal{F}_1(t))} + \max_{[0,t]} |(\hat{h}, \hat{\theta}, \hat{\ell}, \hat{r})(t)|^2 \Big] \, dt.$$

We get as before

$$\frac{d}{dt} \left(|\hat{h}|^2 + |\hat{\theta}|^2 \right) \leq C \left(|\hat{\ell}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2 \right).$$

Hence using $\mathcal{B}(t) \in L^1$ and Gronwall's lemma concludes the proof.