On the Steady Motion of a Coupled System Solid-Liquid

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Preliminary Considerations

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Liquid–solid interaction (LSI) is a relatively new and fascinating branch of applied mathematics. Actually, systematic (analytical and numerical) studies started less than 15 years ago.

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Late start and current increasing interest are probably due to the following reasons:

- The intrinsic difficulties related to this type of problems. In fact, the presence of the solid (rigid or elastic) affects the flow of the liquid, and this, in turn, affects the motion of the solid, so that the problem of determining the flow characteristics is highly coupled.
- A rapidly increasing attention that, over the past decade, these questions have acquired in many fields of applied sciences, like bioengineering, animal locomotion, damage of structures, etc.

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C1 In the case of a rigid (undeformable) body, the interaction between the body and the liquid is nonlocal: forces and torques exerted by the liquid on the body are given through integral quantities.

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Liquid $\mathcal{L}$:

$$\left.\begin{aligned} &\rho\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = \operatorname{div}\boldsymbol{\mathcal{T}}(\boldsymbol{v},p) \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\}\ \text{in} \bigcup_{t>0}[\widetilde{\mathcal{D}}(t)\times\{t\}]$$

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{\eta} + \boldsymbol{\Omega}\times\boldsymbol{x}, \quad (\boldsymbol{x},t)\in\bigcup_{t>0}[\partial\widetilde{\mathcal{D}}(t)\times\{t\}]$$

Rigid Body $\mathcal{B}$:

$$m\,\frac{d\boldsymbol{\eta}}{dt} = \boldsymbol{F} - \int_{\partial\widetilde{\mathcal{D}}(t)} \boldsymbol{\mathcal{T}}(\boldsymbol{v},p)\cdot\boldsymbol{N}$$

$$\frac{d(\boldsymbol{J}\cdot\boldsymbol{\Omega})}{dt} = \boldsymbol{M}_C - \int_{\partial\widetilde{\mathcal{D}}(t)} (\boldsymbol{x}-\boldsymbol{x}_C)\times\boldsymbol{\mathcal{T}}(\boldsymbol{v},p)\cdot\boldsymbol{N}$$

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C2 In the case of an elastic (deformable) body, the deformation of the body due to the action of the liquid becomes a further unknown. Moreover, the motion of the elastic body is naturally described in the Lagrangean formalism, while that of the liquid requires the Eulerian formalism.

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- Steady-state and time-periodic problems may lack of the corresponding uniqueness property, at arbitrarily small (even zero!) value of the relevant physical parameters (e.g. Reynolds number);
- Dynamics can be very rich, also at relatively small Reynolds number. Multiple bifurcation phenomena (steady and time-periodic) may occur.

Therefore,

- Stability and/or Control Analysis of the solutions is of the utmost importance, to find out which solution is "physically meaningful".

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- Stability and/or Control Analysis of the solutions is of the utmost importance, to find out which solution is "physically meaningful".

Most of the above phenomena remain basically unresolved from a mathematical viewpoint.

Body in a Viscous Liquid subject to a Constant Body Force

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Josef BEMELMANS (RWTH Aachen)
&
Mads KYED (TU Darmstadt)

Archive Ratl Mech. Anal. (2011), Memoirs of the AMS (2012)

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Hans WEINBERGER, Proc. Symp. Pure Math. (1973)

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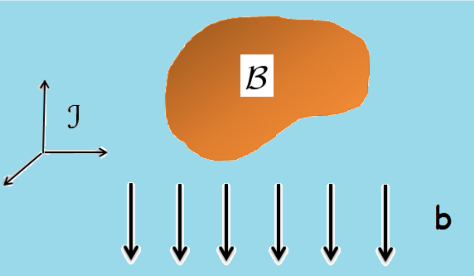
$\mathcal{B}$ is an elastic (deformable) body, that with respect to the inertial frame $\mathcal{I}$ moves in a viscous liquid filling the exterior of $\mathcal{B}$, under the action of a constant (time-independent) body force $\boldsymbol{b}$.



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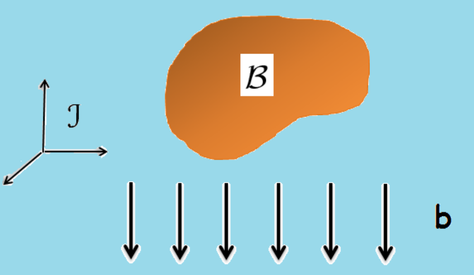

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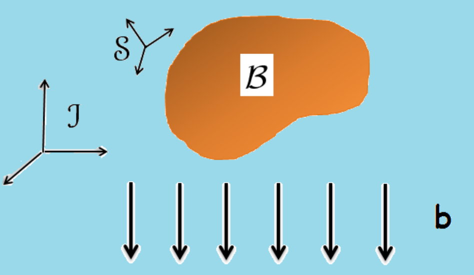

After a "transient time", $\mathcal{B}$ may reach a "terminal" equilibrium state.

Namely, there exists a frame, $\mathcal{S}$, with respect to which the displacement field (and so, the deformation) evaluated from a given reference configuration is time-independent.

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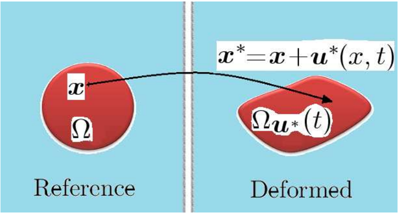

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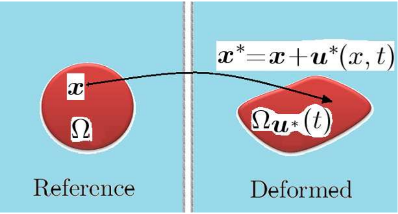

ρ_E is the constant density (in the reference configuration),

$\boldsymbol{\sigma}$ is the (first) Piola-Kirchhoff tensor.

The Solid-Liquid Problem in the Frame $\mathcal{I}$

To fix the ideas, we will consider St.-Venant–Kirchhoff elastic bodies, for which

$$\boldsymbol{\sigma}(\boldsymbol{u}^*) = (\boldsymbol{I} + \nabla\boldsymbol{u}^*)(\lambda_E \mathsf{Tr}\boldsymbol{E}(\boldsymbol{u}^*)\boldsymbol{I} + 2\mu_E\boldsymbol{E}(\boldsymbol{u}^*))$$

$$\boldsymbol{E}(\boldsymbol{u}^*) = \frac{1}{2}\left(\nabla\boldsymbol{u}^* + \nabla\boldsymbol{u}^{*\top} + \nabla\boldsymbol{u}^{*\top}\nabla\boldsymbol{u}^*\right)$$

$\boldsymbol{I}$ = identity matrix, $\mu_E > 0$ and $\lambda_E > -\frac{2}{3}\mu_E$ are the Lame constants.

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$$\Omega_{\boldsymbol{u}^*}(t) = \{\boldsymbol{x}^* \in \mathbb{R}^3 : \ \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{u}^*(x,t)\,,\ \boldsymbol{x} \in \overline{\Omega}\}\,,\quad t > 0\,,$$

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The liquid occupies the region, $\mathcal{E} = \mathcal{E}(t)$, exterior to $\Omega_{\boldsymbol{u}^*}(t)$, that is,

$$\mathcal{E}(t) := \mathbb{R}^3 - \Omega_{\boldsymbol{u}^*}(t)\,.$$

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We thus have

$$\left.\begin{array}{l} \rho(\partial_t \boldsymbol{v}^* + (\nabla \boldsymbol{v}^*)\boldsymbol{v}^*) = \mu \Delta \boldsymbol{v}^* - \nabla p^* \\ \operatorname{div} \boldsymbol{v}^* = 0 \end{array}\right\} \text{ in } \cup_{t>0} \left[\mathcal{E}(t) \times \{t\}\right].$$

$\boldsymbol{v}^*$ is the velocity, p^* is the pressure,
ρ is the density, μ is the shear viscosity coefficient

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For sufficiently "regular" $\boldsymbol{u}^*(\cdot, t)$ ($|\nabla \boldsymbol{u}^*(\cdot, t)| < 1$), we have

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No-Slip Boundary Conditions:

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Continuity of the Stress:

$$\boldsymbol{T}_L \cdot \boldsymbol{n} = \boldsymbol{T}_E \cdot \boldsymbol{n} \quad \text{at } \cup_{t>0}\left[\partial\Omega_{\boldsymbol{u}^*}(t) \times \{t\}\right],$$

where

$\boldsymbol{T}_E$ is the Cauchy stress tensor of the elastic body
$\boldsymbol{T}_L$ is the Cauchy stress tensor of the liquid
$\boldsymbol{n}$ is the outer unit normal to $\partial\Omega_{\boldsymbol{u}^*}(t)$.

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Let $\boldsymbol{\omega}$ be the unknown *constant* angular velocity of the frame $\mathcal{S}$ with respect to the inertial frame $\mathcal{I}$, and set

$$\widehat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

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If $\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{u}^*(x, t)$, we make the following change of variables

$$\boldsymbol{y} = e^{-\widehat{\boldsymbol{\omega}}t} \cdot (\boldsymbol{x}^* - \boldsymbol{x}_c^*)\,, \quad x \in \Omega\,; \quad e^{-\widehat{\boldsymbol{\omega}}t} \in SO(3)\,, \quad t \geq 0\,.$$

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Thus, with respect to $\mathcal{S}$, the displacement field is given by:

$$\boldsymbol{u}(x, t) = \boldsymbol{y} - \boldsymbol{x} \,, \quad x \in \Omega \,,$$

and the velocity of the center of mass:

$$\boldsymbol{\xi} = e^{-\widehat{\boldsymbol{\omega}} t} \cdot \partial_t \boldsymbol{x}_c^* \,.$$

The Solid-Liquid Problem in the Frame $\mathcal{S}$

Equations of Motion for the Elastic Body

$$\rho_E[\partial_t^2 \boldsymbol{u} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\boldsymbol{x} + \boldsymbol{u})) + 2\boldsymbol{\omega} \times \partial_t \boldsymbol{u}]$$

$$+\rho_E(\boldsymbol{\omega} \times \boldsymbol{\xi} + \partial_t \boldsymbol{\xi}) = \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) + \rho_E \, e^{-\widehat{\boldsymbol{\omega}} t} \cdot \boldsymbol{b} \,, \quad \text{in } \Omega \times (0, \infty).$$

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However, this term may still depend on time. It is time-independent if and only if

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The direction of the vector $\mathfrak{b}$ becomes a further unknown.

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$$\mathfrak{b} \times \boldsymbol{\omega} = \mathbf{0}$$

where

$\boldsymbol{\omega}, \boldsymbol{\xi}, \mathfrak{b}$ and $\boldsymbol{u} = \boldsymbol{u}(x)$, $x \in \Omega$, are unknown,

and

$|\mathfrak{b}| = |\boldsymbol{b}|$ is given.

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Set

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and require that $(\boldsymbol{v},p)$ is steady.

We then obtain the following equations

$$\left.\begin{aligned}&\rho[\nabla\boldsymbol{v}(\boldsymbol{v}-(\boldsymbol{\omega}\times\boldsymbol{y}+\boldsymbol{\xi}))+\boldsymbol{\omega}\times\boldsymbol{v}] = \mu\Delta\boldsymbol{v}-\nabla p\\&\operatorname{div}\boldsymbol{v} = 0\end{aligned}\right\}\ \text{in } \mathcal{Y}$$

where

$$\mathcal{Y} := \mathbb{R}^3 - \{\boldsymbol{z}\in\mathbb{R}^3:\ \boldsymbol{z} = \boldsymbol{x}+\boldsymbol{u}(x)\,,\ \ x\in\overline{\Omega}\}\,.$$

The Solid-Liquid Problem in the Frame $\mathcal{S}$

Equations of Motion for the Liquid

Set

$$\boldsymbol{v}(\cdot,t) = e^{-\widehat{\boldsymbol{\omega}}t}\cdot\boldsymbol{v}^*(e^{\widehat{\boldsymbol{\omega}}t}\cdot+\boldsymbol{x}_c^*,t)\,,\quad p(\cdot,t) = e^{-\widehat{\boldsymbol{\omega}}t}\,p^*(e^{\widehat{\boldsymbol{\omega}}t}\cdot+\boldsymbol{x}_c^*,t)$$

and require that $(\boldsymbol{v},p)$ is steady.

We then obtain the following equations

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No-slip boundary condition becomes

$$\boldsymbol{v} = \boldsymbol{\xi}+\boldsymbol{\omega}\times\boldsymbol{y}\ \ \text{at } \partial\mathcal{Y}.$$

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The final step is to rewrite the liquid equations in the exterior of the reference (undeformed) configuration, $\mathbb{R}^3-\Omega$.

This can be done if $\boldsymbol{u}$ is "sufficiently regular".

The Solid-Liquid Problem in the Frame $\mathcal{S}$

Lemma *Let* $\boldsymbol{u} \in W^{2,q}(\Omega)$, $q > 3$, *with*

$$\|\boldsymbol{u}\|_{W^{2,q}(\Omega)} \leq M \quad \text{“sufficiently small”}$$

Then there is a C^1*-diffeomorphism,* $\boldsymbol{\chi_u}$*, from* $\mathbb{R}^3$ *onto itself satisfying the following properties.*

(i) $\boldsymbol{\chi_u}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \overline{\Omega}$;

(ii) $\boldsymbol{\chi_u}(\boldsymbol{x}) = \boldsymbol{x}$, *for all* $\boldsymbol{x}$ *with* $|\boldsymbol{x}| \geq R$, *some* $R > 0$.

In particular, $\boldsymbol{\chi_u}$ *is a* C^1*-diffeomorphism from* $\mathbb{R}^3 - \Omega$ *onto* $\mathcal{Y}$.

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Using the diffeomorphism $\boldsymbol{\chi_u}$ we can rewrite the liquid equations in $\mathbb{R}^3 - \Omega$ and end up with the following complete set of equations

The Solid-Liquid Problem in the Frame $\mathcal{S}$

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where

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$$J_{\boldsymbol{u}} := \det(\operatorname{grad} \boldsymbol{\chi}_{\boldsymbol{u}})\,, \quad \boldsymbol{\Phi}_{\boldsymbol{u}} := J_{\boldsymbol{u}}\, (\operatorname{grad} \boldsymbol{\chi}_{\boldsymbol{u}})^{-1}$$

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Problem. Given $\rho_E, \rho, \mu, \lambda_E, \mu_E$, $\boldsymbol{b}$ and a reference configuration Ω for $\mathcal{B}$, find $\boldsymbol{u}, \boldsymbol{v}, p, \boldsymbol{\xi}, \boldsymbol{\omega}$ and $\mathfrak{b}$ satisfying above conditions.

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This problem can be viewed as a nonlinear eigenvalue problem in a suitable sense.

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Unknowns: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}, \lambda$ and $\mathfrak{b} \in S^2$.

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Therefore, for the existence, we have to find $\lambda \in \mathbb{R}$ in such a way that (1)–(2) has a solution with a normalized $\mathfrak{b} \neq \mathbf{0}$ (for example, $\mathfrak{b} \in S^2$).

Sketch of the Strategy of Proof and Main Results

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The main idea develops according to the following steps:

- Linearize the problem suitably
- Find a (suitable) solution to the linearized problem
- Iterate around this solution
- Find a solution to the original problem (for small data)

Sketch of the Strategy of Proof and Main Results

Linearization

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\begin{aligned}
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&\nu := \frac{\mu_E}{\lambda_E + \mu_E)}\\
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Eigenvalue Problem:

$$\begin{aligned} \mathbb{A} \cdot \mathfrak{b}_0 &= \lambda_0 \mathfrak{b}_0 \\ \boldsymbol{\xi}_0 &= \boldsymbol{F}(\mathfrak{b}_0, \lambda_0) \end{aligned} \tag{4}$$

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Theorem 1

Suppose the reference configuration Ω is such that (4) has at least one simple eigenvalue λ_0. Then, there is $\epsilon_0 > 0$ such that if

$$\rho_E D_0 |\boldsymbol{b}| \leq \epsilon_0 (\mu_E + \lambda_E)$$

the nonlinear problem has at least one solution.

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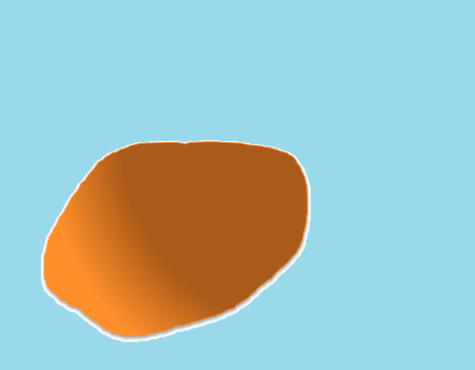
Consider a homogeneous rigid body, $\mathcal{R}$, in the shape of Ω in a Stokes liquid, $\mathcal{L}$, under the action of a constant force.



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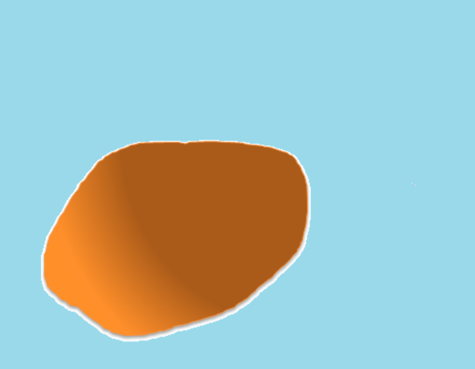

The solvability of the linearized problem implies that the system $\mathcal{R} - \mathcal{L}$ can always execute a steady motion

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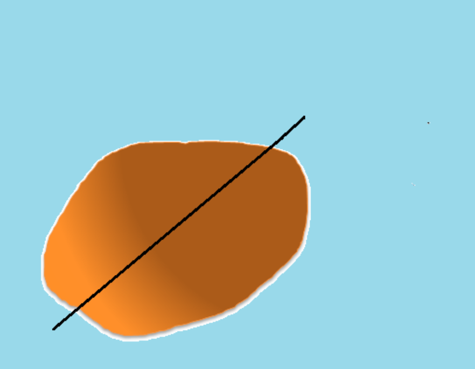

The solvability of the linearized problem implies that the system $\mathcal{R} - \mathcal{L}$ can always execute a steady motion

Sketch of the Strategy of Proof and Main Results

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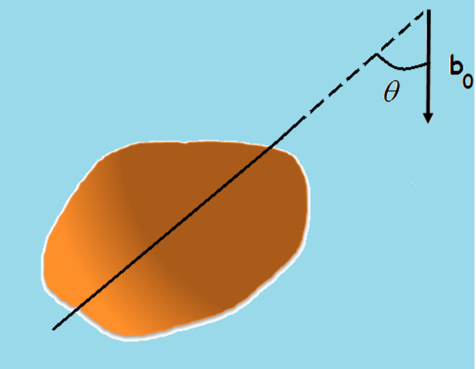

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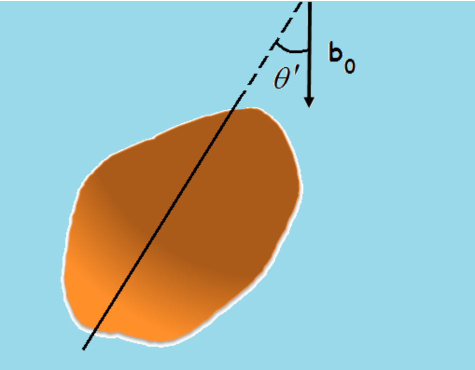

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OPEN QUESTION: Do steady-state regimes exist for reference configurations of arbitrary shape?

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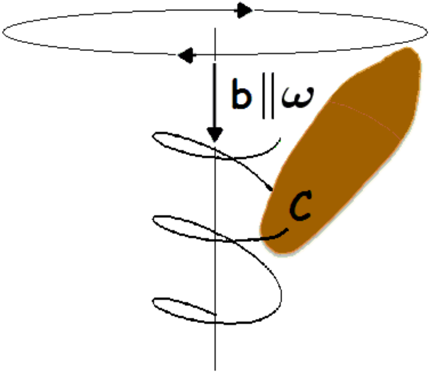

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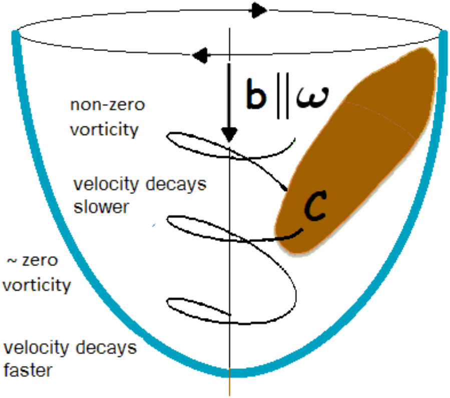

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Questions 2–4 are open also in the case of a rigid body.