A Lipschitz stability estimate for the Stokes system with Robin boundary condition

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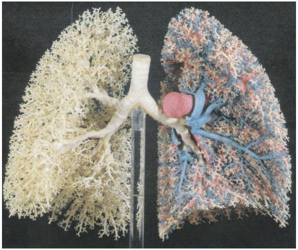
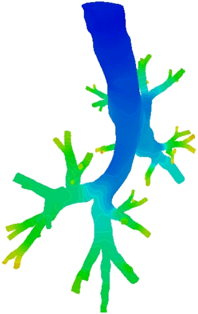
Motivations



Molding of human lung realized by E. R. Weibel.



Reconstructed bronchial tree,

[Baffico, Grandmont, Maury '10].

- airflow in the lungs,

  [Baffico, Grandmont, Maury '10],

- blood flow in the cardiovascular system,

  [Quarteroni, Alessandro '03],
  [Vignon-Clementel, Figueroa , Jansen, Taylor '06].

Our problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded connected open set.

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}. \end{array}\right. \qquad (P_q)$$

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Let $\Gamma \subseteq \Gamma_0$ and (u_j, p_j) be solution of (P_{q_j}) for $j = 1, 2$.

- **Uniqueness:** Does $\mathcal{M}_\Gamma(u_1, p_1) = \mathcal{M}_\Gamma(u_2, p_2)$ implies $q_1 = q_2$?
- **Stability:** Is it possible to obtain stability estimate like

$$\|(q_1 - q_2)_{|\Gamma_{out}}\| \leq f\left(\|(u_1 - u_2)_{|\Gamma}\| + \|(p_1 - p_2)_{|\Gamma}\|\right),$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $\lim_{x\to 0} f(x) = 0$?

State of the art

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ div\ u &= 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}. \end{cases} \qquad (\mathcal{P}_q)$$

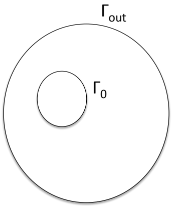

Figure: Example of such an open set Ω in dimension 2.

- $\Gamma_0 \cup \Gamma_{out} = \partial\Omega$,
- $\overline{\Gamma}_0 \cap \overline{\Gamma}_{out} = \emptyset$.

State of the art: logarithmic stability estimates

$\rightsquigarrow$ Logarithmic stability estimates (Boulakia, E., Grandmont).

We point out the main differences between the logarithmic stability estimates obtained:

Regularity on Ω	Regularity needed on (u,p)	Valid in dimension
$\mathcal{C}^{3,1}$	$(u,p) \in H^4(\Omega) \times H^3(\Omega)$	2
locally $\mathcal{C}^\infty$	$(u,p) \in H^{2+k}(\Omega) \times H^{1+k}(\Omega)$ for $k \in \mathbb{N}^*$ such that $k+2 > \frac{d}{2}$	in any dimension

An identifiability result

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}. \end{array}\right. \qquad (P_q)$$

As a corollary of a unique continuation result proved by Fabre and Lebeau '96, we obtain the following proposition.

Proposition

Let $x_0 \in \Gamma_0$, $r > 0$ and $(u_j, p_j) \in H^1(\Omega) \times L^2(\Omega)$ be solution of (P_{q_j}) for $j = 1, 2$. We assume:

- $u_1 = u_2$ *on* $\Gamma_0 \cap \mathcal{B}(x_0, r)$.

Then, $q_1 = q_2$.

Main result: a Lipschitz stability estimate

Let $\eta > 0$, $\Gamma \subseteq \{x \in \Gamma_0 \backslash d(x, \partial\Omega \backslash \Gamma_0) > \eta\}$ be a non empty set. Under some regularity assumption

- locally on the open set Ω,
- on the data,
- and under the *a priori* assumption that the **Robin coefficient** q is **piecewise constant**,

we prove that there exists $C > 0$ such that

$$\|q_1 - q_2\|_{L^\infty(\Gamma_{out})} \leq C \left(\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right).$$

Main tools

The main tools are:

- estimates for the unique continuation properties of the Stokes system,
- a sequence of balls $(B_k)_{k\in\mathbb{N}}$ whose center approach the boundary,
- local Hölder regularity.

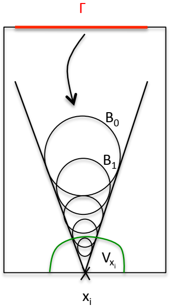

Figure: Figure illustrating how informations spread.

Thank you!