Exact Controllability for a System of Coupled Wave Equations on a Compact Manifold[1]

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[1] Joint work with M.Leautaud and J. Le Rousseau

Position of the problem

$$\begin{cases} \Box u_1 + b(x) u_2 = 0 & \text{in} \quad]0,+\infty[\times M \\ \Box u_2 = b_\omega(x) f & \text{in} \quad]0,+\infty[\times M \\ \text{Initial Data in } (H^2 \times H^1) \times (H^1 \times L^2) \end{cases} \qquad \text{(S)}$$

$\rightarrow$ (M, g) compact Riemannian manifold without boundary.

$\rightarrow$ $\Box = \partial_t^2 - \Delta_g$.

- f is the control.
- $b(x)$, $b_\omega(x)$ both real and smooth, and $b(x) \geq 0$.
- Notice the shift between the two energy levels.

Denote

$$O = \{x \in M, b(x) > 0\} \qquad \text{coupling set}$$

$$\omega = \{x \in M,\ b_\omega(x) \neq 0\} \qquad \text{control set}$$

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The Goal

- Exact controllabilty at (some fixed) time T.
- Analysis (quantization) of the HUM optimal control operator.

Some References

- F. Alabau-Boussouira (starting from 99').
- F.Alabau-Boussouira-M.Leautaud (11'): symmetric systems, long control time.
- L.Rosier - L.de Teresa (11'):1-D , geometric but not sharp control time.
- A.Benabdallah et al (thermo-elasticity 96', reaction-diffusion 07', systems of parabolic equations 10')
- D-Lebeau (09'): quantization of the HUM control operator for the scalar wave equation.

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The crucial point: what are the geometric constraints on the open sets O and ω and the control time T ?

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$\gamma^2 \neq 1$,

$$\begin{cases} (\partial_t^2 - \Delta) u_1 = -b(x) u_2 & H^2 \times H^1 \\ (\partial_t^2 - \gamma^2 \Delta) u_2 = \ b_\omega(x) f & H^1 \times L^2 \end{cases}$$

For $f \in L^2\ (]0, T[\times M)$, $\quad u_2 \in H^2$ outside $\{\tau^2 = \gamma^2 \, |\xi|^2\}$.
Therefore, starting from $(0,0)$, $u_1 \in H^3(]0, T[\times M)$.
Actually, $u_1 \in C(]0, T[, H^3) \cap C^1(]0, T[, H^2)$.

$\Rightarrow$ **We can not reach any given state in $H^2 \times H^1$!**

Theorem

Assume that $\overline{\omega} \cap \overline{O}$ *does not satisfy GCC and* $\gamma \neq 1$.
For all $s \geq 1$ *and all* $T > 0$, *there exists* $(u_1^0, u_1^1) \in H^{s+1}(M) \times H^s(M)$ *such that for all* $f \in L^2(]0, T[\times M)$, *the solution to system*

$$\begin{cases} (\partial_t^2 - \Delta)u_1 + b(x)u_2 = 0 \\ (\partial_t^2 - \gamma^2\Delta)u_2 = \ b_\omega(x)f \\ (u_1, \partial_t u_1)_{|t=0} = (u_1^0, u_1^1) \\ (u_2, \partial_t u_2)_{|t=0} = (0, 0) \end{cases}$$

satisfies $(u_1(T), \partial_t u_1(T), u_2(T), \partial_t u_2(T)) \neq (0, 0, 0, 0)$.

Remark: No control in any energy space !

Theorem

For any bicharacteristic curve Γ *of the wave operator* $\Box$ *and any* $s > 1$, *there exists a finite energy solution* u *of*

$$\begin{cases} \Box u = 0 & \text{in }]0, T[\times M \\ (u(0), \partial_t u(0)) \in H^1 \times L^2 \end{cases}$$

such that $WFu = \Gamma$ *(resp.* $WF^s u = \Gamma$*).*

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Theorem

There exists a sequence of solutions u^k , *such that*

$$\liminf \left\| (u^k(0)\ , \partial_t u^k(0)) \right\|^2_{H^1 \times L^2} \geq 1, \qquad u^k \rightharpoonup 0 \quad in\ H^1(]0, T[\times M)$$

and the H^1- *microlocal defect measure* μ *of* u^k *satisfies supp*(μ)$\subset \Gamma$.

Assumption

$$(O, T_O) \quad \text{and} \quad (\omega, T_\omega) \quad \text{satisfy (GCC)}$$

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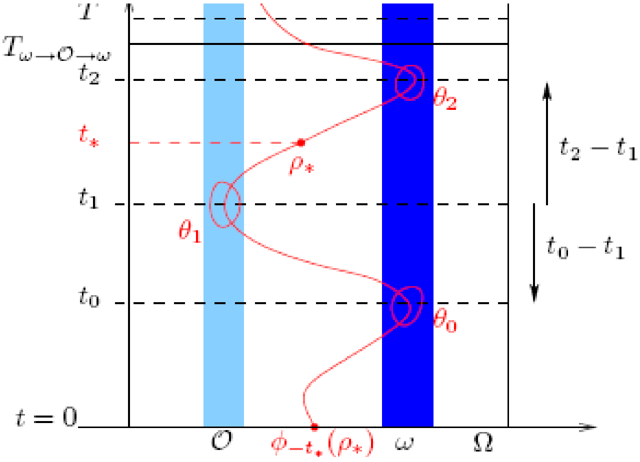
The optimal control time

Definition

$T_{\omega\to O\to\omega}$ is the infimum of the times $T > 0$ satisfying the following: Every geodesic travelling with speed 1 in M meets ω in a time $t_0 < T$, then meets O in a time $t_1 \in (t_0, T)$ and meets again ω in a time $t_2 \in (t_1, T)$.



Remarks

In general

- $T_{\omega \to O \to \omega} \neq T_{O \to \omega \to O}$.
- $\max(T_\omega, T_O) \leq T_{\omega \to O \to \omega} \leq 2T_\omega + T_O$.

Theorem

(D-Le Rousseau-Leautaud)
Assume that ω and O both satisfy (GCC). Then system (S) is controlable

if $T > T_{\omega \to O \to \omega}$ and is not controlable if $T < T_{\omega \to O \to \omega}$.

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Remarks

- No condition on the set $O \cap \omega$.
- The optimal control time only depends on the geometry of (Ω, ω, O).
- Case of a smooth domain of $\mathbb{R}^d$: work in progress.

The observability estimate

The adjoint system

$$\begin{cases} \Box v_1 = 0 & \text{in} \quad]0, T[\times M \\ \Box v_2 + b(x) v_1 = 0 & \text{in} \quad]0, T[\times M \\ \text{I. D} & \text{in } (H^{-1} \times H^{-2}) \times (L^2 \times H^{-1}) \end{cases} \qquad (S^*)$$

$$E_{-1}(v_1) + E_0(v_2) \leq c \int_0^T \int_M | \; b_\omega v_2|^2 \, dxdt$$

where

$$E_{-k}(v) = \|(v, \partial_t v)(0)\|^2_{H^{-k} \times H^{-k-1}}$$

Change of functions

$$w_1 = (1-\Delta)^{-1/2} v_1, \qquad w_2 = v_2$$

$$\begin{cases} \Box w_1 = 0 \quad \text{in} \quad]0,+T[\times M \\ \Box w_2 + b(x)(1-\Delta)^{1/2} w_1 = 0 \quad \text{in} \quad]0,T[\times M \\ \text{I. D} \quad \text{in } (L^2 \times H^{-1}) \times (L^2 \times H^{-1}) \end{cases}$$

Observability Estimate

$$E_0(w_1) + E_0(w_2) \leq c \int_0^T \int_M |\, b_\omega w_2|^2 \, dxdt$$

Sketch of the proof

Relaxed Observability Estimate

$$E_0(w_1) + E_0(w_2) \leq c_1 \int_0^T \int_M |\, b_\omega w_2|^2 \, dxdt + c_2 (E_{-1}(w_1) + E_{-1}(w_2))$$

Sketch of the proof

Relaxed Observability Estimate

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Contradiction argument

$$\begin{cases} E_0(w_1^k) + E_0(w_2^k) = 1 \\ \int_0^T \int_M |\, b_\omega w_2^k|^2 \, dxdt + E_{-1}(w_1^k) + E_{-1}(w_2^k) \leq 1/k \end{cases}$$

(w_j^k) is bounded in $L^2(]0, T[\times M)$ and converges to 0 in $H^{-1}(]0, T[\times M)$. Hence

$$w_j^k \rightharpoonup 0 \qquad \text{in} \qquad L^2(]0, T[\times M) \quad \text{weakly}$$

It remains to:

a) Prove the strong convergence (propagation of the m.d.m's).

b) Drop the compact term in the RHS of the relaxed observabilty estimate. (unique continuation).

Proof of b) assuming a).

$$E_0(w_1) + E_0(w_2) \leq c \int_0^T \int_M |\, b_\omega w_2|^2 \, dxdt \qquad \textbf{???}$$

Again by contradiction, one gets that the weak limit (w_1, w_2) satisfies

$$\begin{cases} \Box w_1 = 0 \quad \text{in} \quad]0,T[\times M \\ \Box w_2 + b(x)(1-\Delta)^{1/2} w_1 = 0 \quad \text{in} \quad]0,T[\times M \\ w_2 = 0 \quad \text{in} \quad]0,T[\times \omega \end{cases} \qquad \text{(L.S)}$$

$$\mathcal{N}(T) = \{(w_1, \partial_t w_1, w_2, \partial_t w_2) \in (L^2 \times H^{-1})^2, \text{solution of (L.S)}\}$$

$\mathcal{N}(T)$ is of finite dimension and stable by the action of $\partial/\partial t$.

$$\begin{cases} \Delta w_1 = \lambda^2 w_1 \\ \\ \Delta w_2 - b(x)(1-\Delta)^{1/2} w_1 = \lambda^2 w_2 \end{cases}$$

$$\int_M b(x) \left| (1-\Delta)^{1/2} w_1 \right|^2 = 0$$

So $w_1 = w_2 \equiv 0$. Contradiction since the weak limit satisfies the relaxed estimate

$$1 \leq c_2(E_{-1}(w_1) + E_{-1}(w_2)) = 0$$

Proof of a)

$$\begin{cases} \Box w_1^k = 0 \\ \Box w_2^k + b(x)(1-\Delta)^{1/2} w_1^k = 0 \\ w_2^k \to 0 \quad \text{in} \quad L^2(]0,T[\times\omega) \end{cases}$$

μ_1 a microlocal defect measure attached to (w_1^k) in $L^2(]0,T[\times M)$
μ_2 .... to (w_2^k) in ....
μ_{12} .... to (w_1^k, w_2^k) in ....

$$\begin{cases} H_p \mu_1 = 0 \\ H_p \mu_2 = 2b\,|\eta|\, \mathrm{Im}\,\mu_{12} \\ H_p\, \mathrm{Im}\,\mu_{12} = b\,|\eta|\, \mu_1 \\ H_p\, \mathrm{Re}\,\mu_{12} = 0 \end{cases}$$

And

$$\mu_2 = \mu_{12} = 0 \quad \text{over }]0, T[\times\omega$$

Take $\rho \in B \subset S^*(]0, T[\times O)$ where B is a small borelian set of $S^*(]0, T[\times M)$.

$$\mathrm{Im}\,\mu_{12}(\Phi_{-T_1}(B)) - \mathrm{Im}\,\mu_{12}(\Phi_{T_2}(B))) = \int_{-T_1}^{T_2} b\,|\eta|\,\mu_1(\Phi_s(B))ds = 0$$

Hence $\rho \notin$supp(μ_1).....and conclude by (GCC).

An abstract control

→ $\exp(itA)$, $t \geq 0$, a semi-group of contractions on a hilbert space H .
→ B bounded operator on H and $g \in L^1([0,T],H)$

$$(\partial_t - iA)f = Bg, \qquad f(0) = 0$$

We choose g solution of

$$(\partial_t - iA^*)g = 0, \qquad g(T) = g_0$$

Observation

$$\int_0^T \left\| B^* e^{-itA^*} h \right\|_H^2 dt \geq C \left\| h \right\|_H^2$$

The inverse of the HUM optimal control operator (**Gramian**) is given by

$$M_T = \int_0^T e^{itA} BB^* e^{-itA} dt = \Lambda^{-1}$$

Notations

$(e_j, \omega_j^2)_{j\geq 0}$ the spectral elements of M, $\quad (\omega_0 = 0)$,

$$-\Delta e_j = \omega_j^2 e_j, \qquad \|e_j\|_{L^2(\Omega)} = 1$$

$$H^s(M) = \{u = \sum_j a_j e_j, \quad \sum_j (1+\omega_j^2)^s |a_j|^2 < \infty\} = \mathcal{D}((-\Delta)^{s/2})$$

$$\lambda = \lambda(x, D_x) = \sqrt{-\Delta}, \qquad \lambda(x, D_x) \sum_{j\geq 0} a_j e_j = \sum_{j\geq 0} \omega_j a_j e_j$$

$$\Pi_+ \sum_{j\geq 0} a_j e_j = \sum_{j\geq 1} a_j e_j, \qquad \Pi_0 \sum_{j\geq 0} a_j e_j = a_0 e_0$$

$$L^2_+(M) = \Pi_+ L^2(M)$$

λ, Π_0 and Π_+ are pseudodifferential operators of order 1 and 0 (use the Helffer-Sjöstrand formula).

Change of space, splitting and identification

$$\begin{cases} (H^{-1} \times H^{-2}) \times (L^2 \times H^{-1}) \longleftrightarrow (L^2 \times H^{-1}) \times (L^2 \times H^{-1}) \\ (v_1, \partial_t v_1, v_2, \partial_t v_2) \longleftrightarrow ((1-\Delta)^{-1/2} v_1, \partial_t (1-\Delta)^{-1/2} v_1, v_2, \partial_t v_2) \end{cases}$$

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For $(w_1^0, w_1^1), (w_2^0, w_2^1) \in L^2 \times H^{-1} = L^2_+ \times H^{-1}_+ \oplus \mathbb{C}^2$, wet set

$$\begin{cases} w_1^0 = h_+ + h_- + c_0 e_0 \\ w_1^1 = i\lambda(h_+ - h_-) + c_1 e_0 \end{cases}$$

with $h_\pm \in L^2_+$, and $(c_0, c_1) \in \mathbb{C}^2$. And similarly

$$\begin{cases} w_2^0 = g_+ + g_- + d_0 e_0 \\ w_2^1 = i\lambda(g_+ - g_-) + d_1 e_0 \end{cases}$$

This gives an identification between $(L^2 \times H^{-1})^2$ and $(L^2_+(M))^4 \oplus \mathbb{C}^4$ by an elliptic pseudodifferential operator.

More precisely, we use the identification

$$\begin{array}{rcl} (L^2 \times H^{-1})^2 & \longrightarrow & (L^2_+(M))^4 \oplus \mathbb{C}^4 \\ (w_1^0, w_1^1, w_2^0, w_2^1) & \longrightarrow & (h_+, g_+, h_-, g_-, c_0, c_1, d_0, d_1) \end{array}$$

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We will express the Gramian M_T and the HUM control operator Λ in the space

$$H = (L^2_+(M))^4 \oplus \mathbb{C}^4$$

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For $(x, \xi, t, \tau,) \in T^*M \backslash 0$, $\tau = |\xi|_x$, denote

$$s \rightarrow \gamma_s^{\pm} = (\gamma_{(x,\xi)}(\pm s), t - s\tau, \tau) \quad s \in \mathbb{R}$$

the bicharacteristic curve of the wave operator issued from $(x, \xi, t, \tau = |\xi|_x)$.

Theorem

Assume that ω and O satisfy GCC and $T > T_{\omega\to O\to\omega}$; then in the splitting above, the Gramian operator takes the form

$$M_T = \Pi_+ \begin{pmatrix} M_T^+ & 0 \\ 0 & M_T^- \end{pmatrix} \Pi_+ + R_T$$

where R_T is a 1-smoothing operator and $M_T^\pm$ is an elliptic pseudo-differential operator of order 0.
In addition, the principal symbol $\sigma_0(M_T^+)$ of M_T^+ is given by

$$\begin{pmatrix} \frac{1}{4}\int_0^T b_\omega^2(\gamma_t^+)(\int_0^t b((\gamma_s^+)ds)^2 dt & -\frac{1}{2i}\int_0^T b_\omega^2(\gamma_t^+)(\int_0^t b((\gamma_s^+)ds)dt \\ \frac{1}{2i}\int_0^T b_\omega^2(\gamma_t^+)(\int_0^t b((\gamma_s^+)ds)dt & \int_0^T b_\omega^2(\gamma_t^+)dt \end{pmatrix}$$

and

$$\det(\sigma_0(M_T^+)) = \frac{1}{8}\int_0^T\int_0^T b_\omega^2(\gamma_{t_1}^+)b_\omega^2(\gamma_{t_2}^+)(\int_{t_1}^{t_2} b((\gamma_s^+)ds)^2 dt_1 dt_2$$

Corollary

Under the assumptions above and with $L_\pm = (M_T^\pm)^{-1}$, the HUM control operator Λ takes the form

$$\Lambda = \Pi_+ \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} \Pi_+ + \widetilde{R}_T$$

where $\widetilde{R}_T$ is a 1-smoothing operator.
In particular, Λ is an isomorphism on $(H^s \times H^s)^2$ for every $s \in \mathbb{R}$.

Remarks

- Notice the explicit reading of the geometric condition $(\omega, O, T_{\omega\to O\to\omega})$.
- Λ preserves the WF set(s) of the data to be controlled.
- Λ "commutes" to spectral localization...
- $\widetilde{R}_T$ is a Fourier Integral Operator of order (-1).
- **Key**: Egorov Theorem.

Egorov Theorem

$$\begin{cases} \partial_t u = iA(x;D_x)u & \text{in } \mathbb{R}\times M \\ u(0) = u_0 \end{cases}$$

$\rightarrow A = A_1 + A_0, \qquad A_1(x;\xi) \in S^1_{cl}$ real, $\quad A_0(x;\xi) \in S^0_{cl}$
$\rightarrow A_1(x;\xi)$ homogeneous in ξ for $|\xi| \geq 1$

$$u(t,x) = \exp(itA)u_0$$

$\exp(itA)$ is bounded on each $H^\sigma(M)$, with inverse $\exp(-itA)$.

Egorov Theorem

If $P_0 = p_0(x,D) \in OPS^m_{1,0}$, then for every t, the operator

$$P(t) = \exp(itA)P_0\exp(-itA)$$

belongs to $OPS^m_{1,0}$, modulo a smoothing operator. The principal symbol of $P(t)$ (mod $S^{m-1}_{1,0}$) at (x_0,ξ_0) is equal to $p_0(\gamma(t))$ where γ is the bicharacteristic of A_1 issued from (x_0,ξ_0).

The case of two different speeds

Theorem

Let $\gamma \neq 1$*. For every initial data* $(u_1^0, u_1^1) \in H^3 \times H^2$, $(u_2^0, u_2^1) \in H^1 \times L^2$ *and* $F \in L^1(0, T; L^2)$, *the usual solution of system*

$$\begin{cases} (\partial_t^2 - \Delta) u_1 + b(x) u_2 = 0 \\ (\partial_t^2 - \gamma^2 \Delta) u_2 = F \end{cases} \tag{S_γ}$$

satisfies the additional regularity

$$u_1 \in \cap_{k=0}^3 C^k(0, T; H^{3-k})$$

and we have the continuity estimate

$$\sum_{k=0}^{3} \left\| \partial_t^k u_1 \right\|_{L^\infty(0,T;H^{3-k})} + \sum_{k=0}^{1} \left\| \partial_t^k u_2 \right\|_{L^\infty(0,T;H^{1-k})} \leq C \left\| Data \right\|$$

Lemma

Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, *and* $b(x, D)$ *a 0-order pseudodifferential operator. Then the operator defined by*

$$A(t) = \int_0^t \exp(-i\alpha s\lambda) b(x, D_x) \exp(i\beta s\lambda) ds$$

satisfies $A(t) \in C^0(\mathbb{R}, \mathcal{L}(H^\sigma, H^{\sigma+1}))$, *for every* $\sigma \in \mathbb{R}$.
In particular, $A(t)$ *is 1-smoothing.*

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satisfies $A(t) \in C^0(\mathbb{R}, \mathcal{L}(H^\sigma, H^{\sigma+1}))$, for every $\sigma \in \mathbb{R}$. In particular, $A(t)$ is 1-smoothing.

Integration by parts:

$$\begin{aligned}(\alpha - \beta)\lambda A(t) &= \beta \int_0^t \exp(-i\alpha s\lambda)[b, \lambda] \exp(i\beta s\lambda) ds - ib(x, D) \\ &\quad + i \exp(-i\alpha t\lambda) b(x, D) \exp(i\beta t\lambda)\end{aligned}$$

Theorem

Assume that $\omega \cap O$ satisfies GCC; than system (S_γ) is exactly controlable

in the space $(H^3 \times H^2) \times (H^1 \times L^2)$, *in a time* $T > \max(T^1_{\omega\cap O}, T^\gamma_{\omega\cap O})$.