Inverse problems associated with linear and non-linear parabolic systems

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Two parts

I. Stable reconstruction of coefficients in linear and weakly nonlinear coupled systems.
Joint works with A. Benabdallah, L. De Teresa, P. Gaitan, H. Ramoul and M. Yamamoto (Aix-Marseille, Kemchela, Mexico, Tokyo)

II. Determination of multi coefficients in a system with strong nonlinear coupled parabolic models (Lotka Volterra).
Joint work with L. Roques (INRA France).

Part I

Stable reconstruction of coefficients in linear and weakly nonlinear coupled systems.

Carleman Estimates and Inverse Problems

- Bukhgeim, Klibanov, 81
  Hölder stability results with local Carleman estimates.

- Imanuvilov, Yamamoto, 98
  Lipschitz stability results with global Carleman estimates.
  Parabolic problems.

Carleman Estimates and Inverse Problems

Imanuvilov-Yamamoto (98) : **Sources (or potentials) identification.**

$$\begin{cases} y' - \Delta y + q(x)y = f & \text{in} \quad \Omega \times (0,T), \\ y = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

Let $\omega \subset \Omega$ a non empty subset,

$$\exists\, C > 0,\ \|f\|^2_{L^2(\Omega)} \leq C \left(\|y(.,\theta)\|^2_{H^2(\Omega)} + \|y\|^2_{H^1(0,T;L^2(\omega))} \right)$$

In particular, if $y(.,\theta) = 0$:

$$\|f\|^2_{L^2(\Omega)} \leq C \|y\|^2_{H^1(0,T;L^2(\omega))}$$

2×2 Reaction-Diffusion Systems

Joint work with P. Gaitan and H. Ramoul

Inverse Problems (2006)

Reaction-diffusion systems

- $\Omega \subset \mathbb{R}^n$ ($n \leq 3$) bounded domain. $\Gamma = \partial\Omega$.
- $T > 0$, $Q_T = \Omega \times (0, T)$, $\Sigma_T = \Gamma \times (0, T)$

$$\begin{cases} \partial_t u = \Delta u + a(x)u + b(x)v & \text{in} \quad Q_T, \\ \partial_t v = \Delta v + c(x)u + d(x)v & \text{in} \quad Q_T, \\ u = g, \;\; v = h & \text{on} \quad \Sigma_T, \\ u(0) = u_0 \;\; \text{et} \;\; v(0) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

- $a, b, c, d \in \Lambda(R) = \{\Phi \in L^\infty(\Omega);\; \|\Phi\|_{L^\infty(\Omega)} \leqslant R\}$,

$$\begin{cases} r > 0, \;\; r_0 > 0 \text{ such that } a \geq r_0, \;\; c \geq r_0, \;\; \widetilde{u}_0 \geq r, \;\; \widetilde{v}_0 \geq r, \\ \widetilde{b} > 0, \;\; \widetilde{c} > 0, \;\; ar + b \geq 0, \;\; c + dr \geq 0 \text{ and } g \geq r, \;\; h \geq r. \end{cases}$$

Observability estimate for reaction-diffusion systems

Recall

$$I(\tau, q) = \int_0^T \int_\Omega e^{-2s\eta}(s\varphi)^{\tau-1} \left(|\partial_t q|^2 + |\Delta q|^2 + (s\lambda\varphi)^2 |\nabla q|^2 + (s\lambda\varphi)^4 |q|^2 \right) dx\, dt.$$

$$\text{with } \varphi(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{t(T-t)}$$

$$\beta \in \mathcal{C}^2(\Omega),\ \partial_\nu \beta < 0 \text{ on } \partial\Omega,\ |\nabla\beta| \neq 0 \text{ on } \overline{\Omega \setminus \omega}.$$

Then,

$$I(\tau, \varphi) \leq \widetilde{C}_0 \left(\iint_{\Omega_T} (s\rho)^\tau e^{-2s\eta} |\partial_t \varphi - \Delta\varphi|^2\, dxdt + \lambda^4 \iint_{\omega_T} (s\rho)^{\tau+3} e^{-2s\eta} |\varphi|^2\, dxdt \right)$$

for every $\varphi \in L^2(0, T; H_0^1(\Omega))$, s and λ sufficiently large.

Linearized Inverse Problem

Let (u, v) and $(\widetilde{u}, \widetilde{v})$ be solutions to

$$\begin{cases} \partial_t u = \Delta u + au + bv & \text{in } Q_T, \\ \partial_t v = \Delta v + cu + dv & \text{in } Q_T, \\ u = g, \ \ v = h & \text{on } \Sigma_T, \\ u(0) = u_0 \ \text{ and } \ v(0) = v_0 & \text{in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \widetilde{u} = \Delta \widetilde{u} + a\widetilde{u} + \widetilde{b}\widetilde{v} & \text{in } Q_T, \\ \partial_t \widetilde{v} = \Delta \widetilde{v} + c\widetilde{u} + d\widetilde{v} & \text{in } Q_T, \\ \widetilde{u} = g, \ \ \widetilde{v} = h & \text{on } \Sigma_T, \\ \widetilde{u}(0) = \widetilde{u}_0 \ \text{ and } \ \widetilde{v}(0) = \widetilde{v}_0 & \text{in } \Omega. \end{cases}$$

Denote $U = u - \widetilde{u}$, $V = v - \widetilde{v}$, $y = \partial_t(u - \widetilde{u})$, $z = \partial_t(v - \widetilde{v})$, $\gamma_1 = b - \widetilde{b}$.
Then (y, z) is solution to

$$\begin{cases} \partial_t y = \Delta y + ay + bz + \gamma_1 \partial_t \widetilde{v} & \text{in } Q_T, \\ \partial_t z = \Delta z + cy + dz & \text{in } Q_T, \\ y = z = 0 & \text{on } \Sigma_T, \\ y(0) = \Delta U(0) + aU(0) + bV(0) + \gamma_1 \widetilde{v}(0), & \text{in } \Omega, \\ z(0) = \Delta V(0) + cU(0) + dV(0) & \text{in } \Omega. \end{cases} \tag{2}$$

Global Carleman Estimate

$$s^3\lambda^4 \int_0^T\int_\omega e^{-2s\eta}\varphi^3|y|^2\,dx\,dt \le Cs^7\lambda^8 \int_0^T\int_\omega e^{-2s\eta}\varphi^7|z|^2\,dx\,dt$$

Theorem

We assume $a,\ b,\ c,\ d\ \in \Lambda(R)$ and that exists $c_0 > 0$ such that $c \ge c_0$ in ω. Then there exist $\lambda_1 = \lambda_1(\Omega,\omega) \ge 1$, $s_1 = s_1(\lambda_1, T) > 1$ and a positive constant $C_1 = C_1(\Omega,\omega,c_0,R,T)$ such that, for any $\lambda \ge \lambda_1$ and any $s \ge s_1$, the following inequality holds:

$$I(y)+I(z) \le C_1 \left[s^7\lambda^8 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^7|z|^2\,dx\,dt + \iint_Q e^{-2s\eta}\,|\gamma\partial_t\widetilde{v}|^2\,dx\,dt \right], \tag{3}$$

for any solution (y, z) of (2).

Stability Theorem

We obtain the following stability result:

Theorem

Suppose there exists $\theta \in (0, T)$ *such that* $u(\theta, \cdot) = \widetilde{u}(\theta, \cdot)$, $v(\theta, \cdot) = \widetilde{v}(\theta, \cdot)$. *For* u_0, v_0, $\widetilde{u}_0$, $\widetilde{v}_0$ *in* $H^2(\Omega)$, *there exists a constant* $C > 0$,

$$C = C(\Omega, \omega, c_0, T, r, R)$$

such that

$$\|b - \widetilde{b}\|^2_{L^2(\Omega)} \leq C\|\partial_t v - \partial_t \widetilde{v}\|^2_{L^2(\omega_T)}$$

2 $\times$ 2 Reaction-Diffusion-Convection Systems

Joint works with

1. **A. Benabdallah, P. Gaitan, M. Yamamoto**
   Applicable Analysis, Vol 88 (5), 2009
2. **A. Benabdallah, L. De Teresa, P. Gaitan**
   CRAS, 2010 and submission to MCRF, 2012

2 $\times$ 2 Reaction-Diffusion-Convection Systems

- $\Omega \subset \mathbb{R}^n$ smooth bounded domain, $\omega \subset\subset \Omega$ subdomain, $\theta \in (0, T)$,
- $\Omega_T := \Omega \times (0, T)$, $\Sigma_T := \partial\Omega \times (0, T)$, $\omega_T := \omega \times (0, T)$.

(u, v) solution to

$$\begin{cases} \partial_t u = \Delta u + au + bv + A \cdot \nabla u + B \cdot \nabla v \\ \partial_t v = \Delta v + cu + dv + C \cdot \nabla u + D \cdot \nabla v \quad \text{in} \quad \Omega_T \\ u = h_1, \; v = h_2 \quad \text{on} \quad \Sigma_T, \\ u(0) = u_0, \; v(0) = v_0 \quad \text{in} \quad \Omega, \end{cases}$$

$(\widetilde{u}, \widetilde{v})$ solution to system where b, c are replaced by $\widetilde{b}, \widetilde{c}$.

Inverse Problem

Determine $b(x), c(x)$ from $u|_{\omega_T}$ and $(u, v)|_{\Omega \times \{\theta\}}$

Assumptions

- All the coefficients are in $C^2(\overline{\Omega})$ (for simplicity)
- $\|b\|_{L^\infty}, \|\widetilde{b}\|_{L^\infty}, \|c\|_{L^\infty}, \|\widetilde{c}\|_{L^\infty} \leq M$ fixed constant
- $\|\widetilde{u}\|_{C(\overline{\Omega_T})}, \ \|\widetilde{v}\|_{C(\overline{\Omega_T})}, \|\widetilde{u}\|_{C^3(\omega_T)}, \ \|\widetilde{v}\|_{C^3(\omega_T)} \leq M$
- $\partial\omega \cap \partial\Omega = \gamma$, $|\gamma| \neq 0$
- $|B(x) \cdot \nu(x)| \neq 0$ on γ
  ν: unit outward normal to $\partial\Omega$
- $|\widetilde{u}(\cdot, \theta)| \geq \delta_0 > 0$, $|\widetilde{v}(\cdot, \theta)| \geq \delta_0 > 0$ in $\overline{\Omega}_T$.

2 × 2 Reaction-Diffusion-Convection Systems - First Result

- $y = u - \widetilde{u}$, $z = v - \widetilde{v}$
- $f = (b - \widetilde{b})\widetilde{u}$, $g = (c - \widetilde{c})\widetilde{v}$

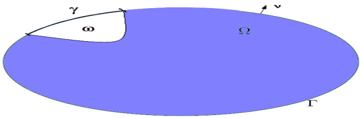
(y, z) solution to

$$\begin{cases} \partial_t y = \Delta y + ay + bz + A \cdot \nabla y + B \cdot \nabla z + f \\ \partial_t z = \Delta z + cy + dz + C \cdot \nabla y + D \cdot \nabla z + g \quad \text{in} \quad \Omega_T \\ y = z = 0 \quad \text{on} \quad \Sigma_T, \\ y(0) = y_0,\ z(0) = z_0 \quad \text{in} \quad \Omega, \end{cases}$$

$$Lz := B \cdot \nabla z + bz = \partial_t y - \Delta y - ay - A \cdot \nabla y - f$$

Local estimate on ω of z with respect to y
Carleman estimate for a first-order operator.

Carleman Estimate for a First-Order Operator



Lemma

$$|B(x)\cdot\nu(x)| \neq 0, \quad x \in \gamma \quad \textit{and} \quad Lz := B\cdot\nabla z + bz.$$

Then, there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega, \omega, T)$ such that for all $\lambda \geq \lambda_1$ and $s \geq s_1$

$$s^2\lambda^2\|z\|^2_{L^2(\omega)} \leq C\|Lz\|^2_{L^2(\omega)}$$

for $z = 0$ on $\gamma \subset \partial\omega$.

Carleman estimate with data of one component

We denote

$$I(\tau,\varphi)=\int_{\Omega_T}(s\rho)^{\tau-1}e^{-2s\eta_\omega}\left(|\varphi_t|^2+|\Delta\varphi|^2+(s\lambda\rho)^2|\nabla\varphi|^2+(s\lambda\rho)^4|\varphi|^2\right)dx\,dt.$$

$$\eta_\omega(x,t):=\frac{\alpha_\omega(x)}{t(T-t)}\quad\text{with}\quad\rho(t):=\frac{1}{t(T-t)},\quad\alpha_\omega(x)=e^{2\lambda K}-e^{\lambda\beta(x)}$$

$$\beta\in\mathcal{C}^2(\Omega),\ \partial_\nu\beta<0\ \text{ on }\ \partial\Omega,\ |\nabla\beta|\neq 0\ \text{ on }\ \overline{\Omega\setminus\omega}.$$

Carleman estimate with data of one component

Theorem

There exist $\alpha_\omega \in C^2(\overline{\Omega})$, > 0 $(\overline{\Omega})$ *and* $s_0, \kappa > 0$ *such that*

$$I(\tau, y) + I(\tau, z) \leq \kappa_1(s,\tau)(\|y\|^2_{W_2^{2,1}(\omega_T)} + \|f\|^2_{L^2(\omega_T)})$$

$$+ \quad \kappa \int_{\Omega_T} (s\rho)^\tau e^{-2s\eta_\omega}(|f|^2 + |g|^2)dxdt$$

for all $s \geq s_0$, $\tau \geq 1$ *and*

$$y, z \in L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

Stability Result

Theorem

Under the previous assumptions and if $(u, v)(., \theta) = (\widetilde{u}, \widetilde{v}))(., \theta)$, *then there exists a constant* $\kappa > 0$ *such that*

$$\begin{aligned} & \|b - \widetilde{b}\|_{L^2(\Omega)} + \|c - \widetilde{c}\|_{L^2(\Omega)} \\ \leq \quad & \kappa \left(\|\partial_t (u - \widetilde{u})\|_{W_2^{2,1}(\omega_T)} + \|u - \widetilde{u}\|_{W_2^{2,1}(\omega_T)} \right) \end{aligned}$$

Here

$$\|u\|_{W_2^{m,\frac{m}{2}}(\Omega_T)} = \sum_{|\alpha| + 2\alpha_{n+1} \leq m} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|_{L^2(\Omega_T)}$$

α: multi-index

2 × 2 Reaction-Diffusion-Convection Systems - Second Result

$$\begin{cases} \partial_t u = \nabla \cdot (H_1 \nabla u) + a\, u + b\, v + A \cdot \nabla u + B \cdot \nabla v + f & \text{in} \quad \Omega_T, \\ \partial_t v = \nabla \cdot (H_2 \nabla v) + c\, u + d\, v + C \cdot \nabla u + D \cdot \nabla v + g & \text{in} \quad \Omega_T, \\ u(\cdot, t) = v(\cdot, t) = 0 & \text{on} \quad \Sigma_T, \\ u(\cdot, 0) = u_0(\cdot), v(\cdot, 0) = v_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{4}$$

$$h^l_{ij} \in W^{1,\infty}(\Omega), h^l_{ij}(x) = h^l_{ji}(x) \text{ a.e. in } \Omega, \text{ and } \sum_{i,j=1}^{n} h^l_{ij}(x)\xi_i\xi_j \geq h_0|\xi|^2, \forall \xi \in \mathbb{R}^n.$$

$$I(\tau, \varphi) = \int_{\Omega_T} (s\rho)^{\tau-1} e^{-2s\eta} \left(|\varphi_t|^2 + \sum_{1\leq i\leq j\leq n} \left| \partial^2_{x_i x_j} \varphi \right|^2 + (s\lambda\rho)^2 |\nabla\varphi|^2 + (s\lambda\rho)^4 |\varphi|^2 \right) dx\, dt$$

New Carleman estimate

Theorem

For $|\tau_1 - \tau_2| < 1$, the following Carleman estimate holds :

$$I(\tau_1, u) + I(\tau_2, v) \leq C \Bigg(\int_{\omega_T} e^{-2s\alpha} \, (s\rho^*)^{\tau^*} \, |u|^2 dx\, dt + \int_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau_2} |Qf|^2 dx\, dt + \int_{\Omega_T} e^{-2s\eta} ((s\rho)^{\tau_1} |f|^2 + (s\rho)^{\tau_2} |g|^2) dx\, dt \Bigg)$$

for all $s \geq s_0$, $\tau^* = 4\tau_2 - 3\tau_1 + 15$ and Q is a bounded operator in $L^2(\omega)$.

$$\eta^* = \max_{\overline{\Omega}} \eta, \quad \eta_- = \min_{\overline{\Omega}} \eta, \quad \alpha = 4\eta_- - 3\eta^*, \quad \rho^* = \max_{\overline{\Omega}} \rho$$

Stability Result

Theorem

Under the previous assumptions and if $(u, v)(., \theta) = (\widetilde{u}, \widetilde{v}))(., \theta)$ *and* **b is known in** ω, *then there exists a constant* $\kappa > 0$ *such that*

$$\|b - \widetilde{b}\|_{L^2(\Omega)} + \|c - \widetilde{c}\|_{L^2(\Omega)} \leq \kappa \|u - \widetilde{u}\|_{H^1(0,T;L^2(\omega))}$$

3 $\times$ 3 Reaction-Diffusion Systems - First Result

$$\begin{cases} \partial_t u = \Delta u + a_{11} u + a_{12} v + a_{13} w + f & \text{in} \quad \Omega_T, \\ \partial_t v = \Delta v + a_{21} u + a_{22} v + a_{23} w + g & \text{in} \quad \Omega_T, \\ \partial_t w = \Delta w + a_{31} u + a_{32} v + a_{33} w + h & \text{in} \quad \Omega_T, \\ u = v = w = 0 & \text{on} \quad \Sigma_T \end{cases}$$

- change of function $z = a_{12} v + a_{13} w$
- Carleman estimate for a 2 $\times$ 2 reaction-diffusion-convection system

Carleman Estimate

Theorem

$$I(0,u)+I(0,v)+I(0,w)$$

$$\leq \kappa_1(s)(\|u\|^2_{W_2^{4,2}(\omega_T)}+\|f\|^2_{W_2^{2,1}(\omega_T)}+\|g\|^2_{L^2(\omega_T)}+\|h\|^2_{L^2(\omega_T)})$$

$$+\kappa\int_{\Omega_T}(|f^2|+|g|^2+|h|^2)e^{-2s\eta_\omega}\,dxdt$$

for all $s \geq s_0$*.*

3 $\times$ 3 Reaction-Diffusion Systems - Second Result

$$\begin{cases} \partial_t u = div(H_1 \nabla u) + a_{11} u + a_{21} v + a_{31} w + f & \text{in} \quad \Omega_T, \\ \partial_t v = div(H_2 \nabla v) + a_{12} u + a_{22} v + a_{32} w + g & \text{in} \quad \Omega_T, \\ \partial_t w = div(H_2 \nabla w) + a_{13} u + a_{23} v + a_{33} w + h & \text{in} \quad \Omega_T, \\ u = v = w = 0 & \text{on} \quad \Sigma_T, \\ u(\cdot, 0) = u_0,\ v(\cdot, 0) = v_0,\ w(\cdot, 0) = w_0, & \text{in } \Omega. \end{cases}$$

Assumptions

1. $\omega \subset \Omega$ is a non-empty subdomain of class C^2 with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$,
2. There exists $j \in \{2, 3\}$ such that $|a_{j1}(x, t)| \geq C > 0$ for all $x \in \omega_T$ and for $k_j = \frac{6}{j}$

$$|H_2 \left(\nabla a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \nabla a_{j1} \right) \cdot \nu| \neq 0, \text{ on } \gamma_T,$$

3. Let $H_2|_{\omega_T} \in (W^{3,\infty}(\omega_T))^{n^2}$.

New Carleman Estimate

Theorem

Under the previous Assumption the following Carleman estimate holds

$$
\begin{aligned}
& I(\tau, u) + I(\tau, v) + I(\tau, w) \leq \\
& C\Big(\lambda^{32} \int\int_{\omega_T} s^{(\tau+33)} (\rho^*)^{\tau+31} e^{(-4s\alpha+2s\eta)} (|u|^2 + |f|^2) dxdt \\
& + \lambda^4 \int\int_{\omega_T} e^{-2s\eta} (s\rho)^{3+\tau} (|Qg|^2 + |Qh|^2) dxdt \\
& + \int\int_{\Omega_T} e^{-2s\eta} (s\rho)^{\tau} (|f|^2 + |g|^2 + |h|^2) dxdt \Big).
\end{aligned}
$$

for $s \geq s_0$ and all (u, v, w) solution of the previous system, and Q is a bounded operator, $\alpha = 4\eta_- - 3\eta^*$, $\eta^* = \max_{\overline{\Omega}} \eta$, $\eta_- = \min_{\overline{\Omega}} \eta$ and $\rho^* = \max_{\overline{\Omega}} \rho$.

Stability Result

- 3×3 **Systems**

  We obtain a Lipschitz stability estimate for **three coefficients** (one in each equation) by the observation of only **one component** on ω assuming the knowledge of these coefficients on ω for any subset ω of Ω.

Generalization

- $n \times n$ **Systems**

  We fix three components (e.g. y_1, y_2, y_3) , the three first associated equations and three coefficients inside to recover. Assuming the knowledge of these three coefficients on the set of observation ω, we derive a stability estimate for n **coefficients**, one in each equation, by the observation of only $n-2$ **components** (y_1, and $(y_i)_{i\geq 3}$).

2 $\times$ 2 Reaction-Diffusion Systems with non-linear terms

Joint work with P. Gaitan, H. Ramoul and M. Yamamoto

Applicable Analysis (2011)

Reaction-Diffusion Systems with non-linear terms

We consider the following 2 × 2 reaction-diffusion system:

$$\begin{cases} \partial_t U = \Delta U + a_{11}(x)U + a_{12}(x)V + a_{13}(x)f(U,V) & \text{in} \quad Q_0, \\ \partial_t V = \Delta V + a_{21}(x)U + a_{22}(x)V & \text{in} \quad Q_0, \\ U(x,t) = k_1(x,t), \; V(x,t) = k_2(x,t) & \text{on} \quad \Sigma_0, \\ U(x,0) = U_0 \quad \text{and} \quad V(x,0) = V_0 & \text{in} \quad \Omega, \end{cases} \tag{5}$$

Hypothesis

1. $a_{ij}, \widetilde{a}_{13}$ and $\widetilde{a}_{21} \in \Lambda(R)$ for $i = 1, 2, j = 1, 2, 3$.
2. There exist constants $r_1 > 0$ and $a_0 > 0$ such that
   $\widetilde{U}_0 \geq r_1, \ \ \widetilde{V}_0 \geq 0, \ \ a_{11} r_1 + a_{12} \widetilde{V}_0 + \widetilde{a}_{13} \, f(r_1, \widetilde{V}_0) \geq 0,$
   $a_{21} \geq a_0, \ \ \widetilde{a}_{21} \geq a_0,$ in Q_0
   $k_1 \geq r_1$ and $k_2 \geq 0$ on Σ_0.
3. $f \in W^{1,\infty}(\mathbb{R}^2)$.
4. There exists a constant $r_2 > 0$ such that $f(\widetilde{U}, \widetilde{V})(T', x) \geq r_2 > 0$, $x \in \Omega$.
5. $\partial_t f(U(x,t), V(x,t)) \in L^2(0, T; H^2(\Omega))$.

Global Carleman estimate

We assume that $a_{11}, a_{12}, a_{21}, a_{22} \in \Lambda(R)$, $a_{21} \geq a_0 > 0$ and we consider the following system:

$$\left\{ \begin{array}{lll} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + H_1, & \text{in} & Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + H_2 & \text{in} & Q_0, \\ Y(x,t) = Z(x,t) = 0 & \text{on} & \Sigma_0, \end{array} \right. \tag{6}$$

where H_1 and H_2 are arbitrary functions.

Global Carleman estimate

Theorem

There exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_1, T) > 1$ *and a positive constant* $C_1 = C_1(\Omega, \omega, R, T, a_0)$ *such that, for any* $\lambda \geq \lambda_1$ *and any* $s \geq s_1$ *and* $\varepsilon > 0$ *fixed, the following estimate holds:*

$$\lambda^{-4+\varepsilon} I(-3, Y) + I(0, Z) \leq C_1 s^4 \lambda^{4+\varepsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt$$

$$+ C_1 \left[s^{-3} \lambda^{-4+\varepsilon} \iint_Q e^{-2s\eta} \varphi^{-3} \, |H_1|^2 \, dx \, dt + \lambda^{2\varepsilon} \iint_Q e^{-2s\eta} \, |H_2|^2 \, dx \, dt \right].$$

Stability estimate

Theorem

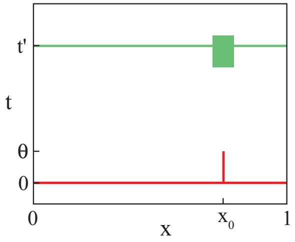
Let ω be a subdomain of a bounded domain Ω in $\mathbb{R}^n$. Under the previous assumptions and if we assume that $(U, V)(\cdot, T') = (\widetilde{U}, \widetilde{V})(\cdot, T')$ in Ω and that $\widetilde{U}_0$, $\widetilde{V}_0$ in $H^2(\Omega)$. Then there exists a constant $C = C(\Omega, \omega, a_0, t_0, T, r_1, r_2, R) > 0$ such that

$$\|a_{21} - \widetilde{a}_{21}\|_{L^2(\Omega)} + \|a_{13} - \widetilde{a}_{13}\|_{L^2(\Omega)} \leq C\|\partial_t V - \partial_t \widetilde{V}\|_{L^2(Q_\omega)}.$$

Part II

Determination of multi coefficients in a system with strong nonlinear coupled parabolic equations (models : Lotka Volterra).

The problem of data



(a) different type of data

In green, data required by the previous methods involving Carleman estimates. In red, data needed by the forthcoming techniques.

The inverse problem of determining several coefficients in a nonlinear Lotka-Volterra system

Joint work with L. Roques

Accepted for publication in Inverse Problems (2012)

An exemple of biological background

Consider a single-species model:

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1(x)\, u - a_{11}(x)\, u^2, \text{for } t > 0,\ x \in (a,b) \subset \mathbb{R},$$

$D_1 > 0$, the intrinsic growth rate r_1 belongs to $C^{0,\eta}([a,b])$, $\eta \in (0,1]$.
Assume that a second species enters in competition with species 1. The two-species system can be modelled by the Lotka-Volterra competition model:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1\, u - a_{11}\, u^2 - a_{12}\, uv, \\ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + r_2\, v - a_{21}\, uv - a_{22}\, v^2, \end{cases} \text{for } t > 0,\ x \in (a,b) \subset \mathbb{R},$$

An exemple of biological background

Consider a single-species model:

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1(x)\, u - a_{11}(x)\, u^2, \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R},$$

$D_1 > 0$, the intrinsic growth rate r_1 belongs to $C^{0,\eta}([a, b])$, $\eta \in (0, 1]$.
Assume that a second species enters in competition with species 1.
The two-species system can be modelled by the Lotka-Volterra competition model:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1\, u - a_{11}\, u^2 - a_{12}\, uv, \\ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + r_2\, v - a_{21}\, uv - a_{22}\, v^2, \end{cases} \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R},$$

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An exemple of biological background

$D_2 > 0$: diffusion coefficient of the second species,
$r_2 \in C^{0,\eta}([a,b])$: 2nd species intrinsic growth rate
$a_{22} > 0$: 2nd species intraspecific competition coefficient
$(a_{22} \in C^{0,\eta}([a,b]))$.
Competitive system $\rightarrow a_{12},\ a_{21} > 0$.
We assume a_{12} is constant and $a_{21} \in C^{0,\eta}([a,b])$.
These two coefficients respectively measure the impact of species 2 upon species 1 (resp. of species 1 upon species 2).

Initial and boundary conditions

the boundary conditions:

$$\begin{cases} \alpha_1 u(t,a) - \beta_1 \dfrac{\partial u}{\partial x}(t,a) = 0, \quad \gamma_1 u(t,b) + \delta_1 \dfrac{\partial u}{\partial x}(t,b) = 0, \\ \alpha_2 v(t,a) - \beta_2 \dfrac{\partial v}{\partial x}(t,a) = 0, \quad \gamma_2 v(t,b) + \delta_2 \dfrac{\partial v}{\partial x}(t,b) = 0, \end{cases} \quad \text{for } t > 0,$$

with

$$\alpha_i^2 + \beta_i^2 > 0 \text{ and } \delta_i^2 + \gamma_i^2 > 0, \text{ for } i = 1,2.$$

These general boundary conditions include the classical Dirichlet case ($\beta_i = \delta_i = 0$, for $i = 1,2$) and Neumann case ($\alpha_i = \gamma_i = 0$, for $i = 1,2$).

$$u_0,\ v_0 \in C^{2,\eta}([a,b]),$$

plus compatibilities conditions.

Main questions

For system of the type:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + r_1\, u - a_{11}\, u^2 - a_{12}\, uv, \\ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + r_2\, v - a_{21}\, uv - a_{22}\, v^2, \end{cases} \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R},$$

- Can the coefficients $a_{12}, r_2(x), a_{21}(x), a_{22}(x)$ be uniquely determined from partial measurements of $u(t, x)$?
- Is it possible to obtain a good approximation of the coefficients given such measurements ?

Partial measurements are measurements of $u(t, x)$ or $v(t, x)$ over a subset of (a, b).

Uniqueness of several coefficients from pointwise measurement

Assumptions

Consider three couples of initial conditions (u_0^1, v_0^1), (u_0^2, v_0^2), (u_0^3, v_0^3)

$$u_0^1, u_0^2, u_0^3, v_0^1, v_0^2, v_0^3 > 0 \text{ in } (a, b),$$

and

$$(u_0^3 - u_0^2)(v_0^2 - v_0^1) - (v_0^3 - v_0^2)(u_0^2 - u_0^1) \neq 0 \text{ in } (a, b).$$

For all $x \in (a, b)$, the points $(u_0^1, v_0^1)(x)$, $(u_0^2, v_0^2)(x)$ and $(u_0^3, v_0^3)(x)$ belong to the positive quadrant and are misaligned.

Assume that the admissible coefficients $a_{12}, r_2(x), a_{21}(x), a_{22}(x)$ belong to:

$$\mu \in M := \{\psi \in C^{0,\eta}([a, b]) \text{ such that } \psi \text{ is piecewise analytic on } (a, b)\}.$$

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Uniqueness of 4 coefficients from pointwise measurement

Consider for $\tilde{a}_{12} \in (0, \infty)$ and $\tilde{r}_2, \tilde{a}_{21}, \tilde{a}_{22}$ belonging to $\mathcal{M}$, $(\tilde{u}, \tilde{v})$ be the solution of:

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = D_1 \frac{\partial^2 \tilde{u}}{\partial x^2} + r_1 \tilde{u} - a_{11} \tilde{u}^2 - \tilde{a}_{12} \tilde{u}\tilde{v}, \\ \frac{\partial \tilde{v}}{\partial t} = D_2 \frac{\partial^2 \tilde{v}}{\partial x^2} + \tilde{r}_2 \tilde{v} - \tilde{a}_{21} \tilde{u}\tilde{v} - \tilde{a}_{22} \tilde{v}^2, \end{cases} \text{for } t > 0,\ x \in (a, b) \subset \mathbb{R},$$

with the initial conditions:

$$\tilde{u}(0, x) = u_0,\ \tilde{v}(0, x) = v_0.$$

Uniqueness of 4 coefficients from a pointwise measurements

Theorem

Let $x_0 \in (a,b)$ and $\varepsilon > 0$. Assume that for each couple of initial conditions (u_0^k, v_0^k), with $k = 1,2,3$, we have:

$$\left\{ \begin{array}{rcl} (u,v)(t,x_0) & = & (\tilde{u},\tilde{v})(t,x_0), \\ \dfrac{\partial u}{\partial x}(t,x_0) & = & \dfrac{\partial \tilde{u}}{\partial x}(t,x_0) \text{ or } \dfrac{\partial v}{\partial x}(t,x_0) \;=\; \dfrac{\partial \tilde{v}}{\partial x}(t,x_0) \end{array} \right. \quad \text{for all } t \in (0,\varepsilon)$$

Then, $\tilde{a}_{12} = a_{12}$ and $\tilde{r}_2 \equiv r_2$, $\tilde{a}_{21} \equiv a_{21}$, $\tilde{a}_{22} \equiv a_{22}$ in (a,b).

Uniqueness of 4 coefficients from a pointwise measurements

Corollary

Let $x_0 \in (a, b)$ *and* $\varepsilon > 0$. *Assume that for each couple of initial conditions* (u_0^k, v_0^k), *with* $k = 1, 2, 3$, *we have:*

$$\begin{cases} u(t, x_0) = \tilde{u}(t, x_0), \\ \dfrac{\partial u}{\partial x}(t, x_0) = \dfrac{\partial \tilde{u}}{\partial x}(t, x_0), \\ \dfrac{\partial^2 u}{\partial x^2}(t, x_0) = \dfrac{\partial^2 \tilde{u}}{\partial x^2}(t, x_0), \end{cases} \quad \text{for all } t \in (0, \varepsilon).$$

Then, $\tilde{a}_{12} = a_{12}$ *and* $\tilde{r}_2 \equiv r_2$, $\tilde{a}_{21} \equiv a_{21}$, $\tilde{a}_{22} \equiv a_{22}$ *in* (a, b).

Still valid if $x_0 = a$ and β_1, $\beta_2 \neq 0$ (resp. $x_0 = b$ and δ_1, $\delta_2 \neq 0$) if : u_0^k, v_0^k are assumed to be positive in $[a, b)$ (resp. $(a, b]$).

Sketch of the proof

We set $U = u - \tilde{u}$, $V = \tilde{v} - v$, $R = r_2 - \tilde{r}_2$ and $A_{ij} = a_{ij} - \tilde{a}_{ij}$ for all $i, j \in \{1, 2\}$ (and $(i, j) \neq (1, 1)$).
Whatever the initial condition (u_0^k, v_0^k), with $k = 1 \ldots 3$, the couple (U, V) verifies:

$$\begin{cases} \frac{\partial U}{\partial t} - D_1 \frac{\partial^2 U}{\partial x^2} = U\,[r_1 - a_{11}(u + \tilde{u}) - \tilde{a}_{12} v] + \tilde{a}_{12}\tilde{u} V - A_{12}\, u\, v, \\ \frac{\partial V}{\partial t} - D_2 \frac{\partial^2 V}{\partial x^2} = V\,[\tilde{r}_2 - \tilde{a}_{22}(v + \tilde{v}) - \tilde{a}_{21} u] + \tilde{a}_{21}\tilde{v} U - v\,[R - A_{21} u - A_{22} v]\,, \end{cases}$$

and $U(0, x) = V(0, x) = 0$ for all $x \in (a, b)$.

Sketch of the proof

We set:

$$\mathcal{A}_+ = \Big\{ x \geq x_0 \text{ s.t. } R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0 \text{ for all } y \in [x_0, x] \Big\},$$

and

$$x_1 := \begin{cases} \sup(\mathcal{A}_+) & \text{if } \mathcal{A}_+ \text{ is not empty,} \\ x_0 & \text{if } \mathcal{A}_+ \text{ is empty.} \end{cases}$$

If $x_1 = b$, then $R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0$ on $[x_0, b]$. If $x_1 < b$.

Step 1. Assume that $x_1 < b$. We prove that there exists an initial condition $(u_0^{k^*}, v_0^{k^*})$, with $k^* \in \{1, 2, 3\}$ such that for $\theta \in (0, T)$, and $x_2 \in (x_1, b)$:
$R(x) - A_{21}(x)\, u(t, x) - A_{22}(x)\, v(t, x) \neq 0$ for all $(t, x) \in [0, \theta] \times (x_1, x_2]$.

Sketch of the proof

Step 2. We show that if $(u_0, v_0) = (u_0^{k^*}, v_0^{k^*})$, $\exists\, \varepsilon' > 0$ *such that :* *either* $U > 0$ *and* $V > 0$ *or* $U < 0$ *and* $V < 0$ *in* $(0, \varepsilon') \times (x_0, x_2)$.

Step 3. Using Hopf's Lemma and assumption of theorem, we get a contradiction with the assumption of Step 1.

Sketch of the proof

Step 2. We show that if $(u_0, v_0) = (u_0^{k^*}, v_0^{k^*})$, $\exists\, \varepsilon' > 0$ *such that :*

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Step 3. Using Hopf's Lemma and assumption of theorem, we get a contradiction with the assumption of Step 1.

Sketch of the proof

Step 4. Conclusion.

Step 3 $\rightarrow x_1 = b \rightarrow R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0$ on $[x_0, b]$.
Setting:

$$\mathcal{A}_- = \Big\{ x \leq x_0 \text{ s.t. } R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0 \text{ for all } y \in [x, x_0] \Big\},$$

and

$$y_1 := \begin{cases} \inf(\mathcal{A}_-) & \text{if } \mathcal{A}_- \text{ is not empty,} \\ x_0 & \text{if } \mathcal{A}_- \text{ is empty,} \end{cases}$$

we prove (by applying the same arguments as above) that $y_1 = a$ and $R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0$ on $[a, x_0]$.
Finally,

$$R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv 0 \text{ on } [a, b].$$

Reconstruction of four coefficients

We fixed :
$D_1 = 0.1; D_2 = 0.2; r_1 = a_{11} = 1$ and
$(u_0^k, v_0^k) = (0.1; 0.5); (0.5; 0.2); (0.9; 0.5)$.
The solution is measured at the point $x_0 = 2/3$ for $t \in (0, 0.3)$.
We look for $a_{12}, r_2(x), a_{21}(x), a_{22}(x)$ as the minimum of :

$$G(a_{12}, r_2(x), a_{21}(x), a_{22}(x)) =$$

$$\sum_{k=1}^{3} \left(\|(u^k, v^k)(\cdot, x_0) - (\tilde{u}^k, \tilde{v}^k)(\cdot, x_0)\|_{(L^2(0,\varepsilon))^2} + \| \frac{\partial u^k}{\partial x}(\cdot, x_0) - \frac{\partial \tilde{u}^k}{\partial x}(\cdot, x_0)\|_{L^2(0,\varepsilon)} \right)$$

We obtain the following average values for 30 samples:
$\|r_2 - r_2^*\|_{L^2(0,1)} = 0.1$, $\|a_{21} - a_{21}^*\|_{L^2(0,1)} = 0.2$, $\|a_{22} - a_{22}^*\|_{L^2(0,1)} = 0.2$, and
$|a_{12} - a_{12}^*| = 3 \cdot 10^{-5}$.

Reconstruction of four coefficients

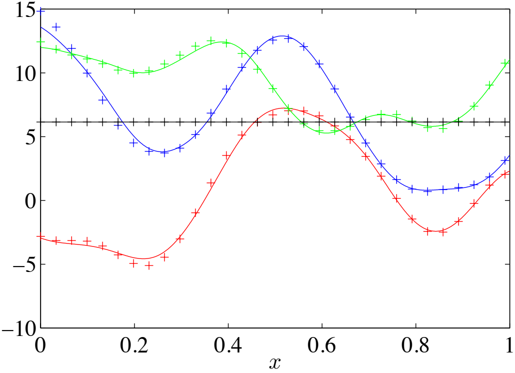

Figure: Plain lines: functions $r_2(x)$ (red line), $a_{21}(x)$ (blue line), $a_{22}(x)$ (green line) and a_{12} (black line) in E. Crosses: the functions $r_2^*(x)$ (in red), $a_{21}^*(x)$ (in blue), $a_{22}^*(x)$ (in green) and a_{12}^* (in black). In this particular example, we have $\|r_2 - r_2^*\|_{L^2(0,1)} = 0.2$, $\|a_{21} - a_{21}^*\|_{L^2(0,1)} = 0.3$, $\|a_{22} - a_{22}^*\|_{L^2(0,1)} = 0.2$, and $|a_{12} - a_{12}^*| = 4 \cdot 10^{-5}$.

Numerical stability issue

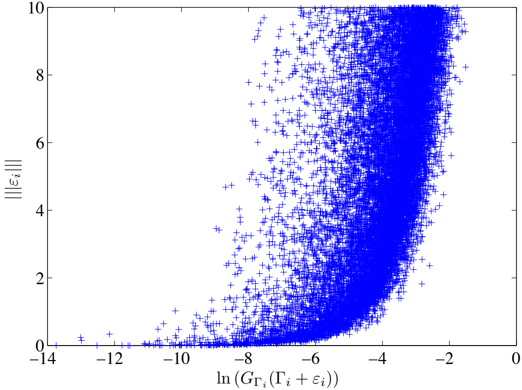

Figure: Scatter plot of the couples $(\ln(G_{\Gamma_i}(\Gamma_i + \varepsilon_i)), |||\varepsilon_i|||)$, for $i = 1, \ldots, 10^4$.

Counter-example

Counter-example : Number of measurements smaller than 3 or "initials conditions aligned".

Assume:

- All the coefficients are constant and exists λ such that
$$r_2 = \lambda r_1,\ a_{12} = \lambda a_{11} \text{ and } a_{22} = \lambda a_{12}$$
- Neumann boundary conditions.

Take for $k = 1, 2, 3$, (u_0^k, v_0^k) constants and positive initial conditions aligned on the straight line $\{(x, y),\ |\ a_{11}\, x + a_{12}\, y = r_1\}$. Then they are stationary solutions of the problem studied associated to the coefficients $r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}$:

$$\begin{cases} r_1\, u_0^k - a_{11}\, (u_0^k)^2 - a_{12}\, u_0^k\, v_0^k = 0, \\ r_2\, v_0^k - a_{21}\, u_0^k\, v_0^k - a_{22}\, (v_0^k)^2 = 0, \end{cases}$$

Counter-example

Counter-example : Number of measurements smaller than 3 or "initials conditions aligned".

Let $\tilde{\lambda} > 0$ verify $\tilde{\lambda} \neq \lambda$, and take $\tilde{r}_2 = \tilde{\lambda}\, r_1$, $\tilde{a}_{21} = \tilde{\lambda}\, a_{11}$ and $\tilde{a}_{22} = \tilde{\lambda}\, a_{12}$. Again, the three couples (u_0^k, v_0^k) verify the system:

$$\begin{cases} r_1\, u_0^k - a_{11}\,(u_0^k)^2 - a_{12}\, u_0^k\, v_0^k = 0, \\ \tilde{r}_2\, v_0^k - \tilde{a}_{21}\, u_0^k\, v_0^k - \tilde{a}_{22}\,(v_0^k)^2 = 0, \end{cases}$$

The assumptions of our theorem and corollary are fulfilled at any point $x_0 \in (a, b)$ and for all $\varepsilon > 0$.
However, $\tilde{r}_2 \neq r_2$, $\tilde{a}_{21} \neq a_{21}$ and $\tilde{a}_{22} \neq a_{22}$.

Conclusion

- Here, contrarily to previous results, **there are some regions in** (a, b) **where** u **is never measured**
- The subset M of $C^{0,\sigma}[a, b]$ made of piecewize analytic functions is much larger than the set of analytic functions on $[a, b]$. It indeed contains some functions which regularity is not higher than $C^{0,\sigma}$, and some functions which are constant on some subsets of $[a, b]$

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Perspectives

- Time dependant coefficients . . .

Thank you for attention