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Stabilization of a coupled Fluid-Solid System, by the Deformation of the Self-Propelled Solid

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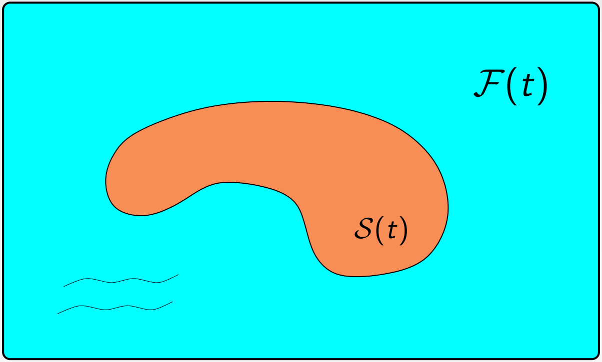
Presentation

$\mathcal{O} = \mathcal{F}(t) \cup \overline{S(t)} \subset \mathbb{R}^2$ or $\mathbb{R}^3$.



The Lagrangian mapping $X_{\mathcal{S}}$ of the solid defines

$$\mathcal{S}(t) \quad = \quad X_{\mathcal{S}}(\mathcal{S}(0), t),$$

and we denote

$$\mathcal{F}(t) \quad = \quad \mathcal{O} \setminus \overline{\mathcal{S}(t)}.$$

Presentation

The mapping $X_{\mathcal{S}}$ is decomposed into two parts, as follows

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t), \qquad y \in \mathcal{S}(0).$$

- $h(t)$: the position of the solid's center of mass.
- $\mathbf{R}(t)$: the rotation associated to its angular velocity ω, satisfying

$$\begin{aligned} \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} &= \mathbb{S}(\omega)\,\mathbf{R} \\ \mathbf{R}(0) &= \mathrm{I}_{\mathbb{R}^3}, \end{aligned} \qquad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

- We define

$$\begin{aligned} w^*(x^*,t) &= \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in X^*(\mathcal{S}(0),t), \\ w(x,t) &= \mathbf{R}(t)w^*(\mathbf{R}(t)^T(x-h(t)),t), \quad x \in \mathcal{S}(t). \end{aligned}$$

The main system

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \quad x \in \mathcal{F}(t), \quad t \in (0, T), \\ \operatorname{div} u &= 0, \quad x \in \mathcal{F}(t), \quad t \in (0, T), \end{aligned}$$

$$\begin{aligned} u = 0, &\qquad x \in \partial\mathcal{O}, \quad t \in (0, T), \\ u = h'(t) + \omega(t) \wedge (x - h(t)) + w(x, t), &\qquad x \in \ \partial\mathcal{S}(t), \quad t \in (0, T), \end{aligned}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u, p) n \mathrm{d}\Gamma, \qquad t \in (0, T),$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge \sigma(u, p) n \mathrm{d}\Gamma, \qquad t \in (0, T),$$

$$\begin{aligned} u(y, 0) = u_0(y), \ y \in \mathcal{F}(0), \quad \omega(0) = \omega_0 \in \mathbb{R}^3, \\ h(0) = h_0 \in \mathbb{R}^d, \quad h'(0) = h_1 \in \mathbb{R}^d, . \end{aligned}$$

The regularity on the control X^*

We denote $\mathcal{S} = \mathcal{S}(0)$ and $\mathcal{F} = \mathcal{F}(0)$.
We consider $\mathcal{W}_{0,m}(0, T; \mathcal{S})$ - with $m > 2$ if $d = 2$ and $m > 5/2$ if $d = 3$ - defined as

$$X^* \in \mathcal{W}_{0,m}(0, T; \mathcal{S}) \Leftrightarrow$$
$$\begin{cases} \dfrac{\partial X^*}{\partial t} \in \mathrm{L}^2(0, T; \mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^{m/2}(0, T; \mathbf{L}^2(\mathcal{S})), \\ \\ X^*(y, 0) = y - h_0, \ \dfrac{\partial X^*}{\partial t}(y, 0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

This class of functions allows in particular to consider controls X^* lying in $C^1(\mathcal{S})$.

The constraints that X^* must satisfy

- The deformation $X^*(\cdot, t)$ must be a C^1-diffeomorphism from $\mathcal{S}(0)$ onto $\mathcal{S}^*(t)$.
- The conservation of the volume :

$$\int_{\partial \mathcal{S}} (\mathrm{cof}\ \nabla X^*) \frac{\partial X^*}{\partial t} \cdot n \mathrm{d}\Gamma = 0.$$

- The conservation of momenta (to guarantee the *self-propelled* nature of the deformation) :

$$\begin{aligned} \int_{\mathcal{S}} X^*(y,t) \mathrm{d}y &= 0, \\ \int_{\mathcal{S}} X^*(y,t) \wedge \frac{\partial X^*}{\partial t} \mathrm{d}y &= 0. \end{aligned}$$

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Rewriting the main system in fixed domains

We extend the Lagrangian mapping induced by X^* to the whole domain $\mathcal{O}$, in a mapping denoted $\tilde{X}$.
We set

$$X(y,t) \quad = \quad h(t) + \mathbf{R}(t)\tilde{X}(y,t), \quad y \in \mathcal{F}(0),$$

and we consider the following change of unknowns

$$\begin{aligned} \tilde{u}(y,t) &= \mathbf{R}(t)^T u(X(y,t),t), & \qquad \tilde{h}'(t) &= \mathbf{R}(t)^T h'(t), \\ \tilde{p}(y,t) &= p(X(y,t),t), & \qquad \tilde{\omega}(t) &= \mathbf{R}(t)^T \omega(t). \end{aligned}$$

We also set

$$\begin{aligned} \hat{u}(y,t) &= e^{\lambda t}\tilde{u}(y,t), & \qquad \hat{h}'(t) &= e^{\lambda t}\tilde{h}'(t), \\ \hat{p}(y,t) &= e^{\lambda t}\tilde{p}(y,t), & \qquad \hat{\omega}(t) &= e^{\lambda t}\tilde{\omega}(t). \end{aligned}$$

The linearized system

$$\frac{\partial \hat{\mathrm{U}}}{\partial t} - \lambda \hat{\mathrm{U}} - \nu \Delta \hat{\mathrm{U}} + \nabla \hat{\mathrm{P}} = 0, \qquad \text{in } (0,T) \times \mathcal{F}(0),$$

$$\mathrm{div}\ \hat{\mathrm{U}} = 0, \qquad \text{in } (0,T) \times \mathcal{F}(0),$$

$$\hat{\mathrm{U}} = 0, \qquad \text{in } (0,T) \times \partial\mathcal{O},$$

$$\hat{\mathrm{U}} = \hat{\mathrm{H}}'(t) + \hat{\Omega}(t) \wedge y + e^{\lambda t} \frac{\partial X^*}{\partial t}(y,t), \qquad y \in \partial\mathcal{S}(0), \quad t \in (0,T),$$

$$M\hat{\mathrm{H}}''(t) - \lambda M \hat{\mathrm{H}}'(t) = -\int_{\partial\mathcal{S}} \sigma(\hat{\mathrm{U}}, \hat{\mathrm{P}}) n \mathrm{d}\Gamma, \quad t \in (0,T),$$

$$I_0 \hat{\Omega}'(t) - \lambda I_0 \hat{\Omega}(t) = -\int_{\partial\mathcal{S}} y \wedge \sigma(\hat{\mathrm{U}}, \hat{\mathrm{P}}) n \mathrm{d}\Gamma, \quad t \in (0,T),$$

$$\hat{\mathrm{U}}(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad \hat{\mathrm{H}}'(0) = h_1 \in \mathbb{R}^d, \quad \hat{\Omega}(0) = \omega_0 \in \mathbb{R}^3.$$

The main result

Let us define

$$\mathbf{H}_{cc} = \Big\{(u_0, h_1, \omega_0) \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3 \mid \operatorname{div} u_0 = 0,\ u_0 = h_1 + \omega_0 \wedge y \text{ on } \partial\mathcal{S}\Big\}.$$

Theorem

For all $\lambda > 0$, and all $(u_0, h_1, \omega_0) \in \mathbf{H}_{cc}$, there exists a control $X^* \in \mathcal{W}_{0,m}(0,\infty;\mathcal{S})$ with $m \geq 3$, satisfying the constraints aforementioned, such that the solution to the linear system above obeys

$$\|(\hat{\mathrm{U}}, \hat{\mathrm{H}}', \hat{\Omega})\|_{\mathrm{L}^2(0,\infty;\mathbf{H}_{cc})} < \infty.$$

References

- J. San Martín, J.-F. Scheid, T. Takahashi, M. Tucsnak, *An initial and boundary value problem modeling of fish-like swimming,* Arch. Rational Mech. Anal., 188 (2008), pp. 429–455.

- J.-P. Raymond, *Feedback stabilization of a fluid-structure model*, SIAM J. Control Optim., 48 (2010), pp. 5398–5443.