Controllability and Lipschitz stability for Grushin-type operators

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Control of Fluid-Structure Systems and Inverse Problems

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Outline

1. Introduction to Baouendi-Grushin parabolic operators

2. Controllability for Baouendi-Grushin operators
   - review of controllability for parabolic operators
   - controllability for degenerate parabolic operators
   - controllability of Baouendi-Grushin operators

3. Inverse source problem for Baouendi-Grushin operators

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Grushin-type operators

$\Omega = (-1,1) \times (0,1)$
$T > 0$
$\Omega_T = (0,T) \times \Omega$

$$\gamma > 0 \qquad \begin{cases} \partial_t u - \underbrace{\left(\partial_x^2 u + |x|^{2\gamma} \partial_y^2 u\right)}_{G_\gamma u} = f & \text{in } \Omega_T \\ u(t, \pm 1, y) = 0 & 0 < y < 1 \\ u(t,x,0) = 0 = u(t,x,1) & -1 < x < 1 \\ u(0,x,y) = u_0(x,y) & (x,y) \in \Omega \end{cases}$$

- $u_0 \in L^2(\Omega)$
- $f \in L^2(\Omega_T)$

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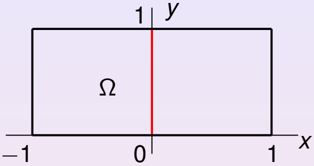

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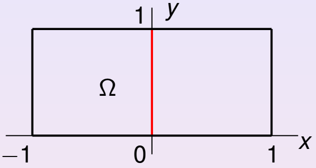

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hypoellipticity of Baouendi-Grushin operator

- sum of squares of vector fields

$$G_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2 = X_1^2 + X_2^2$$

- hypoelliptic

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \; [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \; \dots$$

satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$

- related to almost riemannian structures

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associated diffusion process ($\gamma = 1$)

diffusion process

$$\begin{cases} dX(t) = dW_1(t)dt\,, & X(0) = x \\ dY(t) = X(t)dW_2(t)\,, & Y(0) = y \end{cases}$$

W_1, W_2 independent 1D Brownian motions

$$X(t,x,y) = x + W_1(t)\,, \quad Y(t,x,y) = y + xW_2(t) + \int_0^t W_1(s)dW_2(s)$$

Cauchy problem for Kolmogorov equation

$$\begin{cases} \partial_t u - (\partial_x^2 + x^2\partial_y^2)u = 0 & \text{in} \quad (0,+\infty)\times\mathbb{R}^2 \\ u(0,x,y) = u_0(x,y) & \text{on} \quad \mathbb{R}^2 \end{cases}$$

solved by transition semigroup

$$u(t,x,y) = \mathbb{E}\Big[u_0\big(X(t,x,y), Y(t,x,y)\big)\Big]$$

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existence and uniqueness of solutions

$$\begin{cases} \partial_t u - \left(\partial_x^2 u + |x|^{2\gamma}\partial_y^2 u\right) = f & \text{in } \Omega_T \\ u(t,\pm 1,y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) & \text{in } \partial\Omega \\ u(0,x,y) = u_0(x,y) & (x,y)\in\Omega \end{cases}$$

- $H = L^2(\Omega)$ and $V = \overline{C_0^\infty(\Omega)}$ with respect to $(f,g) = \int_\Omega \left(f_x g_x + |x|^{2\gamma} f_y g_y\right) dxdy$
- $D(A) = \{f \in V : \exists\, c > 0$ such that $|(f,h)| \le c\|h\|_H\ \forall h \in V\}$
  $\langle Af, g\rangle = -(f,g) \quad \forall g \in V$
- $A : D(A) \subset H \to H$ generator of a semigroup e^{tA} of contractions in H

Theorem

$T > 0\,,\ u_0 \in L^2(\Omega)\,,\ f \in L^2(\Omega_T)$

$\Longrightarrow \exists!\ u \in C([0,T]; L^2(\Omega))\ :\ \forall t \in (0,T)\,,\ \phi \in C^2([0,T]\times\Omega)$

$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega u(\partial_t\phi + \partial_x^2\phi + |x|^{2\gamma}\partial_y^2\phi) + f\phi$$

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control of uniformly parabolic equations

$\omega \subset\subset \Omega \qquad T > 0 \qquad A(x) = \big(a_{ij}(x)\big)_{i,j=1}^{n}$ positive definite in $\overline{\Omega}$

$$u^f \leftrightarrow \begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) = \chi_\omega(x) f(t,x) & \text{in} \quad \Omega_T = (0,T)\times\Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ u(t,\cdot) = 0 & \text{on } \Gamma \end{cases}$$

- f locally distributed control (χ_ω = characteristic function of ω)

- also of interest boundary control $\Gamma_1 \subset \Gamma$

$$u(t,x) = g(t,x) \qquad (t,x) \in (0,T)\times\Gamma_1$$

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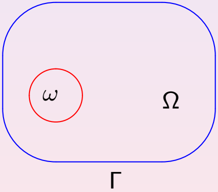

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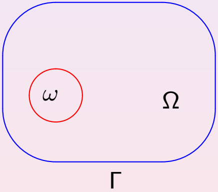

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- approximately controllable in time $T > 0$

$$\forall u_0, u_1 \in L^2(\Omega) \ \ \forall \varepsilon > 0 \quad \exists f \in L^2(\Omega_T) \ : \ \|u^f(\cdot, T) - u_1\| < \varepsilon$$

by duality equivalent to

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$$\begin{cases} \partial_t v + \mathrm{div}(A(x)\nabla v) = 0 & \text{in} \quad \Omega_T \\ v(t,\cdot) = 0 & \text{on } \Gamma \end{cases}$$

satisfies

$$v \equiv 0 \quad \text{on} \quad (0,T)\times\omega \qquad \Longrightarrow \qquad v \equiv 0 \text{ in} \quad \Omega_T$$

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roadmap to observability

- Fattorini and Russell (1971), Russell (1978)
  by a Riesz basis approach
- Lebeau and Robbiano (1995)
  by a combination of Riesz basis techniques and local Carleman estimates
- Fursikov and Emanouilov (1996)
  by a **global Carleman estimate**

$$\begin{cases} \partial_t v + \text{div}(A(x)\nabla v) = 0 & \text{in} \quad \Omega_T \\ v(t,\cdot) = 0 & \text{on } \Gamma \end{cases}$$

satisfies for $\tau >> 0$

$$\iint_{\Omega_T} \underbrace{\tau^3\theta^3(t)v^2}_{+\tau\theta(t)|Dv|^2+\cdots} \; e^{2\tau\phi(x,t)}\,dxdt \le C\int_0^T\int_\omega v^2\,dxdt$$

where $\begin{cases} \theta(t) = \frac{1}{t(T-t)} \\ \phi(x,t) = \theta(t)\left[e^{\psi(x)} - e^{2\|\psi\|_\infty}\right] \end{cases}$ with $\quad D\psi(x) \neq 0$ in $\Omega \setminus \omega$

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what needs to be changed

- observability ($\Rightarrow$ null controllability) may fail (for violent degeneracies)
- ϕ in Carleman must be adapted to degeneracy
- Hardy's inequality can be useful

$$\int_\Omega d_\Gamma^{\alpha-2} w^2\, dx \le C_\alpha \int_\Omega d_\Gamma^{\alpha} |\nabla w|^2\, dx \qquad (\alpha \neq 1)$$

- take advantage of dissipation

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the simplest example of degeneracy

$\omega = (a,b) \subset\subset (0,1)$ $\quad \boxed{a(x) = x^{2\gamma}} \quad (\gamma > 0)$

$$u_t - \left(x^{2\gamma} u_x\right)_x = \chi_\omega f \,, \qquad u(0,x) = u_0(x)$$

Theorem (C – Martinez – Vancostenoble, 2008)

null controllability $\begin{cases} \textit{false} & \gamma \geq 1 \\ \textit{true} & 0 \leq \gamma < 1 \end{cases}$

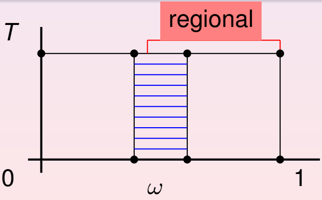
the simplest example of degeneracy

$\omega = (a,b) \subset\subset (0,1)$ $\quad$ $\boxed{a(x) = x^{2\gamma}}$ $\quad$ $(\gamma > 0)$

$$u_t - \left(x^{2\gamma} u_x\right)_x = \chi_\omega f \,, \qquad u(0,x) = u_0(x)$$

Theorem (C – Martinez – Vancostenoble, 2008)

null controllability $\begin{cases} \text{false} & \gamma \geq 1 \\ \text{true} & 0 \leq \gamma < 1 \end{cases}$



references

- divergence form
  - Martinez – Vancostenoble (2006) $u_t - (a(x)u_x)_x = \chi_\omega f$
  - Alabau – C – Fragnelli (2006) $u_t - (a(x)u_x)_x + g(u) = \chi_\omega f$
  - Flores – de Teresa (2010) $u_t - (x^\theta u_x)_x + x^\sigma b(x,t)u_x = \chi_\omega f$
- non-divergence form C – Fragnelli – Rocchetti (2007, 2008)

$$u_t - a(x)u_{xx} - b(x)u_x = \chi_\omega f$$

- degenerate/singular problems
  - Vancostenoble – Zuazua (2008), Vancostenoble (2009)

$$u_t - (x^\theta u_x)_x - \frac{\lambda}{x^\sigma} u = \chi_\omega f$$

- systems
  - C – de Teresa (2009) cascade 2×2
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- higher dimension $\partial_t u - \operatorname{div}(a(t,x)\nabla u) + g(t,x,u) = \chi_\omega f$
  - null controllability C – Marinez – Vancostenoble (CRAS, 2009)
  - approximate controllability Wang (2009)

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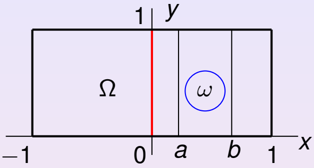
Outline

1. Introduction to Baouendi-Grushin parabolic operators

2. Controllability for Baouendi-Grushin operators
   - review of controllability for parabolic operators
   - controllability for degenerate parabolic operators
   - controllability of Baouendi-Grushin operators

3. Inverse source problem for Baouendi-Grushin operators

problem set-up



$$u^f \longleftrightarrow \begin{cases} \partial_t u - (\partial_x^2 u + |x|^{2\gamma}\partial_y^2 u) = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1, y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$

- $u_0 \in L^2(\Omega)$, $\quad f \in L^2(\Omega_T)$ control
- $\omega \subset (a,b) \times (0,1)$ with $0 < a < b < 1$

want to study

- approximate controllability in time $T > 0$
- null controllability in time $T > 0$

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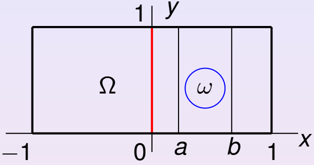

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  - Martinez – Raymond – Vancostenoble (2003) ($a \equiv 1$)
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approximate controllability $\Longleftrightarrow$ unique continuation

Proposition

Let $T > 0$, $\gamma > 0$, let $\omega \subset (0,1) \times (0,1)$, and let v be a solution of

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma} \partial_y^2 v = 0 & (t,x,y) \in (0,T) \times \Omega \\ v(t,x,y) = 0 & (t,x,y) \in (0,T) \times \partial\Omega \end{cases}$$

If $v \equiv 0$ on $(0,T) \times \omega$, then $v \equiv 0$ on $(0,T) \times \Omega$.

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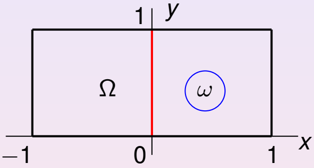

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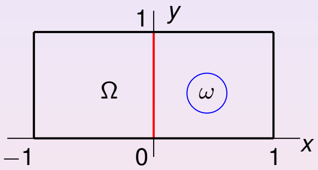

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Fourier decomposition

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(t,\pm 1, y) = 0\,,\;\; v(t,x,0) = 0 = v(t,x,1) \\ v(0,x,y) = v_0(x,y) \end{cases} \qquad (G^*)$$

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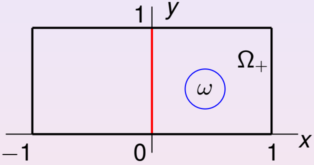

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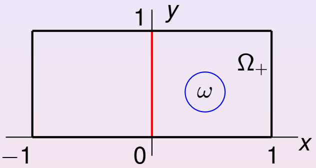

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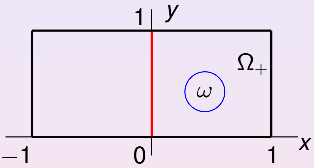

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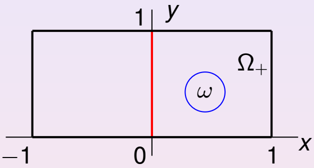

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adjoint problem

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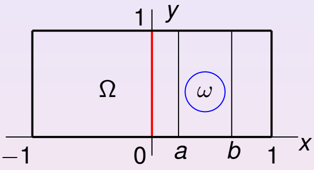

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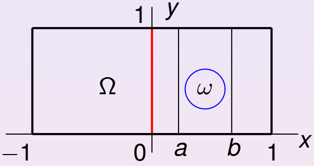

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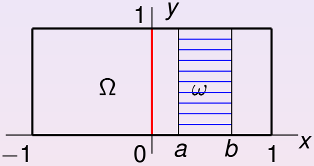

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observability for (G^*) in ω $\iff$ uniform observability for (G_n^*) in (a,b)

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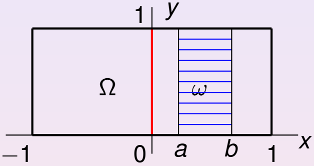

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dissipation rate

- define $G_{\gamma,n} : D(G_{\gamma,n}) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$D(G_{\gamma,n}) := H^2 \cap H^1_0(-1,1)\,, \qquad G_{\gamma,n}\varphi := -\varphi'' + (n\pi)^2 |x|^{2\gamma}\varphi$$

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(ub) $\forall \gamma > 0 \quad \exists c^* > 0$ *such that* $\boxed{\lambda_n \le c^* n^{\frac{2}{1+\gamma}}}$

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- $\gamma > 1 \implies (G^*)$ *not observable*
- $\gamma = 1 \implies \exists T^* \geqslant a^2/2$ *such that* (G^*) *not observable* $\forall\, T < T^*$

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proof of negative results

- take eigenfunctions ϕ_n of $G_{\gamma,n}$ associated with λ_n

$$\begin{cases} -\phi_n''(x) + \big[(n\pi)^2|x|^{2\gamma} - \lambda_n\big]\phi_n(x) = 0 & x \in (-1,1) \\ \phi_n(\pm 1) = 0\,, \quad \phi_n \geq 0\,, \quad \|\phi_n\|_{L^2(-1,1)} = 1 \end{cases}$$

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$$\gamma = 1 \quad \Longrightarrow \quad \begin{cases} \lambda_n \sim n\pi \\ \int_a^b \phi_n(x)^2 dx \sim \dfrac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}} \end{cases} \qquad as\ n \to \infty$$

proof by comparison with

$$\psi_n(x) = \frac{\sqrt[4]{n\pi}\,U(\sqrt{n\pi}\,x) - \sqrt[4]{n}e^{-\frac{n\pi}{2}}\theta(x)}{C_n}$$

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$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma} \partial_y^2 u = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1, y) = 0\,, \ u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$

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- $\boxed{0 < \gamma < 1} \implies (G)$ *null controllable* $\forall T > 0$
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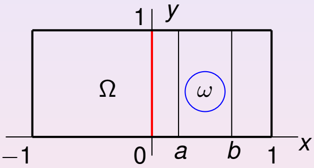

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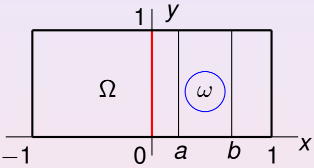

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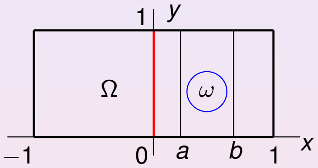

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Carleman estimate

Theorem

Let $\boxed{\gamma \in (0,1]}$ *and let* $w \in C^0([0,T];L^2(-1,1)) \cap L^2(0,T;H^1_0(-1,1))$

$$\begin{cases} \partial_t w - \partial_x^2 w + (n\pi)^2 |x|^{2\gamma} w = g & (t,x) \in (0,T)\times(-1,1) \\ w(t,\pm 1) = 0 & t \in (0,T) \end{cases}$$

Then $\exists \beta \in C^1([-1,1])$ *positive and constants* $\mathcal{C}_1, \mathcal{C}_2 > 0$ *such that*

$$\mathcal{C}_1 \int_0^T \int_{-1}^1 \left(\frac{M_n}{t(T-t)} |\partial_x w|^2 + \frac{M_n^3}{(t(T-t))^3} |w|^2 \right) e^{-\frac{M_n \beta(x)}{t(T-t)}} dxdt$$
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$$M_n = \mathcal{C}_2 \max\{T + T^2; nT^2\}$$

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Theorem

- $\boxed{\gamma \in (0,1)}$ $\Longrightarrow$ $\exists C > 0$ *such that* $\forall T > 0,\ n \geqslant 1$

$$\int_{-1}^{1} v_n(T,x)^2 dx \leqslant CT^2 e^{C\left(1+T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_a^b v_n(t,x)^2 dxdt$$

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null controllability for general ω and $0 < \gamma < 1$

apply technique by Benabdallah-Dermenjian-Le Rousseau (2007)

- $e_n(y) := \sqrt{2}\sin(n\pi y) \qquad y \in [0,1]\,,\ n \geqslant 1$

recall

Proposition (Lebeau-Robbiano)

Let $c, d \in \mathbb{R}$ be such that $c < d$
There exists $C > 0$ such that, for every $n \geqslant 1$ and $(b_k)_{1\leqslant k\leqslant n} \in \mathbb{R}^n$,

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approximation by closed subspaces

study observability for adjoint problem on finite dimensional subspaces

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- $H_n := L^2(-1,1) \otimes e_n \quad (n \geqslant 1)$
- $E_j := \oplus_{n \leqslant 2^j} H_n \quad (j \geqslant 0)$

by Carleman estimate and Lebeau-Robbiano lemma

Proposition

Let $\gamma \in (0,1)$, and let $a,b,c,d \in \mathbb{R}$ be such that $0 < a < b < 1$ and $0 < c < d < 1$ Then there exists $C > 0$ such that for every $T > 0$ and $v_0 \in E_j$ $(j \geqslant 1)$

$$\int_\Omega v(T,x,y)^2 dxdy \leqslant e^{C\left(2^j + T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_\omega v(t,x,y)^2 dxdydt$$

where $\omega := (a,b) \times (c,d)$

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Let $\gamma \in (0,1)$, and let $a,b,c,d \in \mathbb{R}$ be such that $0 < a < b < 1$ and $0 < c < d < 1$ Then there exists $C > 0$ such that for every $T > 0$ and $v_0 \in E_j$ $(j \geqslant 1)$

$$\int_\Omega v(T,x,y)^2 dxdy \leqslant e^{C\left(2^j + T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_\omega v(t,x,y)^2 dxdydt$$

where $\omega := (a,b) \times (c,d)$

construction of the control

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1, y) = 0\,, \;\; u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$

- fix $0 < \rho < \frac{1-\gamma}{1+\gamma}$ and let $K = K(\rho) > 0$ be such that $K \sum_{j=1}^{\infty} 2^{-j\rho} = T$
- let $T_j := K2^{-j\rho}$ and let $(a_j)_{j\in\mathbb{N}}$ be defined by

$$a_0 = 0\,, \quad a_{j+1} = a_j + 2T_j$$

- on $[a_j, a_j + T_j]$ apply control f such that $\Pi_{E_j} u(a_j + T_j, \cdot) = 0$ and

$$\|f\|_{L^2(a_j, a_j+T_j; L^2(\Omega))} \leqslant \mathcal{C}_j \|u(a_j,\cdot)\|_{L^2(\Omega)}$$

with $\mathcal{C}_j := e^{C\left(2^j + T_j^{-\frac{1+\gamma}{1-\gamma}}\right)}$ and $\|u(a_j+T_j,\cdot)\|_{L^2(\Omega)} \leqslant (1 + \sqrt{T_j}\mathcal{C}_j)\|u(a_j,\cdot)\|_{L^2(\Omega)}$

- no control on $[a_j + T_j, a_{j+1}]$ $\Rightarrow$ $\|u(a_{j+1},\cdot)\|_{L^2(\Omega)} \leqslant e^{-\lambda_{2^j} T_j} \|u(a_j + T_j,\cdot)\|_{L^2(\Omega)}$
- combining above inequalities to conclude

$$\|u(a_{j+1},\cdot)\|_{L^2(\Omega)} \leqslant \exp\Big(\underbrace{\sum_{k=1}^{2^j}\Big[\ln(1 + \sqrt{T_k}\mathcal{C}_k) - C(2^k)^{\frac{2}{1+\gamma}} T_k\Big]}_{\to -\infty \text{ as } j\to\infty}\Big)\|u_0\|_{L^2(\Omega)}$$

inverse source problem for Grushin-type operators

$$0 < \gamma \leqslant 1 \qquad \begin{cases} \partial_t u - \underbrace{\left(\partial_x^2 u + |x|^{2\gamma}\partial_y^2 u\right)}_{G_\gamma u} = f(x,y)R(t,x) \\ u(t,\pm 1, y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) \end{cases}$$

want to determine source term $f(x,y)$ by

- locally distributed measurement over $[T_0, T_1]$
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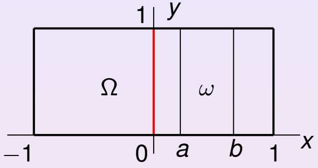

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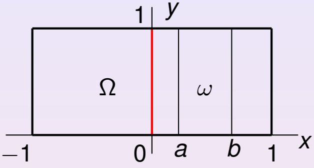

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$$\boxed{\Omega_T = (0,T)\times\Omega \qquad Q_T = (0,T)\times(-1,1) \qquad \omega = (a,b)\times(0,1)}$$

Theorem (Beauchard – C – Yamamoto)

- $\boxed{0 < \gamma < 1}$
- $R, \partial_t R \in C(\overline{Q_T})$ *and* $\exists r_0 >, T_1 \in (0,T]$ *such that* $R(T_1,x) \geqslant r_0\ \forall x \in [-1,1]$

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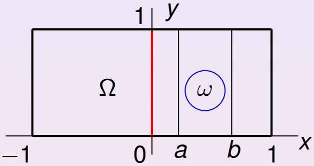

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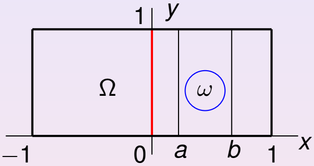

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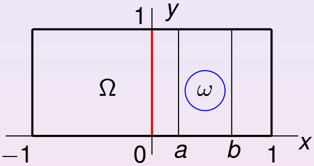

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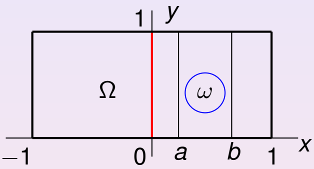

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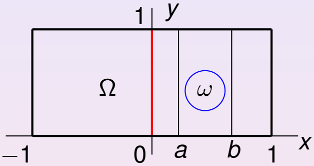
- holds in any positive time when $\gamma \in (0,1)$ and $\omega \subset (0,1) \times (0,1)$
- holds in large time when $\gamma = 1$ and $\omega = (a,b) \times (0,1)$
- does not hold when
  - degeneracy is too strong, i.e. $\gamma > 1$
  - time is too short, i.e. $\gamma = 1$ and $T < a^2/2$

approximate controllability, inverse source problem

conclusions

$$\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t)$$



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approximate controllability,   inverse source problem

extensions

- Lipschitz stability for $\boxed{0 < \gamma < 1}$

$$\partial_t u - (\partial_x^2 u + |x|^{2\gamma} \partial_y^2 u) = f(x, y)R(t, x)$$

true with general ω

- null controllability and Lipschitz stability results can be extended to multi-dimensional Grushin-type operators

$$G_\gamma u = \Delta_x u + |x|^{2\gamma} \Delta_y u$$

with $x \in \Omega_1 \subset \mathbb{R}^{N_1}$, $y \in \Omega_2 \subset \mathbb{R}^{N_2}$ bounded and

  - $\boxed{0 < \gamma < 1}$ $\quad \omega \subset \Omega_1 \times \Omega_2$
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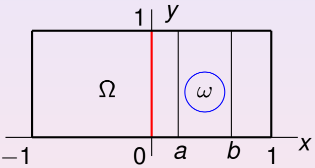

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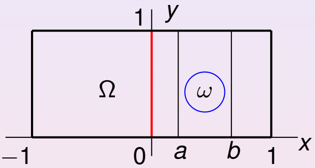

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open problems

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  - sharp estimate of T^* for $\gamma = 1$ ($T^* = a^2/2$? Miller)
- study

$$\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u + \frac{c}{x^2}u = 0 \qquad (0 < \gamma < 1)$$

merci de votre attention

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