An “exterior approach” to solve the inverse obstacle problem for the Stokes system

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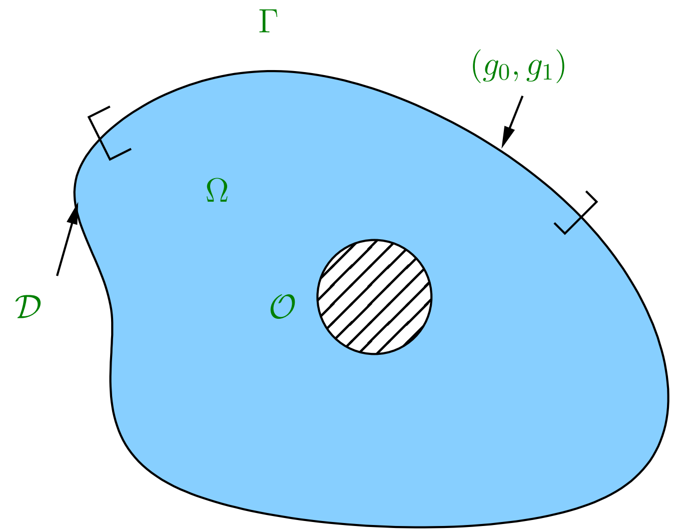
Inverse obstacle problem (Dirichlet)

- $\mathcal{O} \Subset \mathcal{D}$ (bounded) $\subset \mathbb{R}^d$ $(d = 2, 3)$
- $\Gamma \subset \partial\mathcal{D}$ (Γ open with measure > 0)
- $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ connected



For $(\boldsymbol{g}_0, \boldsymbol{g}_1)$, find $\mathcal{O}$ s.t. $(\boldsymbol{u}, p) \in (H^1(\Omega))^d \times L^2(\Omega)$ and

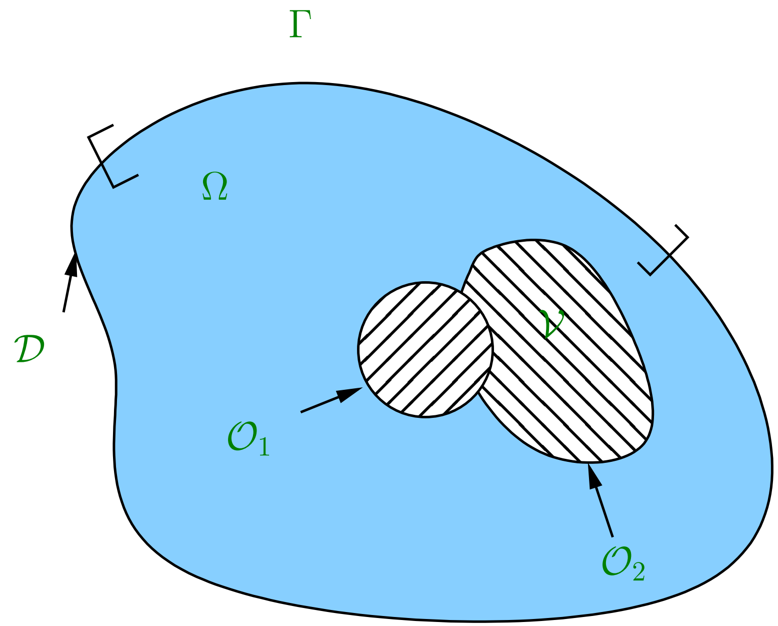
$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_0 & \text{on } \Gamma \\ \sigma(\boldsymbol{u}, p)\cdot n = \boldsymbol{g}_1 & \text{on } \Gamma \\ \boldsymbol{u} = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

$$e(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$$

$$\sigma(\boldsymbol{u}, p) = 2\nu e(\boldsymbol{u}) - pI$$

Uniqueness : for $(\boldsymbol{g}_0, \boldsymbol{g}_1)$ with $\boldsymbol{g}_0 \neq 0$ and $\boldsymbol{u} \in (C^0(\overline{\Omega}))^d$, there exists at most one obstacle $\mathcal{O}$

Assume that two obstacles $\mathcal{O}_1$ et $\mathcal{O}_2$ are compatible with $(\boldsymbol{g}_0, \boldsymbol{g}_1)$



- $\tilde{\mathcal{D}}$:= connected component of $\mathcal{D} \setminus \overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2}$ in contact with Γ
- $\mathcal{V} := \mathcal{D} \setminus (\overline{\mathcal{O}_1} \cup \overline{\tilde{\mathcal{D}}})$

We have $\boldsymbol{u}_1 = 0$ on $\partial\mathcal{O}_1$

Unique continuation implies $\boldsymbol{u}_1 = \boldsymbol{u}_2$ on $\partial\tilde{\mathcal{D}}$

Hence $\boldsymbol{u}_1 = 0$ on $\partial\mathcal{O}_2 \cap \partial\tilde{\mathcal{D}}$

Finally $\boldsymbol{u}_1 = 0$ on $\partial\mathcal{V}$, that is ($\boldsymbol{u}_1$ continuous) $\boldsymbol{u}_1 \in (H_0^1(\mathcal{V}))^d$ and

$$\int_{\mathcal{V}} (-\nu\Delta\boldsymbol{u}_1 + \nabla p_1) \cdot \boldsymbol{u}_1 \, dx = 0 = \int_{\mathcal{V}} \nu|\nabla\boldsymbol{u}_1|^2 \, dx$$

Hence $\boldsymbol{u}_1 = 0$ in $\mathcal{V}$, and $\Rightarrow = 0$ in $\mathcal{D} \setminus \overline{\mathcal{O}_1}$ $\rightarrow$ incompatible with $\boldsymbol{g}_0 \neq 0$

Some contributions on IP with Stokes system

- **Unique continuation for Stokes** (with nonsmooth potential) : Fabre & Lebeau (96)

- **Quantification of unique continuation for Stokes** : Lin, Uhlmann & Wang (10), Boulakia, Egloffe & Grandmont (12)

- **Uniqueness and stability of the inverse obstacle problem for fluids** : Alvarez, Conca, Fritz, Kavian & Ortega (05), Ballerini (10), Conca, Malik, Munnier (10), Conca, Schwindt, Takahashi (12)

- **Data completion for Stokes** : Ben Abda, Ben Saad, Hassine (09)

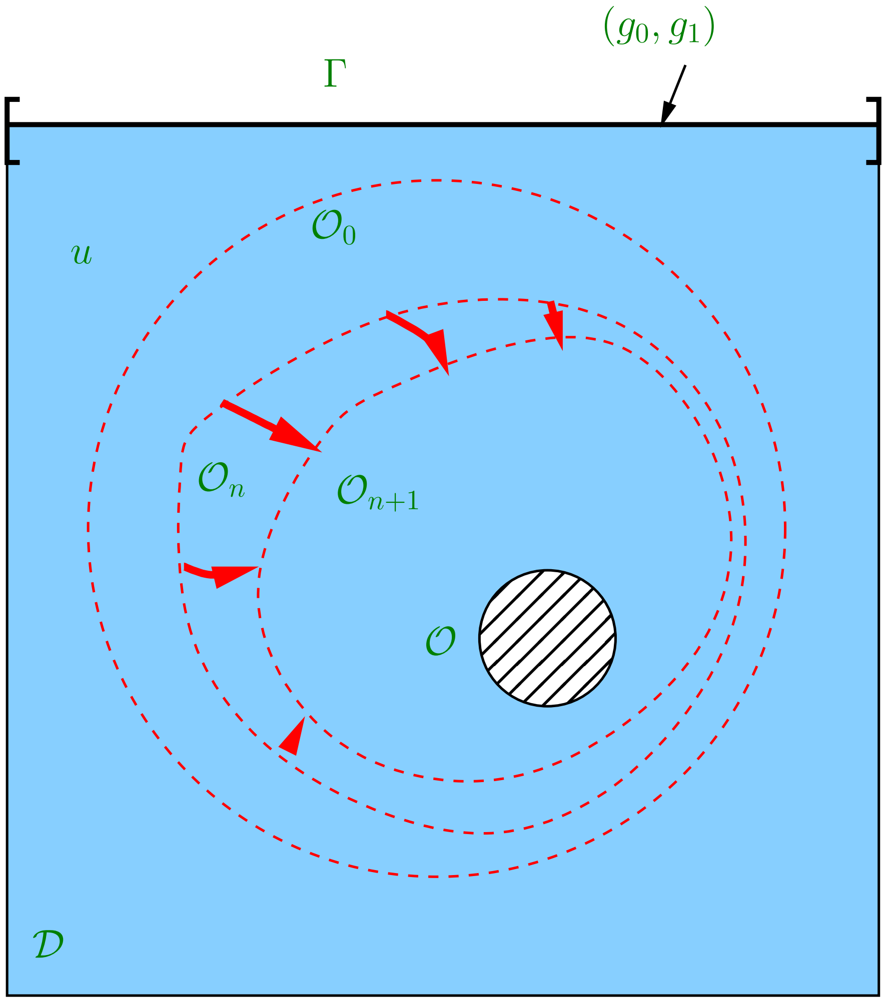
- **Numerical methods for the inverse obstacle problem** : Martins & Silvestre (08) (parametrization), Alvarez, Conca, Lecaros & Ortega (08), Badra, Caubet & Dambrine (11), Caubet, Dambrine, Kateb & Timimoun (12)(shape derivative), Ben Abda, Hassine, Jaoua, Masmoudi (10) (topological gradient)

The exterior approach

An iterative approach that couples a quasi-reversibility method and a level set method $\rightarrow$ no optimization

- **Step 1** : given the current obstacle $\mathcal{O}_n$, find an approximation $\boldsymbol{u}_n$ of solution $\boldsymbol{u}$ in $\Omega_n = \mathcal{D} \setminus \overline{\mathcal{O}}_n$ with the **method of quasi-reversibility**

- **Step 2** : given the approximate solution $\boldsymbol{u}_n$ in Ω_n, update the obstacle $\mathcal{O}_n$ with a **level set method**



A new level set method

Velocity $V \in H^1(\mathcal{D})$ s.t. $V|_{\mathcal{O}} \in H^1_0(\mathcal{O})$

$$\begin{cases} V = |\boldsymbol{u}| = \sqrt{\sum_{i=1}^d u_i^2} & \text{in} \quad \Omega \\ V \leq 0 & \text{in} \quad \mathcal{O} \end{cases}$$

For $f \geq \Delta V$ in $H^{-1}(\mathcal{D})$

$$\begin{cases} \mathcal{O} \subset \mathcal{O}_0 \Subset \mathcal{D} \\ \mathcal{O}_{n+1} = \{x \in \mathcal{O}_n, \quad \phi_n(x) < 0\} \end{cases} \qquad \begin{cases} \Delta\phi_n = f & \text{in} \quad \mathcal{O}_n \\ \phi_n = V & \text{on} \quad \partial\mathcal{O}_n \end{cases}$$

Theorem (convergence of level sets) : if the $\mathcal{O}_n$ are uniformly Lipschitz, for the Hausdorff distance

$$\overbrace{\bigcap_n \mathcal{O}_n}^{\circ} = \mathcal{O}$$

A new level set method (cont.)

Proposition : if $\mathcal{O} \subset \mathcal{O}_0$, the sequence of $\mathcal{O}_n$ converges (Hausdorff distance) to $\tilde{\mathcal{O}}$:= interior of $\cap_n \mathcal{O}_n$, with $\mathcal{O} \subset \tilde{\mathcal{O}}$

Proof : The sequence of $\mathcal{O}_n$ is decreasing $\Rightarrow$ convergence of $\mathcal{O}_n$ (Hausdorff distance) to $\tilde{\mathcal{O}}$.

We show by induction that $\mathcal{O} \subset \mathcal{O}_n$:

$\psi_n = \phi_n - V$ is solution in $H^1_0(\mathcal{O}_n)$ of $\Delta\psi_n = f - \Delta V$.

Weak maximun principle : $f - \Delta V \geq 0 \Rightarrow \psi_n \leq 0$ in $\mathcal{O}_n$.

Hence $\phi_n = \psi_n + V \leq V$ in $\mathcal{O}_n$.

Since $V \leq 0$ in $\mathcal{O} \subset \mathcal{O}_n$, we have $\phi_n \leq 0$ in $\mathcal{O}$, then $\mathcal{O} \subset \mathcal{O}_{n+1}$.

Inclusion is conserved by Hausdorff convergence $\Rightarrow \mathcal{O} \subset \tilde{\mathcal{O}}$.

A new level set method (cont.)

Lemma : $\psi_n \overset{H_0^1(\mathcal{D})}{\longrightarrow} \psi$ when $n \to +\infty$ with

$$\begin{cases} \psi_n \in H_0^1(\mathcal{O}_n) & \Delta\psi_n = f - \Delta V \\ \psi \in H_0^1(\tilde{\mathcal{O}}) & \Delta\psi = f - \Delta V \end{cases}$$

Proof : "$\mathcal{O}_n \overset{\mathcal{H}}{\to} \tilde{\mathcal{O}}$" + "$\mathcal{O}_n$ uniformly Lipschitz", see (Henrot & Pierre, 05)

End of the proof of the theorem : we alreary have $\mathcal{O} \subset \tilde{\mathcal{O}}$, assume that $\mathcal{R} := \tilde{\mathcal{O}} \setminus \overline{\mathcal{O}} \neq \emptyset$. From the lemma, by passing to the limit on sequence (ψ_n), $\phi := \psi + V \leq 0$ in $\tilde{\mathcal{O}}$. We hence have $V = |\boldsymbol{u}| \leq V - \phi = -\psi$ in $\mathcal{R}$. But $\psi = 0$ on $\partial\tilde{\mathcal{O}}$ and $\boldsymbol{u} = 0$ on $\partial\mathcal{O}$. Hence $\boldsymbol{u} = 0$ on $\partial\mathcal{R}$. We conclude as for uniqueness.

The standard method of quasi-reversibility: the case of Laplacian

A regularization technique to solve ill-posed Cauchy problems (Lattès & Lions, 67)

$$V_g = \left\{ v \in H^2(\Omega) |\ v = g_0,\ \partial_n v = g_1 \text{ on } \Gamma \right\}$$

$$V_0 = \left\{ v \in H^2(\Omega) |\ v = 0,\ \partial_n v = 0 \text{ on } \Gamma \right\}$$

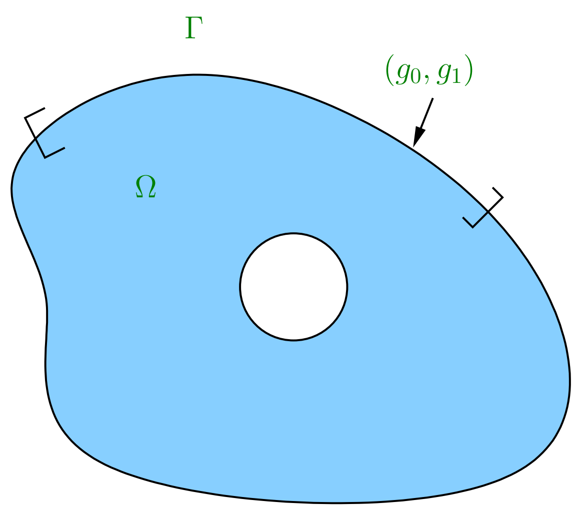
- Second-order ill-posed problem :

  find $u \in V_g$ s.t. $\Delta u = 0$

- Fourth-order well-posed problem :

  find $u_\varepsilon \in V_g$ s.t. for all $v \in V_0$

  $(\Delta u_\varepsilon, \Delta v)_{L^2(\Omega)} + \varepsilon (u_\varepsilon, v)_{H^2(\Omega)} = 0$



Theorem (convergence of QR) : $\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{H^2(\Omega)} = 0$

A mixed formulation of quasi-reversibility

- Ill-posed Cauchy problem : find $u \in H^1(\Omega)$ s.t.

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ u|_\Gamma = g_0 \quad \partial_n u|_\Gamma = g_1 \end{cases}$$

$$W_g = \{v \in H^1(\Omega),\, v|_\Gamma = g_0\}, \quad W_0 = \{v \in H^1(\Omega),\, v|_\Gamma = 0\}$$

$$\tilde{W}_0 = \{\mu \in H^1(\Omega),\, \mu|_{\tilde{\Gamma}} = 0\}, \quad \tilde{\Gamma} = \partial\Omega \setminus \overline{\Gamma}$$

- Well-posed mixed problem : find $(u_{\varepsilon,\gamma}, \lambda_{\varepsilon,\gamma}) \in W_g \times \tilde{W}_0$ s.t.

$$\begin{cases} \varepsilon \int_\Omega \nabla u_{\varepsilon,\gamma} \cdot \nabla v \, dx + \int_\Omega \nabla v \cdot \nabla \lambda_{\varepsilon,\gamma} \, dx = 0, \quad \forall v \in W_0 \\ \int_\Omega \nabla u_{\varepsilon,\gamma} \cdot \nabla \mu \, dx - \gamma \int_\Omega \nabla \lambda_{\varepsilon,\gamma} \cdot \nabla \mu \, dx = \int_\Gamma g_1 \mu \, d\Gamma, \quad \forall \mu \in \tilde{W}_0 \end{cases}$$

Theorem : if $\lim_{\varepsilon \to 0} \varepsilon / \gamma(\varepsilon) = 0$, $\lim_{\varepsilon \to 0} (u_{\varepsilon,\gamma}, \lambda_{\varepsilon,\gamma}) = (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$

A mixed formulation of QR (cont.)

Advantages : the mixed formulation enables us

- to solve the ill-posed problem for standard regularity $H^1(\Omega)$ (instead of $H^2(\Omega)$) for exact solution
- to use standard Lagrange (instead of Hermite) finite elements

Drawback : we have to introduce a second regularization term with $\gamma > 0$: the bilinear form

$$\begin{cases} \{v \in H^1(\Omega),\, v|_\Gamma = 0\} \times \{\mu \in H^1(\Omega),\, \mu|_{\tilde{\Gamma}} = 0\} \to \mathbb{R} \\ (v, \mu) \mapsto \displaystyle\int_\Omega \nabla v \cdot \nabla \mu \, dx \end{cases}$$

does not satisfy the inf-sup condition (because the Cauchy problem is ill-posed !)

A mixed formulation of QR (cont.)

Proof (convergence of mixed formulation) : notice that u is the exact solution iff $u \in W_g$ and

$$\int_\Omega \nabla u \cdot \nabla \mu \, dx = \int_\Gamma g_1 \mu \, d\Gamma, \quad \forall \mu \in \tilde{W}_0$$

The approximate solution $(u_\varepsilon, \lambda_\varepsilon) := (u_{\varepsilon,\gamma(\varepsilon)}, \lambda_{\varepsilon,\gamma(\varepsilon)})$ solves

$$\begin{cases} \varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx + \int_\Omega \nabla v \cdot \nabla \lambda_\varepsilon \, dx = 0, & \forall v \in W_0 \\ \int_\Omega \nabla (u_\varepsilon - u) \cdot \nabla \mu \, dx - \gamma \int_\Omega \nabla \lambda_\varepsilon \cdot \nabla \mu \, dx = 0, & \forall \mu \in \tilde{W}_0 \end{cases}$$

Choose $v = u_\varepsilon - u$ and $\mu = \lambda_\varepsilon$, we obtain

$$\varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla (u_\varepsilon - u) \, dx + \gamma(\varepsilon) \int_\Omega |\nabla \lambda_\varepsilon|^2 \, dx = 0,$$

$$\Rightarrow \quad ||u_\varepsilon||_{H^1} \leq ||u||_{H^1} \quad ||\lambda_\varepsilon||_{H^1} \leq \sqrt{\varepsilon/\gamma(\varepsilon)} \, ||u||_{H^1}$$

A mixed formulation of QR (cont.)

Proof (continuation) :

- we extract a subsequence $u_\varepsilon \rightharpoonup w$ in $H^1(\Omega)$, with $w \in W_g$ since W_g is weakly closed
- $\lambda_\varepsilon \to 0$ in $H^1(\Omega)$ when $\varepsilon \to 0$

Passing to the limit $\varepsilon \to 0$ in the second equation of QR formulation,

$$\int_\Omega \nabla w \cdot \nabla \mu \, dx = \int_\Gamma g_1 \mu \, d\Gamma, \quad \forall \mu \in \tilde{W}_0$$

In conclusion, $w = u$.
From identify

$$||u_\varepsilon - u||^2_{H^1} = (u_\varepsilon, u_\varepsilon - u)_{H^1} - (u, u_\varepsilon - u)_{H^1} \leq -(u, u_\varepsilon - u)_{H^1},$$

weak convergence implies strong convergence and $u_\varepsilon \to u$ in $H^1(\Omega)$.

The mixed formulation : extension to the Stokes system

The ill-posed Stokes problem :

For $(\boldsymbol{g}_0, \boldsymbol{g}_1)$, find $(\boldsymbol{u}, p) \in (H^1(\Omega))^d \times L^2(\Omega)$ s.t.

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &= 0 & \text{in } \Omega \\ \text{div}\boldsymbol{u} &= 0 & \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g}_0 & \text{on } \Gamma \\ \sigma(\boldsymbol{u}, p) \cdot n &= \boldsymbol{g}_1 & \text{on } \Gamma \end{cases}$$

$$e(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$$

$$\sigma(\boldsymbol{u}, p) = 2\nu e(\boldsymbol{u}) - pI$$

Theorem (uniqueness property) :

$$(\boldsymbol{g}_0, \boldsymbol{g}_1) = (0, 0) \text{ implies } (\boldsymbol{u}, p) = (0, 0)$$

The mixed formulation for the Stokes system

$$W_g = \{\boldsymbol{v} \in (H^1(\Omega))^d,\, \boldsymbol{v}|_\Gamma = \boldsymbol{g}_0\}, \quad W_0 = \{\boldsymbol{v} \in (H^1(\Omega))^d,\, \boldsymbol{v}|_\Gamma = 0\}$$

$$\tilde{W}_0 = \{\boldsymbol{\mu} \in (H^1(\Omega))^d,\, \boldsymbol{\mu}|_{\tilde{\Gamma}} = 0\}$$

Well-posed mixed problem : find $(\boldsymbol{u}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}) \in W_g \times \tilde{W}_0$ s.t.

$$\begin{cases} 2\nu\varepsilon \int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{v})\, dx + \int_\Omega \operatorname{div} \boldsymbol{u}_{\varepsilon,\gamma} \operatorname{div} \boldsymbol{v}\, dx \\ +2\nu \int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{\lambda}_{\varepsilon,\gamma})\, dx = 0, \quad \forall \boldsymbol{v} \in V_0 \\ 2\nu \int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\, dx - \frac{1}{\varepsilon} \int_\Omega \operatorname{div} \boldsymbol{\lambda}_{\varepsilon,\gamma} \operatorname{div} \boldsymbol{\mu}\, dx \\ -\gamma \int_\Omega e(\boldsymbol{\lambda}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\, dx = \int_\Gamma \boldsymbol{g}_1 \cdot \boldsymbol{\mu}\, d\Gamma, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0. \end{cases}$$

Theorem : for $p_{\varepsilon,\gamma} := \operatorname{div} \boldsymbol{\lambda}_{\varepsilon,\gamma}/\varepsilon$, if $\lim_{\varepsilon\to 0} \varepsilon/\gamma(\varepsilon) = 0$,

$$\lim_{\varepsilon\to 0} (\boldsymbol{u}_{\varepsilon,\gamma}, p_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}) = (\boldsymbol{u}, p, 0) \in (H^1(\Omega))^d \times L^2(\Omega) \times (H^1(\Omega))^d$$

Back to the exterior approach

The algorithm :

1. Initial guess $\mathcal{O}_0$: $\mathcal{O} \subset \mathcal{O}_0 \Subset \mathcal{D}$
2. First step : for $\mathcal{O}_n$ given, compute the solution of quasi-reversibility $\boldsymbol{u}_n$ in $\Omega_n := \mathcal{D} \setminus \overline{\mathcal{O}_n}$
3. Second step : for $\boldsymbol{u}_n$ given in Ω_n, compute the Poisson solution ϕ_n in $\mathcal{O}_n$ with velocity $V_n = |\boldsymbol{u}_n|$

$$\begin{cases} \Delta\phi_n = C & \text{in} \quad \mathcal{O}_n \\ \phi_n = |\boldsymbol{u}_n| & \text{on} \quad \partial\mathcal{O}_n \end{cases}$$

   and update $\mathcal{O}_{n+1}$

4. Back to the first step until stopping criteria is reached

Numerical experiments

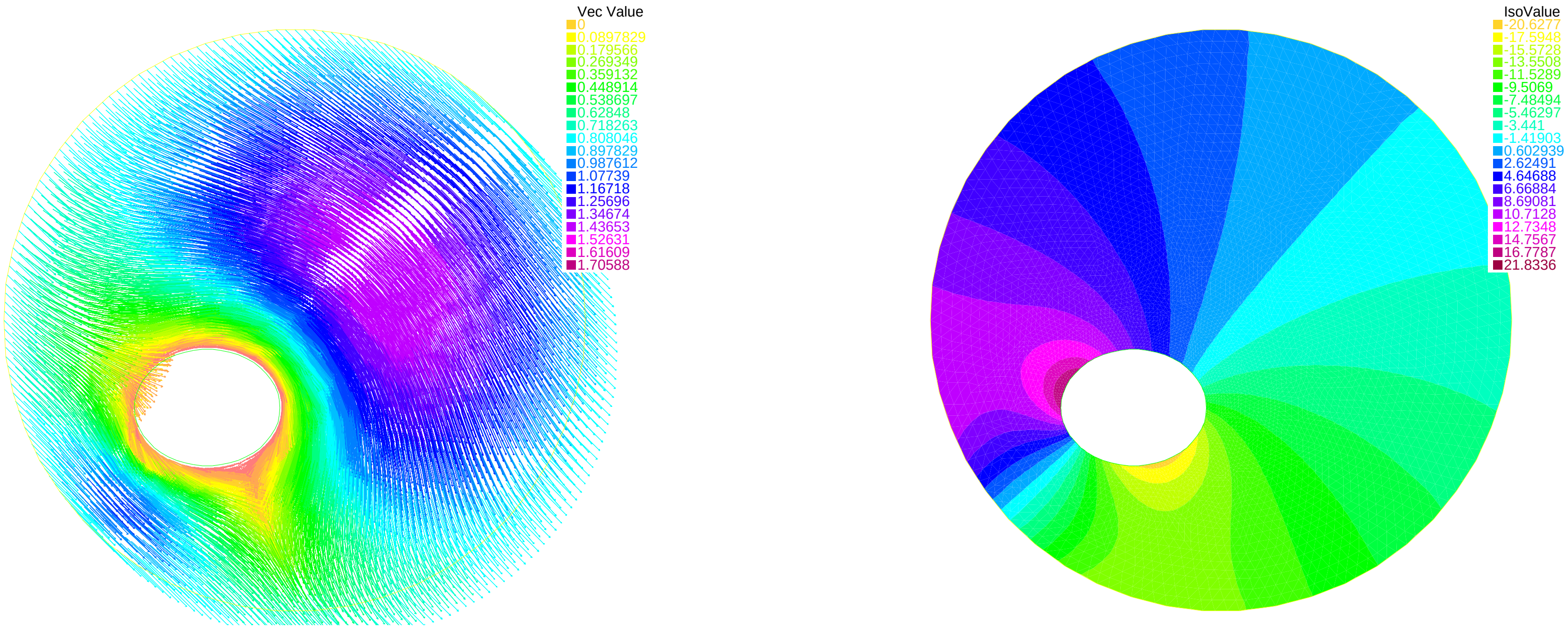
Artificial data obtained for :

- $\mathcal{D} = B(0,1) \subset \mathbb{R}^2$
- Dirichlet data $\boldsymbol{u} = (1,-1)/\sqrt{2}$ on $\partial\mathcal{D}$
- **First obstacle :**

$$\begin{cases} x(t) = 0.25\cos(t) - 0.3 \\ y(t) = 0.2\sin(t) - 0.3 \end{cases}$$

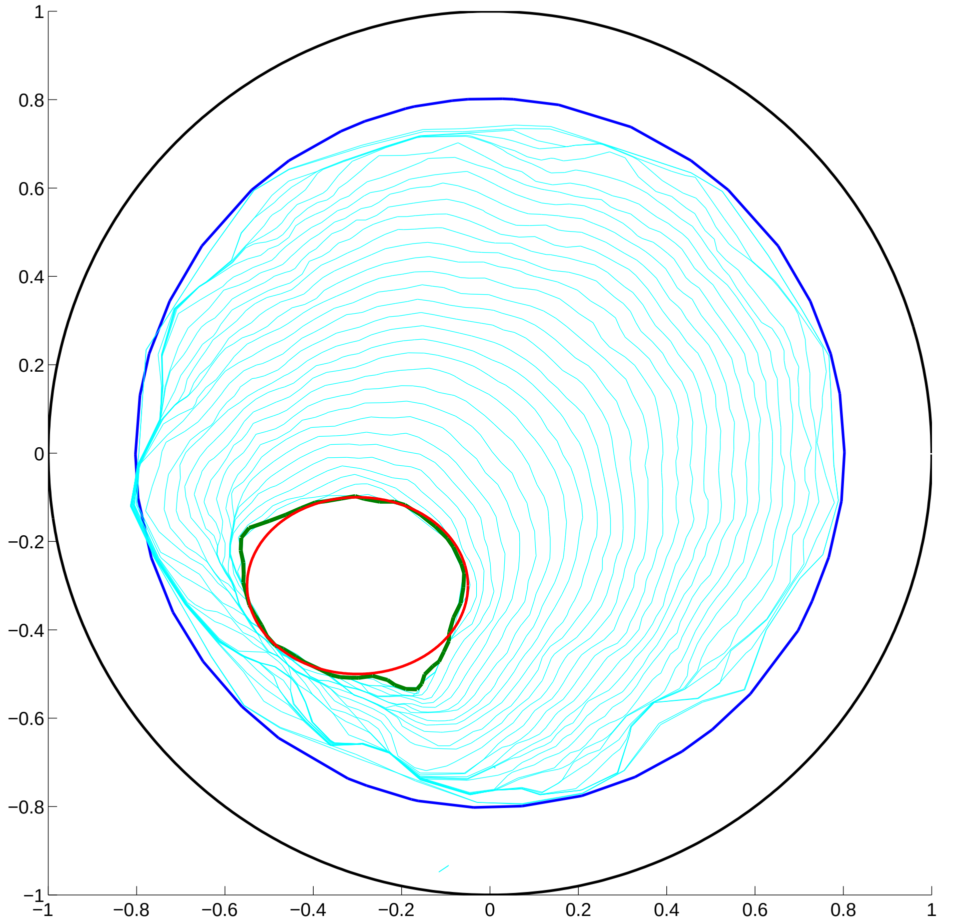
Second obstacle : Two discs of radius $R = 0.2$, of center $(0.4, 0.4)$ and $(-0.3, -0.3)$

Artificial data



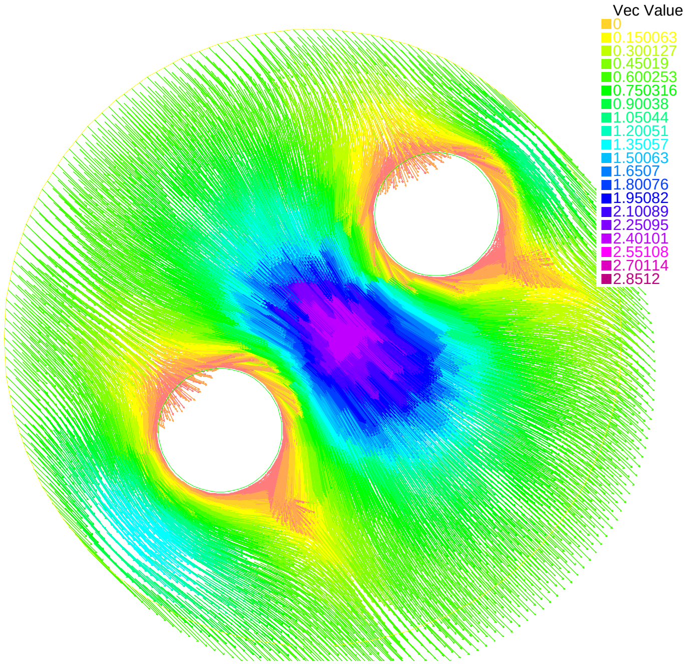
First obstacle : velocity field and pressure field

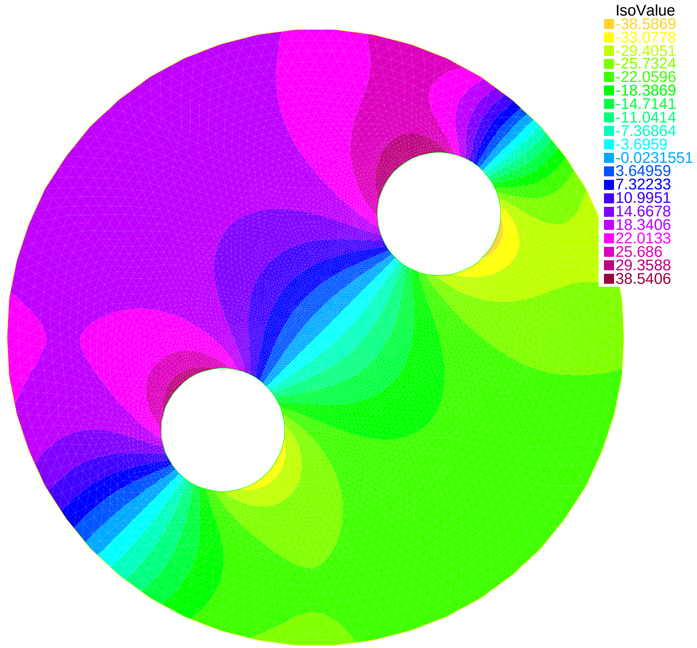
Identification results



First obstacle

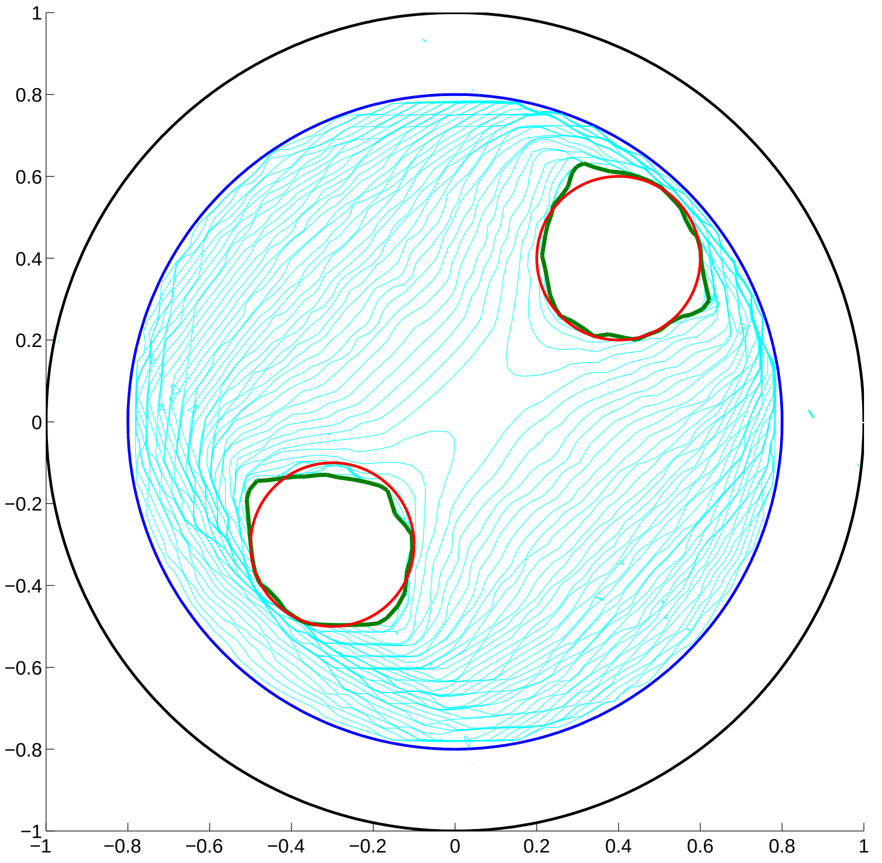
Artificial data





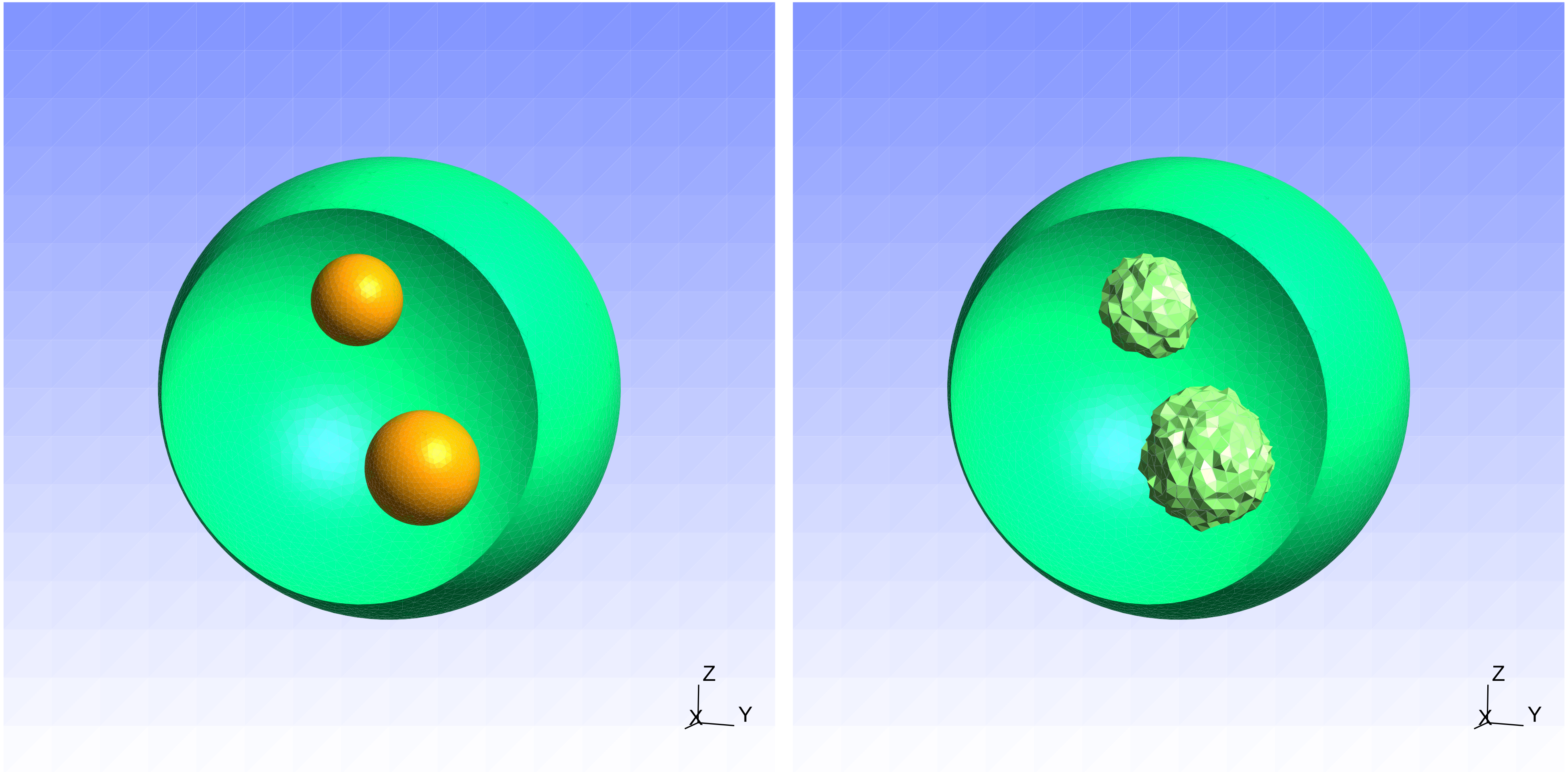
Second obstacle : velocity field and pressure field

Identification results



Second obstacle

3D example with Laplace equation



Conclusions

Main advantages of the exterior approach :

- No optimization
- The number of obstacle is a priori unknown
- Few iterations
- A single mesh for “exterior” and “interior” problems
- Partial Cauchy data

Extensions :

- Other boundary conditions : $|\nabla u| = 1$ (detection of plastic zone)
- Other level set method : eikonal equation
- The unsteady case for the heat equation and the Stokes system