Parameter identification for a simplified model of the respiratory tract

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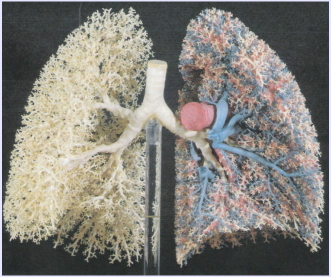
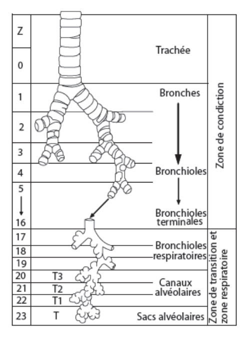
Introduction



Moulding of a human lung (Weibel)



Different compartments of the lung

Model

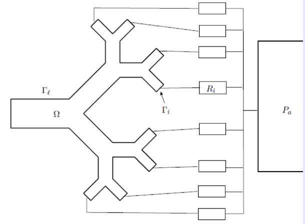
$$\begin{cases} \rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u + \nabla p = 0, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot u = 0, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\Gamma_l, \\ \mu\nabla u\cdot n - pn = -P_e n, & \text{on } (0,T)\times\Gamma_e, \\ \mu\nabla u\cdot n - pn = -\pi_i n, & \text{on } (0,T)\times\Gamma_i,\ 1\le i\le N, \\ u(0) = u_0, & \text{in } \Omega. \end{cases}$$

Model

$$\left\{\begin{array}{rcll} \rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u + \nabla p &=& 0, & \text{in } (0,T)\times\Omega,\\ \nabla\cdot\ u &=& 0, & \text{in } (0,T)\times\Omega,\\ u &=& 0, & \text{on } (0,T)\times\Gamma_l,\\ \mu\nabla u\cdot n - pn &=& -P_e n, & \text{on } (0,T)\times\Gamma_e,\\ \mu\nabla u\cdot n - pn &=& -\pi_i n, & \text{on } (0,T)\times\Gamma_i,\ 1\leq i\leq N,\\ u(0) &=& u_0, & \text{in } \Omega. \end{array}\right.$$

$$\pi_i - P_a = R_i \int_{\Gamma_i} u\cdot n$$

- R_i : resistance of each bronchial tube
- P_a : alveolar pressure (known)



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\nabla\cdot u = 0, & \text{in } (0,T)\times\Omega,\\
u = 0, & \text{on } (0,T)\times\Gamma_l,\\
\mu\nabla u\cdot n - pn = -P_e n, & \text{on } (0,T)\times\Gamma_e,\\
\mu\nabla u\cdot n + R_i \displaystyle\int_{\Gamma_i} u\cdot n - pn = -P_a n, & \text{on } (0,T)\times\Gamma_i,\ 1\le i\le N,\\
u(0) = u_0, & \text{in } \Omega.
\end{cases}
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Model

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\end{cases}
$$

- More complex model : coupling with the diaphragm
  Grandmont, Maday, Maury (2005)
- Same kind of model for the blood flow in the cardiovascular system
  Quarteroni, Veneziani (2003), Vignon-Clementel, Figueroa, Jansen, Taylor (2006)
- asthma : larger resistance

Model

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- Mixed boundary conditions Dirichlet / Neumann / Robin
- Not much regularity for the domain
- Energy estimate : boundary terms

$$\sum_{i=1}^{N}\int_{\Gamma_i}|u|^2(u\cdot n) + \int_{\Gamma_e}|u|^2(u\cdot n)$$

Model

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Baffico, Grandmont, Maury (2010)

- angle $\frac{\pi}{2}$ between Γ_i and Γ_l and between Γ_e and Γ_l
- dimension 2 or 3
- existence and uniqueness of solution defined locally in time

A linear model

$$\left\{\begin{array}{rcll} \partial_t u - \mu\Delta u + \nabla p & = & 0, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\ u & = & 0, & \text{in } (0,T)\times\Omega, \\ u & = & 0, & \text{on } (0,T)\times\Gamma_l, \\ \mu\nabla u\cdot n - pn & = & g, & \text{on } (0,T)\times\Gamma_e, \\ \mu\nabla u\cdot n - pn + qu & = & 0, & \text{on } (0,T)\times\Gamma_0, \\ u(0) & = & u_0, & \text{in } \Omega. \end{array}\right.$$

q does not depend on time

A linear model

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q does not depend on time

Objective : evaluate q from measurements of u and p on Γ_e.

A linear model

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q does not depend on time

Let (u_1, p_1) be the solution for $q = q_1$ and (u_2, p_2) be the solution for $q = q_2$.

- Identifiability:

$$u_1 = u_2 \text{ on a part of } \Gamma_e \Rightarrow q_1 = q_2?$$

- Stability:

$$u_1 \sim u_2 \text{ on a part of } \Gamma_e \Rightarrow q_1 \sim q_2?$$

- Stability inequality:

$$\|q_1 - q_2\| \leq f(\|u_1 - u_2\|_{\Gamma_e} + \|p_1 - p_2\|_{\Gamma_e})$$

Identifiability

For $s > 0$ and $p \geq 1$, we define $B_{s,p}(\mathbb{R}^d) = \{w \in \mathcal{S}'(\mathbb{R}^d)/(1+|\xi|^2)^{s/2}\hat{w} \in L^p(\mathbb{R}^d)\}$.

$q_1,\ q_2 \in B_{1/2,1}(\Gamma_0)$, $g \in H^1(0,T;H^{1/2}(\Gamma_e))$ non identically null

$$
i = 1,2, \quad \left\{
\begin{array}{rcll}
\partial_t u_i - \mu\Delta u_i + \nabla p_i & = & 0, & \text{in } (0,T)\times\Omega, \\
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u_i & = & 0, & \text{on } (0,T)\times\Gamma_l, \\
\mu\nabla u_i\cdot n - p_i n & = & g, & \text{on } (0,T)\times\Gamma_e, \\
\mu\nabla u_i\cdot n - p_i n + q_i u_i & = & 0, & \text{on } (0,T)\times\Gamma_0.
\end{array}
\right.
$$

We assume that $u_1 = u_2$ on $(0,T)\times K, \quad K\subset\Gamma_e$.

Identifiability

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$$i = 1,2, \quad \begin{cases} \partial_t u_i - \mu\Delta u_i + \nabla p_i = 0, & \text{in } (0,T)\times\Omega, \\ \nabla \cdot u_i = 0, & \text{in } (0,T)\times\Omega, \\ u_i = 0, & \text{on } (0,T)\times\Gamma_l, \\ \mu\nabla u_i \cdot n - p_i n = g, & \text{on } (0,T)\times\Gamma_e, \\ \mu\nabla u_i \cdot n - p_i n + q_i u_i = 0, & \text{on } (0,T)\times\Gamma_0. \end{cases}$$

We assume that $u_1 = u_2$ on $(0,T)\times K$, $\quad K \subset \Gamma_e$. We define $u = u_1 - u_2$, $p = p_1 - p_2$. Then

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla p = 0, & \text{in } (0,T)\times\Omega, \\ \nabla \cdot u = 0, & \text{in } (0,T)\times\Omega, \\ \mu\nabla u \cdot n - pn = 0, & \text{on } (0,T)\times K, \\ u = 0, & \text{on } (0,T)\times K, \end{cases}$$

Unique continuation property (**Fabre, Lebeau (1996)**): $u = 0$ and $p = 0$ in $(0,T)\times\Omega$. Thus

$$(q_1 - q_2)u_1 = 0 \text{ on } (0,T)\times\Gamma_0.$$

Identifiability

$q_1,\ q_2 \in B_{1/2,1}(\Gamma_0),\ g \in H^1(0,T;H^{1/2}(\Gamma_e))$ non identically null

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$$(q_1-q_2)u_1 = 0 \text{ on } (0,T)\times\Gamma_0.$$

If there exists $x_0\in\Gamma_0$ such that $q_1(x_0)\neq q_2(x_0)$ on Γ_0, then, $\exists\kappa\subset\Gamma_0$ such that $|q_1-q_2|>0$ on κ. So $u_1=0$ on $(0,T)\times\kappa$ and thus :

$$\left\{\begin{array}{rcll} \partial_t u_1 - \mu\Delta u_1 + \nabla p_1 & = & 0, & \text{in } (0,T)\times\Omega,\\ \nabla\cdot\ u_1 & = & 0, & \text{in } (0,T)\times\Omega,\\ \mu\nabla u_1\cdot n - p_1 n & = & 0, & \text{on } (0,T)\times\kappa.\\ u_1 & = & 0, & \text{on } (0,T)\times\kappa. \end{array}\right.$$

Unique continuation: $u_1=0$, $p_1=0$ in $(0,T)\times\Omega$ and thus $g\equiv 0$. Absurd : $q_1\equiv q_2$.

Stability result

$\Omega \subset \mathbb{R}^2$ bounded connected open set of class $\mathcal{C}^{3,1}$, $\partial\Omega = \Gamma_e \cup \Gamma_0$ with $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$.

$$\left\{ \begin{array}{rcll} -\mu\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla \cdot \ u & = & 0, & \text{in } \Omega, \\ \mu\nabla u \cdot n - pn & = & g, & \text{on } \Gamma_e, \\ \mu\nabla u \cdot n - pn + qu & = & 0, & \text{on } \Gamma_0. \end{array} \right.$$

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Hypotheses :

- $g \in H^{5/2}(\Gamma_e)$, $\|g\|_{H^{5/2}(\Gamma_e)} \leq M_1$, g non identically null
- $q_1,\ q_2 \in B_{5/2,1}(\Gamma_0)$, $\|q_1\|_{B_{5/2,1}(\Gamma_0)}, \|q_2\|_{B_{5/2,1}(\Gamma_0)} \leq M_2$ and $q_1, q_2 \geq \alpha > 0$ on Γ_0

Let K be a compact set of Γ_0 such that $|u_1| \geq m > 0$ on K. Then

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C(m, M_1, M_2, \alpha)}{\left(\ln \left(\frac{C_1}{\|u_1 - u_2\|_{L^2(\Gamma_e)} + \|\nabla(u_1 - u_2)\cdot n\|_{L^2(\Gamma_e)^2} + \|p_1 - p_2\|_{L^2(\Gamma_e)} + \|\nabla(p_1 - p_2)\cdot n\|_{L^2(\Gamma_e)}} \right) \right)^{\frac{1}{2}}}.$$

M.B., A.-C. Egloffe, C. Grandmont, submitted

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M.B., A.-C. Egloffe, C. Grandmont, submitted

Remark : same result for the Laplace equation **Cheng, Choulli, Lin (2008)**

Sketch of the proof

$$\text{On } \Gamma_0,\ (q_2 - q_1)u_1 = q_2(u_1 - u_2) + \nabla(u_1 - u_2)\cdot n - (p_1 - p_2)n.$$

Thus, if we set $u = u_1 - u_2$, $p = p_1 - p_2$,

$$\|q_1 - q_2\|_{L^2(K)} \leq C(m, M_2)(\|u\|_{L^2(\Gamma_0)} + \|\nabla u \cdot n\|_{L^2(\Gamma_0)} + \|p\|_{L^2(\Gamma_0)})$$

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Lemma (**Bukhgeim (1993)**) : let $\Psi \in \mathcal{C}^2(\overline{\Omega})$. For u and $p \in H^3(\Omega)$

$$\int_\Omega (\Delta\Psi|u|^2 + (\Delta\Psi - 1)|\nabla u|^2)e^\Psi \leq \int_\Omega |\Delta u|^2 e^\Psi + \int_{\partial\Omega} \nabla\Psi\cdot n(|u|^2 + |\nabla u|^2 + 2|\partial_\tau|\nabla u|^2|)e^\Psi.$$

$$\int_\Omega (\Delta\Psi|p|^2 + (\Delta\Psi - 1)|\nabla p|^2)e^\Psi \leq \int_\Omega |\Delta p|^2 e^\Psi + \int_{\partial\Omega} \nabla\Psi\cdot n(|p|^2 + |\nabla p|^2 + 2|\partial_\tau|\nabla p|^2|)e^\Psi.$$

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$$\int_\Omega (\Delta\Psi|p|^2 + (\Delta\Psi - 1)|\nabla p|^2)e^\Psi \leq \int_{\partial\Omega} \nabla\Psi\cdot n(|p|^2 + |\nabla p|^2 + 2|\partial_\tau|\nabla p|^2|)e^\Psi.$$

We take Ψ such that $\Delta\Psi = \lambda$, $\lambda \geq 2$.

Sketch of the proof

$$\int_{\partial\Omega} \nabla\Psi \cdot n\Big(|u|^2 + |\nabla u|^2 + 2|\partial_\tau|\nabla u|^2| + |p|^2 + |\nabla p|^2 + 2|\partial_\tau|\nabla p|^2|\Big)e^{\Psi} \geq 0$$

We choose $\Psi = \Psi_1 + s\Psi_0$. After some computations, we get, for $s > 0$,

$$\int_{\Gamma_0} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right) \leq C\left(e^{ks}\int_{\Gamma_e} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right) + \frac{1}{s}\right)$$

Sketch of the proof

$$\int_{\partial\Omega} \nabla\Psi \cdot n \Big(|u|^2 + |\nabla u|^2 + 2|\partial_\tau |\nabla u|^2| + |p|^2 + |\nabla p|^2 + 2|\partial_\tau |\nabla p|^2| \Big) e^{\Psi} \geq 0$$

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Sketch of the proof

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$$\int_{\Gamma_0} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right) \leq \frac{C}{\ln\left(\dfrac{C_1}{\int_{\Gamma_e} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right)}\right)}.$$

Sketch of the proof

$$\int_{\partial\Omega} \nabla\Psi \cdot n\Big(|u|^2 + |\nabla u|^2 + 2|\partial_\tau|\nabla u|^2| + |p|^2 + |\nabla p|^2 + 2|\partial_\tau|\nabla p|^2|\Big)e^{\Psi} \geq 0$$

We choose $\Psi = \Psi_1 + s\Psi_0$. After some computations, we get, for $s > 0$,

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Remarks

- Bukhgeim inequality satisfied only in dimension 2
- We have to take u and p in $H^3(\Omega)$
- Same kind of result for the non-stationary problem with measurements in $(0, +\infty)$ **Bellassoued, Cheng, Choulli (2008)**

Stability result

$\Omega \subset \mathbb{R}^d$ connected bounded open set, $\partial\Omega = \Gamma_e \cup \Gamma_0$ with $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$.

$$\left\{\begin{array}{rcll} -\mu\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla \cdot \ u & = & 0, & \text{in } \Omega, \\ \mu\nabla u \cdot n - pn & = & g, & \text{on } \Gamma_e, \\ \mu\nabla u \cdot n - pn + qu & = & 0, & \text{on } \Gamma_0. \end{array}\right.$$

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Hypotheses (for $d = 2$ or 3):

- Ω of class $\mathcal{C}^\infty$
- $g \in H^{3/2}(\Gamma_e)$, $\|g\|_{H^{3/2}(\Gamma_e)} \leq M_1$, g non identically null
- q_1, $q_2 \in B_{3/2,1}(\Gamma_0)$, $\|q_1\|_{B_{3/2,1}(\Gamma_0)}, \|q_2\|_{B_{3/2,1}(\Gamma_0)} \leq M_2$ and $q_1, q_2 \geq \alpha > 0$ on Γ_0

Stability result

$\Omega \subset \mathbb{R}^d$ connected bounded open set, $\partial\Omega = \Gamma_e \cup \Gamma_0$ with $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$.

$$\begin{cases} -\mu\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla \cdot \; u & = & 0, & \text{in } \Omega, \\ \mu\nabla u \cdot n - pn & = & g, & \text{on } \Gamma_e, \\ \mu\nabla u \cdot n - pn + qu & = & 0, & \text{on } \Gamma_0. \end{cases}$$

Hypotheses (for $d = 2$ or 3):

- Ω of class $\mathcal{C}^\infty$
- $g \in H^{3/2}(\Gamma_e), \|g\|_{H^{3/2}(\Gamma_e)} \leq M_1$, g non identically null
- q_1, $q_2 \in B_{3/2,1}(\Gamma_0)$, $\|q_1\|_{B_{3/2,1}(\Gamma_0)}, \|q_2\|_{B_{3/2,1}(\Gamma_0)} \leq M_2$ and $q_1, q_2 \geq \alpha > 0$ on Γ_0

Let K be a compact set of Γ_0 such that $|u_1| \geq m > 0$ on K. Then $\exists d_0 > 0$, $\forall \beta \in (0,1)$, $\forall d > d_0$, $\exists C(m, M_1, M_2, \alpha)$, $C_1(m, M_1, M_2, \alpha)$

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C(m, M_1, M_2, \alpha)}{\left(\ln\left(\frac{dC_1(M_1,M_2,\alpha)}{\|u_1-u_2\|_{L^2(\Gamma_e)}+\|p_1-p_2\|_{L^2(\Gamma_e)}+\|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\frac{3\beta}{4}}}.$$

M.B., A.-C. Egloffe, C. Grandmont, in preparation

Remarks

Remarks

- same result for the Laplace equation **Bellassoued, Cheng, Choulli (2008)**
- The data are less regular than in the previous result
- The power in the inequality depends on the regularity of the solution
- Boundaries are very regular locally
- Same kind of result for the non-stationary problem with measurements in $(0, +\infty)$

Sketch of the proof

$$\text{On } \Gamma_0,\ (q_2 - q_1)u_1 = q_2(u_1 - u_2) + \nabla(u_1 - u_2)\cdot n - (p_1 - p_2)n.$$

Thus, if we set $u = u_1 - u_2$, $p = p_1 - p_2$, we have

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leq C(m, M_2)(\|u\|_{L^2(K)} + \|\nabla u \cdot n\|_{L^2(K)} + \|p\|_{L^2(K)}) \\ &\leq C(m, M_2)(\|u\|_{H^{3/2+\epsilon}(\Omega)} + \|p\|_{H^1(\Omega)}) \end{aligned}$$

Then, since $\|u\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)} \leq C$,

$$\|q_1 - q_2\|_{L^2(K)} \leq C(m, M_2)(\|u\|^{\theta}_{H^1(\Omega)} + \|p\|^{\theta}_{H^1(\Omega)})$$

Sketch of the proof: quantification of the unique continuation property

Theorem : Let Ω be a bounded open set of class $\mathcal{C}^\infty$.

There exists $d_0 > 0$, $\forall \gamma \in (0,1)$, $\forall d > d_0$, $\exists c(M) > 0$ such that

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq \frac{c(M)}{\left(\ln\left(\frac{d\, c(M)}{\|u\|_{L^2(\Gamma_e)} + \|p\|_{L^2(\Gamma_e)} + \|\nabla u \cdot n\|_{L^2(\Gamma_e)} + \|\nabla p \cdot n\|_{L^2(\Gamma_e)}}\right)\right)^\gamma}.$$

for all $(u,p) \in H^{3/2+\nu}(\Omega) \times H^{3/2+\nu}(\Omega)$ solution of

$$\left\{ \begin{array}{rcll} -\mu \Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla \cdot u & = & 0, & \text{in } \Omega. \end{array} \right.$$

such that

$$\|u\|_{H^{3/2+\nu}(\Omega)} + \|p\|_{H^{3/2+\nu}(\Omega)} \leq M$$

Proof of the quantification of the unique continuation property

Phung (2003)

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^{\beta}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right),$$

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leq e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^{\beta}(\|u\|_{H^{3/2+\nu}(\Omega)} + \|p\|_{H^{3/2+\nu}(\Omega)}),$$

Hörmander (1985), Lebeau, Robbiano (1995)

Proof of the quantification of the unique continuation property

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$$\begin{aligned}\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \leq\ & \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma_e)} + \|p\|_{H^1(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)}\right)\\ & + \epsilon^\beta\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right).\end{aligned}$$

Hörmander (1985), Lebeau, Robbiano (1995)

Proof of the quantification of the unique continuation property

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$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^{\beta}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right),$$

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$$\begin{aligned}\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \leq\ & \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma_e)} + \|p\|_{H^1(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)}\right) \\ & +\epsilon^{\beta}(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).\end{aligned}$$

Hörmander (1985), Lebeau, Robbiano (1995)

$$\begin{aligned}\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq\ & e^{\frac{c}{\epsilon}}\left(\|u\|_{H^1(\Gamma_e)} + \|p\|_{H^1(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)}\right) \\ & +\epsilon^{\beta}(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).\end{aligned}$$

Proof of the quantification of the unique continuation property

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$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^{\beta}\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right),$$

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$$\begin{aligned}\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \leq \quad & \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma_e)} + \|p\|_{H^1(\Gamma_e)} + \|\nabla u \cdot n\|_{L^2(\Gamma_e)} + \|\nabla p \cdot n\|_{L^2(\Gamma_e)}\right) \\ & + \epsilon^{\beta}(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).\end{aligned}$$

Hörmander (1985), Lebeau, Robbiano (1995)

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq \frac{c(M)}{\left(\ln\left(\frac{d\, c(M)}{\|u\|_{L^2(\Gamma_e)} + \|p\|_{L^2(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^{\gamma}}.$$

Proof of the quantification of the unique continuation property

Phung (2003)

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^\beta \left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right),$$

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leq e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta(\|u\|_{H^{3/2+\nu}(\Omega)} + \|p\|_{H^{3/2+\nu}(\Omega)}),$$

$$\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma_e)} + \|p\|_{H^1(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)}\right) + \epsilon^\beta(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Hörmander (1985), Lebeau, Robbiano (1995)

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq \frac{c(M)}{\left(\ln\left(\frac{d\, c(M)}{\|u\|_{L^2(\Gamma_e)}+\|p\|_{L^2(\Gamma_e)}+\|\nabla u\cdot n\|_{L^2(\Gamma_e)}+\|\nabla p\cdot n\|_{L^2(\Gamma_e)}}\right)\right)^\gamma}.$$

Remark: K compact set of Ω

$$\|u\|_{H^1(K)} + \|p\|_{H^1(K)} \leq C(\|u\|_{L^2(\Gamma_e)} + \|p\|_{L^2(\Gamma_e)} + \|\nabla u\cdot n\|_{L^2(\Gamma_e)} + \|\nabla p\cdot n\|_{L^2(\Gamma_e)})^\beta$$

$$\int_{\Gamma_0} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right) \leq \frac{C}{\ln\left(\dfrac{C_1}{\displaystyle\int_{\Gamma_e} \left(|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2\right)}\right)}.$$

- Bad estimate from the numerical point of view
- Cauchy problem is ill-posed for the Stokes equation
- If we assume that q is piecewise constant, we get Lipschitz estimate

  A.-C. Egloffe, article in preparation, poster in the next session
- Lipschitz stability in the general case ?

  Alessandrini, Del Piero, Rondi (2003), Chaabane, Fellah, Jaoua, Leblond (2004), Sincich (2007), Cornilleau-Robbiano (2011)

Perspectives

- Domain regularity.

  Alessandrini, Beretta, Rosset, Vessela (2000) Bourgeois-Dardé (2010)

- Logarithmic inequality with mixed conditions Dirichlet/Neumann/Robin ?

  Regularity of the solution ?

  Other types of boundary conditions on Γ_0 :

$$\begin{cases} (\mu \nabla u \cdot n - pn + qu) \cdot n &= 0 \\ u \cdot \tau = 0 \end{cases}$$