Index of condensation and controllability

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Toulouse-Juin-2012

GOAL:

Give necessary and sufficient conditions which characterize the boundary controllability properties of a class of systems.

We will only deal with

“Simple" Parabolic Systems:

Coupling Matrices of Constant Coefficients

Even for this simple class of systems, it appears new behaviors totally differents from the scalar case!

Contents

1. The parabolic scalar case: The heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open non empty subset, $\gamma \subseteq \partial\Omega$ a relative non empty open subset and let us fix $T > 0$.

We consider the **linear** problems for the **heat equation**:

(1)
$$\begin{cases} \partial_t y - \Delta y = v 1_\omega & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

(2)
$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = v 1_\gamma \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

In (1) and (2), 1_ω and 1_γ represent resp. the characteristic function of the sets ω and γ, $y(x, t)$ is the state, y_0 is the initial datum and is given in an appropriate space, and v is the control function (which is localized in ω -**distributed control**- or in γ -**boundary control**-).

1. The parabolic scalar case: The heat equation

Theorem (Distributed Controllability Results)

Fix $\omega \subseteq \Omega$ and $T > 0$. Then,

1. *System (1) is* ***approximately controllable*** *at time T, i.e., for any $\varepsilon > 0$ and $y_0, y_d \in L^2(\Omega)$ there is $v \in L^2(Q)$ s.t. the solution y to (1) satisfies*

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon.$$

2. *System (1) is* ***null controllable*** *at time T, i.e., for any $y_0 \in L^2(\Omega)$ there is $v \in L^2(Q)$ s.t. the solution y to (1) satisfies*

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

Theorem (Boundary Controllability Results)

Let $\gamma \subseteq \partial\Omega$ and $T > 0$ be given. Then, for any $y_0 \in H^{-1}(\Omega)$ there exists $v \in L^2(\Sigma)$ s.t. the solution y to (2) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

Four IMPORTANT REFERENCES

1. H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
2. S. DOLECKI , *Observability for the one-dimensional heat equation*, Studia Mathematica. 48 (1973), 292-305.
3. G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
4. A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

1.Distributed controls and boundary controls of parabolic systems : known results

Distributed Controllability:

$$\begin{cases} \dfrac{\partial y}{\partial t} = (M\Delta + A)\, y + 1_{\omega} B v, & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \\ v \in L^2\left(Q, \mathbb{R}^m\right), \ \omega \Subset \Omega. & \end{cases}$$

The null-controllability is equivalent to the "Kalman rank condition"

$$\ker \begin{bmatrix} B^* \\ B^*\left(M\Delta + A^*\right) \\ \vdots \\ B^*\left(M\Delta + A^*\right)^{n-1} \end{bmatrix} = \{0\}.$$

- F. Ammar Khodja, C. Dupaix, M. Gonzalez-Burgos & A.B, (J.E.E, 2009).

Does there exist a similar formulation for the boundary controllability?

1. In N space dimension, there is not yet an answer.
2. In 1 space dimension, a necessary and sufficient condition of null-controllability was obtained in the case $M = I_d$:
   - $n = \mathbf{2}$: E. Fernandez-Cara, M. Gonzalez-Burgos & L. de Teresa (J. Funct. Anal. **2010**),
   - $n \geq \mathbf{2}$: F. Ammar Khodja , M. Gonzalez-Burgos, L. de Teresa & A.B (J.M.P.A. **2011**).

1. A simple case of two coupled parabolic equations

Let $Q := (0,T) \times (0,\pi)$, $y = (y_1, y_2)$

$$\begin{cases} y' = \left(M \dfrac{\partial^2}{\partial x^2} + A \right) y, \\[2ex] y(0,\cdot) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} v, \quad y(\pi,\cdot) = 0, \\[2ex] y(\cdot,0) = y^0 \in \mathbb{L}^2(0,\pi). \end{cases}$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \ (d > 0), \ A = (a_{ij}), \ B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

GOAL:

Does there exist $v \in L^2(0,T)$ such that $y(T) = 0$?

1. A simple case of two coupled parabolic equations

- What is about the case $M \neq I_d$?
- The answer seems to be more intricate than for the internal controllability and different from the case $M = I_d$: besides conditions on A, B, there is the way the eigenvalues of the operator $M\frac{\partial^2}{\partial x^2} + A^*$ (with homogeneous Dirichlet conditions) *condense* at infinity.
- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0, +\infty]$ associated with this sequence and which "measures" the condensation at infinity.
- This notion has been :
  - introduced by Vl. Bernstein in 1933: (Leçons sur les progrès récents de la théorie des séries de Dirichlet) for real sequences,
  - extended by J. R. Shackell in 1967 for complex sequences.

Index of condensation : some background

Let $\Lambda = (\lambda_k) \subset \mathbb{C}$ be a sequence with pairwise distinct elements and:

$$\exists \delta > 0 : \Re(\lambda_k) \geq \delta |\lambda_k| > 0, \ \forall k \geq 1,$$
$$\textstyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty.$$

and the interpolating function:

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \ E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j \neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)$$

Definition

The index of condensation of Λ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{\ln \left|\frac{1}{E'(\lambda_k)}\right|}{|\lambda_k|} \in [0, +\infty].$$

Fact

$|\lambda_k - \lambda_l| \geq |k - l| \Rightarrow c(\Lambda) = 0$

The results

- Let $A_k^* := -k^2 M + A^*, \sigma(A_k) = \{\lambda_{k,1}, \lambda_{k,2}\}$, $\Lambda = \{\lambda_{k,i}, i = 1, 2,\ k \geq 1\}$ and $c(\Lambda)$ the *index of condensation of* Λ.

Theorem

Suppose $d \neq 1$. Then the problem is null-controllable at any $T > 0$ if and only if the following three conditions are satisfied:

$$\lambda_{k,i} \neq \lambda_{l,j}\ \forall (k,i) \neq (l,j), \tag{3}$$

$$\left(\begin{array}{c} \left(-k^2 M + A^*\right) V = \lambda V \\ V \neq 0 \end{array} \Rightarrow B^* M V \neq 0 \right), \forall k \geq 1 \tag{4}$$

$$c(\Lambda) = 0. \tag{5}$$

- The two first conditions are equivalent to the approximate controllability: it does not depend on $c(\Lambda)$.
- If $d = 1$, then $c(\Lambda) = 0 \Rightarrow$ FC-GB-DeT.

The results

GOAL:

What happens if $c(\Lambda) > 0$?

Theorem

Suppose $d \neq 1$ and

$$\lambda_{k,i} \neq \lambda_{l,j} \ \forall (k,i) \neq (l,j),$$

$$\left(\begin{array}{c} \left(-k^2 M + A^*\right) V = \lambda V \\ V \neq 0 \end{array} \Rightarrow B^* M V \neq 0 \right), \ \forall k \geq 1$$

Then the problem is :

Null-controllable at any $T > c(\Lambda)$,

Not null-controllable for any $T < c(\Lambda)$.

The results

GOAL:

Is it possible that $c(\Lambda) > 0$?

Consider the case $A = 0$. Then we have:

$$\lambda_{k,1} = k^2,\ \lambda_{k,2} = dk^2,\ V_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\left[\lambda_{k,i} \neq \lambda_{l,j}\ \forall (k,i) \neq (l,j)\right] \Leftrightarrow \sqrt{d} \notin \mathbb{Q},$$

$$B^* M V_{k,i} \neq 0 \Leftrightarrow b_1 b_2 \neq 0.$$

Theorem

Let $\Lambda = \left(k^2,\ dk^2\right)$. For any $\delta \in [0, \infty]$, there exists $\sqrt{d} \in \mathbb{R} \backslash \mathbb{Q}$ such that $\delta = c(\Lambda)$.

- **A particular consequence:** there exists d such that $c(\Lambda) = \infty$ and, for such a number, the problem is never null-controllable while it is always approximately controllable!

The method of Fattorini-Russell

- Let φ be a solution of the adjoint problem:

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} = \left(M \dfrac{\partial^2}{\partial x^2} + A^* \right) \varphi, & \text{in } Q = (0,\pi) \times (0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi^0 \ \in \mathbb{H}_0^1 (0,\pi) . \end{cases}$$

- If y is a solution of the direct problem, then:

$$\left\langle y(T), \varphi^0 \right\rangle_{H^{-1},H_0^1} - \left\langle y^0, \varphi(0) \right\rangle_{H^{-1},H_0^1} = \int_0^T v.B^* M \frac{\partial \varphi}{\partial x} \ |_{x=0}$$

Thus $y(T) = 0$ if, and only if, there exists v such that

$$\int_0^T v.B^* M \frac{\partial \varphi}{\partial x} \ |_{x=0} = - \left\langle y^0, \varphi(0) \right\rangle_{H^{-1},H_0^1}, \ \forall \varphi^0 \in \mathbb{H}_0^1 (0,\pi) .$$

The method of Fattorini-Russell

Besides, let $\sigma\left(-k^2M + A^*\right) = \{-\lambda_{k,1}, -\lambda_{k,2}\}$ and $V_{k,1}$ and $V_{k,2}$ the associated eigenvectors. Then

$$\sigma\left(M\frac{\partial^2}{\partial x^2} + A^*\right) = \cup_{k\geq 1}\{-\lambda_{k,1}, -\lambda_{k,2}\}$$

$\Phi_{k,i} = \sqrt{\frac{2}{\pi}}V_{k,i}\sin kx$, $i = 1, 2$, are the normalized eigenfunctions.

- By assumption, A^* is such that $\lambda_{k,i} \neq \lambda_{l,j}$ for all $(k,i) \neq (l,j)$: otherwise, the system is neither null nor approximately controllable.
- Last, $(\Phi_{k,i})$ is a (Riesz) basis of $\mathbb{L}^2(0,\pi)$.

The method of Fattorini-Russell

- Choosing $\varphi^0 = \Phi_{k,i}$, we have $\varphi(x,t) = e^{-\lambda_{k,i}(T-t)}\Phi_{k,i}$ and

$$\varphi(0) = e^{-\lambda_{k,i}T}\Phi_{k,i}, \quad \frac{\partial \varphi}{\partial x}\mid_{x=0} = \sqrt{\frac{2}{\pi}} k e^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} k B^* M V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i}t} \\ = \quad & -e^{-\lambda_{k,i}T} \left\langle y^0, \Phi_{k,i} \right\rangle_{H^{-1},H_0^1}, \ \forall (k,\ i). \end{aligned}$$

- The assumption implies $B^* M V_{k,i} \neq 0$ for all $k \geq 1$, $i = 1,2$ and:

$$\int_0^T v(T-t) e^{-\lambda_{k,i}t} = -\frac{\left\langle y^0, \Phi_{k,i} \right\rangle_{H^{-1},H_0^1}}{B^* M V_{k,i} \sqrt{\frac{2}{\pi}} k} e^{-\lambda_{k,i}T} = m_{k,i}, \forall (k,\ i).$$

The method of Fattorini-Russell

- If $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family $(q_{k,i})$ in $L^2\left(0,T\right)$, i.e.:

$$\int_0^T e^{-\lambda_{k,i}t} q_{l,j} dt = \begin{cases} 1, & \text{if } (k,i) = (l,j) \\ 0, & \text{if } (k,i) \neq (l,j) \end{cases}$$

  Then v is given by:

$$v(T-t) = \sum_{k\geq 1} m_{k,i} q_{k,i}(t).$$

- The question is:

$$\sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2\left(0,T\right)?$$

- But this question itself amounts to:

$$\|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

  Here comes the index of condensation of Λ.

The method of Fattorini-Russell

- Going back to the question:

$$\|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

- The family $q_{k,i}$ can be constructed (L. Schwartz, Fattorini-Russell, etc) such that

$$\|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} \left|\frac{1}{E'\left(\lambda_{k,i}\right)}\right| \underset{k\to\infty}{\sim} Ce^{Re(\lambda_{k,i})(\varepsilon+c(\Lambda))}.$$

- $$|m_{k,i}|\ \|q_{k,i}\|_{L^2(0,T)} \underset{k\to\infty}{\sim} \frac{\left|\left\langle y^0, \Phi_{k,i}\right\rangle_{H^{-1},H_0^1}\right|}{\sqrt{\frac{2}{\pi}}k\left|B^*MV_{k,i}\right|}e^{-Re(\lambda_{k,i})(\mathbf{T}-\mathbf{"}-\mathbf{c}(\Lambda))}$$

- Conclusion: after checking that $k\left|B^*MV_{k,i}\right| \geq e^{-\varepsilon Re(\lambda_{k,i})}$,

$$T > c\left(\Lambda\right) \Rightarrow v(T-t) = \sum_{k\geq 1} m_{k,i}q_{k,i}(t) \in L^2\left(0,T\right).$$

No controllability

- The null controllability property is equivalent to the observability inequality:

$$\|\varphi(0)\|_{H_0^1}^2 \leq C_T \int_0^T \left| B^* M \frac{\partial \varphi}{\partial x}(t,0) \right|^2 dt.$$

- Let $0 < T < c(\Lambda)$

Lemma

There exists $\Delta = (G_k)_{k\geq 1} = \left(\left(\lambda_{n_{k,j}} \right)_{0\leq j\leq p_k} \right)_{k\geq 1}$ *such that:*

$$\overline{\lim_{k\to\infty}} \max_{0\leq j\leq p_k} \left\{ \frac{1}{\left|\lambda_{n_{k,j}}\right|} \log \frac{p_k!}{\left|P'_{G_k}(\lambda_n)\right|} \right\} = c(\Lambda),$$

where $\left|P'_{G_k}(\lambda_n)\right| = \prod_{\substack{\lambda_m \in G_k \\ \lambda_m \neq \lambda_n}} |\lambda_n - \lambda_m|$.

Over convergence of Dirichlet Series

The idea is to construct a sequence (φ_k) of solutions of the adjoint problem such that

$$\int_0^T \left| B^* M \frac{\partial \varphi_k}{\partial x}(t,0) \right|^2 dt \to 0, \ \ \|\varphi_k(0)\|^2_{H^1_0} \geq \delta > 0.$$

Theorem (Shackell-Theorem 9)

Suppose that the sequence (λ_n) has maximum density $D := \lim \frac{k}{|\lambda_k|} < +\infty$ and index of condensation δ. We can choose a sequence of of integers $(n(k))$ and a sequence of complex numbers a_n, such that the following conditions are satisfied

1. $\frac{n(k+1)-n(k)}{n(k)} \to_{k \to +\infty} 0,$
2. *If R denotes the region of convergence of the $\sum a_n e^{-ns}$, then the series $\sum_{k=1}^{+\infty} (\sum_{n(k)}^{n(k+1)-1} a_n e^{-\lambda_n s})$ converges in the region $R + \delta E$.*

$$\varphi_k(0) = \sum_{\lambda_n \in G_k} \frac{p_k!}{\overline{b}_n P'_{G_k}(\lambda_n)}, e^{-\lambda_n T} \Phi_n.$$

$$\begin{cases} \|\varphi_k(0)\|^2 = \sum_{\lambda_n \in G_k} \left| \frac{p_k!}{\overline{b}_n P'_{G_k}(\lambda_n)} e^{-\lambda_n T} \right|^2 \geq \left| \frac{p_k!}{\overline{b}_{n_k} P'_{G_k}(\lambda_{n_k})} e^{-\lambda_{n_k} T} \right|^2 \\ \quad = e^{2\Re(\lambda_{n_k}) \left[\frac{1}{\Re(\lambda_{n_k})} \left(\log\left| \frac{1}{b_{n_k}} \right| + \log\left| \frac{p_k!}{P'_{G_k}(\lambda_{n_k})} \right| \right) - T \right]} \sim e^{\Re(\lambda_{n_k})(c(\Lambda) - T)} \longrightarrow +\infty . \end{cases}$$

Generalization to controls that depend only on time

$$(6) \qquad \begin{cases} y' = \mathcal{A}y + \mathcal{B}v & \text{on } (0,T) \\ y(0) = y_0 \in \mathbb{X}. \end{cases}$$

$$v \in L^2(0,T;\mathbb{C}).$$

Generalization to controls that depend only on time

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$$v \in L^2(0,T;\mathbb{C}).$$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \left(M\dfrac{\partial^2}{\partial x^2} + A\right) y = \delta\,(x - x_0)\,Bv(t), & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0\,, & \text{in } (0,\pi), \end{cases}$$

Scalar case: S. Dolecki , *Observability for the one-dimensional heat equation*, Studia Mathematica (1973)

Generalization to controls that depend only on time

Theorem

Under some assumptions on $\mathcal{B}$, $\Lambda := (\lambda_k)$ the sequence of eigenvalues of $(\mathcal{A}, D(\mathcal{A}))$, let

$$T_0 = \limsup \max_{i=1,2} \left(\frac{\log \frac{1}{|b_{k,i}|}}{\Re\left(\lambda_{k,i}\right)} + \frac{\log \frac{1}{|E'(\lambda_{k,i})|}}{\Re\left(\lambda_{k,i}\right)} \right),$$

where

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \quad z \in \mathbb{C}, \quad b_{k,i} := \mathcal{B}^* \psi_{k,i} = \varphi_k\left(x_0\right) B^* V_{k,i}.$$

Then:

1. *System* (**??**) *is null controllable for $T > T_0$;*
2. *System* (**??**) *is not null controllable for $T < T_0$.*

Comments

- For the first attempts to generalize the Kalman rank condition:
  - R. Triggiani. *Extensions of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators*. SIAM J. of Control (1976).
  - See also Russell's review paper (1978') and the references therein.
- Fattorini has also used the method of moments to prove the first null-controllability result for the heat equation by mean of boundary controls in higher dimension of space for parallelepipedon. A work in progress is going in this way for systems.
- In higher dimension, partial results F. Alabau & M. Leautaud (J. Math. Pures Appl (2012)
- The non constant coefficient case is widely open, even if some recent works of M.Gonzalez-Burgos & L. de Teresa (2010, cascade systems), M.Cristofol, P. Gaitan, L. de Teresa & A.B (2011) and K. Mauffrey (2012) for 3×3 systems , offer interesting perspectives.

Thank you for your attention