Reduced number of controls for N-coupled systems of PDE's

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Outline

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Coupled systems appear naturally in models for:

- building insensitizing controls for scalar equations
- reaction-diffusion systems with applications in medecine, biology, population dynamics . . .
- mechanics: Timoshenko beams, acoustic models
- fluid-structure systems . . .

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For applications, it is important to

- control, observe or stabilize these systems
- identify sources or coefficients of these systems

Further, it is also important either for cost reasons or for practical realization to have a reduced number of controls acting on the system,

that is when the number of equations (or components of the state-vector) > number of controls, this is also called *indirect control*.

We shall consider this case.

The control/observability/stabilization of scalar wave type equations, either by a locally distributed control or a boundary control is by now quite well-understood.

What happens for non scalar equations, that is in case of coupled systems?

Indeed further properties have to be understood since the coupling, the nature of the equations, the geometry, the type of control . . . influence the answer.

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An example: insensitizing control for the wave equation

We consider either the scalar wave equation with a locally distributed control v:

$$\begin{cases} y_{tt} - \Delta y = \xi + bv \quad \text{in } (0, T) \times \Omega\,, \\ y = 0 \quad \text{in } (0, T) \times \Gamma\,, \\ y(0, .) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0, .) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases} \tag{1}$$

the location of the control depending on the the support of the coefficient function b

or the scalar wave equation with a boundary control v:

$$\begin{cases} y_{tt} - \Delta y = \xi \quad \text{in } (0,T) \times \Omega\,, \\ y = bv \quad \text{in } (0,T) \times \Gamma\,, \\ y(0,.) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0,.) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases} \tag{2}$$

the location of the control depending on the the support of the coefficient function b in Γ

where

- $\xi \in L^2((0,T) \times \Omega)$,
- the initial data y^0, y^1 are given known functions in $H_0^1(\Omega) \times L^2(\Omega)$ or in $L^2(\Omega) \times H^{-1}(\Omega)$,
- the perturbations z^0, z^1 are supposed to be unknown of norm 1 in the appropriate spaces
- the real numbers τ_0, τ_1 are assumed to be small and to measure the amplitudes of the unknown perturbations of the initial data.

The goal of insensitizing controls is to build controls that are such that a given measure of the unknown is "robust" to small perturbations on the initial data.

This notion has been introduced by J.-L. Lions (1989)

Let us consider the following functional associated to the solutions y defined by

$$\Phi(y, \tau_0, \tau_1) = \frac{1}{2} \int_0^T \int_\Omega c y^2 \, dxdt \, ,$$

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$$\Phi(y, \tau_0, \tau_1) = \frac{1}{2} \int_0^T \int_\Omega c y^2 \, dx dt \,,$$

c has a support localized in the neighbourhood of a subset O which is a given subset of Ω.

The functional Φ consists in an observation of the solution on the set O during a length of time T.

The control bv is said to insensitize the observation Φ if for all (z^0, z^1) the corresponding solution y satisfies

$$\frac{\partial \Phi}{\partial \tau_0}(y, 0, 0) = \frac{\partial \Phi}{\partial \tau_1}(y, 0, 0) = 0 .$$

One can show that this problem is equivalent to an exact controllability result for an associated system of two wave equations coupled in cascade, namely

Localized control

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x) y_2 = 0 \quad \text{in } (0,T)\times\Omega\,, \\ y_{2,tt} - \Delta y_2 = \xi + bv \quad \text{in } (0,T)\times\Omega\,, \\ y_1 = y_2 = 0 \quad \text{in } (0,T)\times\Gamma\,, \\ \text{Initial data } Y^0 \text{ given}\,. \end{cases}$$

Boundary control

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x) y_2 = 0 \quad \text{in } (0,T)\times\Omega\,, \\ y_{2,tt} - \Delta y_2 = \xi \quad \text{in } (0,T)\times\Omega\,, \\ y_1 = 0\,, y_2 = bv \quad \text{in } (0,T)\times\Gamma\,, \\ \text{Initial data } Y^0 \text{ given}\,. \end{cases}$$

Däger 2006 proved in one-dimension with $\Omega = (0,1)$, the following results

Locally distributed control:

$b = \mathbb{1}_\omega$, $c = \mathbb{1}_O$:

Let ω and O be any open nonempty subsets of Ω and $T \geq 4$. Let $\varepsilon > 0$ be given. Then for any $\xi \in L^2((0,T); L^2(\Omega))$, any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control v_ε in L^2 that $\varepsilon-$insensitizes Φ along the solutions, i.e. for any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ of norm 1 one has:

$$|\frac{\partial \Phi}{\partial \tau_0}(y,0,0)| \leq \varepsilon \, , |\frac{\partial \Phi}{\partial \tau_1}(y,0,0)| \leq \varepsilon \, .$$

Boundary control:

$b = 1$ and the control is at the boundary $x = 1$, $c = \mathbb{1}_O$, $O \neq \emptyset$ arbitrary :

Let $T \geq 4$. Then for any $\xi \in L^2((0,T); L^2(\Omega))$, any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control v in L^2 that insensitizes Φ along the solutions, i.e. for any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ of norm 1 one has:

$$\frac{\partial \Phi}{\partial \tau_0}(y, 0, 0) = \frac{\partial \Phi}{\partial \tau_1}(y, 0, 0) = 0 \,.$$

The proof relies on the property that the semigroup generated by the free wave equation is periodic $\rightsquigarrow$ valid only in 1-D domains.

Tebou 2011 multi-D, boundary observation (does not work for localized observation), localized control, the control region $\supset$ observation region.

The general problem is open:

multi-D, localized or boundary control, localized observation, optimal conditions on the supports of b and c.

More precisely, in both problems for the wave equation, the challenging question is to have results for which

$$supp\{b\} \cap supp\{c\} = \emptyset\,.$$

Indeed one can solve this problem (A.-B. 2012):

- with exact insensitivity,
- in multi-D,
- for both locally distributed and boundary controls,
- with control region $\cap$ coupling region $= \emptyset$, that is when

$$supp\{b\} \cap supp\{c\} = \emptyset.$$

In the same way, building insensitizing controls for a scalar heat equation is equivalent to a null controllability result for a system of two heat equations –one forward in time, the other backward in time– coupled in cascade.

Results for the heat equation:

Bodart and Fabre 1995, de Teresa 2000 . . . , Bodart and Gonzalez-Burgos and Perez-Garcia 2004 for heat. A restrictive result by de Teresa and Zuazua 2009 . . .

Other models arise for

- combined mechanical and temperature effects as for thermoplates or thermoelasticity. In this case one wants to determine if the dissipation effect of the heat equation is sufficient to stabilize the thermo-mechanical system Lebeau Zuazua 1999, Burq Lebeau 2001, Zhang Zuazua 2003, ...
- Coupling effects arise also in fluid-structure interactions, in electro-magnetism...

Here, we are interested in a general setting, that is identifying large classes of systems for which it is possible to

- build a general,
- robust and,
- flexible methodology

to answer the above observation/control questions at least for these classes.

We shall deal with two classes:

1. *symmetric 2-coupled systems* $\longleftrightarrow y'' + \mathcal{A}y + \mathcal{C}y = \mathbf{B}v$, with

$$y = (y_1, y_2)^t, \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \mathcal{C} = \begin{pmatrix} 0 & C \\ C^\star & 0 \end{pmatrix}, \mathbf{B}v = (Bv, 0)^t$$

where A is an unbounded self-adjoint coercive operator on an Hilbert space H (pivot space) and C is a bounded operator in H, B is the control operator bounded or unbounded.

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2. *cascade N-coupled systems* $\longleftrightarrow y'' + \mathcal{A}y + \mathcal{C}y = \mathbf{B}v$, with $y = (y_1, \ldots, y_N)^t$ and

$$\mathcal{A} = \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & A \end{pmatrix}, \mathcal{C} = \begin{pmatrix} 0 & C_{21}^\star & C_{31}^\star & \ldots & C_{N1}^\star \\ 0 & 0 & C_{32}^\star & \ldots & C_{N2}^\star \\ \vdots & & & & \\ 0 & 0 & \ldots & \ldots & C_{NN-1}^\star \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

$\mathbf{B}v = (0, 0, \ldots, B_{p+1}v_{p+1}, \ldots, B_N v_N)^t$, with $v = (v_{p+1}, \ldots, v_N)$ are the $N - p$ controls, $N \geq 2$, $p \in \{1, \ldots, N-1\}$.

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For the first class of systems, the coupling matrix operator is symmetric for the control problem.

For the second class (cascade systems), the coupling matrix operator is an upper triangular matrix for the control problem.

The dual homogeneous observability problems will involve respectively symmetric and lower triangular matrix operators.

Furthermore:

The coupling operators C or C_{ij} may be either coercive in the underlying pivot space H, that is there exists $\eta > 0$ such that

$$\langle Cu, u\rangle_H \geq \eta |u|_H^2 \quad \forall\, u \in H\,.$$

or only partially coercive in H, that is there exists $\eta > 0$ such that

$$\langle Cu, u\rangle_H \geq \eta |\Pi u|_H^2 \quad \forall\, u \in H\,,$$

where $\Pi \in \mathcal{L}(H)$ is a nonnegative projection operator.

Model dual observability examples of 2-symmetric coupled systems

$$
\begin{cases}
u_{1\,,tt} - \Delta u_1 + c u_2 = 0 \text{ in } \Omega \times (0, T)\,, \\
u_{2\,,tt} - \Delta u_2 + c u_1 = 0 \text{ in } \Omega \times (0, T)\,, \\
u_1 = 0 \text{ on } \Sigma = \Gamma \times (0, T)\,, u_2 = 0 \text{ on } \Sigma\,, \\
u_i(0) = u_i^0\,, u_{i,t}(0) = u_i^1\,.
\end{cases}
$$

where $c \geq 0$ in Ω is a coupling function (it may also be nonpositive).

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Model dual observability examples of 2-cascade coupled systems

$$\begin{cases} u_{1\,,tt} - \Delta u_1 = 0 \text{ in } \Omega \times (0,T)\,, \\ u_{2\,,tt} - \Delta u_2 + c u_1 = 0 \text{ in } \Omega \times (0,T)\,, \\ u_1 = 0 \text{ on } \Sigma = \Gamma \times (0,T)\,, u_2 = 0 \text{ on } \Sigma\,, \\ u_i(0) = u_i^0\,, u_{i,t}(0) = u_i^1\,. \end{cases}$$

where $c \geq 0$ in Ω is a coupling function (it may also be nonpositive).

The coercive case corresponds to

$$c \geq c_- > 0 \text{ in } \Omega \,.$$

The partially coercive case corresponds to

$$c \geq c_- > 0 \text{ in } O \subset \Omega \,,$$

with $|\Omega \backslash O| > 0$.

Let us associate an observation $B^\star(u,u')$ to these dual homogeneous problems

- either locally distributed with

$$B^\star(u,u') = bu' ,$$

with $\{b > 0\} \supset \overline{\omega}\,, \omega \subset \Omega$

- or localized on a part of the boundary

$$B^\star(u,u') = b\frac{\partial u}{\partial \nu} ,$$

with $\{b > 0\} \supset \overline{\Gamma_1}\,, \Gamma_1 \subset \Gamma$

We look for the following type of observability inequality (for sufficiently large time T)

- Locally distributed observation:

$$\int_0^T \int_\Omega |bu_1'|^2 \, d\gamma \, dt \geq c\left(e_1(u_1(0)) + e_2(u_2(0))\right),$$

- Boundary observation:

$$\int_0^T \int_\Gamma |b\frac{\partial u_1}{\partial \nu}|^2 \, d\gamma \, dt \geq c\left(e_1(u_1(0)) + e_2(u_2(0))\right),$$

where $e_i(u_i(t))$ stands for some energy of the corresponding component of the unknown.

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where $e_i(u_i(t))$ stands for some energy of the corresponding component of the unknown.

If $c \equiv 0$, the two waves are uncoupled – in the symmetric as well as in the cascade case – so that we cannot hope to get such a result by a perturbation argument for α small.

What can be said for $c \neq 0$?

We are also interested in similar questions for coupled parabolic, diffusive or Schrödinger systems.

The coercive coupling case:

We assume the following multiplier geometric conditions

Ω is a non-empty bounded open set in $\mathbb{R}^N$ having a boundary Γ of class C^2.

Moreover, $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and x_0 is a point in $\mathbb{R}^N$ such that $m \cdot \nu \leq 0$ on Γ_0 and $m \cdot \nu > 0$ on Γ_1, where $m(x) = x - x_0$.

We denote by $| \ |$ the L^2-norm on Ω. Then, we prove

Theorem (A.-B. 2001, 2003, coercive case)

There exists $c^\star > 0$ such that for all $0 < ||c||_{L^\infty(\Omega)} < c^\star$, there exists $T_0 = T_0(c^\star) > 0$ such that for all $T > T_0$ and all $U^0 = (u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ the solution (u_1, u_2) satisfies

$$\int_0^T \int_{\Gamma_1} |\frac{\partial u_1}{\partial \nu}|^2 \geq c_1 \left(|u_1^1|^2 + |\nabla u_1^0|^2\right) + c_2 \left(|u_2^1|^2_{H^{-1}(\Omega)} + |u_2^0|^2\right) ,$$

where the constants c_1, c_2 depend on an explicit way on $c^\star$ and T_0 behaves as $1/c^\star$ as $c^\star$ goes to zero.

By duality, using the HUM method we deduce an exact *indirect controllability result* for the following control problem:

For given initial data, determine a L^2 control v such that the solution of

$$\begin{cases} y_{1,tt} - \Delta y_1 + c y_2 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2,tt} - \Delta y_2 + c y_1 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = v \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,T)\,, y_1 = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T)\,, \\ y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1,t})(0) = (y_1^0, y_1^1), (y_2, y_{2,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

satisfies

$$(y_1, y_2, y_{1,t}, y_{2,t})(T) = 0 \text{ on } \Omega\,.$$

Theorem (A.-B. SICON 2003)

Under the multiplier geometric condition: there exists $c^\star > 0$ such that for all $0 < ||c||_{L^\infty(\Omega)} < c^\star$, such that for all $T > T_0$ (the observability time) and all $Y^0 = (y_1^0, y_1^1, y_2^0, y_2^1) \in L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ there exists a control $v \in L^2([0, T]; L^2(\Gamma_1))$ such that the solution $Y(t) = (y_1, y_1', y_2, y_2')$ satisfies

$$y_i(., T) = \partial_t y_i(., T) = 0 \text{ in } \Omega\,, \text{ for } i = 1\,, 2\,.$$

Note that the initial data for the uncontrolled component have to be taken in a smaller space than the space of the controlled one.

Based on the two-level energy method (A.-B. 2001, 2003): compensate the lack of observation of the second component by a balance effect between the natural energy of the observed component and the weakened energy of the unobserved one.
Ingredients for the proof:

- a key estimate due to the coercivity properties of the coupling.
- observability assumption for a single wave equation with a forcing source term uniform with respect to sufficiently large times T.
- energy type estimates (several ones are required).
- conservation of the total natural and weakened energies and suitable balance of energies.

These results extend to indirect observability (resp. controllability) for adjoint (resp. direct) abstract coupled systems, with applications to coupled waves, plates, that is for instance for observability for

$$\begin{cases} u_1'' + A_1 u_1 + C u_2 = 0 \quad \text{in } V_1' \,, \\ u_2'' + A_2 u_2 + C^{\star} u_1 = 0 \quad \text{in } V_2' \,, \\ (u_1, u_1')(0) = (u_1^0, u_1^1) = U_1^0 \in V_1 \times H \,, \\ (u_2, u_2')(0) = (u_2^0, u_2^1) = U_2^0 \in V_2 \times H \,, \end{cases}$$

Can be extended to the case of bounded (i.e. localized from PDE's point of view) observation (easier case).

The above results are valid only for bounded coercive coupling operators C.

That is under the assumption

$$\exists \eta > 0 \text{ such that } \langle Cu, u \rangle \geq \eta |u|_H^2 , \quad \forall\, u \in H \, .$$

where H is the natural pivot space.

What can be said in the situation of noncoercive coupling operators.

or equivalently

for systems of coupled PDE's when the coupling coefficient is localized on some part of the domain and vanishes outside a a subset of Ω?

Positive results: this is a joint work with Matthieu Léautaud on abstract coupled wave systems.

The partially coercive coupling case:

Model example of 2 symmetric coupled systems with partial coercive couplings

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(.)y_2 = b\,v \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2,tt} - \Delta y_2 + c(.)\,y_1 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1,t})(0) = (y_1^0, y_1^1), (y_2, y_{2,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

where v is the control, and b and c are respectively the control and coupling coefficients with:

$$\begin{cases} b \geq 0 \text{ on } \Omega \text{ (resp. on } \Gamma) , \ c \geq 0 \text{ on } \Omega , \ \text{both smooth} , \\ b > 0 \text{ on } \overline{\omega} \text{ (resp. on } \overline{\Gamma}_1) , \ \omega \subset \Omega \text{ (resp. } \Gamma_1 \subset \Gamma) \\ c > 0 \text{ on } \overline{O} , \ \ O \subset \Omega . \end{cases}$$

We say that $\omega \subset \Omega$ satisfies GCC if every ray of geometric optics (or generalized geodesics) traveling at speed one in Ω meets ω (resp. meets Γ on a non-diffractive point) in finite time

due to Bardos Lebeau Rauch 1992.

We follow the methodology of the two-level energy method A.-B. 2003, but this requires non trivial extensions to handle noncoercive coupling operators.

We are especially interested in situations for which $\overline{\omega} \cap \overline{O} = \emptyset$.

- a key estimate due to the coercivity properties of the coupling : here localized coupling $\Rightarrow$ weaker coercivity property. Extension of the two-level energy method.
- observability assumption for a forcing source term, uniform with respect to sufficiently large times: still works however if we prove it by a multiplier method as in A.-B. 2003, we have observation and coupling regions which necessarily meet in dimensions $\geq$ 2. We modify this assumption to handle this situation.
- energy type estimates $\longleftrightarrow$ same methodology
- conservation of the total natural and weakened energies and balance of energies $\longleftrightarrow$ same methodology.

Theorem (A.-B.-Léautaud CRAS 2011, JMPA 2012)

Assume that $\omega \subset \Omega$ (resp. $\Gamma_1 \subset \Gamma$) satisfies GCC and O satisfies GCC. Then there exists a constant $c_* > 0$ such that for all $||c||_\infty < c_*$, there exists a time $T_* > 0$ such that for all $T > T_*$, and all initial data $(y_1^0, y_2^0, y_1^1, y_2^1) \in H_0^1(\Omega) \times H^2 \cap H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ (resp. $(y_1^0, y_2^0, y_1^1, y_2^1) \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$), there exists a control $v \in L^2((0,T) \times \Omega)$ (resp. $v \in L^2((0,T) \times \Gamma)$), such that the solution satisfies $(y_1, y_2, y_1', y_2')|_{t=T} = 0$.

Very localized coupling effects: no conditions in 1-D except that O and ω are non-empty open sets.

Hence if we consider Ω a bounded open set of $\mathbb{R}$ and the uncoupled control problem

$$\begin{cases} y_{1,tt} - y_{1,xx} = b\,v \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2,tt} - y_{2,xx} = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1,t})(0) = (y_1^0, y_1^1), (y_2, y_{2,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

It is not possible, controlling only the first equation to control the second component y_2.

Now, if c is any nonnegative function on Ω, bounded away from 0 on any open nonempty subset of Ω, then considering the localized control problem

$$\begin{cases} y_{1,tt} - y_{1,xx} + c(.)y_2 = b\,v \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2,tt} - y_{2,xx} + c(.)\,y_1 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1,t})(0) = (y_1^0, y_1^1), (y_2, y_{2,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,, \end{cases}$$

it becomes possible by controlling only the first equation, to control the second component y_2. Hence a **very localized interaction between the two components** y_1 and y_2 is sufficient to get a positive answer to a controllability result by a single control.

Applications to parabolic systems

For each initial data in a suitable space, determine a L^2 control v such the solution of

$$\begin{cases} e^{i\theta} y_1' - \Delta y_1 + c(.) y_2 = bv & \text{in } (0,T)\times\Omega, \\ e^{i\theta} y_2' - \Delta y_2 + c(.) y_1 = 0 & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\Omega, \\ (y_1, y_2)|_{t=0} = (y_1^0, y_2^0) & \text{in } \Omega, \end{cases}$$

satisfies $(y_1, y_2)(T) = 0$ on Ω.

Here $\theta = 0$ (heat case), $\theta \in (0, \pi/2)$ (diffusive case) or $\theta = \pi/2$ (Schrödinger case).

Combining the above controllability result for the coupled wave system, we can deduce, using the transmutation method Phung 2001, Miller 2005, Ervedoza and Zuazua 2010 (also Russell in 1973), as a byproduct, a controllability result for heat/diffusive coupled systems

Corollary (Heat-type systems, A.-B. Léautaud 2011)

Suppose that O satisfies GCC and that ω (resp Γ_1) satisfies GCC. Then, there exists a constant $c_* > 0$ such that for all $||c||_\infty < c_*$, for all $T > 0$, $\theta \in (-\pi/2, \pi/2)$, for all initial data $(y_1^0, y_2^0) \in \left(L^2(\Omega)\right)^2$ (resp $(y_1^0, y_2^0) \in \left(H^{-1}(\Omega)\right)^2$), there exists a control $v \in L^2((0,T) \times \Omega)$ (resp $v \in L^2((0,T) \times \Gamma)$) such that the solution of heat coupled type systems satisfies $(y_1, y_2)|_{t=T} = 0$.

Corollary (Schrödinger-type systems, A.-B. Léautaud 2011)

Assume $\theta = \pm\pi/2$. Under the above conditions, the same null-controllability result holds for any $T > 0$, taking initial data $(y_1^0, y_2^0) \in L^2(\Omega) \times H_0^1(\Omega)$ (resp. $(y_1^0, y_2^0) \in H^{-1}(\Omega) \times L^2(\Omega)$) for a suitable L^2 control v.

Applications to parabolic systems

About direct methods for parabolic coupled systems:
Main tool for parabolic coupled systems: **Carleman estimates** several positive results based on Kalman type condition have been obtained by Ammar Khodja, Benabdallah, Dupaix, Gonzalez-Burgos, de Teresa in a series of paper (survey paper MCRF 2011).

Main assumption: either constant couplings, or the coupling region meets the control region

2 coupled cascade systems

Rosier and De Teresa 2011 proved a null controllability result for heat coupled cascade systems, that is

$$\begin{cases} e^{i\theta} y_1' - \Delta y_1 + \mathbb{1}_O y_2 = 0 & \text{in } (0,T)\times\Omega, \\ e^{i\theta} y_2' - \Delta y_2 = bv & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\Omega, \\ (y_1, y_2)|_{t=0} = (y_1^0, y_2^0) & \text{in } \Omega, \end{cases}$$

Their method is based on Däger's approach and a controllability result for the corresponding cascade wave system.

It requires that the semigroup generated by the free wave equation is periodic, so that it is valid only in 1-D domains.

They also have a positive null controllability result for Schrödinger cascade coupled systems in the torus (multi-D) and for sufficiently large time T. They also have a sharper geometric condition for Schrödinger case.

Their result does not require smoothness of the coupling coefficient.

There is a recent result for 2 coupled cascade systems with localized control by Dehman Léautaud Le Rousseau 2011 in a $\mathcal{C}^\infty$ compact connected Riemannian manifold without boundary with characterization of the minimal control time using micro-local analysis.

Uses the idea of the two-level energy method of working with a weakened energy space.

Gives the minimal control time, but it is not constructive.

In general situations:

multi-D cases, boundary control , N-coupled cascade systems, empty intersection between control and coupling regions, ...

the problem is open.

We deal with 2-coupled cascade systems in multi-D, with locally distributed or boundary control/observation.

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multi-D cases, boundary control , N-coupled cascade systems, empty intersection between control and coupling regions, ...

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Consider the following either locally/boundary cascade hyperbolic control systems

$$\begin{cases} y_{1,tt} - \Delta y_1 + cy_2 = 0 \, , \text{in } Q_T = \Omega \times (0, T) \, , \\ y_{2,tt} - \Delta y_2 = bv \, , \text{in } Q_T = \Omega \times (0, T) \, , \\ y_1 = y_2 = 0 \, , \text{on } \Sigma_T = \partial\Omega \times (0, T) \, , \\ (y_i, y_{i,t})(0, .) = (y_i^0, y_i^1)(.) \, , \text{in } \Omega \, , i = 1, 2 \, , \end{cases}$$

$$\begin{cases} y_{1,tt} - \Delta y_1 + cy_2 = 0 \, , \text{in } Q_T = \Omega \times (0, T) \, , \\ y_{2,tt} - \Delta y_2 = 0 \, , \text{in } Q_T = \Omega \times (0, T) \, , \\ y_1 = 0 \, , y_2 = bv \, , \text{on } \Sigma_T = \partial\Omega \times (0, T) \, , \\ (y_i, y_{i,t})(0, .) = (y_i^0, y_i^1)(.) \, , \text{in } \Omega \, , i = 1, 2 \, , \end{cases}$$

Let O stands for the localization region of the coupling function c and ω (resp. Γ_1) be the region on which the control is active.

Then we prove

Theorem (A.-B. 2011, CRAS 2012)

Assume that the subsets O and ω (resp. O and Γ_1) satisfy (GCC). Then, there exists $T^\star > 0$ such that for all $T > T^\star$, for all initial data $Y^0 = (y_1^0, y_2^0, y_1^1, y_2^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)$ (resp. $Y^0 \in H_0^1(\Omega) \times L^2(\Omega)^2 \times H^{-1}(\Omega)$), there exists a control $v \in L^2((0,T) \times \Omega)$ (resp. $v \in L^2((0,T) \times \Gamma_1)$) such that the solution of the above internal (resp. control) system satisfies $(y_1, y_2, y_{1,t}, y_{1,tt})(T,.) = 0$ in Ω.

Indeed we give for all the above cases a necessary and sufficient condition for controllability by a single control to hold. Based on suitable observability estimates for the adjoint homogeneous problem.

We also solve the insensitizing control problem for the scalar wave equation, giving a sharp condition on the control coefficient b and the coupling coefficient c, which lead to optimal condition on the supports of b and c.

The proof is based on the two-level energy method (A.-B. 2001, 2003), its recent extension (A.-B. Léautaud 2011) and some new ideas for the corresponding cascade hyperbolic systems.

The result holds for systems in abstract form as well for bounded as well as unbounded control operators.

As for the results with Matthieu Léautaud for 2-symmetric systems, we can handle various situations for which the control and coupling regions do not intersect.

The results for 2-cascade systems do not allow to recover results for 2-symmetric cases and vice-versa.

Insensitizing controls

We go back to the insensitizing control problem.

An example: insensitizing control for the wave equation

We consider the scalar wave equation with a locally distributed control v:

$$\begin{cases} y_{tt} - \Delta y = \xi + bv \quad \text{in } (0,T) \times \Omega\,, \\ y = 0 \quad \text{in } (0,T) \times \Gamma\,, \\ y(0,.) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0,.) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases}$$

the location of the control depending on the the support of the coefficient function b

or the scalar wave equation with a boundary control v:

$$\begin{cases} y_{tt} - \Delta y = \xi \quad \text{in } (0,T) \times \Omega\,, \\ y = bv \quad \text{in } (0,T) \times \Gamma\,, \\ y(0,.) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0,.) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases}$$

the location of the control depending on the the support of the coefficient function b in Γ

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We want controls bv that insensitize the observation Φ, that is such that for all (z^0, z^1) the corresponding solution y satisfies

$$\frac{\partial \Phi}{\partial \tau_0}(y, 0, 0) = \frac{\partial \Phi}{\partial \tau_1}(y, 0, 0) = 0 \, .$$

where

$$\Phi(y, \tau_0, \tau_1) = \frac{1}{2} \int_0^T \int_\Omega c y^2 \, dxdt \, ,$$

and c has a support localized in the neighborhood of a subset O which is a given subset of Ω.

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Theorem (A.-B. 2012)

Assume that c satisfies

$$\begin{cases} c \in W^{1,\infty}(\Omega)\,, c \geq 0 \text{ on } \Omega\,, \\ \{c > 0\} \supset \overline{O} \text{ for some open subset } O \subset \Omega\,. \end{cases}$$

We have the following properties

- *Locally distributed control. Let $b \in L^\infty(\Omega)$ in Ω be given such that*

$$\begin{cases} b \in L^{\infty}(\Omega)\,, b \geq 0 \text{ on } \Omega\,, \\ \{b > 0\} \supset \overline{\omega} \text{ for some open subset } \omega \subset \Omega\,. \end{cases}$$

Theorem (continued)

Assume that O and ω satisfy (GCC). Then for any given $\xi \in L^2((0,T);L^2(\Omega))$ and $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control $v \in L^2((0,T);L^2(\Omega))$ that insensitizes Φ along the solutions.

- *Boundary control. Let $b \in L^\infty(\Gamma)$ in Γ be given such that*

$$\begin{cases} b \in L^\infty(\Gamma)\,, b \geq 0 \text{ on } \Gamma\,, \\ \{b > 0\} \supset \overline{\Gamma_1} \text{ for some subset } \Gamma_1 \subset \Gamma\,. \end{cases}$$

  Assume that O and Γ_1 satisfy (GCC).

Theorem (continued)

Then for any given $\xi \in L^2((0,T);L^2(\Omega))$ *and* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, there exists a control* $v \in L^2((0,T);L^2(\Gamma))$ *that insensitizes* Φ *along the solutions.*

- *Moreover the above condition on* O *and* ω *(resp. on* O *and* Γ_1*) for the case of locally (resp. boundary) distributed control is sharp.*

We generalize these results and the two-level energy method in an involved way in A.-B. 2012 to the case of:

- N-coupled cascade hyperbolic, diffusive or Schrödinger systems
- controlled by $N - p$ controls with p ranging from 1 to $N - 1$
- localized or boundary dampings
- with the control/observation region which do not intersect any of the localized coupling regions.

That is for systems of the form

$$\begin{cases} y_1'' + Ay_1 + C_{21}^\star y_2 + \dots C_{N1}^\star y_N = 0\,, \\ y_2'' + Ay_2 + C_{32}^\star y_3 + \dots C_{N2}^\star y_N = 0\,, \\ \vdots \\ y_p'' + Ay_p + C_{p+1p}^\star y_p + \dots C_{Np}^\star y_N = 0\,, \\ y_{p+1}'' + Ay_{p+1} C_{p+2p+1}^\star y_{p+1} + \dots C_{Np+1}^\star y_N = B_{p+1} v_{p+1}\,, \\ \vdots \\ y_{N-1}'' + Ay_{N-1} + C_{NN-1}^\star y_N = B_{N-1} v_{N-1}\,, \\ y_N'' + Ay_N = B_N v_N\,, \\ (y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, \dots N\,, \end{cases}$$

Further properties have to be understood: the coupling, the nature of the equations, the geometry, the type of control ...

$\leadsto$ influence the answer to controllability by a reduced number of controls.

Goal $\leadsto$ get an insight and further on $\leadsto$ a classification of systems at least with certain properties.

Works in progress in several directions:

- to understand the "limits" of what make things work,
- having in mind several applications to control of mechanical structures, ...
- inverse problems,

⋮

Thanks for your attention