Analysis of the upwind finite volume method for general initial and boundary value transport problems

FRANCK BOYER †
Aix-Marseille Université, Laboratoire d'Analyse, Topologie et Probabilités,
FST Saint-Jérôme, Av. Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France.

This paper is devoted to the convergence analysis of the upwind finite volume scheme for the initial and boundary value problem associated with the linear transport equation in any dimension, on general unstructured meshes. We are particularly interested in the case where the initial and boundary data are in L^∞ and the advection vector field v has low regularity properties, namely $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, with suitable assumptions on its divergence.
In this general framework, we prove *uniform in time* strong convergence in $L^p(\Omega)$ with $p < +\infty$, of the approximate solution towards the unique weak solution of the problem as well as the strong convergence of its trace. The proof relies, in particular, on the Friedrichs' commutator argument, which is classical in the renormalized solutions theory. Note that this result remains valid if the data are suitably approximated in L^1. This is nothing but the discrete counterpart of the nice compactness properties deduced from the renormalized solution theory.
We conclude by some numerical experiments showing that the convergence rate seems to be $\frac{1}{2}$, like in the case of smoother advection fields, but this is still an open question up to now.

Keywords: Finite volume methods - Transport equation - Renormalized solutions.

1. Introduction

This paper is devoted to the analysis of the upwind finite volume scheme for solving a general linear transport-reaction problem in any dimension. We are interested here in a low regularity framework for the data, still leading to existence and uniqueness of weak solutions, namely the one of renormalized solutions first introduced and studied in DiPerna & Lions (1989). More precisely, we consider here the case where the transport vector-field may not be characteristic at the boundary of the domain. It is thus needed to use the trace theorems and the well-posedness results for the associated initial and boundary value problems given in Boyer (2005).

Our main result in the present paper is the proof of the uniform in time strong convergence in $L^p(\Omega), p < +\infty$, of the approximate solution given by the finite volume scheme towards the unique weak solution of the continuous problem with minimal assumptions on the data, and the meshes.

This work is a first step in the analysis of finite volume methods for coupled systems in which transport-like equations play a key role. We can think for instance of transport in porous medium models (in which the advection field is coupled with transport equation through some Darcy equation), non-homogeneous incompressible Navier-Stokes problems, Vlasov-like systems, ... In each of these situations, it can appear that the advection field is not smooth (for instance, we can only expect that $v \in L^2(]0,T[,(H^1(\Omega))^d)$ for the Navier-Stokes problem). This is one of the main motivation of the present study, to prove that upwind finite volume discretisation is robust enough to handle such situations, at least from a given advection field.

†Email: fboyer@latp.univ-mrs.fr

GENERAL NOTATION. We shall adopt the following notation.

- $\mathscr{L}(f)$ will denote the Lipschitz constant of any Lipschitz continuous function f.
- For any real number x we define its positive and negative parts by $x^+ = (x+|x|)/2$, $x^- = (|x|-x)/2$, and we will often use that $x = x^+ - x^-$ and $|x| = x^+ + x^-$.
- For any $a, b \in \mathbb{R}$, we define $[\![a,b]\!] = [a,b] \cap \mathbb{N}$.
- The characteristic function of a set A will be denoted by $\mathbb{1}_A$.

THE CONTINUOUS PROBLEM. Let $d \geqslant 1$, $\Omega \subset \mathbb{R}^d$ a bounded polygonal (or polyhedral) domain, and $T > 0$ given. We are interested here in the following initial and boundary value problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) + c\rho = 0, & \text{in }]0,T[\times\Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ \rho = \rho^{in}, & \text{on }]0,T[\times\Gamma, \text{ where } (v\cdot\boldsymbol{\nu}) < 0. \end{cases} \tag{1.1}$$

The general existence and uniqueness theory given in Boyer (2005); Boyer & Fabrie (2011) relies on the following assumptions

$$c \in L^1(]0,T[\times\Omega), \tag{1.2}$$

$$\begin{cases} v \in L^1(]0,T[,(W^{1,1}(\Omega))^d), & (1.3a) \\ (v\cdot\boldsymbol{\nu}) \in L^\alpha(]0,T[\times\Gamma), \text{ for some } \alpha > 1, & (1.3b) \end{cases}$$

$$\begin{cases} (c + \operatorname{div} v)^- \in L^1(]0,T[,L^\infty(\Omega)), & (1.4a) \\ \quad (\operatorname{div} v)^+ \in L^1(]0,T[,L^\infty(\Omega)). & (1.4b) \end{cases}$$

The case where $c = \operatorname{div} v = 0$ and where Ω is a smooth domain is treated in Boyer (2005) and the extension to general data c, v and piecewise smooth domains is given in Boyer & Fabrie (2011). Associated to the vector field v, we introduce the measure $\mathrm{d}\mu_v = (v\cdot\boldsymbol{\nu})\,dx\,dt$ on $]0,T[\times\Gamma$ and we denote by $\mathrm{d}\mu_v^+$ (resp. $\mathrm{d}\mu_v^-$) its positive (resp. negative) part in such a way that $|\mathrm{d}\mu_v| = \mathrm{d}\mu_v^+ + \mathrm{d}\mu_v^-$. The support of $\mathrm{d}\mu_v^+$ (resp. $\mathrm{d}\mu_v^-$) is the outflow (resp. inflow) part of the boundary.

This problem is the conservative form of the linear transport-reaction equation. As an example, for $c = -\operatorname{div} v$, we recover the usual non-conservative transport equation $\partial_t \rho + v\cdot\nabla\rho = 0$.

THEOREM 1.1 (Existence and uniqueness, Boyer (2005); Boyer & Fabrie (2011)) We assume that assumptions (1.2), (1.3), (1.4) hold.

For any $\rho^0 \in L^\infty(\Omega)$ and $\rho^{in} \in L^\infty(]0,T[\times\Gamma,\mathrm{d}\mu_v^-)$, there exists a unique weak solution $(\rho,\gamma(\rho)) \in L^\infty(]0,T[\times\Omega)\times L^\infty(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)$ of (1.1) in the sense that

$$\int_0^T\int_\Omega \rho(\partial_t\phi + v\cdot\nabla\phi - c\phi)\,dx\,dt - \int_0^T\int_\Gamma \gamma(\rho)\phi(v\cdot\boldsymbol{\nu})\,dx\,dt + \int_\Omega \rho^0\phi(0,.)\,dx = 0,\ \forall\phi\in\mathscr{C}_c^1([0,T[\times\overline{\Omega}), \tag{1.5}$$

the boundary condition being satisfied in the following sense

$$\gamma(\rho) = \rho^{in},\ \ \mathrm{d}\mu_v^-\text{-almost everywhere.}$$

Moreover, the following properties are also proven in the same references:

- **L^∞ bound:**
$$\|\rho\|_{L^\infty(]0,T[\times\Omega)} \leqslant \max(\|\rho_0\|_{L^\infty}, \|\rho^{in}\|_{L^\infty}) e^{\int_0^T \|(c+\operatorname{div} v)^-\|_{L^\infty(\Omega)}\,dt}. \tag{1.6}$$

- **Time regularity:** ρ lies in $\mathscr{C}^0([0,T], L^p(\Omega))$ for any $p < +\infty$ and $\rho(0) = \rho_0$.

- **Renormalization property:** For any smooth function $\beta : \mathbb{R} \mapsto \mathbb{R}$, the function $\beta(\rho)$ satisfies in the weak sense the problem

$$\begin{cases} \partial_t \beta(\rho) + \operatorname{div}(\beta(\rho)v) + c\beta'(\rho)\rho + (\operatorname{div} v)(\beta'(\rho)\rho - \beta(\rho)) = 0, & \text{in }]0,T[\times\Omega, \quad (1.7a)\\ \beta(\rho)(0,.) = \beta(\rho^0), & (1.7b)\\ \gamma(\beta(\rho)) = \beta(\gamma(\rho)), \quad \text{on }]0,T[\times\Gamma. & (1.7c) \end{cases}$$

  Note that this property still holds for any continuous piecewise smooth function β.

Assumption (1.4a) clearly plays a fundamental role to obtain the L^∞ bound above. However, assumption (1.4b) is only useful in order to deduce the uniqueness property from the renormalization property through a Gronwall-like argument. Note that this last assumption can be slightly relaxed (see Desjardins (1996)) allowing to use Osgood's Lemma instead of Gronwall's Lemma. For instance, all the above results still hold if we assume the weaker condition that $e^{C(\operatorname{div} v)^+} \in L^1(]0,T[\times\Omega)$, for some $C > 0$.

PREVIOUSLY KNOWN RESULTS. The upwind finite volume method is the most classical linear, stable and monotone method for the numerical approximation of linear transport problems (see for instance Eymard *et al.* (2000); LeVeque (2002)). The method is formally first order but it is well-known that, for non smooth initial data (say in $\mathrm{BV}(\Omega)$ or in some Sobolev space), the optimal convergence rate falls down to $1/2$, see for instance Kuznecov (1976); Peterson (1991); Vila & Villedieu (2003); Desprès (2004a,b); Merlet & Vovelle (2007); Merlet (0708); Cockburn *et al.* (2010); Delarue & Lagoutière (2011). As shown in Bouche *et al.* (2005) for instance, contrary to what can be thought at first sight, the irregularity of the mesh is not the main reason for this behavior. In fact, this loss of convergence rate is mainly due to the numerical dissipation of the scheme which implies that discontinuities in the solution are smoothed along time even on regular grids thus leading to suboptimal convergence rate.

In all the results quoted above, the transport vector field v is assumed to be at least Lipschitz-continuous (in some of them, v is even supposed to be constant) in order for the associated characteristic flow to be well defined and smooth enough, which is often one of the main tools in these analysis. Moreover, to our knowledge, the analysis of finite volume schemes for boundary value problems for linear hyperbolic equations is only addressed in Coudière *et al.* (2000); Bouche *et al.* (2005) in the case of a constant vector field v (see for instance Ohlberger & Vovelle (2006) for the case of nonlinear conservation laws).

The present study extends those results by accounting for less regular general vector fields and for L^∞ initial/boundary data. This framework was already considered in Walkington (2005) (see also Fettah (2011)) where convergence of the P^k Discontinuous Galerkin method was analyzed. In the case $k = 0$, the scheme which is considered in this reference reduces to the one we study in the present work, with the addition of a stabilisation term that we do not need here. The results we present in this paper extend Walkington's ones in different directions: we take into account boundary data, the meshes we consider are more general than simplicial meshes and, more importantly, the convergence we prove here is much stronger since it is uniform in time.

Note that, to the best of our knowledge, no convergence rate is known in this general framework. Since the renormalized solution theory allows to define a suitable weak notion of characteristic flows for vector fields satisfying (1.3) (the so-called *regular Lagrangian flow*, see De Lellis (2007)), it should be possible to extend some of the results mentioned above concerning the convergence rate of the scheme to the current framework. We give in Section 7 some numerical results which seem to show that the convergence rate in the L^1 norm seems to be the same as for smooth advection fields, that is $\frac{1}{2}$.

2. The implicit upwind finite volume scheme

2.1 *Notation*

We introduce here the main notation we need in order to define and analyse the finite volume method. A finite volume mesh (see Fig. 1) of the domain Ω is a set $\mathcal{T} = (K)_{K\in\mathcal{T}}$ of closed connected polygonal subsets of $\mathbb{R}^d$, with disjoint interiors and such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}} K$.

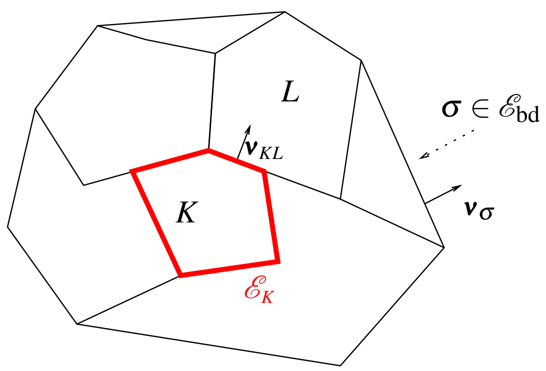


FIG. 1. A finite volume mesh

The boundary of each control volume $K \in \mathcal{T}$ can be written as the union of a finite number of edges/faces (we will often use the word “edge” even if $d > 2$) which are closed connected sets of dimension $d-1$ contained into hyperplanes. We denote by $\mathcal{E}_K$ the set of the faces/edges of K. We assume that for any K,L such that $K \neq L$ and $K\cap L$ is of co-dimension 1, then $K\cap L \in \mathcal{E}_K \cap \mathcal{E}_L$, in that case the corresponding face is denoted by $K|L$.

The set of all the faces in the mesh is denoted by $\mathcal{E}$ and $\mathcal{E}_{\mathrm{bd}}$ denote the subset of the faces which are included in the boundary $\partial\Omega$, $\mathcal{E}_{\mathrm{int}} = \mathcal{E} \setminus \mathcal{E}_{\mathrm{bd}}$ the set of the interior faces.

- For each $K \in \mathcal{T}$, and $\sigma \in \mathcal{E}_K$, we denote by $\boldsymbol{\nu}_{K\sigma}$ the unit outward normal vector to K on σ. If $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, we shall sometimes use the notation $\boldsymbol{\nu}_{KL} = \boldsymbol{\nu}_{K\sigma} = -\boldsymbol{\nu}_{L\sigma}$. If $\sigma \in \mathcal{E}_{\mathrm{bd}}$, there is a unique $K \in \mathcal{T}$ such that $\sigma \in \mathcal{E}_K$ and then $\boldsymbol{\nu}_{K\sigma}$ is nothing but the unit outward normal to $\partial\Omega$ and we may also write $\boldsymbol{\nu}_\sigma$ or $\boldsymbol{\nu}$ if no confusion is possible.

- We will denote by $|K|$ (resp. $|\sigma|$) the d-dimensional Lebesgue measure of the control volume K (resp. the $d-1$ dimensional measure of the face σ).

- The diameter of a control volume K (resp. of an edge σ) shall be denoted by d_K (resp. d_σ) and the size of the mesh is defined by $h_{\mathscr{T}} = \max_{K\in\mathscr{T}} d_K$.

We need to measure the regularity of the mesh. To this end, we denote by $\mathrm{reg}(\mathscr{T})$, the smallest positive number such that

$$\|f\|_{L^1(\partial K)} \leqslant \frac{\mathrm{reg}(\mathscr{T})}{d_K}\|f\|_{W^{1,1}(K)}, \quad \forall K\in\mathscr{T}, \forall f\in W^{1,1}(K). \tag{2.1}$$

In the convergence results given below we shall assume that $\mathrm{reg}(\mathscr{T})$ remains bounded as $h_{\mathscr{T}}\to 0$, which amounts to assume that the control volumes are not allowed to degenerate. In the case of simplexes, then the above assumption is nothing but the usual regularity assumption used in the finite element framework. The assumption is also satisfied for convex control volumes such that the ratio $d_K/|\sigma|$ is uniformly bounded for any edge of K and which are non flat in the sense that they contain a ball of radius r_K with d_K/r_K uniformly bounded. Note finally, that (2.1) implies in particular (take $f=1$) that

$$\sum_{\sigma\in\mathscr{E}_K} d_K|\sigma| \leqslant \mathrm{reg}(\mathscr{T})|K|, \quad \forall K\in\mathscr{T}.$$

It will be useful to associate a point $x_K\in K$ to each control volume $K\in\mathscr{T}$. We may for instance choose x_K to be the mass center of K, if K is convex. These points actually do not enter the definition of the scheme, they are only used as a tool in the analysis.

2.2 *Definition of the scheme*

Let us first define the discretisation of the data needed to define our finite volume method (see Section 6.1 for further comments on this point).

- For any $K\in\mathscr{T}$, $n\in[\![0,N-1]\!]$, we define

$$c_K^n = \frac{1}{\delta t|K|}\int_{t^n}^{t^{n+1}}\int_K c\,dx\,dt, \ \text{ and } \ v_{K\sigma}^n = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot \boldsymbol{\nu}_{K\sigma})\,dx\,dt, \ \forall\sigma\in\mathscr{E}_K.$$

  Furthermore, if $\sigma\in\mathscr{E}_{\mathrm{int}}$, with $\sigma=K|L$ we shall use the notation $v_{KL}^n = v_{K\sigma}^n = -v_{L\sigma}^n$, and if $\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}$ we will note $v_\sigma^n = v_{K\sigma}^n$. We will often use the fact that, by Stokes' formula, we have

$$\sum_{\sigma\in\mathscr{E}_K}|\sigma|v_{K\sigma}^n = |K|(\mathrm{div}\,v)_K^n = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_K(\mathrm{div}\,v)\,dx\,dt. \tag{2.2}$$

- For any boundary edge $\sigma\in\mathscr{E}_{\mathrm{bd}}$ and any $n\in[\![0,N-1]\!]$, we define

$$\rho_\sigma^{in,n+1} = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma \rho^{in}\,dx\,dt. \tag{2.3}$$

  Notice that ρ^{in} is *a priori* only given $d\mu_v^-$-almost everywhere so that in this formula we need, in fact, to consider an extension of ρ^{in} in $L^\infty(]0,T[\times\Gamma)$.

To simplify a little the notation, let us introduce $v_{K\sigma}^{n+} = (v_{K\sigma}^n)^+$ and $v_{K\sigma}^{n-} = (v_{K\sigma}^n)^-$. The implicit finite volume scheme we consider is the following: Find $(\rho_K^n)_{\substack{n\in[\![0,N]\!]\\ K\in\mathscr{T}}}$ such that

$$\begin{cases} |K|\dfrac{\rho_K^{n+1}-\rho_K^n}{\delta t} + \displaystyle\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\rm int}} |\sigma|\big(v_{K\sigma}^{n+}\rho_K^{n+1} - v_{K\sigma}^{n-}\rho_L^{n+1}\big) + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\rm bd}} |\sigma| v_{K\sigma}^n \rho_\sigma^{n+1} \\ \qquad\qquad + |K| c_K^n \rho_K^{n+1} = 0, \ \ \forall n\in[\![0,N-1]\!], \forall K\in\mathscr{T}, \\ \rho_K^0 = \dfrac{1}{|K|}\displaystyle\int_K \rho^0\,dx, \ \ \forall K\in\mathscr{T}, \\ \rho_\sigma^{n+1} = \rho_\sigma^{in,n+1}, \ \ \forall n\in[\![0,N-1]\!], \forall\sigma\in\mathscr{E}_{\rm bd}, \ \text{ s.t. } \ v_{K\sigma}^n \leqslant 0, \\ \rho_\sigma^{n+1} = \rho_K^{n+1}, \ \ \forall n\in[\![0,N-1]\!], \forall\sigma\in\mathscr{E}_{\rm bd}, \ \text{ s.t. } \ v_{K\sigma}^n > 0. \end{cases} \tag{2.4}$$

REMARK 2.1
- For the pure advection equation, that is when $c = -\mathrm{div}\, v$, with (2.2), the scheme reads

$$|K|\frac{\rho_K^{n+1}-\rho_K^n}{\delta t} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\rm int}} |\sigma| v_{K\sigma}^{n-}\big(\rho_K^{n+1}-\rho_L^{n+1}\big) + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\rm bd}} |\sigma| v_{K\sigma}^{n-}(\rho_K^{n+1}-\rho_\sigma^{in,n+1}) = 0,$$

which is a more usual formulation of the scheme for the pure advection equation.

- We only consider here the implicit version of the scheme in order to avoid the introduction of a stability CFL condition but all the results given below are valid for the explicit scheme. Note that the CFL condition for the explicit upwind scheme reads

$$\delta t\left(c_K^n + \frac{1}{|K|}\sum_{\sigma\in\mathscr{E}_K} |\sigma| v_{K\sigma}^{n+}\right) \leqslant 1, \ \ \forall K\in\mathscr{T}.$$

In general, this condition should be difficult to fulfill since v and c can be unbounded.

2.3 *Outline of the paper*

In Section 3, we prove existence and uniqueness of the approximate solution, for small enough time steps (this condition on δt being independent of the mesh $\mathscr{T}$), then we establish *a priori* estimates on those solutions and their traces: a uniform L^∞-bound and a weak $L^2(H^1)$ estimate which will be useful in the convergence analysis. In Section 4, we prove the weak-$\star$ convergence in L^∞ of the approximate solution towards the unique weak solution of the problem, as well as for the traces. In Section 5, we prove the main result of this paper, which says that this convergence is in fact strong in $L^\infty(]0,T[,L^p(\Omega))$, for any $p<+\infty$, together with a suitable strong convergence result for the traces. The proof of this result is based on the same tools than those used to prove existence and uniqueness of weak solutions for the problem in the framework of renormalized solutions, namely the Friedrichs commutator lemma. Note that the strong convergence of the approximate solutions in $L^p(]0,T[\times\Omega)$ is easier to obtain (see Walkington (2005) for instance); the difficult point here is to prove a convergence which is uniform in time. We conclude the paper by some extensions and remarks concerning the scheme under study and by numerical illustrations of the actual accuracy of the method.

2.4 *A technical result*

Notice that we do not assume that the boundary edges of the mesh are completely included in the inflow or outflow part of the boundary, in particular because these sets can be very complicated objects due to the low regularity of the advection field. As a consequence, we shall need the following technical result to deal with these particular edges. The proof can be done by usual density arguments and is left to the reader.

LEMMA 2.1 1. For any $1 \leqslant p < +\infty$, $f \in L^p(]0,T[\times\Gamma)$ we have

$$\sum_{n=0}^{N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}\int_{t^n}^{t^{n+1}}\int_\sigma \left|f(t,x)-f_\sigma^{n+1}\right|^p\,dx\,dt \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0,$$

where f_σ^{n+1} is the mean-value of f on $]t^n,t^{n+1}[\times\sigma$.

2. For any $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, we have

$$\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v_{K\sigma}^n\leqslant 0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dx\,dt+\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v_{K\sigma}^n\geqslant 0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0.$$

3. Existence and uniqueness. *A priori* estimates

3.1 *Existence and uniqueness. First properties*

Since the scheme we are studying is implicit in time, it is needed to prove that the approximate solution actually exists and is unique. This is the goal of the first result of this paper.

THEOREM 3.1 Assume that (1.2),(1.3a) and (1.4a) hold. There exists $\delta t_{\max} > 0$ (depending only on $(c+\operatorname{div} v)^-$) such that for any initial and boundary data $\rho^0 \in L^\infty(\Omega)$, $\rho^{in} \in L^\infty(]0,T[\times\Gamma)$, any mesh $\mathscr{T}$ and any time step such that

$$\delta t \leqslant \delta t_{\max},$$

there exists an unique solution of the scheme (2.4).

Moreover in that case, the scheme is monotone, that is

$$\left(\rho^0 \geqslant 0 \text{ and } \rho^{in} \geqslant 0\right) \implies \left(\rho_K^n \geqslant 0, \forall K \in \mathscr{T}, \forall n \in [\![0,N]\!]\right).$$

Finally, the following L^∞ bound holds:

$$|\rho_K^n| \leqslant \max(\|\rho^0\|_{L^\infty},\|\rho^{in}\|_{L^\infty})\exp\left(2\int_0^{t^n}\|(c+\operatorname{div} v)^-\|_{L^\infty}\,dt\right),\ \ \forall K\in\mathscr{T},\forall n\in[\![0,N]\!].$$

It will be clear in the proof that, in the case where $c+\operatorname{div} v \geqslant 0$ (in particular for the pure advection equation where $c = -\operatorname{div} v$), we have $\delta t_{\max} = +\infty$.

Proof. The initial data $(\rho_K^0)_{K\in\mathscr{T}}$ is directly defined from ρ^0. Assume now that $(\rho_K^n)_{K\in\mathscr{T}}$ is known at time t^n, $n \leqslant N-1$ and let us show that $(\rho_K^{n+1})_{K\in\mathscr{T}}$ is uniquely defined.

The set of equations being linear with the same number of unknowns as that of equations, it is enough to show that, if $\rho^{in} = 0$, and $\rho_K^n = 0$ for any $K \in \mathscr{T}$, then any solution of the system satisfies $\rho_K^{n+1} = 0$ for any $K \in \mathscr{T}$. To this end, we will in fact prove the monotony of the scheme which will imply its well-posedness.

By assumption $t \mapsto \|(c+\operatorname{div} v)^-(t)\|_{L^\infty}$ is integrable on $]0,T[$, hence there exists $\delta t_{\max} > 0$ such that

$$\int_I \|(c+\operatorname{div} v)^-(t)\|_{L^\infty}\, dt \leqslant \frac{1}{2}, \quad \forall I \subset]0,T[, \text{ s.t. } |I| \leqslant \delta t_{\max}. \tag{3.1}$$

- **Step 1. Change of variables.** For any $n \in [\![0,N-1]\!]$, let us define

$$\gamma^n = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \|(c+\operatorname{div} v)^-\|_{L^\infty}\, dt, \tag{3.2}$$

and $(\alpha^n)_n$ by

$$\alpha^0 = 1, \ \ \alpha^{n+1} = (1-\delta t\gamma^n)\alpha^n, \ \ \forall n \in [\![0,N-1]\!]. \tag{3.3}$$

Using the following basic inequality

$$\frac{1}{1-x} \leqslant 1+2x \leqslant e^{2x}, \ \ \forall x \in [0,1/2],$$

and the property (3.1) defining $\delta t_{\max}$, it is easily seen that we have

$$\forall n \in [\![0,N-1]\!], \ \ 0 \leqslant \exp\left(-2\int_0^{t^n} \|(c+\operatorname{div} v)^-\|_{L^\infty}\, dt\right) \leqslant \alpha^n \leqslant 1. \tag{3.4}$$

We can now perform the following change of variables

$$\tilde{\rho}_K^n = \alpha^n \rho_K^n, \ \ \forall K \in \mathscr{T}, \ \forall n \in [\![0,N]\!],$$

$$\tilde{\rho}_\sigma^n = \alpha^n \rho_\sigma^n, \text{ and } \tilde{\rho}_\sigma^{in,n} = \alpha^n \rho_\sigma^{in,n}, \ \ \forall \sigma \in \mathscr{E}_{\mathrm{bd}}, \ \forall n \in [\![0,N]\!].$$

and we get

$$\begin{cases} |K|\dfrac{\tilde{\rho}_K^{n+1}-\tilde{\rho}_K^n}{\delta t} + \dfrac{\alpha^n}{\alpha^{n+1}}\displaystyle\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma|\big(v_{K\sigma}^{n+}\tilde{\rho}_K^{n+1} - v_{K\sigma}^{n-}\tilde{\rho}_L^{n+1}\big) + \dfrac{\alpha^n}{\alpha^{n+1}}\displaystyle\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}} |\sigma| v_{K\sigma}^n \tilde{\rho}_\sigma^{n+1} \\ \qquad\qquad + |K|\dfrac{\alpha^n}{\alpha^{n+1}}(c_K^n+\gamma^n)\tilde{\rho}_K^{n+1} = 0, \ \ \forall n \in [\![0,N-1]\!], \forall K \in \mathscr{T}, \\ \tilde{\rho}_K^0 = \rho_K^0, \ \ \forall K \in \mathscr{T}, \\ \tilde{\rho}_\sigma^{n+1} = \tilde{\rho}_\sigma^{in,n+1}, \ \ \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathscr{E}_{\mathrm{bd}}, \ \ \text{s.t. } \ v_{K\sigma}^n \leqslant 0, \\ \tilde{\rho}_\sigma^{n+1} = \tilde{\rho}_K^{n+1}, \ \ \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathscr{E}_{\mathrm{bd}}, \ \ \text{s.t. } \ v_{K\sigma}^n > 0. \end{cases} \tag{3.5}$$

By using the formula $v_{K\sigma}^{n+} = v_{K\sigma}^n + v_{K\sigma}^{n-}$, and the boundary conditions we consider in the scheme, we write

$$\begin{aligned} &\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma| v_{K\sigma}^{n+}\tilde{\rho}_K^{n+1} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}} |\sigma| v_{K\sigma}^n \tilde{\rho}_\sigma^{n+1} \\ &\qquad = \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma| v_{K\sigma}^n\tilde{\rho}_K^{n+1} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma| v_{K\sigma}^{n-}\tilde{\rho}_K^{n+1} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}} |\sigma| v_{K\sigma}^n \tilde{\rho}_\sigma^{n+1} \\ &\qquad = \underbrace{\sum_{\sigma\in\mathscr{E}_K} |\sigma| v_{K\sigma}^n\tilde{\rho}_K^{n+1}}_{=|K|(\operatorname{div} v)_K^n\tilde{\rho}_K^{n+1}} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma| v_{K\sigma}^{n-}\tilde{\rho}_K^{n+1} + \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}} |\sigma| \underbrace{\big(v_{K\sigma}^n\tilde{\rho}_\sigma^{n+1} - v_{K\sigma}^n\tilde{\rho}_K^{n+1}\big)}_{=-v_{K\sigma}^{n-}(\tilde{\rho}_\sigma^{n+1}-\tilde{\rho}_K^{n+1})}. \end{aligned}$$

Hence we may write the first equation in (3.5) in the following equivalent form

$$|K|\frac{\tilde\rho_K^{n+1}-\tilde\rho_K^n}{\delta t}+\frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}}|\sigma|v_{K\sigma}^{n-}(\tilde\rho_K^{n+1}-\tilde\rho_L^{n+1})+\frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}}|\sigma|v_{K\sigma}^{n-}(\tilde\rho_K^{n+1}-\tilde\rho_\sigma^{in,n+1})$$
$$+|K|\frac{\alpha^n}{\alpha^{n+1}}(c_K^n+(\operatorname{div} v)_K^n+\gamma^n)\tilde\rho_K^{n+1}=0. \quad (3.6)$$

- **Step 2. Monotonicity. Existence and uniqueness.** We assume that $\rho^0\geqslant 0$ and $\rho^{in}\geqslant 0$. We are going to show that any super-solution to (3.5) (that is to say by replacing the $=$ symbol in the first three equation in (3.5) by the symbol $\geqslant$, the fourth equation being unchanged) is non-negative.

  By induction we assume that n is such that $\rho_K^n\geqslant 0,\forall K\in\mathscr{T}$ and we want to show that $\rho_K^{n+1}\geqslant 0,\forall K\in\mathscr{T}$. Let us assume, by contradiction, that there is a $K\in\mathscr{T}$ such that

  $$\rho_K^{n+1}=\min_{L\in\mathscr{T}}\rho_L^{n+1}<0.$$

  In formula (3.6) (with a $\geqslant$ sign instead of $=$) for this particular control volume K, we see that the two sums over edges are non-positive since $\tilde\rho_K^{n+1}\leqslant\tilde\rho_L^{n+1}$ for any $L\in\mathscr{T}$, and since $\tilde\rho_\sigma^{in,n+1}\geqslant 0$ and $\tilde\rho_K^{n+1}<0$. Furthermore, by the definition of γ^n in formula (3.2), we see that

  $$c_K^n+(\operatorname{div} v)_K^n+\gamma^n\geqslant 0. \quad (3.7)$$

  Since we assumed that $\tilde\rho_K^{n+1}<0$, it finally remains the inequality $\tilde\rho_K^{n+1}\geqslant\tilde\rho_K^n$. This implies $\tilde\rho_K^n<0$, which is impossible since we assumed that the approximate solution is non-negative at time t^n.

  Thus, any super-solution (and consequently any solution) associated with non-negative data is non-negative. The scheme (3.5) being a linear set of equations with the same number of equations as that of unknowns, it is well known that the monotonicity property implies existence and uniqueness of the approximate solution for any data.

- **Step 3. L^∞-bound.** We define $M=\max(\|\rho^0\|_{L^\infty},\|\rho^{in}\|_{L^\infty})$ and we observe, thanks to (3.7), that the set of constant values $\overline{\rho_K^n}=M,\forall K\in\mathscr{T},\forall n\in[\![0,N]\!]$ is a super-solution to (3.5).

  As a consequence, the difference $M-\tilde\rho_K^n=\overline{\rho_K^n}-\tilde\rho_K^n$ is a super-solution to (2.4) associated with the data $M-\rho^0$ and $M-\tilde{\rho^{in}}$. By the choice of M, these data are non-negative and then we can apply the monotonicity result above to deduce that $\tilde\rho_K^n\leqslant M$ for any $K\in\mathscr{T}$ and any $n\in[\![0,N]\!]$. The claim finally follows from (3.4).

$\square$

REMARK 3.1 In the particular case of the so-called mass conservation equation, that is for $c=0$, we can prove (by a duality argument) existence, uniqueness and monotonicity without any additional assumption on $(c+\operatorname{div} v)^-=(\operatorname{div} v)^-$ and without any condition on the time-step. These conditions are however mandatory, even in that case, to obtain the L^∞ bound on the approximate solution.

REMARK 3.2 We can also prove a bound from below for the solutions of our finite volume scheme associated with non-negative data ρ^0 and ρ^{in}. Indeed, under assumptions (1.2), (1.3a) and $(c+\operatorname{div} v)\in L^1(]0,T[,L^\infty(\Omega))$, then for any mesh $\mathscr{T}$ and any time step $\delta t\leqslant\delta t_{\max}$, the unique solution to the finite volume scheme (2.4) satisfies

$$\rho_K^n\geqslant\exp\left(-\int_0^{t^n}\|(c+\operatorname{div} v)^+\|_{L^\infty}\,dt\right)\min\left(\inf_\Omega\rho^0,\inf_{]0,T[\times\Gamma}\rho^{in}\right),\ \ \forall K\in\mathscr{T},\forall n\in[\![0,N]\!].$$

This kind of result is important in the applications, for instance in fluid mechanics, where the fact that the fluid density remains far from zero uniformly in the discretisation parameters may be crucial. Note that the assumption on $c+\operatorname{div} v$ we need is stronger than (1.4a).

We shall now define the approximate solution to be the piecewise constant function $\rho_{\mathscr{T},\delta t} \in L^\infty(]0,T[\times\Omega)$ defined as follows

$$\rho_{\mathscr{T},\delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}} \rho_K^{n+1}\mathbb{1}_{]t^n,t^{n+1}[\times K}.$$

We define the trace $\gamma\rho_{\mathscr{T},\delta t} \in L^\infty(]0,T[\times\Gamma)$ of this approximate solution as follows

$$\gamma\rho_{\mathscr{T},\delta t} = \sum_{n=0}^{N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}} \rho_K^{n+1}\mathbb{1}_{]t^n,t^{n+1}[\times\sigma},$$

where in this sum K is the unique control volume such that $\sigma\in\mathscr{E}_K$. Note that this definition is nothing but the trace, in the BV sense, of the function $\rho_{\mathscr{T},\delta t}$. We also need to introduce the discretisation of the initial data

$$\rho^0_{\mathscr{T}} = \sum_{K\in\mathscr{T}} \rho_K^0 \mathbb{1}_K.$$

REMARK 3.3 By standard approximation arguments, we know that $\rho^0_{\mathscr{T}}$ converges towards ρ^0 in $L^\infty(\Omega)$ weak-$\star$ and in $L^p(\Omega)$-strong for any $p<+\infty$.

With these notations, the L^∞ bound given in Theorem 3.1 leads to the inequalities

$$\|\rho_{\mathscr{T},\delta t}\|_{L^\infty(]0,T[\times\Omega)} \leqslant \rho_{\max}, \quad \|\gamma\rho_{\mathscr{T},\delta t}\|_{L^\infty(]0,T[\times\Gamma)} \leqslant \rho_{\max}, \tag{3.8}$$

where $\rho_{\max}$ does not depend on δt and $\mathscr{T}$ and is defined by

$$\rho_{\max} = \max(\|\rho^0\|_{L^\infty(\Omega)}, \|\rho^{in}\|_{L^\infty(]0,T[\times\Gamma)})e^{2\int_0^T \|(c+\operatorname{div} v)^-\|_{L^\infty(\Omega)}\,dt}.$$

Notice that, by (1.6), we know that the exact solution ρ satisfies similar estimates

$$\|\rho\|_{L^\infty(]0,T[\times\Omega)} \leqslant \rho_{\max}, \quad \|\gamma\rho\|_{L^\infty(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)} \leqslant \rho_{\max}.$$

3.2 *Weak $L^2(H^1)$ estimate*

In the following proposition, we derive a kind of energy estimate for the solution of the finite volume scheme.

PROPOSITION 3.2 Assume that (1.2), (1.3a) and (1.4a) hold. There exists $M>0$ depending only on c, v, ρ^0 and ρ^{in}, such that for any $\delta t \leqslant \delta t_{\max}$ and any mesh $\mathscr{T}$ we have the following bound

$$\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}|\sigma||v^n_{K\sigma}|(\rho_K^{n+1}-\rho_\sigma^{n+1})^2 + \sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}}|\sigma||v^n_{KL}|(\rho_L^{n+1}-\rho_K^{n+1})^2 \leqslant M. \tag{3.9}$$

This estimate can be understood as a weak $L^2(]0,T[,H^1(\Omega))$ estimate since, if the mesh is quasi-uniform, we can write (for the interior edges for instance)

$$\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}}|\sigma|d_{KL}|v^n_{KL}|\left|\frac{\rho_L^{n+1}-\rho_K^{n+1}}{d_{KL}}\right|^2 \leqslant \frac{M}{h_{\mathscr{T}}}, \tag{3.10}$$

where d_{KL} is the distance between x_K and x_L. Hence, for a smooth exact solution ρ, if we think ρ_K^{n+1} as an approximation of $\rho(t^{n+1},x_K)$ (which it is not !), then the left-hand side of this inequality looks like the square of a weighted discrete $L^2(H^1)$ norm, the weight being proportional to the mean-value of the flow across each edge. In particular, this estimate provide useful information only on the parts of $]0,T[\times\Omega$ where the vector-field v does not vanish. Such kind of property is also known as a weak BV estimate, in the framework of nonlinear scalar conservation laws, see Champier *et al.* (1993); Eymard *et al.* (2000).

As shown in Boyer (2005); Boyer & Fabrie (2011), if one consider the following parabolic approximation of the original problem

$$\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v) + c\rho_\varepsilon - \varepsilon\Delta\rho_\varepsilon = 0, \tag{3.11}$$

with the initial data $\rho_\varepsilon(0)=\rho^0$ and the Fourier boundary condition $\varepsilon\frac{\partial\rho_\varepsilon}{\partial\boldsymbol{\nu}} + (\rho_\varepsilon - \rho^{in})(v\cdot\boldsymbol{\nu})^- = 0$, the corresponding estimate reads

$$\|\rho_\varepsilon\|_{L^2(]0,T[,H^1(\Omega))} \leqslant \frac{C}{\sqrt{\varepsilon}}.$$

Here, (3.10), the size of mesh $h_{\mathscr{T}}$ plays the role of the approximation parameter ε and, moreover, the numerical diffusion tensor is isotropic, heterogeneous and depends on v.

It was shown in Boyer (2005); Boyer & Fabrie (2011) that the solution to (3.11) strongly converges towards the solution ρ of the transport equation in $\mathscr{C}^0([0,T],L^p(\Omega))$ for any $p<+\infty$. We will use the same kind of idea in the present paper in order to show the uniform in time strong convergence of our finite volume approximate solution in Section 5.

Proof. First of all, for any interior edge $\sigma = K|L \in \mathscr{E}_{\text{int}}$ we define $\rho_\sigma^{n+1} = (\rho_K^{n+1}+\rho_L^{n+1})/2$. Recall that for boundary edges the value of ρ_σ^{n+1} is already given in the definition of the scheme.

By simple algebraic manipulations, for any $K\in\mathscr{T}$ and $n\in[\![0,N-1]\!]$ the finite volume scheme (2.4) gives

$$|K|\frac{\rho_K^{n+1}-\rho_K^n}{\delta t} + \sum_{\sigma\in\mathscr{E}_K}|\sigma|v_{K\sigma}^n\rho_\sigma^{n+1} + |K|c_K^n\rho_K^{n+1} - \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\text{int}}}|\sigma||v_{K\sigma}^n|\frac{\rho_L^{n+1}-\rho_K^{n+1}}{2} = 0. \tag{3.12}$$

We multiply (3.12) by $\delta t\rho_K^{n+1}$, we sum over n and K and we finally use the algebraic identity $ab = \frac{1}{2}a^2+\frac{1}{2}b^2-\frac{1}{2}(a-b)^2$, to obtain

$$\sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}}|K|(\rho_K^{n+1}-\rho_K^n)\rho_K^{n+1} + \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}|K|c_K^n|\rho_K^{n+1}|^2 - \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\text{int}}}|\sigma||v_{K\sigma}^n|\frac{\rho_L^{n+1}-\rho_K^{n+1}}{2}\rho_K^{n+1}$$
$$+\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K}|\sigma|v_{K\sigma}^n\left(\frac{1}{2}(\rho_\sigma^{n+1})^2+\frac{1}{2}(\rho_K^{n+1})^2-\frac{1}{2}(\rho_\sigma^{n+1}-\rho_K^{n+1})^2\right) = 0.$$

Note that we have

$$v_{K\sigma}^n = -v_{L\sigma}^n = v_{KL}^n, \text{ and } (\rho_\sigma^{n+1}-\rho_K^{n+1})^2 = (\rho_\sigma^{n+1}-\rho_L^{n+1})^2,\ \forall\sigma = K|L\in\mathscr{E}_{\text{int}},$$

so that, reorganizing the sums on the edges and using (2.2) and the values of ρ_σ^{n+1} prescribed on the boundary of the domain, we get

$$\frac{1}{2}\|\rho_{\mathcal{T}}^N\|_{L^2}^2 + \frac{1}{2}\sum_{n=0}^{N-1}\delta t\|\rho_{\mathcal{T}}^{n+1} - \rho_{\mathcal{T}}^n\|_{L^2}^2 + \underbrace{\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{T}}|K|\left(c_K^n + \frac{1}{2}(\operatorname{div} v)_K^n\right)|\rho_K^{n+1}|^2}_{=T_1}$$

$$+\underbrace{\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_\sigma^n<0}}|\sigma||v_\sigma^n|(\rho_\sigma^{in,n+1} - \rho_K^{n+1})^2 + \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}}|\sigma||v_{KL}^n|(\rho_L^{n+1} - \rho_K^{n+1})^2}_{=T_2}$$

$$+\frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_\sigma^n\geqslant 0}}|\sigma|v_\sigma^n(\rho_\sigma^{n+1})^2 = \frac{1}{2}\|\rho_{\mathcal{T}}^0\|_{L^2}^2 + \frac{1}{2}\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_\sigma^n<0}}|\sigma||v_\sigma^n|(\rho_\sigma^{in,n+1})^2. \quad (3.13)$$

All the terms in this identity are non-negative, except possibly the term T_1. Nevertheless, using the L^∞ bound on $\rho_{\mathcal{T},\delta t}$, we can bound this term as follows

$$|T_1| \leqslant (\rho_{\max})^2\left\|\left(c + \frac{1}{2}\operatorname{div} v\right)^-\right\|_{L^1(]0,T[\times\Omega)},$$

and we finally obtain a bound on the term T_2, which is the expected result. □

4. Weak convergence result

In this section, we are going to prove the weak convergence of the solution of the finite volume scheme towards the unique weak solution of the initial and boundary value problem (1.1). This is the first step towards the strong convergence result that we shall prove in the next section.

THEOREM 4.1 Assume that (1.2), (1.3) and (1.4) hold. Let $\mathrm{reg}_{\max} > 0$ be given and consider a family of meshes and time steps, such that $(\delta t, h_{\mathcal{T}}) \to 0$ and satisfying the bound

$$\max\left(\mathrm{reg}(\mathcal{T}), \max_{K\in\mathcal{T}}\frac{\delta t}{d_K}\right) \leqslant \mathrm{reg}_{\max}. \quad (4.1)$$

Then, we have

$$\rho_{\mathcal{T},\delta t} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \rho, \text{ in } L^\infty(]0,T[\times\Omega) \text{ weak-}\star,$$

$$\gamma\rho_{\mathcal{T},\delta t} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \gamma\rho, \text{ in } L^\infty(]0,T[\times\Gamma, |\mathrm{d}\mu_v|) \text{ weak-}\star,$$

where ρ and $\gamma\rho$ solves (1.5).

Note that a proof of a similar result is given in Walkington (2005); Fettah (2011) in the case when the vector field v is tangent to the boundary of $\partial\Omega$ and for meshes made of simplexes.

Proof. Notice first that assumption (1.4b) is only used to ensure uniqueness of the weak solution and of its trace (see Theorem 1.1) and is not directly used in the following computations.

Thanks to the L^∞ bounds (3.8), we can find subsequences of $(\rho_{\mathcal{T},\delta t})$ and $(\gamma\rho_{\mathcal{T},\delta t})$ which weak-$\star$ converge in the spaces given above. In fact, up to another extraction of a subsequence, we may also assume that $\gamma\rho_{\mathscr{T},\delta t}$ weak-$\star$ converges in $L^\infty(]0,T[\times\Gamma)$. We denote by ρ and g the respective limits of these two subsequences. We will show that ρ and g satisfy the weak formulation of the problem. Since this weak solution is unique, we will then deduce the claim.

TRACE IDENTIFICATION. Let us first show that $g=\rho^{in}$, $d\mu_v^-$-almost everywhere by writing

$$\int_0^T\int_\Gamma|\gamma\rho_{\mathscr{T},\delta t}-\rho^{in}|(v\cdot\boldsymbol{\nu})^-\,dt\,dx$$
$$\leqslant 2\rho_{\max}\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}>0}}\int_{t^n}^{t^{n+1}}\int_\sigma(v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt+\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}\leqslant 0}}\int_{t^n}^{t^{n+1}}\int_\sigma\left|\rho^{in,n+1}_\sigma-\rho^{in}\right|(v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt,$$
$$\leqslant 2\rho_{\max}\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}>0}}\int_{t^n}^{t^{n+1}}\int_\sigma(v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt+\|v\cdot\boldsymbol{\nu}\|_{L^\alpha(]0,T[\times\Gamma)}\left(\sum_{n=0}^{N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}\int_{t^n}^{t^{n+1}}\int_\sigma\left|\rho^{in,n+1}_\sigma-\rho^{in}\right|^{\alpha'}dx\,dt\right)^{\frac{1}{\alpha'}}.$$

The two terms in the right-hand side tend to zero when δt and $h_{\mathscr{T}}$ tend to zero by Lemma 2.1, so that we proved that $\gamma\rho_{\mathscr{T},\delta t}$ strongly converges towards ρ^{in} in $L^1(]0,T[\times\Gamma,d\mu_v^-)$ which implies, in particular, that the weak-$\star$ limit g of $\gamma\rho_{\mathscr{T},\delta t}$ coincides with ρ^{in} $d\mu_v^-$-almost everywhere.

WEAK FORMULATION. Let $\phi\in\mathscr{C}^\infty_c([0,T[\times\overline{\Omega})$ be a smooth test function. We want to show that the weak limits ρ and g obtained above solve the weak formulation of the problem.

We define $\phi^n_K=\phi(t^n,x_K)$, where we recall that $x_K\in K$ is a point arbitrarily chosen in each control volume. We multiply the first equation of (2.4) by $\delta t\phi^n_K$ and we sum over $n\in[\![0,N-1]\!]$ and $K\in\mathscr{T}$. It follows

$$\underbrace{\sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}}|K|(\rho^{n+1}_K-\rho^n_K)\phi^n_K}_{=T_1}+\underbrace{\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}|K|c^n_K\rho^{n+1}_K\phi^n_K}_{=T_2}$$
$$+\underbrace{\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}}|\sigma|(v^{n+}_{K\sigma}\rho^{n+1}_K-v^{n-}_{K\sigma}\rho^{n+1}_L)\phi^n_K+\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}|\sigma|v^n_{K\sigma}\rho^{n+1}_\sigma\phi^n_K}_{=T_3}=0. \quad (4.2)$$

Since ϕ has a compact support in time in $[0,T[$, we have $\phi^n_K=0$ for any $K\in\mathscr{T}$, for $n=N$. Thus, the term T_1 can be expressed as follows

$$T_1=\sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}}|K|\rho^{n+1}_K(\phi^n_K-\phi^{n+1}_K)-\sum_{K\in\mathscr{T}}|K|\rho^0_K\phi^0_K$$
$$=-\sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}}\rho^{n+1}_K\int_{t^n}^{t^{n+1}}|K|\partial_t\phi(t,x_K)\,dt-\sum_{K\in\mathscr{T}}|K|\rho^0_K\phi^0_K.$$

Since ϕ is smooth, and $\|\rho_{\mathscr{T},\delta t}\|_{L^\infty}\leqslant\rho_{\max}$, we get that

$$\left|T_1+\int_0^T\int_\Omega\rho_{\mathscr{T},\delta t}\partial_t\phi\,dx\,dt+\int_\Omega\rho^0_{\mathscr{T}}(x)\phi(0,x)\,dx\right|\leqslant C_{\phi,\rho_{\max}}(\delta t+h_{\mathscr{T}}).$$

By weak convergence of $\rho_{\mathcal{T},\delta t}$ and $\rho^0_{\mathcal{T}}$, we deduce that

$$T_1 \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} -\int_0^T \int_\Omega \rho \partial_t \phi \, dx\, dt - \int_\Omega \rho^0 \phi(0,.)\, dx. \tag{4.3}$$

For the term T_2, we easily see that

$$\left| T_2 - \int_0^T \int_\Omega \rho_{\mathcal{T},\delta t} c \phi \, dx\, dt \right| \leqslant C_{\phi,\rho_{\max}} \|c\|_{L^1} (\delta t + h_{\mathcal{T}}),$$

so that

$$T_2 \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} \int_0^T \int_\Omega \rho c \phi \, dx\, dt. \tag{4.4}$$

Let us now concentrate on the term T_3 in (4.2). We first write $v^{n+}_{K\sigma} = v^n_{K\sigma} + v^{n-}_{K\sigma}$ so that we get

$$T_3 = \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} |K| (\operatorname{div} v)^n_K \rho^{n+1}_K \phi^n_K}_{=T_{31}}$$
$$+ \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}}} |\sigma| v^{n-}_{K\sigma} (\rho^{n+1}_K - \rho^{n+1}_L)\phi^n_K + \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\text{bd}}} |\sigma| v^n_{K\sigma} (\rho^{n+1}_\sigma - \rho^{n+1}_K)\phi^n_K}_{=T_{32}}$$

The term T_{31} can be treated in the very same way as the term T_2, let us concentrate on the term T_{32}. Reordering the summation on the interior edges by using the conservativity property $v^n_{KL} = v^n_{K\sigma} = -v^n_{L\sigma}$, we get

$$T_{32} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} |\sigma| \left(v^{n-}_{KL} (\rho^{n+1}_K - \rho^{n+1}_L)\phi^n_K + v^{n+}_{KL} (\rho^{n+1}_L - \rho^{n+1}_K)\phi^n_L \right)$$
$$+ \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\text{bd}}} |\sigma| v^n_{K\sigma} (\rho^{n+1}_\sigma - \rho^{n+1}_K)\phi^n_K.$$

We will now stress on the fact that, the test function ϕ and the solution $\rho_{\mathcal{T},\delta t}$ of the finite volume scheme being fixed, the term T_{32} actually depends on the velocity field v, so that we shall in fact denote this term by $T_{32}(v)$.

The behavior of $T_{32}(v)$ when δt and $h_{\mathcal{T}}$ go to 0 is then given by the following lemma and its corollary. The main difficulty comes from the fact that the sign of the normal part of the advection field may change along some edges leading to difficulties to justify that one can take the limit in those terms. If the advection field is smooth, this difficulty is handled by using the fact that, on such edges, $v \cdot \boldsymbol{\nu}$ is necessarily small. As a consequence, we prove the general result by approximating v by a smooth field in a way which is uniform with respect to the approximation parameters.

LEMMA 4.1 Let $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$.

1. For any $w \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ we have

$$|T_{32}(v) - T_{32}(w)| \leqslant C_{\phi,\rho_{\max}} \|v - w\|_{L^1(W^{1,1})}.$$

2. For any $w \in (\mathscr{C}^\infty([0,T]\times\overline{\Omega}))^d$, there exists $R_{32}(v,w)$ (depending on $v, w, \mathscr{T}, \delta t, \rho^{\mathscr{T}}, \phi$) such that we have

$$T_{32}(w) + R_{32}(v,w) \xrightarrow[(\delta t, h_{\mathscr{T}})\to 0]{} -\int_0^T\int_\Omega \rho \operatorname{div}(\phi w)\,dx\,dt + \int_0^T\int_\Gamma g\phi(w\cdot \boldsymbol{v})\,dx\,dt, \tag{4.5}$$

and

$$|R_{32}(v,w)| \leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \|v-w\|_{L^1(W^{1,1})}. \tag{4.6}$$

By density of $(\mathscr{C}^\infty([0,T]\times\overline{\Omega}))^d$ in $L^1(]0,T[,(W^{1,1}(\Omega))^d)$, the convergence result for the term T_{32} that we need follows.

COROLLARY 4.1 Assuming only that $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, we have the following convergence

$$T_{32}(v) \xrightarrow[(\delta t, h_{\mathscr{T}})\to 0]{} -\int_0^T\int_\Omega \rho \operatorname{div}(\phi v)\,dx\,dt + \int_0^T\int_\Gamma g\phi(v\cdot \boldsymbol{v})\,dx\,dt.$$

The proof of the corollary is left to the reader. Let us now prove the lemma.
Proof of Lemma 4.1.

1. For any $\sigma \in \mathscr{E}$ and any $n \in [\![0, N-1]\!]$ we define an interface value ϕ_σ^n for ϕ, as follows

$$\phi_\sigma^n = \phi(t^n, x_\sigma),$$

where x_σ is an arbitrary point in σ. We can then get

$$T_{32}(w) = \underbrace{\sum_{n=0}^{N-1}\delta t \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma| w_{KL}^n(\rho_L^{n+1}-\rho_K^{n+1})\phi_\sigma^n + \sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}} |\sigma| w_{K\sigma}^n(\rho_\sigma^{n+1}-\rho_K^{n+1})\phi_K^n}_{=T'_{32}(w)}$$
$$+ \underbrace{\sum_{n=0}^{N-1}\delta t \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|\left(w_{KL}^{n-}(\rho_K^{n+1}-\rho_L^{n+1})(\phi_K^n-\phi_\sigma^n) + w_{KL}^{n+}(\rho_L^{n+1}-\rho_K^{n+1})(\phi_L^n-\phi_\sigma^n)\right)}_{=R_1(w)}.$$

The term $T'_{32}(w)$ can be written as follows (paying attention to the fact that only interior edges are taken into account in the first sum)

$$T'_{32}(w) = \underbrace{-\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}\rho_K^{n+1}\left(\sum_{\sigma\in\mathscr{E}_K}|\sigma| w_{K\sigma}^n\phi_\sigma^n\right)}_{=T_{321}(w)}$$
$$+ \underbrace{\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}|\sigma| w_{K\sigma}^n\rho_K^{n+1}\phi_\sigma^n + \sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}|\sigma| w_{K\sigma}^n(\rho_\sigma^{n+1}-\rho_K^{n+1})\phi_K^n}_{=T'_{321}(w)}.$$

For the boundary terms $T'_{321}(w)$, we see that we can write

$$T'_{321}(w) = \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}} |\sigma| w^n_{K\sigma} \rho^{n+1}_\sigma \phi^n_\sigma}_{=T_{322}(w)} + \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}} |\sigma| w^n_{K\sigma} (\rho^{n+1}_\sigma - \rho^{n+1}_K)(\phi^n_K - \phi^n_\sigma)}_{=R_2(w)}.$$

Finally the term $T_{32}(w)$ we are studying is written

$$T_{32}(w) = T_{321}(w) + T_{322}(w) + R_1(w) + R_2(w),$$

and we shall analyse each term separately as follows. Notice that each term is linear with respect to the vector field w except $R_1(w)$.

- For each n and K, by definition of ϕ^n_K and ϕ^n_σ, and using (2.1) and (2.2), we have

$$\begin{aligned}\left|\sum_{\sigma\in\mathscr{E}_K} |\sigma| w^n_{K\sigma} \phi^n_\sigma\right| &\leqslant \left|\sum_{\sigma\in\mathscr{E}_K} |\sigma| w^n_{K\sigma} \underbrace{(\phi^n_\sigma - \phi^n_K)}_{|.|\leqslant C_\phi d_K}\right| + \left|\sum_{\sigma\in\mathscr{E}_K} |\sigma| w^n_{K\sigma} \phi^n_K\right| \\ &\leqslant C_\phi d_K \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \int_{\partial K} |w\cdot \boldsymbol{\nu}_K| \, dx\, dt + |\phi^n_K||K||(\operatorname{div} w)^n_K| \\ &\leqslant C_{\phi,\mathrm{reg}(\mathscr{T})} d_K \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \frac{1}{d_K} \|w(t)\|_{W^{1,1}(K)} \, dt + C_\phi \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \|\operatorname{div} w(t)\|_{L^1(K)} \, dt,\end{aligned}$$

  so that, multiplying by δt and summing over n and K lead to the estimate

$$|T_{321}(w)| \leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \|w\|_{L^1(W^{1,1})}.$$

- We easily find that

$$|T_{322}(w)| \leqslant \rho_{\max} \|\phi\|_{L^\infty} \int_0^T \int_\Gamma |w\cdot\boldsymbol{\nu}| \, dx\, dt \leqslant C_{\phi,\rho_{\max}} \|w\|_{L^1(W^{1,1})}.$$

- Since $|\phi^n_K - \phi^n_\sigma| \leqslant C_\phi d_K$ and thanks to the L^∞ bound (3.8) on the approximate solution $\rho_{\mathscr{T},\delta t}$ we have (by using the fact that the maps $s\in\mathbb{R}\mapsto s^\pm$ are Lipschitz-continuous),

$$\begin{aligned}|R_1(w) - R_1(v)| &\leqslant \|\rho_{\mathscr{T},\delta t}\|_{L^\infty} C_\phi \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathscr{E}_{\mathrm{int}}} |\sigma| |v^n_{KL} - w^n_{KL}| (d_K + d_L) \\ &\leqslant C_{\phi,\rho_{\max}} \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathscr{T}} \left(\sum_{\sigma\in\mathscr{E}_K} |\sigma| d_K |v^n_{K\sigma} - w^n_{K\sigma}|\right) \\ &\leqslant C_{\phi,\rho_{\max}} \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathscr{T}} \frac{1}{\delta t} d_K \int_{t^n}^{t^{n+1}} \|v(t) - w(t)\|_{L^1(\partial K)} \, dt \\ &\leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \|v - w\|_{L^1(W^{1,1})}.\end{aligned}$$

- The bound on $R_2(w)$ is obtained in a similar way as the one for $T_{322}(w)$.

Collecting all the above estimates, and using the linearity of T_{321}, T_{322} and R_2, the first claim of the lemma is proven.

2. In the second part of this lemma, we consider a smooth vector field w and, as we did before, we split the term $T_{32}(w)$ into the same formal four parts $T_{321}(w)+T_{322}(w)+R_1(w)+R_2(w)$, except that we change the definition of the interface values ϕ_σ^n of ϕ. For a given $n \in [\![0,N-1]\!]$ and $\sigma \in \mathscr{E}$, two cases have to be considered:

   - If $(w \cdot \mathbf{v}_{KL})$ has a constant sign on $[t^n,t^{n+1}] \times \sigma$ then, by the mean-value theorem (recall that ϕ and w are smooth) we find that there exists some point $(\xi_\sigma^n, x_\sigma^n) \in [t^n,t^{n+1}] \times \sigma$ such that
   $$\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w \cdot \mathbf{v}_{KL})\,dx\,dt = \phi(\xi_\sigma^n, x_\sigma^n)\left(\int_{t^n}^{t^{n+1}}\int_\sigma (w \cdot \mathbf{v}_{KL})\,dx\,dt\right), \tag{4.7}$$
   and we then define
   $$\phi_\sigma^n = \phi(\xi_\sigma^n, x_\sigma^n).$$
   - If $(w \cdot \mathbf{v}_\sigma)$ has not a constant sign on $[t^n,t^{n+1}] \times \sigma$, then we choose
   $$\phi_\sigma^n = \phi(t^n, x_\sigma),$$
   for some arbitrary point $x_\sigma \in \sigma$. Notice, in that case, that $(w \cdot \mathbf{v}_{KL})$ necessarily vanishes at some point in $[t^n,t^{n+1}] \times \sigma$ and then we have
   $$|w(t,x) \cdot \mathbf{v}_{KL}| \leqslant \mathscr{L}(w)(\delta t + d_\sigma), \;\; \forall (t,x) \in [t^n,t^{n+1}] \times \sigma.$$
   It follows that
   $$\left|\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w \cdot \mathbf{v}_{KL})\,dx\,dt - \phi_\sigma^n\left(\int_{t^n}^{t^{n+1}}\int_\sigma (w \cdot \mathbf{v}_{KL})\,dx\,dt\right)\right| \leqslant C_{\phi,w}\delta t|\sigma|(\delta t^2 + d_\sigma^2). \tag{4.8}$$

With this particular choice of the interface values for ϕ we can now study all the terms T_{321}, T_{322}, R_1 and R_2 as follows.

   - By (4.7) and (4.8), for any $\sigma \in \mathscr{E}, n \in [\![0,N-1]\!]$, we have
   $$w_{K\sigma}^n \phi_\sigma^n = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w \cdot \mathbf{v}_{K\sigma})\,dx\,dt + O_{\phi,w}(\delta t^2 + d_\sigma^2),$$
   the last term being exactly 0 if the sign of $w \cdot \mathbf{v}_{K\sigma}$ is constant. It follows, by (4.1), that
   $$\begin{aligned} T_{321}(w) = &-\sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}}\rho_K^{n+1}\int_{t^n}^{t^{n+1}}\underbrace{\left(\sum_{\sigma\in\mathscr{E}_K}\int_\sigma \phi(w \cdot \mathbf{v}_{K\sigma})\,dx\right)}_{=\int_K \operatorname{div}(\phi w)\,dx} dt \\ &+ (1+\mathrm{reg}_{\max})\left(\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathscr{T}}\rho_K^{n+1}\underbrace{\sum_{\sigma\in\mathscr{E}_K}|\sigma|d_K}_{\leqslant \mathrm{reg}(\mathscr{T})|K|}\right)O(\delta t + h_{\mathscr{T}}) \\ = &-\int_0^T\int_\Omega \rho_{\mathscr{T},\delta t}\operatorname{div}(\phi w)\,dx\,dt + O_{\mathrm{reg}_{\max},w,\phi}(\delta t + h_{\mathscr{T}}). \end{aligned}$$

Since w and ϕ are fixed, we finally deduce, using the weak-$\star$ convergence of $\rho_{\mathscr{T},\delta t}$, that

$$T_{321}(w) \xrightarrow[(\delta t, h_{\mathscr{T}})\to 0]{} -\int_0^T \int_\Omega \rho \,\mathrm{div}\,(\phi w)\,dx\,dt.$$

- The term $T_{322}(w)$ can be treated in the very same way (in fact this term is even easier) since we can write

$$T_{322}(w) = \int_0^T \int_\Gamma (\gamma \rho_{\mathscr{T},\delta t})\phi(w\cdot \mathbf{v})\,dx\,dt + O(h_{\mathscr{T}}(\delta t + h_{\mathscr{T}})),$$

and then by weak-$\star$ convergence of the trace $\gamma\rho_{\mathscr{T},\delta t}$ we get

$$T_{322}(w) \xrightarrow[(\delta t, h_{\mathscr{T}})\to 0]{} \int_0^T \int_\Gamma g\phi(w\cdot \mathbf{v})\,dx\,dt.$$

- The term $R_1(w)$: Let us write

$$R_1(w) = R_1(v) + \big(R_1(w) - R_1(v)\big).$$

  - We use the weak $L^2(H^1)$ estimate (3.9) and the regularity assumption (4.1) in order to bound $R_1(v)$

$$\begin{aligned}
|R_1(v)| &= \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma||v^n_{KL}||\rho^{n+1}_K - \rho^{n+1}_L|\,(|\phi^n_K - \phi^n_\sigma| + |\phi^n_L - \phi^n_\sigma|)\\
&\leqslant C_{\rho_{\max}} \underbrace{\left(\sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma||v^n_{KL}||\rho^{n+1}_K - \rho^{n+1}_L|^2\right)^{\frac{1}{2}}}_{\leqslant C \text{ by (3.9)}} \left(\sum_{n=0}^{N-1} \delta t \sum_{\substack{K\in\mathscr{T}\\ \sigma\in\mathscr{E}_K}} |\sigma||v^n_{K\sigma}|\underbrace{|\phi^n_K - \phi^n_\sigma|^2}_{\leqslant C_\phi(\delta t + d_K)^2}\right)^{\frac{1}{2}}\\
&\leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \left(\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathscr{T}} \frac{1}{d_K}\|v\|_{W^{1,1}(K)}(\delta t + d_K)^2\right)^{\frac{1}{2}}\\
&\leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \|v\|^{\frac{1}{2}}_{L^1(W^{1,1})}\sqrt{h_{\mathscr{T}}}.
\end{aligned}$$

    Hence, this term tends to zero when $(\delta t, h_{\mathscr{T}}) \to 0$.
  - The term $R_1(w) - R_1(v)$ is bounded as in the first part of the Lemma (even though the choice of ϕ^n_σ is different here), by using the L^∞ bound on $\rho_{\mathscr{T},\delta t}$

$$\begin{aligned}
|R_1(w) - R_1(v)| &\leqslant 2\rho_{\max} \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathscr{T}} \sum_{\sigma\in\mathscr{E}_K} |\sigma||w^n_{K\sigma} - v^n_{K\sigma}|\underbrace{|\phi^n_K - \phi^n_\sigma|}_{\leqslant C_\phi(\delta t + d_K)}\\
&\leqslant C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \sum_{n=0}^{N-1} \sum_{K\in\mathscr{T}} \int_{t^n}^{t^{n+1}} \|v(t) - w(t)\|_{W^{1,1}(K)}\,dt.
\end{aligned}$$

    This term is uniformly controlled by $C\|v - w\|_{L^1(W^{1,1})}$.

- The term $R_2(w)$: We easily find that

$$|R_2(w)| \leqslant 2\|\rho_{\mathcal{T},\delta t}\|_{L^\infty} C_\phi \|w\|_{L^1(]0,T[\times\Gamma)}(\delta t + h_{\mathcal{T}}).$$

As a consequence of the previous estimates, we see that properties (4.5) and (4.6) hold with $R_{32}(v,w) = R_1(v) - R_1(w)$.

□

CONCLUSION. We may now conclude the proof of Theorem 4.1. Indeed, gathering (4.3), (4.4) and the result of Corollary 4.1 we see that the weak limits (ρ, g) solve the weak formulation of the problem and satisfy $g = \rho^{in}$, $\mathrm{d}\mu_v^-$-almost everywhere. According to the uniqueness of such a couple in $L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, |\mathrm{d}\mu_v|)$, the proof is complete. □

5. Strong convergence result

We want now to show that the approximate solution actually strongly converges towards the weak solution of the problem. We emphasize the fact that the convergence we obtain in Theorem 5.1 is uniform with respect to time.

5.1 *An improved Friedrichs-type result*

We first need to adapt a little the classical convolution argument (Friedrichs' lemma) used in the renormalized solution theory DiPerna & Lions (1989); Blouza & Le Dret (2001); Boyer (2005), then we will state and prove our main result.

LEMMA 5.1 Assume that (1.2), (1.3) and (1.4a) hold.
For any $\varepsilon > 0$, there exists a function $\rho^\varepsilon \in W^{1,\infty}(]0,T[\times\Omega)$ satisfying the following properties:

- We have $\|\rho^\varepsilon\|_{L^\infty(]0,T[\times\Omega)} \leqslant \|\rho\|_{L^\infty(]0,T[\times\Omega)}$.
- For any $p < +\infty$, $(\rho^\varepsilon)_\varepsilon$ converges towards ρ in $\mathscr{C}^0([0,T], L^p(\Omega))$ and there exists $C > 0$, which does not depend on ε such that

$$\|\rho^\varepsilon\|_{\mathscr{C}^0([0,T],L^p(\Omega))} \leqslant C\|\rho\|_{L^\infty(]0,T[\times\Omega)}.$$

- For any $p < +\infty$, the traces $(\gamma\rho^\varepsilon)_\varepsilon$ (in the usual sense since ρ^ε is continuous up to the boundary) converge towards $\gamma\rho$ in $L^p(]0,T[\times\Gamma, |\mathrm{d}\mu_v|)$.
- The following equation is satisfied in the distribution sense

$$\partial_t \rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon v) + c\rho^\varepsilon = R^\varepsilon, \text{ in }]0,T[\times\Omega, \tag{5.1}$$

for some $R^\varepsilon \in L^1(]0,T[\times\Omega)$ satisfying

$$\|R^\varepsilon\|_{L^1(]0,T[\times\Omega)} \xrightarrow[\varepsilon\to 0]{} 0.$$

Notice that $\|\rho^\varepsilon\|_{W^{1,\infty}}$ blows up when $\varepsilon \to 0$.

The complete proof is proposed in Boyer & Fabrie (2011) so that we only give here a brief sketch. The usual Friedrichs commutator Lemma, adapted to the case of a non tangential vector field on the boundary in Blouza & Le Dret (2001); Boyer (2005) leads to a family of functions, say $\tilde{\rho}^\varepsilon$, satisfying:

- For any $\varepsilon > 0$, $\tilde{\rho}^\varepsilon \in L^\infty(]0,T[\times W^{1,\infty}(\Omega))$.
- $(\tilde{\rho}^\varepsilon)_\varepsilon$ converges to ρ in $\mathscr{C}^0([0,T],L^p(\Omega))$ for any $p < +\infty$.
- The traces $\gamma\tilde{\rho}^\varepsilon$ converges to $\gamma\rho$ in $L^p(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)$ for any $p < +\infty$.
- There exists $\tilde{R}^\varepsilon \in L^1(]0,T[\times\Omega)$, such that $\tilde{\rho}^\varepsilon$ solves

$$\partial_t\tilde{\rho}^\varepsilon + \mathrm{div}\,(\tilde{\rho}^\varepsilon v) + c\tilde{\rho}^\varepsilon = \tilde{R}^\varepsilon,$$

  with $\|\tilde{R}^\varepsilon\|_{L^1(]0,T[\times\Omega)} \xrightarrow[\varepsilon\to 0]{} 0$.

This family of approximations suffers from a weak regularity in time. The result of the Lemma will thus follows by considering a suitable mollifying procedure in the time variable, in the very spirit of the original Friedrichs lemma.

5.2 *The uniform in time convergence result*

Our main result of this section is the following uniform in time strong convergence result.

THEOREM 5.1 Assume that (1.2), (1.3) and (1.4) hold.

Let $\mathrm{reg}_{\max} > 0$ be given, and consider a family of meshes and time steps, such that $(\delta t, h_{\mathscr{T}}) \to 0$ and satisfying (4.1).

We have the following convergences

$$\|\rho_{\mathscr{T},\delta t} - \rho\|_{L^\infty(]0,T[,L^p(\Omega))} \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0,\ \ \forall p < +\infty,$$

$$\|\gamma\rho_{\mathscr{T},\delta t} - \gamma\rho\|_{L^p(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)} \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0,\ \ \forall p < +\infty.$$

REMARK 5.1

- The previous theorem implies, in particular, that

$$\rho^N_{\mathscr{T}} \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} \rho(T),\ \text{ in } L^p(\Omega),\ \forall p < +\infty. \tag{5.2}$$

- In fact, we can construct an approximate solution which is continuous in time by the formula

$$\tilde{\rho}_{\mathscr{T},\delta t}(t,x) = \sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}} \frac{(t^{n+1}-t)\rho_K^n + (t-t^n)\rho_K^{n+1}}{\delta t} \mathbb{1}_{]t^n,t^{n+1}[\times K}.$$

  Since $\rho \in \mathscr{C}^0([0,T],L^p(\Omega))$ for any $p < +\infty$, the previous theorem implies that

$$\|\tilde{\rho}_{\mathscr{T},\delta t} - \rho\|_{\mathscr{C}^0([0,T],L^p(\Omega))} \xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0,\ \ \forall p < +\infty,$$

In order to simplify the presentation, we will assume in the following proof that

$$c + \frac{1}{2}\operatorname{div} v \geqslant 0, \text{ almost everywhere in }]0,T[\times\Omega. \tag{5.3}$$

The general case can be proved by a change of variables similar to the one we used in section 3.1 by using both assumptions (1.4).

Proof. Notice first that, thanks to the L^∞ bounds (1.6) and (3.8), it is enough to prove the result for $p = 2$.

We now consider the discretisation of the family of approximations given by Lemma 5.1 and defined by

$$\rho_K^{\varepsilon,n} = \rho^\varepsilon(t^n, x_K), \forall K \in \mathscr{T}, \forall n \in [\![0,N]\!], \text{ and } \rho^\varepsilon_{\mathscr{T},\delta t} = \sum_{n=0}^{N-1}\sum_{K\in\mathscr{T}} \rho_K^{\varepsilon,n+1} \mathbb{1}_{]t^n,t^{n+1}[\times K} \in L^\infty(]0,T[\times\Omega).$$

By the triangle inequality, we get

$$\begin{aligned}\|\rho_{\mathscr{T},\delta t} - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} \leqslant \|\rho_{\mathscr{T},\delta t} - \rho^\varepsilon_{\mathscr{T},\delta t}\|_{L^\infty(]0,T[,L^2(\Omega))} \\ + \|\rho^\varepsilon_{\mathscr{T},\delta t} - \rho^\varepsilon\|_{L^\infty(]0,T[,L^2(\Omega))} + \|\rho^\varepsilon - \rho\|_{L^\infty(]0,T[,L^2(\Omega))}.\end{aligned} \tag{5.4}$$

By Lemma 5.1, the third term is known to converge to 0 when $\varepsilon \to 0$ and, since ρ^ε is smooth, the second term can be bounded as follows: for every $t \in]0,T[$, such that $t \in]t^n, t^{n+1}[$ for some n,

$$\|\rho^\varepsilon_{\mathscr{T},\delta t}(t) - \rho^\varepsilon(t)\|^2_{L^2(\Omega))} \leqslant \frac{1}{\delta t}\sum_{K\in\mathscr{T}}\int_{t^n}^{t^{n+1}}\int_K |\rho^\varepsilon(t^{n+1},x_K) - \rho^\varepsilon(t,x)|^2\,dx \leqslant C(\delta t^2 + h^2_{\mathscr{T}})\mathscr{L}(\rho^\varepsilon)^2. \tag{5.5}$$

Most of the sequel of the proof will consist in estimating the first term in (5.4). To this end, we define approximate edge values of ρ^ε by

$$\rho_\sigma^{\varepsilon,n} = \begin{cases} (\rho_K^{\varepsilon,n} + \rho_L^{\varepsilon,n})/2, & \forall \sigma = K|L \in \mathscr{E}_{\mathrm{int}}, \\ \rho_\sigma^{in,n}, & \forall \sigma \in \mathscr{E}_{\mathrm{bd}}, \text{s.t. } v^n_{K\sigma} \leqslant 0, \\ \rho_K^{\varepsilon,n}, & \forall \sigma \in \mathscr{E}_{\mathrm{bd}}, \text{s.t. } v^n_{K\sigma} > 0. \end{cases}$$

For any $\sigma \in \mathscr{E}$, let us choose an arbitrary point $x_\sigma \in \sigma$. By integrating (5.1) over $[t^n, t^{n+1}] \times K$ and putting the result under same form as in (3.12), we obtain

$$\begin{aligned}|K|\frac{\rho_K^{\varepsilon,n+1} - \rho_K^{\varepsilon,n}}{\delta t} + \sum_{\sigma\in\mathscr{E}_K} |\sigma| v^n_{K\sigma}\rho_\sigma^{\varepsilon,n+1} + |K|c^n_K\rho_K^{\varepsilon,n+1} - \sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}} |\sigma||v^n_{K\sigma}|\frac{\rho_L^{\varepsilon,n+1} - \rho_K^{\varepsilon,n+1}}{2} \\ = |K|(\delta_K^{\varepsilon,n+1} - \delta_K^{\varepsilon,n}) + |K|R_K^{\varepsilon,n} + \sum_{\sigma\in\mathscr{E}_K}|\sigma|\delta^{\varepsilon,n}_{K\sigma} + \sum_{\sigma\in\mathscr{E}_K}|\sigma|\gamma^{\varepsilon,n}_{K\sigma}, \;\; \forall K \in \mathscr{T}, \forall n \in [\![0, N-1]\!],\end{aligned} \tag{5.6}$$

where the remainder terms are defined by

$$\delta_K^{\varepsilon,n} = \frac{1}{|K|}\int_K (\rho^\varepsilon(t^n, x_K) - \rho^\varepsilon(t^n, x))\,dx,$$

$$\delta^{\varepsilon,n}_{K\sigma} = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\nu_{K\sigma})(\rho^\varepsilon(t^{n+1}, x_\sigma) - \rho^\varepsilon(t,x))\,dx\,dt,$$

$$\gamma_{K\sigma}^{\varepsilon,n} = \begin{cases} -|v_{K\sigma}^n| \dfrac{\rho_L^{\varepsilon,n+1} - \rho_K^{\varepsilon,n+1}}{2} + v_{K\sigma}^n \left(\rho_\sigma^{\varepsilon,n+1} - \rho^\varepsilon(t^{n+1}, x_\sigma)\right), & \text{for } \sigma = K|L \in \mathscr{E}_{\text{int}} \\ v_{K\sigma}^n \left(\rho_\sigma^{\varepsilon,n+1} - \rho^\varepsilon(t^{n+1}, x_\sigma)\right), & \text{for } \sigma \in \mathscr{E}_{\text{bd}} \end{cases}$$

$$R_K^{\varepsilon,n} = \frac{1}{\delta t |K|} \int_{t^n}^{t^{n+1}} \int_K R^\varepsilon \, dx\, dt + \frac{1}{\delta t |K|} \int_{t^n}^{t^{n+1}} \int_K c(t,x)(\rho^\varepsilon(t^n, x_K) - \rho^\varepsilon(t,x))\, dx\, dt.$$

Note that for any interior edge $\sigma = K|L$ we have the local conservativity properties $\gamma_{K\sigma}^{\varepsilon,n} + \gamma_{L\sigma}^{\varepsilon,n} = 0$ and $\delta_{K\sigma}^{\varepsilon,n} + \delta_{L\sigma}^{\varepsilon,n} = 0$, which are useful to perform the computations which follow. For any $n \in [\![0,N]\!]$, we introduce now the error term

$$E_K^{\varepsilon,n} = \rho_K^{\varepsilon,n} - \rho_K^n, \ \forall K \in \mathscr{T}, \ \text{ and } \ E_{\mathscr{T}}^{\varepsilon,n} = \sum_{K \in \mathscr{T}} E_K^{\varepsilon,n} \mathbb{1}_K,$$

so that the quantity we want to bound reads

$$\|\rho_{\mathscr{T},\delta t}^\varepsilon - \rho_{\mathscr{T},\delta t}\|_{L^\infty(]0,T[,L^2(\Omega))} = \sup_{0 \leqslant n \leqslant N} \|E_{\mathscr{T}}^{\varepsilon,n}\|_{L^2(\Omega)} = \|E_{\mathscr{T}}^{\varepsilon,\tilde{N}}\|_{L^2(\Omega)}, \tag{5.7}$$

for some $\tilde{N} \in [\![0,N]\!]$. Note that, we have the following elementary bounds

$$\sup_{0 \leqslant n \leqslant N} \|E_{\mathscr{T}}^{\varepsilon,n}\|_{L^\infty(\Omega)} \leqslant 2\rho_{\max}, \tag{5.8}$$

$$\|E_{\mathscr{T}}^{\varepsilon,0}\|_{L^2(\Omega)} \leqslant C h_{\mathscr{T}} \mathscr{L}(\rho^\varepsilon) + C\|\rho^\varepsilon(0,.) - \rho^0\|_{L^2(\Omega)}. \tag{5.9}$$

We subtract (5.6) and (3.12) then we multiply the result by $E_K^{\varepsilon,n+1}$ and we sum over $n = 0, ..., \tilde{N}-1$ and $K \in \mathscr{T}$. Then we do exactly the same computations than the ones we used to obtain (3.13) and we get

$$\begin{aligned}
&\frac{1}{2}\|E_{\mathscr{T}}^{\varepsilon,\tilde{N}}\|_{L^2}^2 + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \delta t \|E_{\mathscr{T}}^{\varepsilon,n+1} - E_{\mathscr{T}}^{\varepsilon,n}\|_{L^2}^2 + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{K \in \mathscr{T}} |K| \left(c_K^n + \frac{1}{2}(\operatorname{div} v)_K^n\right) |E_K^{\varepsilon,n+1}|^2 \\
&\qquad + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\text{bd}}} |\sigma||v_\sigma^n|(E_K^{\varepsilon,n+1})^2 + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\substack{\sigma \in \mathscr{E}_{\text{int}} \\ \sigma = K|L}} |\sigma||v_{KL}^n|(E_L^{\varepsilon,n+1} - E_K^{\varepsilon,n+1})^2 \\
&= \frac{1}{2}\|E_{\mathscr{T}}^{\varepsilon,0}\|_{L^2}^2 + \sum_{K \in \mathscr{T}} |K| \delta_K^{\varepsilon,\tilde{N}} E_K^{\varepsilon,\tilde{N}} + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{K \in \mathscr{T}} |K| \delta_K^{\varepsilon,n}(E_K^{\varepsilon,n} - E_K^{\varepsilon,n+1}) - \sum_{K \in \mathscr{T}} |K| \delta_K^{\varepsilon,0} E_K^{\varepsilon,0} \\
&+ \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{K \in \mathscr{T}} |K| R_K^{\varepsilon,n} E_K^{\varepsilon,n+1} + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\substack{\sigma \in \mathscr{E}_{\text{int}} \\ \sigma = K|L}} |\sigma| \gamma_{K\sigma}^{\varepsilon,n}(E_K^{\varepsilon,n+1} - E_L^{\varepsilon,n+1}) + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\text{bd}}} |\sigma| \gamma_{K\sigma}^{\varepsilon,n} E_K^{\varepsilon,n+1} \\
&\qquad + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\substack{\sigma \in \mathscr{E}_{\text{int}} \\ \sigma = K|L}} |\sigma| \delta_{K\sigma}^{\varepsilon,n}(E_K^{\varepsilon,n+1} - E_L^{\varepsilon,n+1}) + \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\text{bd}}} |\sigma| \delta_{K\sigma}^{\varepsilon,n} E_K^{\varepsilon,n+1}.
\end{aligned} \tag{5.10}$$

We denote by $I_{\mathscr{T},\delta t}^\varepsilon$ the left-hand side of this inequality (whose all terms are non-negative thanks to assumption (5.3)) and we have to estimate all the terms $T_i, i = 1, \ldots, 9$ in the right-hand side. We want to point out the fact that numerical diffusion terms (fourth and fifth) in the definition of $I_{\mathscr{T},\delta t}^\varepsilon$ will be of major importance in the following estimates.

- Term T_1: By (5.8) and (5.9) we get

$$|T_1| \leqslant C_{\rho_{\max}} \|E_{\mathscr{T}}^{\varepsilon,0}\|_{L^2(\Omega)} \leqslant C_{\rho_{\max}} \left(\mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} + \|\rho^\varepsilon(0,.) - \rho^0\|_{L^2(\Omega)} \right)$$
$$\leqslant C_{\rho_{\max}} \left(\mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} + \|\rho^\varepsilon - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} \right).$$

- Term T_2: We use the Cauchy-Schwarz inequality and the definition of $\delta_K^{\varepsilon,n}$

$$|T_2| \leqslant \|E_{\mathscr{T}}^{\varepsilon,\tilde{N}}\|_{L^2} \left(\sum_{K\in\mathscr{T}} |K||\delta_K^{\varepsilon,\tilde{N}}|^2 \right)^{\frac{1}{2}} \leqslant \mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} |\Omega| \|E_{\mathscr{T}}^{\varepsilon,\tilde{N}}\|_{L^2} \leqslant C\mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} \sqrt{I_{\mathscr{T},\delta t}^{\varepsilon}}.$$

- Term T_3: By using similar arguments as for T_2 we get

$$|T_3| \leqslant \left(\sum_{n=0}^{\tilde{N}-1} \delta t \|E_{\mathscr{T}}^{\varepsilon,n+1} - E_{\mathscr{T}}^{\varepsilon,n}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{K\in\mathscr{T}} |K||\delta_K^{\varepsilon,n}|^2 \right)^{\frac{1}{2}} \leqslant C\mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} \sqrt{I_{\mathscr{T},\delta t}^{\varepsilon}}.$$

- Term T_4: By (5.8) and the definition of $\delta_K^{\varepsilon,0}$ we get

$$|T_4| \leqslant \mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}} \|E_{\mathscr{T}}^{\varepsilon,0}\|_{L^2} \leqslant 2\rho_{\max} \mathscr{L}(\rho^\varepsilon) h_{\mathscr{T}}.$$

- Term T_5: By definition of $R_K^{\varepsilon,n}$ and (5.8) we have

$$|T_5| \leqslant 2\rho_{\max} \sum_{n=0}^{\tilde{N}-1} \delta t \sum_{K\in\mathscr{T}} |K||R_K^{\varepsilon,n}| \leqslant C_{\rho_{\max}} \|R^\varepsilon\|_{L^1(]0,T[\times\Omega)} + C_{\rho_{\max}} \|c\|_{L^1} \mathscr{L}(\rho^\varepsilon)(\delta t + h_{\mathscr{T}}).$$

- Term T_6: Observing that $|\gamma_{K\sigma}^{\varepsilon,n}| \leqslant (d_K + d_L)\mathscr{L}(\rho^\varepsilon)|v_{K\sigma}^n|$, since we are only concerned with interior edges, we first evaluate this term as follows

$$|T_6| \leqslant \mathscr{L}(\rho^\varepsilon) \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathscr{E}_{\mathrm{int}}} |\sigma||v_{K\sigma}^n|(d_K + d_L)|E_K^{\varepsilon,n+1} - E_L^{\varepsilon,n+1}|.$$

  We then use the Cauchy-Schwarz inequality, (2.1) and the bound (5.8), to obtain

$$|T_6| \leqslant C_{\rho_{\max}} \mathscr{L}(\rho^\varepsilon) \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma\in\mathscr{E}_{\mathrm{int}}} |\sigma||v_{K\sigma}^n||E_K^{\varepsilon,n+1} - E_L^{\varepsilon,n+1}|^2 \right)^{\frac{1}{2}}$$
$$\times \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma\in\mathscr{E}_{\mathrm{int}}} |\sigma||v_{K\sigma}^n|(d_K + d_L)^2 \right)^{\frac{1}{2}}$$
$$\leqslant C_{\rho_{\max}} \sqrt{h_{\mathscr{T}}} \mathscr{L}(\rho^\varepsilon) \sqrt{I_{\mathscr{T},\delta t}^{\varepsilon}} \left(\sum_{n=0}^{\tilde{N}-1} \sum_{K\in\mathscr{T}} |\sigma| \int_{t^n}^{t^{n+1}} d_K \|v\|_{L^1(\partial K)} \, dx\, dt \right)^{\frac{1}{2}}$$
$$\leqslant C_{\rho_{\max},\mathrm{reg}_{\max}} \sqrt{h_{\mathscr{T}}} \mathscr{L}(\rho^\varepsilon) \sqrt{I_{\mathscr{T},\delta t}^{\varepsilon}} \|v\|_{L^1(]0,T[,W^{1,1}(\Omega))}^{\frac{1}{2}}.$$

- Term T_7: For the boundary edges such that $v^n_{K\sigma} \geqslant 0$, we have $\rho^{\varepsilon,n}_\sigma = \rho^{\varepsilon,n}_K = \rho^\varepsilon(t^{n+1}, x_K)$ so that the contribution of this term can be treated in the same way as for the term T_6.

  For the boundary edges such that $v^n_{K\sigma} < 0$, the value of $\rho^{\varepsilon,n}_\sigma$ is given by the boundary data and thus we have to adapt the argument. To this end, we write

$$\gamma^{\varepsilon,n}_{K\sigma} = v^n_{K\sigma}\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma (\rho^{in}(t,x) - \rho^\varepsilon(t,x))\,dx\,dt + v^n_{K\sigma}\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma (\rho^\varepsilon(t,x) - \rho^\varepsilon(t^{n+1}, x_\sigma))\,dx\,dt.$$

  The contribution of the second part of this term can be treated just like in the term T_6, using the fact that ρ^ε is Lipschitz continuous. It remains to evaluate the following contribution

$$T'_7 = \sum_{n=0}^{\tilde N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} E^{\varepsilon,n+1}_K v^n_{K\sigma}\int_{t^n}^{t^{n+1}}\int_\sigma (\rho^{in}(t,x) - \rho^\varepsilon(t,x))\,dx\,dt.$$

  Let us introduce, for any $v, w \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ the notation

$$T''_7(v,w) = -\sum_{n=0}^{\tilde N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} w^n_{K\sigma}\int_{t^n}^{t^{n+1}}\int_\sigma |\rho^{in}(t,x) - \rho^\varepsilon(t,x)|\,dx\,dt,$$

  so that, by (5.8), the term $|T'_7|$ is bounded by $2\rho_{\max}T''_7(v,v)$, which is non-negative. Note that $T''_7(v,w)$ is linear with respect to w but nonlinear with respect to v.

  We consider a smooth vector field w, to be determined later, and we write $T''_7(v,v) = T''_7(v,w) + T''_7(v, v-w)$.

  - Since w is smooth, for any $f \in L^1(]0,T[\times\Gamma)$ we have

$$\left|\sum_{n=0}^{\tilde N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}} w^n_{K\sigma}\int_{t^n}^{t^{n+1}}\int_\sigma f\,dx\,dt - \int_0^{t^{\tilde N}}\int_\Gamma (w\cdot\mathbf{v})f\,dx\,dt\right| \leqslant \mathscr{L}(w)(\delta t + h_{\mathscr{T}})\|f\|_{L^1(]0,T[\times\Gamma)}.$$

    Applying this result to $f = \left(\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} \mathbb{1}_{]t^n,t^{n+1}[\times\sigma}\right)|\rho^{in} - \rho^\varepsilon|$ and using the L^∞ bounds on

ρ^{in} and ρ^{ε}, we then have

$$
\begin{aligned}
|T_7'(v,w)| \leqslant \mathscr{L}(w)(\delta t + h_{\mathscr{T}}) \sum_{n=0}^{\tilde N-1} \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} \int_{t^n}^{t^{n+1}} \int_\sigma \underbrace{|\rho^{in}(t,x)-\rho^{\varepsilon}(t,x)|}_{\leqslant 2\rho_{\max}}\,dx\,dt \\
+ \sum_{n=0}^{N-1} \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} \int_{t^n}^{t^{n+1}} \int_\sigma |w\cdot \boldsymbol{\nu}_\sigma||\rho^{in}(t,x)-\rho^{\varepsilon}(t,x)|\,dx\,dt \\
\leqslant 2\rho_{\max} T|\Gamma|\mathscr{L}(w)(\delta t + h_{\mathscr{T}}) + 2\rho_{\max} \underbrace{\|(v-w)\cdot\boldsymbol{\nu}\|_{L^1(]0,T[\times\Gamma)}}_{\leqslant C\|v-w\|_{L^1(W^{1,1})}} \\
+ \underbrace{\sum_{n=0}^{N-1} \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} \int_{t^n}^{t^{n+1}} \int_\sigma |v\cdot \boldsymbol{\nu}_\sigma||\rho^{in}(t,x)-\rho^{\varepsilon}(t,x)|\,dx\,dt}_{=T_7'''}.
\end{aligned}
$$

By writing $|v\cdot\boldsymbol{\nu}_\sigma| = (v\cdot\boldsymbol{\nu}_\sigma)^+ + (v\cdot\boldsymbol{\nu}_\sigma)^-$, we finally bound the last term as follows

$$
T_7''' \leqslant \int_0^T \int_\Gamma (v\cdot\boldsymbol{\nu})^-|\rho^{in}(t,x)-\rho^{\varepsilon}(t,x)|\,dx\,dt + 2\rho_{\max} \sum_{n=0}^{N-1} \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}} \int_{t^n}^{t^{n+1}} \int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dx\,dt.
$$

– The second part of $T_7''(v,v)$ is classically bounded as follows

$$
|T_7''(v,v-w)| \leqslant C_{\rho_{\max}}\|v-w\|_{L^1(W^{1,1})}.
$$

- Term T_8: this term also needs a particular care. We will also denote it by $T_8(v)$, in order to point out the linear dependence of this term with respect to v. We first estimate $\delta^{\varepsilon,n}_{K\sigma}$ as follows

$$
|\delta^{\varepsilon,n}_{K\sigma}| \leqslant (\delta t + d_\sigma)\mathscr{L}(\rho^\varepsilon)\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_{K\sigma}|\,dx\,dt.
$$

As we have already seen, the difficulty comes from the fact that the sign of $v\cdot\boldsymbol{\nu}_{K\sigma}$ may change inside $[t^n,t^{n+1}]\times\sigma$ and then we can not estimate $\delta^{\varepsilon,n}_{K\sigma}$ by using $|v^n_{K\sigma}|$.

Consider a smooth vector field $w\in\mathscr{C}^1([0,T]\times\overline{\Omega})^d$, and let us write $T_8(v) = T_8(w) + T_8(v-w)$. In the above estimate of $\delta^{\varepsilon,n}_{K\sigma}$ (with w in place of v), since w is smooth we can write

$$
|\delta^{\varepsilon,n}_{K\sigma}(w)| \leqslant (\delta t + d_\sigma)\mathscr{L}(\rho^\varepsilon)\Big(|w^n_{K\sigma}| + \mathscr{L}(w)(\delta t + d_\sigma)\Big).
$$

We recall that $d_\sigma \leqslant d_K$ for any $\sigma\in\mathscr{E}_K$ and that, by (4.1), we have $\delta t \leqslant \mathrm{reg}_{\max} d_K$ for any $K\in\mathscr{T}$.

Therefore, using that $|w^n_{K\sigma}| \leqslant |v^n_{K\sigma}| + |w^n_{K\sigma} - v^n_{K\sigma}|$, we can then write

$$
\begin{aligned}
|T_8(w)| \leqslant C_{\mathrm{reg}_{\max}} \mathscr{L}(\rho^\varepsilon) \Bigg[&\left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\mathrm{int}}} |\sigma| |v^n_{K\sigma}| (d_K + d_L) |E^{\varepsilon,n+1}_K - E^{\varepsilon,n+1}_L| \right) \\
&+ \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\mathrm{int}}} |\sigma| |v^n_{K\sigma} - w^n_{K\sigma}| (d_K + d_L) |E^{\varepsilon,n+1}_K - E^{\varepsilon,n+1}_L| \right) \\
&+ \mathscr{L}(w)(\delta t + h_{\mathscr{T}}) \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathscr{E}_{\mathrm{int}}} |\sigma| (d_K + d_L) |E^{\varepsilon,n+1}_K - E^{\varepsilon,n+1}_L| \right) \Bigg].
\end{aligned}
$$

We use the Cauchy-Schwarz inequality for the first term above and we simply use the bound (5.8) in the other terms. It follows

$$
\begin{aligned}
|T_8(w)| \leqslant C_{\mathrm{reg}_{\max}} \mathscr{L}(\rho^\varepsilon) \Bigg[&\sqrt{h_{\mathscr{T}}} \sqrt{I^\varepsilon_{\mathscr{T},\delta t}} \left(\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \sum_{K \in \mathscr{T}} d_K \|v(t)\|_{L^1(\partial K)} \, dt \right)^{\frac{1}{2}} \\
&+ \left(\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \sum_{K \in \mathscr{T}} d_K \|v(t) - w(t)\|_{L^1(\partial K)} \, dt \right) + \mathscr{L}(w)(\delta t + h_{\mathscr{T}}) T |\Omega| \Bigg],
\end{aligned}
$$

and thus

$$
|T_8(w)| \leqslant C_{\mathrm{reg}_{\max},\rho_{\max}} \mathscr{L}(\rho^\varepsilon) \left(\sqrt{h_{\mathscr{T}}} \sqrt{I^\varepsilon_{\mathscr{T},\delta t}} \|v\|^{\frac{1}{2}}_{L^1(W^{1,1})} + \|v - w\|_{L^1(W^{1,1})} + \mathscr{L}(w) h_{\mathscr{T}} \right).
$$

Finally, the term $T_8(v - w)$ can be estimated just like above by writing

$$
|T_8(v - w)| \leqslant C_{\mathrm{reg}_{\max}} \mathscr{L}(\rho^\varepsilon) \|v - w\|_{L^1(W^{1,1})}.
$$

- Term T_9: This boundary term does not present any new difficulty since, by using (5.8), we can write

$$
|T_9| \leqslant C_{\mathrm{reg}_{\max}} (\delta t + h_{\mathscr{T}}) \mathscr{L}(\rho^\varepsilon) \|v \cdot \boldsymbol{\nu}\|_{L^1(]0,T[\times\Gamma)}.
$$

Collecting all the above estimates in the inequality (5.10) and using Young's inequality, we get

$$
\begin{aligned}
I^\varepsilon_{\mathscr{T},\delta t} \leqslant C_{\mathrm{reg}_{\max},\rho_{\max}} \Bigg(&(\mathscr{L}(\rho^\varepsilon) + \mathscr{L}(w))(1 + \mathscr{L}(\rho^\varepsilon)) h_{\mathscr{T}} + (1 + \mathscr{L}(\rho^\varepsilon)) \|v - w\|_{L^1(W^{1,1})} \\
+ \|\rho^\varepsilon - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} + \|R^\varepsilon\|_{L^1} + &\int_0^T \int_\Gamma (v \cdot \boldsymbol{\nu})^- |\rho^\varepsilon - \rho^{in}| \, dx \, dt + \sum_{n=0}^{N-1} \sum_{\substack{\sigma \in \mathscr{E}_{\mathrm{bd}} \\ v^n_{K\sigma} < 0}} \int_{t^n}^{t^{n+1}} \int_\sigma (v \cdot \boldsymbol{\nu}_\sigma)^+ \, dx \, dt \Bigg).
\end{aligned}
\tag{5.11}
$$

By definition of $I^\varepsilon_{\mathscr{T},\delta t}$, we have $\|E^{\varepsilon,\tilde{N}}_{\mathscr{T}}\|^2_{L^2} \leqslant 2 I^\varepsilon_{\mathscr{T},\delta t}$, and then, by the choice (5.7) of $\tilde{N}$, we see that (5.11) gives an estimate on $\|\rho_{\mathscr{T},\delta t} - \rho^{\varepsilon}_{\mathscr{T},\delta t}\|_{L^\infty(]0,T[,L^2(\Omega))}$. Finally, with (5.5) and (5.4), we obtain

$$\|\rho_{\mathscr{T},\delta t} - \rho\|^2_{L^\infty(]0,T[,L^2(\Omega))} \leqslant C_{\mathrm{reg}_{\max},\rho_{\max}}\Bigg((\mathscr{L}(\rho^\varepsilon) + \mathscr{L}(w))\,(1+\mathscr{L}(\rho^\varepsilon))\,h_{\mathscr{T}} + (1+\mathscr{L}(\rho^\varepsilon))\|v-w\|_{L^1(W^{1,1})} + \|\rho^\varepsilon - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} + \|R^\varepsilon\|_{L^1} + \int_0^T\int_\Gamma (v\cdot\mathbf{v})^-|\rho^\varepsilon - \rho^{in}|\,dx\,dt + \sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{bd}}\\ v^n_{K\sigma}<0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\mathbf{v}_\sigma)^+\,dx\,dt\Bigg). \quad (5.12)$$

Let now $\Delta > 0$ be any small positive number. By Lemma 5.1, we first choose $\varepsilon > 0$ small enough so that

$$\|\rho^\varepsilon - \rho\|^2_{L^\infty(]0,T[,L^2(\Omega))} \leqslant \Delta,\ \ \|R^\varepsilon\|_{L^1} \leqslant \Delta, \text{ and } \|\gamma\rho^\varepsilon - \rho^{in}\|_{L^1(]0,T[\times\Gamma,\mathrm{d}\mu_v^-)} \leqslant \Delta.$$

Then, $\varepsilon > 0$ being fixed, we can find a smooth vector field w such that $(1+\mathscr{L}(\rho^\varepsilon))\|v-w\|_{L^1(0,T,W^{1,1}(\Omega))}$ is less than Δ. This vector field w being now fixed, all the other terms in (5.12) can be made smaller than Δ for $h_{\mathscr{T}}$ and δt small enough satisfying (4.1) (we use Lemma 2.1 for the last term). This proves the first strong convergence property.

It remains to prove the strong convergence of the traces, for $p = 2$ for instance. Using the triangle inequality, we first write

$$\|\gamma\rho_{\mathscr{T},\delta t} - \gamma\rho\|_{L^2(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)} \leqslant \|\gamma\rho_{\mathscr{T},\delta t} - \gamma\rho^\varepsilon_{\mathscr{T},\delta t}\|_{L^2(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)} + \|\gamma\rho^\varepsilon_{\mathscr{T},\delta t} - \gamma\rho^\varepsilon\|_{L^2(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)} + \|\gamma\rho^\varepsilon - \gamma\rho\|_{L^2(]0,T[\times\Gamma,|\mathrm{d}\mu_v|)}.$$

The third term goes to zero when $\varepsilon \to 0$ by 5.1 and the second term is bounded by $C_v\mathscr{L}(\rho^\varepsilon)(\delta t + h_{\mathscr{T}})$. It thus remain to study the first term. To this end, we write

$$\int_0^T\int_\Gamma |\gamma\rho_{\mathscr{T},\delta t} - \gamma\rho^\varepsilon_{\mathscr{T},\delta t}|^2|v\cdot\mathbf{v}|\,dx\,dt = \sum_{n=0}^{N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\mathbf{v}||E_K^{\varepsilon,n+1}|^2\,dx\,dt$$
$$\leqslant 4\rho^2_{\max}\sum_{n=0}^{N-1}\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\mathbf{v} - v^n_\sigma|\,dx\,dt + \sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathscr{E}_{\mathrm{bd}}}|\sigma||v^n_\sigma||E_K^{\varepsilon,n+1}|^2.$$

The first term in the right-hand side tends to 0 when δt and $h_{\mathscr{T}}$ tend to 0 by Lemma 2.1. The second term in the right-hand side is one of the terms in $I^\varepsilon_{\mathscr{T},\delta t}$ (the left-hand side of (5.10)) when $\tilde{N}$ is replaced by N. The above proof shows that this term can be bounded like in (5.11).

Therefore, we can conclude by the same argument as above: choosing first $\varepsilon > 0$ small enough, then choosing a smooth w close enough to v and finally δt and $h_{\mathscr{T}}$ small enough. □

6. Additional properties of the scheme

6.1 *The case of approximate data*

We deal here with the case where the data c and v are not exactly known. This situation occurs, for instance, if the transport equation is coupled with other equations involving c and v like the one listed in the introduction.

We assume that, for each mesh $\mathscr{T}$ and each time step δt, we are given approximate values

$$\left\{c_K^n,\ K\in\mathscr{T},\ n\in[\![0,N-1]\!]\right\},\ \text{ and }\ \left\{v_{K\sigma}^n,\ K\in\mathscr{T},\sigma\in\mathscr{E}_K,\ n\in[\![0,N-1]\!]\right\}.$$

We define the discrete divergence to be

$$(\operatorname{div} v)_K^n = \frac{1}{|K|}\sum_{\sigma\in\mathscr{E}_K}|\sigma| v_{K\sigma}^n,\ \ \forall K\in\mathscr{T},\forall n\in[\![0,N-1]\!].$$

By adapting the proofs given above, we can show that all the results in this paper remain valid provided that we have the following properties:

- Local conservativity :

$$v_{K\sigma}^n = -v_{L\sigma}^n,\quad \forall\sigma = K|L\in\mathscr{E}_{\text{int}},\quad \forall n\in[\![0,N]\!]. \tag{6.1}$$

- Bounds on $(c+\operatorname{div} v)^-$: We assume that there exists $\delta t_{\max}>0$, $0<\gamma<1$ and $M>0$ such that for any mesh and any $\delta t\leqslant\delta t_{\max}$, we have

$$\begin{cases}\delta t\left(\sup\limits_{K\in\mathscr{T}}\left(c_K^n+(\operatorname{div} v)_K^n\right)^-\right)\leqslant\gamma,\ \ \forall n\in[\![0,N-1]\!],\\ \displaystyle\sum_{n=0}^{N-1}\delta t\left(\sup_{K\in\mathscr{T}}\left(c_K^n+(\operatorname{div} v)_K^n\right)^-\right)\leqslant M.\end{cases}$$

  The L^∞ bound we shall obtain will then depend on γ and M.

- L^1 convergence of the reaction coefficient:

$$\sum_{n=0}^{N}\sum_{K\in\mathscr{T}} c_K^n \mathbb{1}_{]t^n,t^{n+1}[\times K}\xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} c,\ \ \text{in } L^1(]0,T[\times\Omega).$$

- L^1 convergence of the divergence:

$$\sum_{n=0}^{N}\sum_{K\in\mathscr{T}}(\operatorname{div} v)_K^n \mathbb{1}_{]t^n,t^{n+1}[\times K}\xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{}\operatorname{div} v,\ \ \text{in } L^1(]0,T[\times\Omega). \tag{6.2}$$

- L^1 convergence of the advection field:

$$\sum_{n=0}^{N}\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K} d_K\left|\delta t|\sigma| v_{K\sigma}^n - \int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_{K\sigma})\,dx\,dt\right|\xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{} 0. \tag{6.3}$$

- L^1 convergence of the normal trace of the advection field on the boundary:

$$\sum_{n=0}^{N}\sum_{\sigma\in\mathscr{E}_{\text{bd}}} v_\sigma^n \mathbb{1}_{]t^n,t^{n+1}[\times\sigma}\xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{}(v\cdot\boldsymbol{\nu}),\ \ \text{in } L^1(]0,T[\times\Gamma).$$

Assumption (6.3) is not so straightforward to interpret. It really has to be understood as a kind of convergence in $(L^1(]0,T[\times\Omega))^d$ and not in $L^1(]0,T[,(W^{1,1}(\Omega))^d)$, even though it is needed for v to belong to this last space in order for the traces of v on edges to be well defined. We will give an alternative formulation of this assumption in a particular case (yet not too far from generality).

Let us assume that each control volume K is convex and that we can choose a point $x_K \in K$ such that $r_K = \sup\{r>0, B(x_K,r)\subset K\}$ satisfies

$$\sup_{K\in\mathscr{T}} \frac{d_K}{r_K} \leqslant \mathrm{reg}_{\max}.$$

This assumption is a usual regularity assumption on the meshes we consider. We then define the so-called half-diamond cell $D_{K\sigma}$, to be the pyramid (triangle if $d=2$) based on σ and with x_K as a principal vertex. The above assumptions lead to the following properties for any $K\in\mathscr{T}$

$$K = \bigcup_{\sigma\in\mathscr{E}} D_{K\sigma}, \text{ and } |\sigma| d_K \leqslant C_{\mathrm{reg}_{\max}} |D_{K\sigma}|, \ \forall \sigma\in\mathscr{E}_K.$$

We also introduce the diamond cells $D_\sigma = D_{K\sigma}$, for $\sigma\in\mathscr{E}_{\mathrm{bd}}\cap\mathscr{E}_K$ and $D_\sigma = D_{K\sigma}\cup D_{L\sigma}$, for $\sigma = K|L \in \mathscr{E}_{\mathrm{int}}$.

With these additional notation and assumption on the meshes, we can treat the following two examples

- First of all, by using for instance Lemma 6.2 in Droniou & Eymard (2006), we get

$$\sum_{n=0}^{N}\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K} d_K|\sigma| \left| \frac{1}{|D_\sigma|}\int_{t^n}^{t^{n+1}}\int_{D_\sigma} (v\cdot \mathbf{v}_{K\sigma})\,dx\,dt - \frac{1}{|\sigma|}\int_{t^n}^{t^{n+1}}\int_{\sigma}(v\cdot\mathbf{v}_{K\sigma})\,dx\,dt\right|$$
$$\leqslant C h_{\mathscr{T}} \sum_{n=0}^{N}\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K} \frac{|\sigma| d_K}{|D_\sigma|}\int_{t^n}^{t^{n+1}}\int_{D_\sigma}|\nabla v|\,dx\,dt \leqslant C_{\mathrm{reg}_{\max}} h_{\mathscr{T}} \int_0^T\int_\Omega |\nabla v|\,dx\,dt. \quad (6.4)$$

  This proves that the choice $v_{K\sigma}^n = \frac{1}{\delta t|D_\sigma|}\int_{t^n}^{t^{n+1}}\int_{D_\sigma}(v\cdot\mathbf{v}_{K\sigma})\,dx\,dt$, satisfies the property (6.3). Furthermore, we also have

$$\sum_{n=0}^{N}\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K} |\sigma| \left| \frac{1}{|D_\sigma|}\int_{t^n}^{t^{n+1}}\int_{D_\sigma} (v\cdot \mathbf{v}_{K\sigma})\,dx\,dt - \frac{1}{|\sigma|}\int_{t^n}^{t^{n+1}}\int_{\sigma}(v\cdot\mathbf{v}_{K\sigma})\,dx\,dt\right|$$
$$\leqslant C h_{\mathscr{T}} \sum_{n=0}^{N}\sum_{K\in\mathscr{T}}\sum_{\sigma\in\mathscr{E}_K} \frac{|\sigma| d_K}{|D_\sigma|}\int_{t^n}^{t^{n+1}}\int_{D_\sigma}|\nabla v|\,dx\,dt \leqslant C_{\mathrm{reg}_{\max}} \int_0^T\int_\Omega |\nabla v|\,dx\,dt,$$

  and thus, by a density argument, we see that (6.2) also holds in that case.

- Suppose now that $v_{K\sigma}^n$ has the following form $v_{K\sigma}^n = V_\sigma^n\cdot\mathbf{v}_{K\sigma}$, for some vector $V_\sigma^n\in\mathbb{R}^d$ (notice that (6.1) is then satisfied). Then, (6.4) shows that (6.3) is satisfied as soon as the following convergence holds

$$\sum_{n=0}^{N}\sum_{\sigma\in\mathscr{E}} V_\sigma^n \mathbb{1}_{]t^n,t^{n+1}[\times D_\sigma} \xrightarrow[(\delta t, h_{\mathscr{T}})\to 0]{} v, \text{ in } (L^1(]0,T[\times\Omega))^d.$$

  The fact that (6.2) is satisfied is an additional condition which depend on the particular choice of V_σ^n.

The two examples above, show that for a given $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, the property (6.3) can be interpreted as a strong L^1 convergence property towards v for some sequence of piecewise constant functions.

REMARK 6.1 The fact that we only need L^1 convergence of the vector field and of its normal trace to obtain strong convergence of the associated weak solution of the transport problem is a well-known feature in the study of the stability of such solutions with respect to variations of the data (see for instance DiPerna & Lions (1989); De Lellis (2007); Boyer & Fabrie (2011)).

6.2 *Remark on the discrete renormalization property*

By using the above results, it is possible to prove that a suitable discrete renormalisation property holds.

THEOREM 6.1 For any $\beta : \mathbb{R} \mapsto \mathbb{R}$ which is continuous and piecewise $\mathscr{C}^1$, the approximate solution $(\rho_K^n)_{\substack{n\in[\![0,N]\!]\\ K\in\mathscr{T}}}$ satisfy the following set of equations

$$|K|\frac{\beta(\rho_K^{n+1})-\beta(\rho_K^n)}{\delta t}+\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{int}}}|\sigma|\big(v_{K\sigma}^{n+}\beta(\rho_K^{n+1})-v_{K\sigma}^{n-}\beta(\rho_L^{n+1})\big)+\sum_{\sigma\in\mathscr{E}_K\cap\mathscr{E}_{\mathrm{bd}}}|\sigma|v_{K\sigma}^n\beta(\rho_\sigma^{n+1})$$
$$+|K|c_K^n\beta'(\rho_K^{n+1})\rho_K^{n+1}+|K|(\operatorname{div} v)_K^n\big(\beta'(\rho_K^{n+1})\rho_K^{n+1}-\beta(\rho_K^{n+1})\big)=|K|R_K^{n+1},\ \forall n\in[\![0,N-1]\!],\forall K\in\mathscr{T}, \tag{6.5}$$

where the remainder term $R_{\mathscr{T},\delta t}=(R_K^{n+1})_{\substack{n\in[\![0,N-1]\!]\\ K\in\mathscr{T}}}$ satisfy

$$\|R_{\mathscr{T},\delta t}\|_{L^1(]0,T[\times\Omega)}\xrightarrow[(\delta t,h_{\mathscr{T}})\to 0]{}0.$$

Furthermore, when β is convex we have

$$R_K^{n+1}\leqslant 0,\ \ \forall K\in\mathscr{T},\forall n\in[\![0,N-1]\!].$$

REMARK 6.2 Since β is only assumed to be piecewise $\mathscr{C}^1$, it is worth to say that this result holds for any value that we choose to assign to β' at the singular points of β.

REMARK 6.3 Applying the previous result with $\beta(s)=s^2$ shows, in particular, that the numerical diffusion terms in the estimate (3.13) which leads to the weak $L^2(H^1)$ estimate (3.9) are not only bounded but in fact tend to zero when $h_{\mathscr{T}}$ and δt tend to 0.

7. Numerical experiments

Let us present in this section some numerical experiments that illustrate the actual behavior of the scheme. In order to be able to compute the exact solution and thus to estimate the accuracy of the scheme, we consider here the case $c=\operatorname{div} v=0$, that is the pure divergence-free advection case.

We consider the domain $\Omega=]0,1[^2$ and we choose a time-independent advection field belonging to $(W^{1,p}(\Omega))^2$ for any $p<4/3$ given by $v(x,y)=\begin{pmatrix}0.3\\ f(x)\end{pmatrix}$, with

$$f(x)=\begin{cases}|x-0.5|^{1/2}, & \text{for } x<0.5\\ |x-0.5|^{1/4}, & \text{for } x>0.5.\end{cases}$$

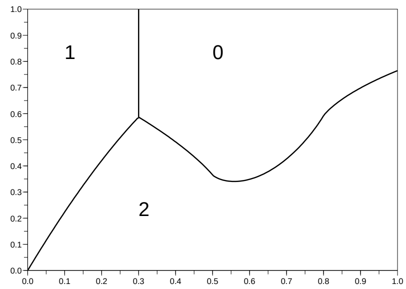


FIG. 2. The exact solution at final time $T = 1$

We consider the initial data $\rho(0,x) = 0$, the time-independent boundary data ρ^{in} is defined by $\rho^{in}(x,y) = 1$ on $\{x = 0\}$ and $\rho^{in}(x,y) = 2$ on $\{y = 0\}$.

One can easily see that the exact solution at time $T > 0$ is given by

$$\rho(t,x,y) = \begin{cases} 1, & \text{for } x < 0.3T \text{ and } y > \frac{1}{0.3}(F(x) - F(0)), \\ 2, & \text{for } x < 0.3T \text{ and } y < \frac{1}{0.3}(F(x) - F(0)), \\ 2, & \text{for } x > 0.3T \text{ and } y < \frac{1}{0.3}(F(x) - F(x-0.3T)), \\ 0, & \text{elsewhere}, \end{cases}$$

where F is a primitive of f. We show the shape of this solution at time $T = 1$ in Figure 2.

We show in Figure 3 some results for three families of meshes: uniform rectangular meshes, unstructured triangular mesh, and locally refined non conformal rectangular meshes. Each plot in this figure shows the behavior of the error at time T in the L^1 norm, as a function of the mesh size $h_{\mathscr{T}}$ (in logarithmic scale) for various values of the time step δt.

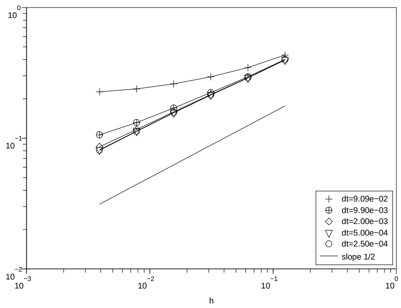
We observe in each case the convergence of the method and that the convergence rate seems to be $\frac{1}{2}$ as in the case of smooth advection fields.

8. Conclusion

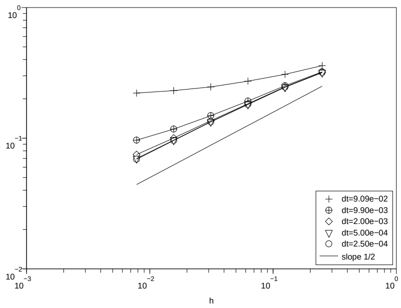
In this paper, we proposed an analysis of the (implicit) upwind finite volume scheme on general unstructured grids in any dimension for initial and boundary value problems of transport type. The framework considered is one of the weaker possible since in particular the regularity of the velocity field is only assumed to be $L^1(]0,T[,(W^{1,1}(\Omega))^d)$. In that case, Cauchy-Lipschitz theory does not apply and our analysis is then directly based on the renormalized solutions theory for the partial differential equation under study. The main result is the strong convergence in $L^\infty(]0,T[,L^p(\Omega))$, $\forall p < +\infty$ of the approximate solutions towards the unique weak solution of the problem.

We conclude by raising two open questions of interest related to this problem:

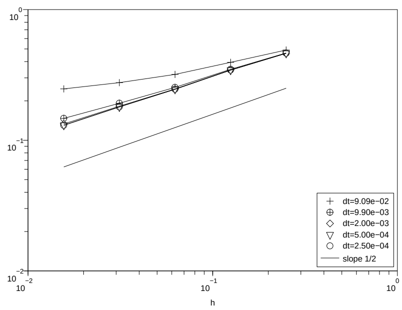
- Is it possible to prove an error estimate for the upwind finite volume method for such regularities of the data? Introducing regular Lagrangian flows (see De Lellis (2007)) instead of usual characteristic flows of the Cauchy-Lipschitz theory should be of some help. Numerical simulations suggest that the rate $\frac{1}{2}$ is still valid in this framework.



(a) Uniform rectangle mesh



(b) Unstructured triangular mesh



(c) Locally refined rectangular mesh

FIG. 3. L^1 error at final time vs. h for various time steps

- It is known (since Ambrosio (2004)), that most of the theoretical results on renormalized solutions of the transport equation remain valid in the case where the vector field lies in $L^1(]0,T[,(\mathrm{BV}(\Omega))^d)$ (and usual additional boundedness assumptions on its divergence). Does the present analysis, including boundary conditions, remains valid in this more general framework ?

Acknowledgements

The author wishes to warmly thank T. Gallouët and R. Herbin for many stimulating discussions on the topic of this paper. I am also very grateful to the anonymous referees for their very interesting comments on this paper.

REFERENCES

AMBROSIO, L. (2004) Transport equation and Cauchy problem for *BV* vector fields. *Invent. Math.*, **158**, 227–260.

BLOUZA, A. & LE DRET, H. (2001) An up-to-the-boundary version of Friedrichs's lemma and applications to the linear Koiter shell model. *SIAM J. Math. Anal.*, **33**, 877–895 (electronic).

BOUCHE, D., GHIDAGLIA, J.-M. & PASCAL, F. (2005) Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation. *SIAM J. Numer. Anal.*, **43**, 578–603 (electronic).

BOYER, F. (2005) Trace theorems and spatial continuity properties for the solutions of the transport equation. *Differential Integral Equations*, **18**, 891–934.

BOYER, F. & FABRIE, P. (2011) *Elements of analysis for the study of some models of incompressible viscous flows*. in preparation. Springer-Verlag.

CHAMPIER, S., GALLOUËT, T. & HERBIN, R. (1993) Convergence of an upstream finite volume scheme for a nonlinear hyperbolic equation on a triangular mesh. *Numer. Math.*, **66**, 139–157.

COCKBURN, B., DONG, B., GUZMÁN, J. & QIAN, J. (2010) Optimal convergence of the original DG method on special meshes for variable transport velocity. *SIAM J. Numer. Anal.*, **48**, 133–146.

COUDIÈRE, Y., VILA, J.-P. & VILLEDIEU, P. (2000) Convergence d'un schéma volumes finis explicite en temps pour les systèmes hyperboliques linéaires symétriques en domaines bornés. *C. R. Acad. Sci. Paris Sér. I Math.*, **331**, 95–100.

DE LELLIS, C. (2007) Notes on hyperbolic systems of conservation laws and transport equations. *Handbook of differential equations: evolutionary equations. Vol. III*. Handb. Differ. Equ. Elsevier/North-Holland, Amsterdam, pp. 277–382.

DELARUE, F. & LAGOUTIÈRE, F. (2011) Probabilistic analysis of the upwind scheme for transport equations. *Archive for Rational Mechanics and Analysis*, **199**, 229–268. 10.1007/s00205-010-0322-x.

DESJARDINS, B. (1996) A few remarks on ordinary differential equations. *Comm. Partial Differential Equations*, **21**, 1667–1703.

DESPRÈS, B. (2004a) Convergence of non-linear finite volume schemes for linear transport. *Notes from the XIth Jacques-Louis Lions Hispano-French School on Numerical Simulation in Physics and Engineering (Spanish)*. Grupo Anal. Teor. Numer. Modelos Cienc. Exp. Univ. Cádiz, Cádiz, pp. 219–239.

DESPRÈS, B. (2004b) Lax theorem and finite volume schemes. *Math. Comp.*, **73**, 1203–1234 (electronic).

DIPERNA, R. & LIONS, P.-L. (1989) Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**, 511–547.

DRONIOU, J. & EYMARD, R. (2006) A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, **105**, 35–71.

EYMARD, R., GALLOUËT, T. & HERBIN, R. (2000) Finite volume methods. *Handbook of numerical analysis, Vol. VII*. Handb. Numer. Anal., VII. Amsterdam: North-Holland, pp. 713–1020.

FETTAH, A. (2011) Analyse de modèles en mécanique des fluides compressibles. *Ph.D. thesis*, Université de Provence. in progress.

KUZNECOV, N. (1976) The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation. *Ž. Vyčisl. Mat. i Mat. Fiz.*, **16**, 1489–1502, 1627.

LEVEQUE, R. (2002) *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge: Cambridge University Press, pp. xx+558.

MERLET, B. (2007/08) L^∞- and L^2-error estimates for a finite volume approximation of linear advection. *SIAM J. Numer. Anal.*, **46**, 124–150.

MERLET, B. & VOVELLE, J. (2007) Error estimate for finite volume scheme. *Numer. Math.*, **106**, 129–155.

OHLBERGER, M. & VOVELLE, J. (2006) Error estimate for the approximation of nonlinear conservation laws on bounded domains by the finite volume method. *Math. Comp.*, **75**, 113–150 (electronic).

PETERSON, T. E. (1991) A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation. *SIAM J. Numer. Anal.*, **28**, 133–140.

VILA, J.-P. & VILLEDIEU, P. (2003) Convergence of an explicit finite volume scheme for first order symmetric systems. *Numer. Math.*, **94**, 573–602.

WALKINGTON, N. J. (2005) Convergence of the discontinuous Galerkin method for discontinuous solutions. *SIAM J. Numer. Anal.*, **42**, 1801–1817 (electronic).