Trace theorems and spatial continuity properties for the solutions of the transport equation

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Abstract. This paper is first concerned with the trace problem for the transport equation. We prove the existence and the uniqueness of the traces as well as the well-posedness of the initial and boundary value problem for the transport equation for any L^p data ($p \in]1, +\infty]$). In a second part, we use our study of the trace problem to prove that any solution to the transport equation is, roughly speaking, continuous with respect to the spatial variable along the direction of the transport vector field with values in a suitable L^q space in the other variables.

We want to emphasize the fact that we do not need to suppose any time regularity on the vector field defining the transport. This point is crucial in view of applications to fluid mechanics for instance. One of the main tools in our study is the theory of renormalized solutions of Di Perna and Lions.

Keywords : Transport equation - Trace theorems - Regularity of weak solutions - Renormalized solutions.

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1 Introduction

This paper is concerned with the study of some properties of the transport equation:

$$\partial_t \rho + v \cdot \nabla \rho = f, \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^d$ ($d \geq 2$), $v(t, x)$ is a given vector field, and $f(t, x)$ is a source term.

In the case where the vector field v is tangential to $\partial\Omega$, that is to say when $v \cdot \boldsymbol{\nu} = 0$ ($\boldsymbol{\nu}$ being the unitary outward normal to $\partial\Omega$), the Cauchy problem for (1.1) was studied by numerous authors for a long time. When v is regular enough (say Lipschitz), the problem can be solved by the characteristics method, that is to say by solving the ordinary differential equation associated to v.

When v is less regular - and this is crucial in the applications - the transport equation (1.1) was studied by Di Perna and Lions in [10] by introducing the fundamental theory of renormalized solutions. In particular, they show that if the vector field v has a bounded divergence and is integrable in time with values in a suitable Sobolev space in x, then any weak solution ρ of the equation (1.1) is a renormalized solution. It means that, for a wide class of regular functions β, $\beta(\rho)$ is also a solution to the corresponding transport equation. This result implies many properties of the weak solutions of (1.1) and in particular, the well-posedness of the Cauchy problem and the continuity of these solutions with respect to the time t and with values in suitable $L^q(\Omega)$ space.

Many variants and improvements of these results were provided by numerous authors, always in the case of a tangential velocity field (or in the case of the whole space $\Omega = \mathbb{R}^d$), in particular by weakening the requirements on the vector field v. For instance, it is shown in [7] that the boundedness of the divergence of v is not necessary and that a more general assumption on $\operatorname{div}(v)$ is sufficient. In [12] the case of "piecewise" $W^{1,1}$ vector fields is adressed. The uniqueness of solutions in the class $L^\infty(]0, T[, \mathcal{C}_b^0(\mathbb{R}^d))$ for vector fields in $L^1(]0, T[, (BV_{loc}(\mathbb{R}^d))^d)$ with bounded divergence and suitable behavior at infinity is proved in [6]. Finally, it is shown in [1] that for the same "BV in space" vector fields the uniqueness of weak solutions still holds in the class $L^\infty(]0, T[\times\mathbb{R}^d)$.

The aim of this paper is twofold. In a first part, we study the transport equation (1.1) in the case where $v \cdot \boldsymbol{\nu} \neq 0$ on the boundary of Ω. We first give the trace theorems for the weak solutions to this equation, then we solve the corresponding initial and boundary value problem. In the second part, we prove a new continuity property of the solutions to (1.1) with respect to one spatial variable with values in appropriate Lebesgue spaces in the other variables. It turns out that the trace results given in the first part of this paper are the main tools in order to prove this regularity result.

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1.1 Trace problem and Cauchy problem for the transport equation

In the first part of this work, we are interested in the study of (1.1) in the case where the vector field v is not necessarily tangential to the boundary of Ω, that is to say that we do not assume that $v \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$.

The key of the problem is to define and to study the properties of the trace of the weak solutions of (1.1) on the boundary $\partial\Omega$. In particular, in order to solve the initial and boundary value problem

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = f, \ \text{ in } \Omega, \\ \rho(0) = \rho_0, \\ \rho = \rho^i, \ \text{ where } v \cdot \boldsymbol{\nu} < 0, \end{cases} \tag{1.2}$$

it is needed to give a precise sense to the boundary condition $\rho = \rho^i$ imposed only on the part of $\partial\Omega$ where $v \cdot \boldsymbol{\nu} < 0$.

For time independent and Lipschitz vector fields v, this problem has been treated by Bardos in [2] by using the characteristics method and the semi-group theory. This problem was also addressed by different authors for other equations: in the case of the neutronic transport equation by Cessenat in [4, 5] using also the characteristics method, and for the Vlasov equation by Mischler in [13] using the renormalized solutions theory.

The detailed outline of this part is the following. In section 2, we introduce the notations used throughout this work. We also recall some results concerning the normal and tangential coordinates near the boundary of a smooth bounded domain, and we give the precise assumptions on the velocity field v and on the data f that we will consider.

In section 3, we prove (Theorem 3.1) the existence and uniqueness of the traces of any solution $\rho \in L^\infty(]0,T[, L^p(\Omega))$ of the transport equation (1.1), for any $1 < p \leq +\infty$. We also prove that these traces enjoy the renormalization property. The main tool in the proof of this result is a regularization procedure and a commutator estimate originally introduced by Di Perna and Lions in [10] and generalized for a bounded domain by Mischler in [13] for the study of the Vlasov equation. We also prove that any such weak solution ρ is continuous in time with values in $L^q(\Omega)$ for any $q < p$.

Section 4 is devoted to the proof of the well-posedness of the initial and boundary value problem for (1.1) in the case of bounded data (Theorem 4.1). We also show the strong convergence of the approximated solutions obtained by a suitable parabolic regularization of the problem (see (4.2)).

The fifth section is concerned with the description of the precise trace space in which the traces of the solutions $\rho \in L^\infty(]0,T[, L^p(\Omega))$ are lying in when p is finite. As it was noticed in [2, 4, 5, 13], it is needed to introduce the notion of life time associated to a vector field in Ω so that we can characterize the trace space (see Theorem 5.1).

Finally, using the results of the previous sections, we are able to prove in section 6, the well-posedness of the initial and boundary value problem with L^p data, for finite values of p (Theorem 6.1).

We want to emphasize here that one of the main challenge of this paper is to provide results which do not require any time regularity property for the vector field v just like in the tangential case (see [10]). Indeed, one of the main applications we have in mind for this problem is the study of outflow boundary conditions for the non-homogeneous Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div}(2\mu(\rho) D(v)) + \nabla p = f, \\ \operatorname{div}(v) = 0, \\ \rho(0) = \rho_0, \ \ v(0) = v_0, \\ \rho = \rho^i, \ \text{ where } v \cdot \boldsymbol{\nu} < 0, \\ + \text{Boundary condition for } v \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

The boundary conditions for v have to be chosen so that the problem is well-posed. We dot not enter the details here, since the study of this problem is proposed in [3]. Nevertheless, it is clear that weak solutions to (1.3) will classically lie in the energy space, *i.e.* $v \in L^2(]0,T[, (H^1(\Omega))^d)$, and that $\partial_t v$ has no reason to be integrable on $]0,T[\times\Omega$ (see [8, 9, 11]). Hence, it is fundamental to provide results which do not need v to be regular in time.

Notice that if one assumes that v is more regular in time (for instance if v is time independent), some of the proofs given below can be simplified.

1.2 Spatial regularity of the solutions to the transport equation

In section 7, we investigate the spatial regularity of the weak solutions to the transport equation. More precisely, we prove that any solution of the transport equation is continuous along the direction of the vector field v in a sense to be precised.

Let us see on a simple example why such a result has to be true. Let $\rho_0 \in L^\infty(\mathbb{R}^2)$ with compact support and let $v(t,x,y) = (\varphi(t), 0), \forall (t,x,y) \in]0,T[\times\mathbb{R}^2$, where $\varphi \in L^1(]0,T[)$ is a positive function such that $\min_{]0,T[} \varphi > 0$. We obviously find that the weak solution to (1.1) in the whole space $\mathbb{R}^2$ with initial data ρ_0 is given by

$$\rho(t,x,y) = \rho_0(x - \Phi(t), y),$$

with $\Phi(t) = \int_0^t \varphi(s)\,ds$. It is clear that ρ is continuous with respect to t with values in $L^1_{x,y}(\mathbb{R}^2)$ (this point is well known since the works by Di Perna and Lions [10]). Furthermore, we see that ρ is also continuous with respect to x (which is the direction of v) and with values in $L^1_{t,y}(]0,T[\times\mathbb{R})$. Indeed, using the change of variable $\tau = \Phi(t)$ we find

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |\rho(t,x,y) - \rho(t,x_0,y)|\,dt\,dy &= \int_0^T \int_{\mathbb{R}} |\rho_0(x-\Phi(t),y) - \rho_0(x_0 - \Phi(t), y)|\,dt\,dy \\ &= \int_0^{\Phi(T)} \int_{\mathbb{R}} |\rho_0(x-\tau,y) - \rho_0(x_0-\tau,y)| \frac{1}{\varphi(\Phi^{-1}(\tau))}\,d\tau\,dy.\end{aligned}$$

Since $\frac{1}{\varphi\circ\Phi^{-1}} \in L^\infty(]0,\Phi(T)[)$, the above quantity tends to zero when x converges to x_0 thanks to the continuity of the translation operators in L^1. Notice of course that ρ is not continuous in any sense with respect to y which is transverse to the vector field v.

Suppose now that φ is any function in $L^1(]0,T[)$. If φ vanishes on an interval $]t_0,t_1[$, then for any $t \in]t_0,t_1[$ and any $x,y \in \mathbb{R}$ the solution of (1.1) is given by

$$\rho(t,x,y) = \rho_0(x - C, y),$$

with $C = \int_0^{t_0} \varphi$. Hence, without any regularity assumption on the initial data ρ_0 we see that ρ cannot be continuous with respect to the variable x with values in $L^1_{t,y}$.

Nevertheless, the same computations than above show that the function

$$\rho(t,x,y)\,(v(t,x,y)\cdot e_x) = \rho(t,x,y)\,\varphi(t),$$

is continuous with respect to x with values in $L^1_{t,y}(]0,T[\times\mathbb{R})$ even if φ vanishes.

In the general case, where v is integrable in time with a Sobolev regularity in the spatial variables, we will give a general result which is similar to the one of the previous example. We shall see that the results are slightly different in the case of bounded solutions (Theorems 7.2 and 7.3) and for L^p solutions with finite values of p (Theorem 7.5). Once more, we emphasize the fact that our proofs are valid without any time regularity assumption on v.

The main idea in the proofs of these results is, roughly speaking, to show that the traces of a solution to the transport equation (defined in the first part of this paper) depend continuously on the domain we consider (since any solution in Ω is also solution in any subdomain of Ω). Then, such a continuity property of the traces is shown to imply a continuity property of the solution itself near the boundary of Ω. Finally, since we can apply all these results to any smooth subdomain in Ω, we can state in fact a continuity property in the whole domain Ω and not only near the boundary.

2 Notations and preliminaries

Throughout this paper, we will denote by $\mathcal{C}^1_b(\mathbb{R})$ the set of bounded $\mathcal{C}^1$ functions with a bounded derivative.

2.1 Elementary differential geometry

We suppose that Ω is a smooth (say $\mathcal{C}^2$) bounded domain in $\mathbb{R}^d$ (with $d \geq 2$) and we denote by Γ the boundary of Ω and by $\boldsymbol{\nu}$ the outward unitary normal to Γ. The distance between any $x \in \mathbb{R}^d$ and Γ is denoted by $d(x, \Gamma)$.

For any $\xi \geq 0$, we introduce the open sets

$$\mathcal{O}_\xi = \{x \in \Omega,\ d(x,\Gamma) < \xi\}, \text{ and } \Omega_\xi = \{x \in \Omega,\ d(x,\Gamma) > \xi\},$$

and for any $0 \leq \xi_1 < \xi_2$, we define

$$\mathcal{O}_{\xi_1,\xi_2} = \{x \in \Omega,\ \xi_1 < d(x,\Gamma) < \xi_2\}.$$

Since Ω is bounded and regular, there exists $\xi_\Omega > 0$ such that the maps $d(\cdot,\Gamma)$ (distance to Γ) and P_Γ (projection on Γ) are well defined and smooth in $\overline{\mathcal{O}_{\xi_\Omega}}$.

In $\overline{\mathcal{O}_{\xi_\Omega}}$ it is possible to use $(d(x,\Gamma), P_\Gamma(x)) \in [0,\xi_\Omega] \times \Gamma$ as a coordinate system. For any function $f : \overline{\Omega_{\xi_\Omega}} \mapsto \mathbb{R}$ we will use the notation

$$f(\xi,\sigma) \equiv f(\sigma - \xi\boldsymbol{\nu}(\sigma)),\ \ \forall (\xi,\sigma) \in [0,\xi_\Omega] \times \Gamma. \tag{2.1}$$

Notice that for any $\sigma \in \Gamma$, we have $f(0,\sigma) = f(\sigma)$. The reverse formula obviously reads

$$f(x) = f(d(x,\Gamma), P_\Gamma(x)),\ \ \forall x \in \overline{\mathcal{O}_{\xi_\Omega}}.$$

In particular, if $f \in W^{1,q}(\mathcal{O}_{\xi_\Omega})$ with $q \geq 1$, we deduce that

$$\frac{\partial}{\partial \xi} f(\xi,\sigma) = -(\nabla f)(\xi,\sigma) \cdot \boldsymbol{\nu}(\sigma) \in L^q(]0,\xi_\Omega[\times\Gamma),$$

and therefore we have

$$\xi \mapsto f(\xi,\sigma)\ \ \in \mathcal{C}^0([0,\xi_\Omega], L^q(\Gamma)). \tag{2.2}$$

We remark that (2.2) also holds for any $f \in W^{\frac{1}{q}+\varepsilon,q}(\mathcal{O}_{\xi_\Omega})$, with $\varepsilon > 0$.

For any $\xi \in [0,\xi_\Omega[$, we denote by $\Gamma_\xi = \{x \in \overline{\Omega},\ d(x,\Gamma) = \xi\}$. Notice that for any $\xi \in [0,\xi_\Omega]$ and $0 \leq \xi_1 < \xi_2 \leq \xi_\Omega$ we have

$$\partial\mathcal{O}_\xi = \Gamma \cup \Gamma_\xi,\ \ \partial\Omega_\xi = \Gamma_\xi,\ \ \partial\mathcal{O}_{\xi_1,\xi_2} = \Gamma_{\xi_1} \cup \Gamma_{\xi_2}.$$

For any $\xi \leq \xi_\Omega$, the manifolds Γ and Γ_ξ are isomorphic through the parallel transport with respect to the vector field $-\xi\boldsymbol{\nu}$. Let $J_\xi(\sigma), \sigma \in \Gamma$ be the Jacobian determinant of the isomorphism between the manifolds Γ and Γ_ξ. For any $g \in L^1(\Gamma_\xi)$, we have

$$\int_{\Gamma_\xi} g(\sigma')\,d\sigma' = \int_\Gamma g(\sigma - \xi\boldsymbol{\nu}(\sigma))J_\xi(\sigma)\,d\sigma = \int_\Gamma g(\xi,\sigma)J_\xi(\sigma)\,d\sigma. \tag{2.3}$$

Therefore, for any $f \in L^1(\mathcal{O}_{\xi_\Omega})$ we have the following change of variables formula :

$$\int_{\mathcal{O}_{\xi_\Omega}} f(x)\,dx = \int_0^{\xi_\Omega} \left(\int_\Gamma f(\xi,\sigma)J_\xi(\sigma)\,d\sigma\right) d\xi. \tag{2.4}$$

It is clear that $J_0(\sigma) = 1$ for any $\sigma \in \Gamma$ and that $(\xi,\sigma) \mapsto J_\xi(\sigma)$ is smooth for ξ small enough. Of course, $J_\xi(\sigma)$ can be computed explicitly depending on the two fundamental forms of Γ but it is not useful in this work. We just have to notice that if ξ_Ω is small enough we have $\frac{1}{2} \leq J_\xi(\sigma) \leq \frac{3}{2}$ for any $(\xi,\sigma) \in [0,\xi_\Omega] \times \Gamma$.

Finally, notice that there exists a smooth function $D : \overline{\Omega} \mapsto \mathbb{R}$ such that $D = d(\cdot,\Gamma)$ on $\mathcal{O}_{\xi_\Omega}$ so that we can define for any $x \in \overline{\Omega}$, the vector field $\boldsymbol{\nu}(x) = -\nabla D(x)$ which is regular bounded and which coincides with the normal vector field near the boundary. More precisely, we have

$$\boldsymbol{\nu}(x) = \boldsymbol{\nu}(P_\Gamma(x)),\ \ \text{for any } x \in \mathcal{O}_{\xi_\Omega}.$$

2.2 Regularization procedure in Ω

Let us recall now the regularization procedure which extend the classical smoothing method to the case of bounded domains (see [13]).

Let η be a positive radial and regular function with compact support in $\mathbb{R}^d$, such that $\displaystyle\int_{\mathbb{R}^d} \eta(x)\,dx = 1$. For any $\varepsilon > 0$, let $\eta_\varepsilon(x) = \dfrac{1}{\varepsilon^d}\eta\left(\dfrac{x}{\varepsilon}\right)$.

For any function $f \in L^1(]0,T[\times\Omega)$ we define the following space regularized function :

$$f \star_{\boldsymbol{\nu}} \eta_\varepsilon(t,y) = \int_\Omega f(t,x)\eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\,dx, \;\; \forall (t,y) \in]0,T[\times\Omega.$$

The properties of this regularization procedure are summed up in the following straightforward result.

Lemma 2.1

- *If Ω is a $\mathcal{C}^2$ bounded domain, then for any $f \in L^p(]0,T[,L^1(\Omega))$ we have*

$$f \star_{\boldsymbol{\nu}} \eta_\varepsilon \in L^p(]0,T[,\mathcal{C}^1(\overline{\Omega})).$$

- *For any $p,q \in [1,+\infty]$, and any $f \in L^p(]0,T[,L^q(\Omega))$, we have*

$$\|f \star_{\boldsymbol{\nu}} \eta_\varepsilon\|_{L^p(]0,T[,L^q(\Omega))} \leq C\|f\|_{L^p(]0,T[,L^q(\Omega))},$$

$$\|\nabla(f \star_{\boldsymbol{\nu}} \eta_\varepsilon)\|_{L^p(]0,T[,L^q(\Omega))} \leq \frac{C}{\varepsilon}\|f\|_{L^p(]0,T[,L^q(\Omega))}.$$

  Furthermore, if $p < +\infty$ and $q < +\infty$ then we have

$$f \star_{\boldsymbol{\nu}} \eta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} f, \;\; \text{in } L^p(]0,T[,L^q(\Omega)).$$

2.3 Assumptions on the vector field v

In this paper, p denotes any element of $]1,+\infty]$ and $p' = \frac{p}{p-1} \in [1,+\infty[$ its conjugate exponent. From section 3 to section 6, we are interested in vector fields v satisfying the following assumptions:

$$\begin{cases} v \in L^1(]0,T[,(W^{1,p'}(\Omega))^d), \\ \operatorname{div}(v) = 0, \\ (v\cdot\boldsymbol{\nu}) \in L^\alpha(]0,T[\times\Gamma), \;\; \text{for a given } \alpha > 1. \end{cases} \tag{2.5}$$

In the case where $v \cdot \boldsymbol{\nu} = 0$, the regularity assumption on v is exactly the same than in previous works in this case (see [7, 10]). In particular, one of the main feature of this paper is that we do not need any additional time regularity assumption on the vector field v to prove our results.

Notice that most of the results in this paper can be adapted to vector fields such that $\operatorname{div}(v)$ is only supposed to be bounded (see [7, 10]).

For any such vector field, we can introduce the following measures on $]0,T[\times\Gamma$:

$$d\mu_v = (v\cdot\boldsymbol{\nu})\,d\sigma dt,$$

$$d\mu_v^\alpha = |v\cdot\boldsymbol{\nu}|^\alpha\,d\sigma dt.$$

As usual $d\mu_v^+ = (v\cdot\boldsymbol{\nu})^+\,d\sigma dt$ and $d\mu_v^- = (v\cdot\boldsymbol{\nu})^-\,d\sigma dt$ are the positive and negative parts of $d\mu_v$ so that we have

$$d\mu_v = d\mu_v^+ - d\mu_v^-, \;\; |d\mu_v| = d\mu_v^+ + d\mu_v^-.$$

3 Trace theorem and continuity in time of solutions.

In this section, we prove the existence and the uniqueness of the trace of any weak solution of the transport equation in the domain Ω. We also give the renormalization property for these traces.

Theorem 3.1
Let $p \in]1,+\infty]$, v satisfying (2.5) and $f \in L^1(]0,T[, L^p(\Omega))$. Let $\rho \in L^\infty(]0,T[, L^p(\Omega))$ be a solution, in the distribution sense, to the transport equation

$$\partial_t \rho + v \cdot \nabla \rho = f.$$

1. *The function ρ lies in $\mathcal{C}^0([0,T], L^q(\Omega))$, for any $1 \leq q < p$. Furthermore, ρ is weakly continuous in time with values in $L^p(\Omega)$.*

2. *There exists a function $\gamma\rho$ measurable on $]0,T[\times\Gamma$ such that for any $\beta \in \mathcal{C}_b^1(\mathbb{R})$, for any test function $\varphi \in \mathcal{C}^1([0,T]\times\overline{\Omega})$, and for any $[t_0,t_1] \subset [0,T]$, we have*

$$\begin{aligned} \int_{t_0}^{t_1}\int_\Omega \beta(\rho)(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \beta(\rho(t_0))\varphi(t_0)\,dx - \int_\Omega \beta(\rho(t_1))\varphi(t_1)\,dx \\ - \int_{t_0}^{t_1}\int_\Gamma \beta(\gamma\rho)\varphi(v\cdot\boldsymbol{\nu})\,dt\,d\sigma + \int_{t_0}^{t_1}\int_\Omega \beta'(\rho) f\varphi\,dt\,dx = 0. \end{aligned} \tag{3.1}$$

   Furthermore, the function $\gamma\rho$ satisfying the above properties is unique in the almost-everywhere sense for the measure $|d\mu_v|$ on $]0,T[\times\Gamma$.

Remark 3.1
Suppose that $p = +\infty$ and let us introduce

$$\rho_{\max} = \sup_{]0,T[\times\Omega} \rho(t,x), \quad \rho_{\min} = \inf_{]0,T[\times\Omega} \rho(t,x).$$

Let β be a regular non-negative function in $\mathcal{C}_b^1(\mathbb{R})$ such that $\beta(s) = 0$ if and only if $s \in [\rho_{\min}, \rho_{\max}]$. For this particular choice of β, formula (3.1) gives

$$\int_0^T \int_\Gamma \beta(\gamma\rho)\varphi(v\cdot\boldsymbol{\nu})\,dt\,d\sigma = 0,$$

for any regular test function φ, since the first term in (3.1) vanishes. This implies that $\beta(\gamma\rho)(v\cdot\boldsymbol{\nu}) = 0$ for almost every $(t,\sigma) \in]0,T[\times\Gamma$. Hence, we have $\beta(\gamma\rho) = 0$ for $|d\mu_v|$-almost every $(t,\sigma) \in]0,T[\times\Gamma$.

Therefore, we have $\rho_{\min} \leq \gamma\rho(t,\sigma) \leq \rho_{\max}$ for $|d\mu_v|$-almost every $(t,\sigma) \in]0,T[\times\Gamma$. In particular, we have $\gamma\rho \in L^\infty(]0,T[\times\Gamma, |d\mu_v|)$. Hence, in this case formula (3.1) is true for any $\beta \in \mathcal{C}^1(\mathbb{R})$ not necessarily bounded. In particular, taking $\beta(s) = s$, we deduce that

$$\begin{aligned} \int_{t_0}^{t_1}\int_\Omega \rho(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho(t_0)\varphi(t_0)\,dx - \int_\Omega \rho(t_1)\varphi(t_1)\,dx \\ - \int_{t_0}^{t_1}\int_\Gamma \gamma\rho(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma + \int_{t_0}^{t_1}\int_\Omega f\varphi\,dt\,dx = 0, \end{aligned} \tag{3.2}$$

which gives the weak formulation of the transport equation taking into account the trace $\gamma\rho$ of the solution on $]0,T[\times\Gamma$.

Notice that (3.2), is not true in general in the case $p < +\infty$ since it is possible that $\gamma\rho$ does not lie in the space $L^1(]0,T[\times\Gamma, |d\mu_v|)$ (see the counter-example by Bardos in [2]). We will adress the problem of giving a sense to (3.2) in section 6.

Proof :

1. By assumption, we have for any $\varphi \in \mathcal{C}_c^1(]0,T[\times\Omega)$

$$\int_0^T \int_\Omega \rho(\partial_t \varphi + v \cdot \nabla \varphi)\, dt\, dx + \int_0^T \int_\Omega f\varphi\, dt\, dx = 0.$$

Let $\psi \in \mathcal{C}_c^\infty(]0,T[\times\Omega)$. For any $y \in \Omega$ we choose $\varphi(t,x) = \psi(t,y)\eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))$ in the following formulation. Notice that φ vanishes on the boundary Γ. If now we integrate with respect to $y \in \Omega$ the above identity, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega \rho(t,x)\partial_t \psi(t,y)\eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\, dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega \int_\Omega \rho(t,x)\psi(t,y)v(t,x) \cdot \nabla_x \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\, dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega \int_\Omega f(t,x)\psi(t,y)\eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\, dt\, dx\, dy = 0,
\end{aligned}$$

which gives after integrations by parts

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \rho \star_{\boldsymbol{\nu}} \eta_\varepsilon(t,y)(\partial_t \psi(t,y) + v(t,y) \cdot \nabla_y \psi(t,y))\, dt\, dy \\
&\quad + \int_0^T \int_\Omega \int_\Omega \rho(t,x)\Big(v(t,x) \cdot \nabla_x \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\psi(t,y) - v(t,y) \cdot \nabla_y \psi(t,y)\eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\Big)\, dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega f \star_{\boldsymbol{\nu}} \eta_\varepsilon(t,y)\psi(t,y)\, dt\, dy \\
&= \int_0^T \int_\Omega \rho \star_{\boldsymbol{\nu}} \eta_\varepsilon(t,y)(\partial_t \psi(t,y) + v(t,y) \cdot \nabla_y \psi(t,y))\, dt\, dy \\
&\quad + \int_0^T \int_\Omega \int_\Omega \rho(t,x)\Big(v(t,x) \cdot \nabla_x \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y)) + v(t,y) \cdot \nabla_y \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\Big)\psi(t,y)\, dt\, dx\, dy \\
&\quad + \int_0^T \int_\Omega f \star_{\boldsymbol{\nu}} \eta_\varepsilon(t,y)\psi(t,y)\, dt\, dy.
\end{aligned}$$

If now we let $\rho_\varepsilon = \rho \star_{\boldsymbol{\nu}} \eta_\varepsilon$ and $f_\varepsilon = f \star_{\boldsymbol{\nu}} \eta_\varepsilon$, we have proved that ρ_ε is solution in the distribution sense to the equation

$$\partial_t \rho_\varepsilon + v \cdot \nabla \rho_\varepsilon = R_\varepsilon + f_\varepsilon, \tag{3.3}$$

where

$$R_\varepsilon(t,y) = \int_\Omega \rho(t,x)\Big(v(t,x) \cdot \nabla_x \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y)) + v(t,y) \cdot \nabla_y \eta_\varepsilon(y - x - 2\varepsilon\boldsymbol{\nu}(y))\Big)\, dx.$$

The renormalized solutions theory (see [10]) is based on the fact that this remainder term R_ε, which can be seen as a commutator, tends to 0 as ε goes to 0 in a suitable Lebesgue space. The precise result in our case is the following.

Lemma 3.1
If $\rho \in L^\infty(]0,T[, L^p(\Omega))$ and $v \in L^1(]0,T[, (W^{1,p'}(\Omega))^d)$ with $\operatorname{div}(v) = 0$, then we have

$$R_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0, \quad \textit{in } L^1(]0,T[\times\Omega).$$

Very similar lemmas can be found in [10] and [13] so that we do not give the proof here.

We deduce from (3.3) that ρ_ε is $W^{1,1}$ in time (and hence continuous) with values in $L^1(\Omega)$. But since ρ lies in $L^\infty(]0,T[,L^p(\Omega))$ then for any $\varepsilon > 0$, ρ_ε lies in $L^\infty(]0,T[\times\Omega)$ and is bounded independtly of ε in $L^\infty(]0,T[,L^p(\Omega))$ (Lemma 2.1). In particular, we see that ρ_ε is continuous in time with values in $L^q(\Omega)$ for any $q < +\infty$.

For any $\varepsilon_1, \varepsilon_2 > 0$, we have

$$\partial_t(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) + v \cdot \nabla(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) = (R_{\varepsilon_1} - R_{\varepsilon_2}) + (f_{\varepsilon_1} - f_{\varepsilon_2}). \tag{3.4}$$

Let β be a regular function such that $|\beta(s)| \leq |s|$ and $|\beta'(s)| \leq C$, for any $s \in \mathbb{R}$. Multiplying (3.4) by $\beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})$ we get in the distribution sense

$$\partial_t\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) + v \cdot \nabla\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) = \beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})(R_{\varepsilon_1} - R_{\varepsilon_2}) + \beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})(f_{\varepsilon_1} - f_{\varepsilon_2}). \tag{3.5}$$

Let $h > 0$ be small enough and φ_h be the function in $W^{1,\infty}(\Omega)$ defined by $\varphi_h = 1$ in Ω_h and $\varphi_h = \frac{1}{h}d(x,\Gamma)$ in $\mathcal{O}_h$ (see section 2.1 for the definitions of Ω_h and $\mathcal{O}_h$). We have $\nabla\varphi_h(x) = -\frac{1}{h}\boldsymbol{\nu}(x)$, for $x \in \mathcal{O}_h$ and $\nabla\varphi_h = 0$ in Ω_h.

Let us take $(\frac{t}{T} - 1)\varphi_h$ as a test function in (3.5) on $]0,T[\times\Omega$, we get

$$\begin{aligned}\int_\Omega \varphi_h\beta(\rho_{\varepsilon_1}(0) - \rho_{\varepsilon_2}(0))\,dx = \frac{1}{T}\int_0^T \int_\Omega \varphi_h\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})\,dt\,dx& \\ + \int_0^T \int_\Omega \left(\frac{t}{T} - 1\right)\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})(v \cdot \nabla\varphi_h)\,dt\,dx& \\ + \int_0^T \int_\Omega \varphi_h \left(\frac{t}{T} - 1\right)\beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})(R_{\varepsilon_1} - R_{\varepsilon_2} + f_{\varepsilon_1} - f_{\varepsilon_2})\,dt\,dx.&\end{aligned} \tag{3.6}$$

Taking now φ_h as a test function in (3.5), we get

$$\frac{d}{dt}\int_\Omega \varphi_h\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})\,dx - \int_\Omega \beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})v \cdot \nabla\varphi_h\,dx = \int_\Omega \beta'(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})(R_{\varepsilon_1} - R_{\varepsilon_2} + f_{\varepsilon_1} - f_{\varepsilon_2})\varphi_h\,dx,$$

so that:

$$\begin{aligned}\sup_{t\in[0,T]}\int_\Omega \varphi_h\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})\,dx \leq \int_\Omega \varphi_h\beta(\rho_{\varepsilon_1}(0,\cdot) - \rho_{\varepsilon_2}(0,\cdot))\,dx + C\int_0^T \int_\Omega |R_{\varepsilon_1} - R_{\varepsilon_2}|\,dt\,dx& \\ + C\int_0^T \int_\Omega |f_{\varepsilon_1} - f_{\varepsilon_2}|\,dtdx + C\int_0^T \int_\Omega |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}||v \cdot \nabla\varphi_h|\,dt\,dx.&\end{aligned}$$

Using (3.6) and the definition of φ_h, it follows:

$$\begin{aligned}\sup_{t\in[0,T]}\int_{\Omega_h} \beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})\,dx \leq C\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^1_{t,x}} + C\|R_{\varepsilon_1} - R_{\varepsilon_2}\|_{L^1_{t,x}} + C\|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{L^1_{t,x}}& \\ + \frac{C}{h}\int_0^T \int_{\mathcal{O}_h} |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}||v \cdot \boldsymbol{\nu}|\,dt\,dx.&\end{aligned} \tag{3.7}$$

Now we choose $\beta(s) = \frac{s^2}{\sqrt{s^2+\delta}}$ and we perform the limit when δ goes to zero so that finally (3.7) holds with $\beta(s) = |s|$. Using the fact that $\Omega = \overline{\Omega_h} \cup \mathcal{O}_h$, we deduce:

$$\begin{aligned}\sup_{t\in[0,T]}\int_\Omega |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}|\,dx \leq \sup_{t\in[0,T]}\int_{\mathcal{O}_h} |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}|\,dx + C\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^1_{t,x}} + C\|R_{\varepsilon_1} - R_{\varepsilon_2}\|_{L^1_{t,x}}& \\ + C\|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{L^1_{t,x}} + \frac{C}{h}\int_0^T \int_{\mathcal{O}_h} |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}||v \cdot \boldsymbol{\nu}|\,dt\,dx.&\end{aligned}$$

As $|\mathcal{O}_h| \le Ch$ for any $h>0$, and using the bound of $(\rho_\varepsilon)_\varepsilon$ in $L^\infty(]0,T[,L^p(\Omega))$, we get

$$\sup_{t\in[0,T]}\int_\Omega |\rho_{\varepsilon_1}-\rho_{\varepsilon_2}|\,dx \le Ch^{\frac{1}{p'}} + C\|\rho_{\varepsilon_1}-\rho_{\varepsilon_2}\|_{L^1_{t,x}} + C\|R_{\varepsilon_1}-R_{\varepsilon_2}\|_{L^1_{t,x}} + C\|f_{\varepsilon_1}-f_{\varepsilon_2}\|_{L^1_{t,x}} + \frac{C}{h}\int_0^T\int_{\mathcal{O}_h} |\rho_{\varepsilon_1}-\rho_{\varepsilon_2}||v\cdot\boldsymbol{\nu}|\,dt\,dx. \tag{3.8}$$

Let us denote the last term in (3.8) by I. We use in $\mathcal{O}_h$, the tangential and normal coordinates introduced in section 2.1 so that we can write

$$\begin{aligned}|I| \le &\frac{1}{h}\int_0^T\int_0^h\int_\Gamma |\rho_{\varepsilon_1}(t,\xi,\sigma)-\rho_{\varepsilon_2}(t,\xi,\sigma)||v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)-v(t,0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\,J_\xi(\sigma)\,d\sigma\,d\xi\,dt\\ &+\frac{1}{h}\int_0^T\int_0^h\int_\Gamma |\rho_{\varepsilon_1}(t,\xi,\sigma)-\rho_{\varepsilon_2}(t,\xi,\sigma)||v(t,0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\,J_\xi(\sigma)\,d\sigma\,d\xi\,dt \equiv I_1+I_2.\end{aligned} \tag{3.9}$$

The term I_1 can be estimated as follows:

$$\begin{aligned}I_1 &\le \frac{1}{h}\int_0^T\int_0^h\int_\Gamma |\rho_{\varepsilon_1}(t,\xi,\sigma)-\rho_{\varepsilon_2}(t,\xi,\sigma)|\left(\xi\int_0^1 |\nabla v(t,u\xi,\sigma)|\,du\right)\,J_\xi(\sigma)\,d\sigma\,d\xi\,dt\\
&\le \frac{1}{h}\|\rho_{\varepsilon_1}-\rho_{\varepsilon_2}\|_{L^\infty(]0,T[,L^p(\Omega))}\int_0^T\left(\int_0^h\int_\Gamma\int_0^1 \xi^{p'}|\nabla v(t,u\xi,\sigma))|^{p'}\,J_\xi(\sigma)du\,d\sigma\,d\xi\right)^{\frac{1}{p'}}\,dt\\
&\le \frac{C}{h}\int_0^T\left(\int_0^h\int_\Gamma \frac{J_\xi(\sigma)}{J_\tau(\sigma)}\xi^{p'-1}\int_0^\xi |\nabla v(t,\tau,\sigma)|^{p'}J_\tau(\sigma)d\tau\,d\sigma\,d\xi\right)^{\frac{1}{p'}}\,dt\\
&\le C\int_0^T\left(\frac{1}{h}\int_0^h\int_{\mathcal{O}_\xi}|\nabla v(t,x)|^{p'}dx\,d\xi\right)^{\frac{1}{p'}}\,dt.\end{aligned}$$

Since for almost every $t\in]0,T[$ we have

$$\left(\frac{1}{h}\int_0^h\int_{\mathcal{O}_\xi}|\nabla v(t,x)|^{p'}dx\,d\xi\right)^{\frac{1}{p'}} \le \|\nabla v(t)\|_{L^{p'}},\;\;\forall h>0,$$

and

$$\frac{1}{h}\int_0^h\int_{\mathcal{O}_\xi}|\nabla v(t,x)|^{p'}dx\,d\xi \underset{h\to 0}{\longrightarrow} 0,$$

we are able to conclude, using the Lebesgue theorem, that the term I_1 converges to zero when h goes to 0, independently of ε_1 and ε_2.

- In the case $p<+\infty$, assumptions (2.5) imply that there exists $\lambda>0$ such that

  $$(t,\sigma)\mapsto (v(t,0,\sigma)\cdot\boldsymbol{\nu}(\sigma))\;\;\in L^1(]0,T[,W^{1-\frac{1}{p'},p'}(\Gamma))\subset L^1(]0,T[,L^{p'+\lambda}(\Gamma)).$$

  But as we also assume that $(v\cdot\boldsymbol{\nu})\in L^\alpha(]0,T[\times\Gamma)\subset L^\alpha(]0,T[,L^1(\Gamma))$, with $\alpha>1$, classical interpolation inequalities show that there exists $\delta>1$ such that $(v\cdot\boldsymbol{\nu})\in L^\delta(]0,T[,L^{p'}(\Gamma))$. As a consequence, we can estimate the term I_2 defined in (3.9) as follows using the Hölder inequality:

  $$I_2 \le \frac{C}{h^{1-\frac{1}{p'}}}\|\rho_{\varepsilon_1}-\rho_{\varepsilon_2}\|_{L^{\delta'}(]0,T[,L^p(\Omega))}\|v\cdot\boldsymbol{\nu}\|_{L^\delta(]0,T[,L^{p'}(\Gamma))}.$$

- In the case $p = +\infty$, we estimate I_2 as follows

$$I_2 \leq \frac{C}{h^{\frac{1}{\alpha'}}} \|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^{\alpha'}(]0,T[,L^{\alpha'}(\Omega))} \|v \cdot \boldsymbol{\nu}\|_{L^\alpha(]0,T[,L^\alpha(\Gamma))},$$

  where $\alpha' = \frac{\alpha}{\alpha-1}$ is the conjugate exponent of α, which is finite since $\alpha > 1$.

Finally, (3.8) gives in any cases

$$\begin{aligned} \sup_{t\in[0,T]} \int_\Omega |\rho_{\varepsilon_1} - \rho_{\varepsilon_2}|\, dx \leq C h^{\frac{1}{p'}} + C\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^1_{t,x}} + C\|R_{\varepsilon_1} - R_{\varepsilon_2}\|_{L^1_{t,x}} + C\|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{L^1_{t,x}} \\ + I_1 + C(h)\|\rho_{\varepsilon_1} - \rho_{\varepsilon_2}\|_{L^\lambda(]0,T[,L^q(\Omega))}, \end{aligned} \tag{3.10}$$

where $\lambda < +\infty$ and $q \leq p$ is finite.

As a consequence, we deduce that $(\rho_\varepsilon)_\varepsilon$ is a Cauchy sequence in $\mathcal{C}^0([0,T], L^1(\Omega))$. Indeed, we can first choose h so that the terms $Ch^{\frac{1}{p'}}$ and I_1 are small enough independently of ε_1 and ε_2. Using Lemmas 2.1 and 3.1 we know that $(\rho_\varepsilon)_\varepsilon$ converge in $L^\lambda(]0,T[, L^q(\Omega))$ and that $(R_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$ converge in $L^1(]0,T[\times\Omega)$. Hence, h being fixed, all the other terms in (3.10) can be made small as ε_1 and ε_2 are close to 0.

At this point, we proved that $(\rho_\varepsilon)_\varepsilon$ converges towards ρ in $\mathcal{C}^0([0,T], L^1(\Omega))$. As $(\rho_\varepsilon)_\varepsilon$ is known to be bounded in $L^\infty(]0,T[, L^p(\Omega))$ we deduce that $(\rho_\varepsilon)_\varepsilon$ converges towards ρ in $\mathcal{C}^0([0,T], L^q(\Omega))$ for any $q < p$. The weak continuity in time of ρ with values in $L^p(\Omega)$ when p is finite follows immediatly.

2. First of all, the uniqueness property of $\gamma\rho$ is straightforward. Indeed, if $\gamma_1\rho$ and $\gamma_2\rho$ are two measurable functions satisfying (3.1), then we have:

$$\int_0^T \int_\Gamma \beta(\gamma_1\rho)(v \cdot \boldsymbol{\nu})\varphi\, dt\, d\sigma = \int_0^T \int_\Gamma \beta(\gamma_2\rho)(v \cdot \boldsymbol{\nu})\varphi\, dt\, d\sigma,$$

for any suitable β and φ which implies that $\beta(\gamma_1\rho)(v \cdot \boldsymbol{\nu}) = \beta(\gamma_2\rho)(v \cdot \boldsymbol{\nu})$ for almost every (t,σ). It follows that $\gamma_1\rho = \gamma_2\rho$ for $|d\mu_v|$-almost every (t,σ).

Now, let us prove the existence of the trace $\gamma\rho$. Since ρ_ε is regular enough, for any $\beta \in \mathcal{C}^1_b(\mathbb{R})$ we have

$$\partial_t \beta(\rho_\varepsilon) + v \cdot \nabla\beta(\rho_\varepsilon) = \beta'(\rho_\varepsilon)R_\varepsilon + \beta'(\rho_\varepsilon)f_\varepsilon, \tag{3.11}$$

in the distribution sense. Hence, for any $\varepsilon_1, \varepsilon_2 > 0$, we have

$$\partial_t(\beta(\rho_{\varepsilon_1}) - \beta(\rho_{\varepsilon_2})) + v \cdot \nabla(\beta(\rho_{\varepsilon_1}) - \beta(\rho_{\varepsilon_2})) = \beta'(\rho_{\varepsilon_1})(R_{\varepsilon_1} + f_{\varepsilon_1}) - \beta'(\rho_{\varepsilon_2})(R_{\varepsilon_2} + f_{\varepsilon_2}). \tag{3.12}$$

Let $\psi(t,x)$ be a smooth function with compact support in time but not necessarily in space. Let us choose $\beta(\rho_{\varepsilon_1} - \rho_{\varepsilon_2})\psi$ as a test function in (3.12). It follows

$$\begin{aligned} \int_0^T \int_\Gamma |\beta(\rho_{\varepsilon_1}) - \beta(\rho_{\varepsilon_2})|^2 \psi (v \cdot \boldsymbol{\nu})\, dt\, d\sigma = \\ \int_0^T \int_\Omega |\beta(\rho_{\varepsilon_1}) - \beta(\rho_{\varepsilon_2})|^2 (\partial_t\psi + v \cdot \nabla\psi)\, dt\, dx \\ + 2\int_0^T \int_\Omega (\beta'(\rho_{\varepsilon_1})(R_{\varepsilon_1} + f_{\varepsilon_1}) - \beta'(\rho_{\varepsilon_2})(R_{\varepsilon_2} + f_{\varepsilon_2}))(\beta(\rho_{\varepsilon_1}) - \beta(\rho_{\varepsilon_2}))\psi\, dt\, dx. \end{aligned} \tag{3.13}$$

Thanks to assumption (2.5), it is now natural to take $\psi = |v \cdot \boldsymbol{\nu}|^{\alpha-2}(v \cdot \boldsymbol{\nu})$ in this formula, in order for the left-hand side term to be non-negative. Unfortunately, this is not possible since this function ψ is not smooth enough, in particular in the time variable.

Let $g_n \in \mathcal{C}^1_c(]0,T[\times\Gamma)$ be a sequence of smooth functions such that $(g_n)_n$ converges towards $|v\cdot\boldsymbol{\nu}|^{\alpha-2}(v\cdot\boldsymbol{\nu})$ in $L^{\alpha'}(]0,T[\times\Gamma)$. Notice that the condition $\alpha>1$ is crucial here. For any $n\geq 0$, let $G_n \in \mathcal{C}^1_c(]0,T[\times\overline{\Omega})$ be such that g_n is the trace of G_n on $]0,T[\times\Gamma$. Now we take $\psi = G_n$ in (3.13), and we get

$$
\begin{aligned}
&\int_0^T\int_\Gamma |\beta(\rho_{\varepsilon_1})-\beta(\rho_{\varepsilon_2})|^2|v\cdot\boldsymbol{\nu}|^\alpha\,dt\,d\sigma = \\
&\qquad \int_0^T\int_\Gamma |\beta(\rho_{\varepsilon_1})-\beta(\rho_{\varepsilon_2})|^2(v\cdot\boldsymbol{\nu})\Big(|v\cdot\boldsymbol{\nu}|^{\alpha-2}(v\cdot\boldsymbol{\nu})-g_n\Big)\,dt\,d\sigma \\
&\qquad + \int_0^T\int_\Omega |\beta(\rho_{\varepsilon_1})-\beta(\rho_{\varepsilon_2})|^2(\partial_t G_n + v\cdot\nabla G_n)\,dt\,dx \\
&\qquad + 2\int_0^T\int_\Omega (\beta'(\rho_{\varepsilon_1})(R_{\varepsilon_1}+f_{\varepsilon_1})-\beta'(\rho_{\varepsilon_2})(R_{\varepsilon_2}+f_{\varepsilon_2}))(\beta(\rho_{\varepsilon_1})-\beta(\rho_{\varepsilon_2}))G_n\,dt\,dx.
\end{aligned}
\tag{3.14}
$$

This shows that $(\beta(\rho_\varepsilon))_\varepsilon$ is a Cauchy sequence in $L^2(]0,T[\times\Gamma, d\mu_v^\alpha)$. Indeed, as β is a bounded function, the first term can be controlled by $C_\beta\|v\cdot\boldsymbol{\nu}\|_{L^\alpha(]0,T[\times\Gamma)}\||v\cdot\boldsymbol{\nu}|^{\alpha-2}(v\cdot\boldsymbol{\nu})-g_n\|_{L^{\alpha'}(]0,T[\times\Gamma)}$, hence it converges towards zero when n goes to infinity, independently of ε_1 and ε_2. Therefore, we choose n so that this first term is as small as we want. Then the last two terms are small as soon as ε_1 and ε_2 are small enough because G_n is now fixed and regular, $(R_\varepsilon)_\varepsilon$ converges to zero in $L^1(]0,T[\times\Omega)$ and furthermore $(\beta(\rho_\varepsilon))_\varepsilon$ converges to $\beta(\rho)$ in $\mathcal{C}^0([0,T],L^q(\Omega))$ for any $q<+\infty$.

Hence, we have shown that the traces $(\beta(\rho_\varepsilon))_\varepsilon$ are a Cauchy sequence in $L^2(]0,T[\times\Gamma,d\mu_v^\alpha)$. In particular, there exists a subsequence $(\beta(\rho_{\varepsilon_k}))_k$ which converges $d\mu_v^\alpha$-almost everywhere and hence $|d\mu_v|$-almost everywhere. If we choose now β to be injective, we deduce that $(\rho_{\varepsilon_k})_k$ has a limit, denoted by $\gamma\rho$, for $|d\mu_v|$-almost every $(t,\sigma)\in]0,T[\times\Gamma$.

We claim that $\gamma\rho$ is finite $|d\mu_v|$-almost everywhere. Indeed, in the case $p=+\infty$, this point follows from Remark 3.1. In the case $p<+\infty$, it is more difficult to prove it at this point but it is worth to remark that this claim will be a straightforward consequence of Theorem 5.1 and Proposition 5.1.

Now we know that for any suitable β, the traces $(\beta(\rho_{\varepsilon_k}))_k$ converges in $L^2(]0,T[\times\Gamma,d\mu_v^\alpha)$ but converges also $d\mu_v^\alpha$-almost everywhere towards $\beta(\gamma\rho)$ since β is continuous.

Let $\varphi\in\mathcal{C}^1_c(]0,T[\times\overline{\Omega})$ be a function test for (3.11), we get

$$
\int_0^T\int_\Omega \beta(\rho_{\varepsilon_k})(\partial_t\varphi+v\cdot\nabla\varphi)\,dt\,dx - \int_0^T\int_\Gamma \beta(\rho_{\varepsilon_k})(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma = \int_0^T\int_\Omega \beta'(\rho_{\varepsilon_k})(R_{\varepsilon_k}+f_{\varepsilon_k})\,dt\,dx.
$$

The convergences obtained previously as well as Lemma 3.1 allow us to pass to the limit in this formula and we get exactly (3.1).

■

4 Initial and boundary value problem in L^∞

In this section we show that the initial and boundary value problem for the transport equation (1.1) with bounded data is well-posed. The existence of a solution is proved by introducing a parabolic approximation of the problem with mixed boundary conditions whose solutions weakly converge to a solution to our problem. The uniqueness is essentially a consequence of the our trace result (Theorem 3.1) and more precisely of the renormalization property.

At the end of this section, we prove *a posteriori* that the solutions (including the traces of these solutions) of the approximate parabolic problem are in fact strongly convergent towards the solution of the limit problem.

Theorem 4.1
Let $T > 0$, $f \in L^1(]0,T[, L^\infty(\Omega))$ and v satisfying (2.5) (with $p' = 1$). For any initial data $\rho_0 \in L^\infty(\Omega)$, and any boundary data $\rho^i \in L^\infty(]0,T[\times\Gamma, d\mu_v^-)$, there exists a unique couple $(\rho,\rho^o) \in L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, d\mu_v^+)$ such that for any $\varphi \in \mathcal{C}^1([0,T]\times\overline{\Omega})$ with $\varphi(T) = 0$, we have:

$$\int_0^T \int_\Omega \rho(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho_0\varphi(0)\,dx \\ - \int_0^T \int_\Gamma \rho^o\varphi(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma + \int_0^T \int_\Gamma \rho^i\varphi(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + \int_0^T \int_\Omega f\varphi\,dt\,dx = 0. \tag{4.1}$$

Furthermore, ρ is continuous in time with values in $L^q(\Omega)$ for any $q < +\infty$ and (ρ,ρ^o) is a renormalized solution of (4.1) which means that for any real function β in $\mathcal{C}^1(\mathbb{R})$, the couple $(\beta(\rho),\beta(\rho^o))$ is the unique solution of (4.1) with data $(\beta(\rho_0),\beta(\rho^i))$ and source term $\beta'(\rho)f$.

Proof :

- **Existence :**

  For any $\varepsilon > 0$ we consider the following parabolic problem

$$\begin{cases} \partial_t\tilde\rho_\varepsilon + v\cdot\nabla\tilde\rho_\varepsilon - \varepsilon\Delta\tilde\rho_\varepsilon = f, \ \text{ in } \Omega \\ \tilde\rho_\varepsilon(t=0) = \rho_0, \\ \varepsilon\dfrac{\partial\tilde\rho_\varepsilon}{\partial\boldsymbol{\nu}} + (\tilde\rho_\varepsilon - \rho^i)(v\cdot\boldsymbol{\nu})^- = 0, \ \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

  Lemma 4.1
  For any $\rho_0 \in L^\infty(\Omega)$ and $\rho^i \in L^\infty(]0,T[\times\Gamma, d\mu_v^-)$ there exists a unique solution $\tilde\rho_\varepsilon \in L^\infty(]0,T[, L^2(\Omega)) \cap L^2(]0,T[, H^1(\Omega))$ to (4.2). Furthermore, we have the estimates

$$\|\tilde\rho_\varepsilon\|_{L^\infty(]0,T[,L^2(\Omega))} + \|\sqrt{\varepsilon}\nabla\tilde\rho_\varepsilon\|_{L^2(]0,T[,L^2(\Omega))} \le C, \tag{4.3}$$

$$\int_0^T \int_\Gamma |\tilde\rho_\varepsilon|^2(v\cdot\boldsymbol{\nu})^+\,d\sigma\,dt \le C, \tag{4.4}$$

  where C does not depend on ε.

  Proof :
  The weak formulation of the problem (4.2) reads:

$$-\int_0^T \int_\Omega \tilde\rho_\varepsilon(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \varepsilon\int_0^T \int_\Omega \nabla\tilde\rho_\varepsilon\nabla\varphi\,dt\,dx \\ + \int_0^T \int_\Gamma (\tilde\rho_\varepsilon(v\cdot\boldsymbol{\nu})^+\varphi - \rho^i(v\cdot\boldsymbol{\nu})^-\varphi)\,dt\,d\sigma - \int_\Omega \rho_0\varphi(0)\,dx = \int_0^T \int_\Omega f\varphi\,dt\,dx, \tag{4.5}$$

  for any $\varphi \in \mathcal{C}_c^1([0,T[\times\overline{\Omega})$. We can use, for instance, a very classical Galerkin method to solve this problem. The energy estimate is obtained by choosing $\varphi = \tilde\rho_\varepsilon$ in the formulation, it follows :

$$\int_\Omega |\tilde\rho_\varepsilon(T)|^2\,dx + 2\varepsilon\int_0^T \int_\Omega |\nabla\tilde\rho_\varepsilon|^2\,dx\,dt + \int_0^T \int_\Gamma \tilde\rho_\varepsilon^2(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma \\ + \int_0^T \int_\Gamma (\tilde\rho_\varepsilon - \rho^i)^2(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma \le \int_\Omega \rho_0^2\,dx + \int_0^T \int_\Gamma (\rho^i)^2(v\cdot\boldsymbol{\nu})^-\,d\sigma\,dt + 2\int_0^T \int_\Gamma \tilde\rho_\varepsilon f\,dt\,dx. \tag{4.6}$$

Hence, there exists a solution $\tilde\rho_\varepsilon \in L^\infty(]0,T[,L^2(\Omega)) \cap L^2(]0,T[,H^1(\Omega)) \cap \mathcal{C}^0([0,T],L^2(\Omega))$, and we deduce the bounds (4.3)-(4.4) and the uniqueness of the solution. ■

We can now prove a maximum principle for $\tilde\rho_\varepsilon$. More precisely we have :

Lemma 4.2
Let us define

$$\rho_{\max} = \max\left(\operatorname*{esssup}_{\Omega} \rho_0, \operatorname*{esssup}_{]0,T[\times\Gamma, d\mu_v^-} \rho^i\right),$$

and

$$\rho_{\min} = \min\left(\operatorname*{essinf}_{\Omega} \rho_0, \operatorname*{essinf}_{]0,T[\times\Gamma, d\mu_v^-} \rho^i\right).$$

We have

$$\rho_{\min} - \int_0^t \|f(s)\|_{L^\infty}\, ds \le \tilde\rho_\varepsilon(t,x) \le \rho_{\max} + \int_0^t \|f(s)\|_{L^\infty}\, ds, \quad \textit{for almost every } (t,x) \in]0,T[\times\Omega,$$

and

$$\rho_{\min} - \int_0^t \|f(s)\|_{L^\infty}\, ds \le \tilde\rho_\varepsilon(t,\sigma) \le \rho_{\max} + \int_0^t \|f(s)\|_{L^\infty}\, ds, \quad \textit{for } d\mu_v^+\textit{-almost every } (t,\sigma) \in]0,T[\times\Gamma.$$

Proof :
Let us denote by F the non-decreasing and bounded function defined by

$$F(t) = \int_0^t \|f(s)\|_{L^\infty}\, ds.$$

The equation being linear, it is enough to prove that if $\rho_{\min} = 0$, then we have

$$\tilde\rho_\varepsilon(t,x) \ge -F(t), \quad \text{for almost every } (t,x) \in]0,T[\times\Omega, \tag{4.7}$$

$$\tilde\rho_\varepsilon(t,\sigma) \ge -F(t), \quad \text{for } d\mu_v^+\text{-almost every } (t,\sigma) \in]0,T[\times\Gamma. \tag{4.8}$$

Let β be the real function defined by $\beta(s) = s^2$ for $s \le 0$ and $\beta(s) = 0$ for $s \ge 0$. If we multiply the equation (4.2) by the function $\varphi(t,x) = \beta'\left(\tilde\rho_\varepsilon(t,x) + F(t)\right)$, we get after integrations by parts

$$\begin{aligned}
&\partial_t \int_\Omega \beta\left(\tilde\rho_\varepsilon + F(t)\right)\, dx - \int_\Omega \|f(t)\|_{L^\infty} \beta'\left(\tilde\rho_\varepsilon + F(t)\right)\, dx \\
&\quad + \int_\Gamma \beta(\tilde\rho_\varepsilon + F(t))(v\cdot\boldsymbol{\nu})\, d\sigma + \varepsilon \int_\Omega \beta''(\tilde\rho_\varepsilon + F(t))|\nabla\tilde\rho_\varepsilon|^2\, dx + \int_\Gamma (\tilde\rho_\varepsilon - \rho^i)(v\cdot\boldsymbol{\nu})^- \beta'(\tilde\rho_\varepsilon + F(t))\, d\sigma \\
&\quad = \int_\Omega f(t)\beta'(\tilde\rho_\varepsilon + F(t))\, dx.
\end{aligned}$$

Notice that we have $f(t,x) + \|f(t)\|_{L^\infty} \ge 0$ for almost every $(t,x) \in]0,T[\times\Omega$, and that $\beta' \le 0$ and $\beta'' \ge 0$. It follows

$$\begin{aligned}
&\partial_t \int_\Omega \beta\left(\tilde\rho_\varepsilon + F(t)\right)\, dx + \int_\Gamma \beta(\tilde\rho_\varepsilon + F(t))(v\cdot\boldsymbol{\nu})^+\, d\sigma \\
&\quad \le \int_\Gamma \Big(\beta(\tilde\rho_\varepsilon + F(t)) + (\rho^i - \tilde\rho_\varepsilon)\beta'(\tilde\rho_\varepsilon + F(t))\Big)(v\cdot\boldsymbol{\nu})^-\, d\sigma,
\end{aligned}$$

and since β is a convex function we get:

$$\partial_t \int_\Omega \beta\left(\tilde{\rho}_\varepsilon + F(t)\right)\, dx + \int_\Gamma \beta(\tilde{\rho}_\varepsilon + F(t))(v\cdot\boldsymbol{\nu})^+\, d\sigma \le \int_\Gamma \beta(\rho^i + F(t))(v\cdot\boldsymbol{\nu})^-\, d\sigma.$$

Integrating in time we get for any $s \in [0, T]$

$$\int_\Omega \beta\left(\tilde{\rho}_\varepsilon(s) + F(s)\right)\, dx + \int_0^s \int_\Gamma \beta(\tilde{\rho}_\varepsilon + F(t))(v\cdot\boldsymbol{\nu})^+\, d\sigma\, dt \le \int_\Omega \beta(\rho_0)\, dx + \int_0^s \int_\Gamma \beta(\rho^i + F(t))(v\cdot\boldsymbol{\nu})^-\, d\sigma\, dt,$$

but since we have assumed that $\rho_{\min} = 0$, and thanks to the choice of β the right-hand side member of this inequality is zero. It follows that

$$\int_\Omega \beta\left(\tilde{\rho}_\varepsilon(s) + F(s)\right)\, dx = 0, \quad \forall s \in [0, T],$$

$$\int_0^T \int_\Gamma \beta\left(\tilde{\rho}_\varepsilon(t) + F(t)\right)(v\cdot\boldsymbol{\nu})^+\, d\sigma\, dt = 0.$$

Since $\beta(y) = 0$ if and only if $y \ge 0$, inequalities (4.7) and (4.8) follow. ■

The two previous lemmas show that $(\tilde{\rho}_\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0, T[\times\Omega)$ and that the trace of $\tilde{\rho}_\varepsilon$ is bounded in $L^\infty(]0, T[\times\Gamma, d\mu_v^+)$, hence we can find a subsequence $(\tilde{\rho}_{\varepsilon_k})_k$, a function $\rho \in L^\infty(]0, T[\times\Omega)$ and a function $\rho^o \in L^\infty(]0, T[\times\Gamma, d\mu_v^+)$ such that

$$\begin{aligned} &\tilde{\rho}_{\varepsilon_k} \underset{\varepsilon\to 0}{\rightharpoonup} \rho, \ \text{ in } L^\infty(]0, T[\times\Omega) \text{ weak-}\star, \\ &\tilde{\rho}_{\varepsilon_k} \underset{\varepsilon\to 0}{\rightharpoonup} \rho^o, \ \text{ in } L^\infty(]0, T[\times\Gamma, d\mu_v^+) \text{ weak-}\star. \end{aligned} \tag{4.9}$$

Thanks to these convergences and to the estimate (4.3), it is now straightforward to perform the limit in the linear equation (4.5) which proves that (ρ, ρ^o) is a solution to the problem.

The continuity in time of this solution is given by Theorem 3.1.

- **Uniqueness :**

Let (ρ_1, ρ_1^o) and (ρ_2, ρ_2^o) be two solutions of (4.1). Let us introduce $\rho = \rho_1 - \rho_2$ and $\rho^o = \rho_1^o - \rho_2^o$. It is clear that (ρ, ρ^o) is solution of (4.1) with $\rho_0 = 0$, $f = 0$ and $\rho^i = 0$. Hence, we have for any test function $\varphi \in \mathcal{C}^1([0, T]\times\overline{\Omega})$

$$\int_0^T \int_\Omega \rho(\partial_t\varphi + v\cdot\nabla\varphi)\, dt\, dx - \int_\Omega \rho(T)\varphi(T)\, dx - \int_0^T \int_\Gamma \rho^o\varphi(v\cdot\boldsymbol{\nu})^+\, d\sigma\, dt = 0. \tag{4.10}$$

We have seen in Theorem 3.1 (and Remark 3.1) that there exists a unique function $\gamma\rho \in L^\infty(]0, T[\times\Gamma, |d\mu_v|)$ such that for any $\beta \in \mathcal{C}^1(\mathbb{R})$ and any $\varphi \in \mathcal{C}^1([0, T]\times\overline{\Omega})$

$$\int_0^T \int_\Omega \beta(\rho)(\partial_t\varphi + v\cdot\nabla\varphi)\, dx\, dt - \int_\Omega \beta(\rho(T))\varphi(T)\, dx - \int_0^T \int_\Gamma \beta(\gamma\rho)\varphi(v\cdot\boldsymbol{\nu})\, d\sigma\, dt = 0, \tag{4.11}$$

in particular, for $\beta(s) = s$, we deduce by comparing (4.10) and (4.11) that

$$\gamma\rho\,(v\cdot\boldsymbol{\nu}) = \rho^o(v\cdot\boldsymbol{\nu})^+.$$

Taking now $\beta(s) = s^2$ and $\varphi(t, x) = 1$, we get for any $t_1 \in [0, T]$,

$$-\int_\Omega |\rho(t_1)|^2\, dx - \int_0^{t_1} \int_\Gamma |\rho^o|^2(v\cdot\boldsymbol{\nu})^+\, d\sigma\, dt = 0.$$

If follows that $\rho^o(t, \sigma) = 0$, for $d\mu_v^+$-almost every $(t, \sigma) \in]0, T[\times\Gamma$ and that $\rho(t, x) = 0$ for almost every (t, x) in $]0, T[\times\Omega$ and the claim is proved.

- **Renormalization :**

  It is now clear from (4.1) that the trace $\gamma\rho$ introduced in Theorem 3.1 is nothing but the function defined by

$$\gamma\rho\,(v\cdot\boldsymbol{\nu}) = \rho^o(v\cdot\boldsymbol{\nu})^+ - \rho^i(v\cdot\boldsymbol{\nu})^-,$$

  that is to say the function equal to ρ^o where $(v\cdot\boldsymbol{\nu}) > 0$ and equal to ρ^i where $(v\cdot\boldsymbol{\nu}) < 0$. Hence, (3.1) proves exactly that $(\beta(\rho), \beta(\rho^o))$ is the unique solution of the transport equation with data $(\beta(\rho_0), \beta(\rho^i))$ and source term $\beta'(\rho)f$.

■

In fact, the uniqueness property of the solution to the boundary and initial value problem (4.1) is a particular case of the following comparison result which can be proved in a similar way.

Proposition 4.1
Let $\rho_0^1, \rho_0^2 \in L^\infty(\Omega)$, $f_1, f_2 \in L^1(]0,T[, L^\infty(\Omega))$ and $\rho_1^i, \rho_2^i \in L^\infty(]0,T[\times\Gamma, d\mu_v^-)$. Let (ρ_1, ρ_1^o) and (ρ_2, ρ_2^o) be the solutions to (4.1) respectively with data $(\rho_0^1, \rho_1^i, f_1)$ and $(\rho_0^2, \rho_2^i, f_2)$.

If we have $\rho_0^1 \leq \rho_0^2$ and $f_1 \leq f_2$, almost everywhere for the Lebesgue-measure, and $\rho_1^i \leq \rho_2^i$ almost everywhere for the measure $d\mu_v^-$, then we have $\rho_1 \leq \rho_2$, and $\rho_1^o \leq \rho_2^o$ almost everywhere with respect to the Lebesgue measure and to $d\mu_v^+$ respectively.

We can also prove that the product of two solutions to the transport equation is a solution to the same transport equation with the corresponding source term and that the trace of the product is the product of the traces.

Proposition 4.2
If ρ_1 and ρ_2 are two bounded solutions of the transport equation with source terms f_1 and f_2 in $L^1(]0,T[\times\Omega)$ then the product $\rho_1\rho_2$ is solution to the transport equation with the source term $\rho_1 f_2 + \rho_2 f_1$. Furthermore we have $\gamma(\rho_1\rho_2) = \gamma\rho_1 \cdot \gamma\rho_2$.

The sketch of the proof consists in taking $\varphi(\rho_1 \star_{\boldsymbol{\nu}} \eta_\varepsilon)$ as a test function in the equation satisfied by ρ_2 and to pass to the limit when ε tends to zero (see [13] for a similar proof).

To conclude this section, we can now describe more precisely the convergence of the parabolic approximations $(\tilde\rho_\varepsilon)_\varepsilon$ introduced above towards the solution of the initial and boundary value problem for the transport equation. In particular, we show that the trace of the approximate solution strongly converge towards the trace of the solution to the transport equation.

Proposition 4.3
If $\tilde\rho_\varepsilon$ is the solution to (4.2) and (ρ, ρ^o) the solution to (4.1), then we have for any $1 \leq p < +\infty$

$$\tilde\rho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \gamma\rho \text{ in } L^p(]0,T[\times\Gamma, |d\mu_v|),$$

$$\tilde\rho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \rho \text{ in } \mathcal{C}^0([0,T], L^p(\Omega)).$$

Proof :
First of all, since the solution to (4.1) is unique we deduce that the weak convergences (4.9) remain valid for the whole family $(\tilde\rho_\varepsilon)_\varepsilon$. Furthermore, it is clear that for any $t\in[0,T]$, $(\tilde\rho_\varepsilon(t))_\varepsilon$ converges weakly towards $\rho(t)$ in $L^2(\Omega)$.

Using the energy estimate (4.6), we deduce that

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega |\tilde\rho_\varepsilon(T)|^2\,dx + \int_0^T\int_\Gamma \tilde\rho_\varepsilon^2(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma\right) \leq \int_\Omega \rho_0^2\,dx + \int_0^T\int_\Gamma (\rho^i)^2(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + 2\int_0^T\int_\Omega f\rho\,dt\,dx,$$

but thanks to the renormalization property of the solution of (4.1) we have:

$$\int_\Omega |\rho(T)|^2\,dx + \int_0^T\int_\Gamma (\rho^o)^2(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma = \int_\Omega \rho_0^2\,dx + \int_0^T\int_\Gamma (\rho^i)^2(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + 2\int_0^T\int_\Omega f\rho\,dt\,dx.$$

Hence, we get:

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega |\tilde\rho_\varepsilon(T)|^2\,dx + \int_0^T\int_\Gamma \tilde\rho_\varepsilon^2 (v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma\right) \le \int_\Omega |\rho(T)|^2\,dx + \int_0^T\int_\Gamma (\rho^o)^2(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma.$$

Using the weak convergences (4.9), it follows that

$$\tilde\rho_\varepsilon(T) \underset{\varepsilon\to 0}{\longrightarrow} \rho(T), \text{ in } L^2(\Omega),$$

$$\tilde\rho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \rho^o, \text{ in } L^2(]0,T[\times\Gamma, d\mu_v^+).$$

Furthermore, the energy estimate (4.6) gives:

$$2\varepsilon\|\nabla\tilde\rho_\varepsilon\|^2_{L^2(]0,T[\times\Omega)} + \|\tilde\rho_\varepsilon - \rho^i\|^2_{L^2(]0,T[\times\Gamma,d\mu_v^-)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.12}$$

so that the first claim of the proposition follows.

Let us consider $\rho_\varepsilon = \rho \star_{\boldsymbol{\nu}} \eta_\varepsilon$ the family of regularized functions introduced in section 2.2. We have already shown in the proof of Theorem 3.1 that $(\rho_\varepsilon)_\varepsilon$ converge towards ρ in $\mathcal{C}^0([0,T],L^p(\Omega))$ for any $p<+\infty$. The second claim of the proposition will immediately follow if we show that

$$\|\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}\|_{\mathcal{C}^0([0,T],L^2(\Omega))} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Using (3.3), we see that the equation satisfied by $\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}$ is

$$\partial_t(\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}) + v\cdot\nabla(\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}) - \varepsilon\Delta\tilde\rho_\varepsilon = f - f\star_{\boldsymbol{\nu}}\eta_{\sqrt\varepsilon} - R_{\sqrt\varepsilon}.$$

Multiplying this equation by $\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}$ and integrating in space and time, we get for any $s\in[0,T]$

$$\begin{aligned}\frac12\|\tilde\rho_\varepsilon(s) - \rho_{\sqrt\varepsilon}(s)\|^2_{L^2} =& \frac12\|\rho_0 - \rho_0\star_{\boldsymbol{\nu}}\eta_{\sqrt\varepsilon}\|^2_{L^2(\Omega)} + \int_0^s\int_\Omega (f - f\star_{\boldsymbol{\nu}}\eta_{\sqrt\varepsilon} - R_{\sqrt\varepsilon})(\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon})\,dt\,dx\\ &- \varepsilon\int_0^s\int_\Omega \nabla\tilde\rho_\varepsilon\cdot\nabla(\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon})\,dt\,dx - \frac12\int_0^s\int_\Gamma |\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}|^2(v\cdot\boldsymbol{\nu})\,dt\,d\sigma\\ &- \int_0^s\int_\Gamma (\tilde\rho_\varepsilon - \rho^i)(\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon})(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma.\end{aligned}$$

It follows that

$$\begin{aligned}\frac12\|\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}\|^2_{\mathcal{C}^0([0,T],L^2(\Omega))} \le& \frac12\|\rho_0 - \rho_0\star_{\boldsymbol{\nu}}\eta_{\sqrt\varepsilon}\|^2_{L^2(\Omega)} + C\|f - f\star_{\boldsymbol{\nu}}\eta_{\sqrt\varepsilon}\|_{L^1(]0,T[\times\Omega)} + C\|R_{\sqrt\varepsilon}\|_{L^1(]0,T[\times\Omega)}\\ &+ \varepsilon\|\nabla\tilde\rho_\varepsilon\|^2_{L^2} + (\sqrt\varepsilon\|\nabla\tilde\rho_\varepsilon\|_{L^2(]0,T[\times\Omega)})(\sqrt\varepsilon\|\nabla\rho_{\sqrt\varepsilon}\|_{L^2(]0,T[\times\Omega)})\\ &+ C\|\tilde\rho_\varepsilon - \rho^i\|_{L^2(]0,T[\times\Gamma,d\mu_v^-)} + C\|\tilde\rho_\varepsilon - \rho_{\sqrt\varepsilon}\|^2_{L^2(]0,T[\times\Gamma,|d\mu_v|)}.\end{aligned}$$

Using Lemmas 2.1 and 3.1 and the convergence (4.12), we see that the right member of this inequality tends to 0 with ε so that the claim is proved. ■

5 Trace operator in the L^p case ($p<+\infty$)

In the case of Lipschitz vector fields v, or in the case of some kinetic equations similar to the transport equation, it was remarked in [2, 4, 5, 13] that the traces of a solution $\rho\in L^\infty(]0,T[,L^p(\Omega))$ when $p<+\infty$ may not belong to $L^p(]0,T[\times\Gamma,|d\mu_v|)$ but only in a larger space (see in particular a counter-example in [2]). In order to define this *trace space* it is necessary to introduce the life time in Ω associated to the vector field v.

5.1 Life time in Ω

By Theorem 4.1, there exists an unique solution τ_- to the problem

$$\begin{cases} \partial_t \tau_- + v \cdot \nabla \tau_- = 1, \\ \tau_-(0) = 0, \\ \gamma\tau_- = 0, \quad \text{on }]0,T[\times\Gamma \text{ where } (v\cdot\boldsymbol{\nu}) < 0. \end{cases} \tag{5.1}$$

Using the comparison principle (see Proposition 4.1) we easily get that

$$0 \leq \tau_-(t,x) \leq t, \quad \text{for almost every } (t,x) \in]0,T[\times\Omega,$$

$$0 \leq \gamma\tau_-(t,\sigma) \leq t, \quad \text{for } |d\mu_v|\text{-almost every } (t,\sigma) \in]0,T[\times\Gamma.$$

If v is regular enough, then τ_- can be computed explicitly using the characteristics of the velocity field v. Indeed, in this case $\tau_-(t,x)$ is exactly the time that a material point evolving with the flow of v, being at position x at time t has spent in $]0,T[\times\Omega$ since it enters the domain. Hence, $\tau_-(t,x) = t$ if the material point has not touched the boundary between times 0 and t.

Similarly we are able to solve the following final value problem (by reverting the time and changing v in $-v$ for instance)

$$\begin{cases} \partial_t \tau_+ + v \cdot \nabla \tau_+ = -1, \\ \tau_+(T) = 0, \\ \gamma\tau_+ = 0, \quad \text{on }]0,T[\times\Gamma \text{ where } (v\cdot\boldsymbol{\nu}) > 0, \end{cases}$$

and we have

$$0 \leq \tau_+(t,x) \leq T-t, \quad \text{for almost every } (t,x) \in]0,T[\times\Omega,$$

$$0 \leq \gamma\tau_+(t,\sigma) \leq T-t, \quad \text{for } |d\mu_v|\text{-almost every } (t,\sigma) \in]0,T[\times\Gamma.$$

Once again, if v is regular enough, then τ_+ is the time that a material point at the position x at time t will spent in Ω before reaching the boundary of $]0,T[\times\Omega$.

Finally we will denote by $\tau = \tau_+ + \tau_-$ the total life time in Ω. It satisfies

$$0 \leq \tau(t,x) \leq T, \quad \text{for almost every } (t,x) \in]0,T[\times\Omega,$$

$$0 \leq \gamma\tau(t,\sigma) \leq T, \quad \text{for } |d\mu_v|\text{-almost every } (t,\sigma) \in]0,T[\times\Gamma.$$

Hence, $\tau(t,x)$ is the total time that the trajectory passing through (t,x) in the phase plane has spent in Ω without reaching the boundary of $]0,T[\times\Omega$.

In fact, we can show that τ and $\gamma\tau$ are positive almost everywhere. More precisely, we have the following result.

Proposition 5.1
We have

$$\tau_+(t_0,x) > 0, \quad \forall t_0 \in [0,T[, \quad \text{for almost every } x \in \Omega,$$

$$\tau_-(t_0,x) > 0, \quad \forall t_0 \in]0,T], \quad \text{for almost every } x \in \Omega,$$

$$\gamma\tau(t,\sigma) > 0, \quad \text{for } |d\mu_v|\text{-almost every } (t,\sigma) \in]0,T[\times\Gamma.$$

As a consequence, the non-negative measures $|d\mu_v|$, $d\mu_v^\alpha$ and $\gamma\tau|d\mu_v|$ have the same zero measure sets on $]0,T[\times\Gamma$.

Proof :

1. Suppose that there exists $t_0 < T$ and a set $E \subset \Omega$ of positive Lebesgue measure such that $\tau_+(t_0, x) = 0$ for any $x \in E$. Let us consider ζ the unique solution on $[t_0, T]$ to the problem

$$\begin{cases} \partial_t \zeta + v \cdot \nabla \zeta = 0, \\ \zeta(t_0) = \mathbf{1}_E, \\ \gamma\zeta = 0, \ \text{ on }]t_0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) < 0, \end{cases}$$

where $\mathbf{1}_E$ stands for the characteristic function of the set E.

Using Proposition 4.2 and since $\tau_+(t_0)$ vanishes on the set E, we know that the function $\tau_+\zeta$ is solution of the problem

$$\begin{cases} \partial_t(\tau_+\zeta) + v \cdot \nabla(\tau_+\zeta) = -\zeta, \\ \tau_+\zeta(t_0) = 0, \\ \tau_+\zeta(T) = 0, \\ \gamma(\tau_+\zeta) = 0, \ \text{ on }]t_0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) > 0, \\ \gamma(\tau_+\zeta) = 0, \ \text{ on }]t_0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) < 0. \end{cases}$$

We deduce that we have

$$0 = -\int_{t_0}^{T} \int_\Omega \zeta \, dx \, dt,$$

and so ζ, which is non-negative, must be identically zero. This is impossible since $\zeta(t_0) = \mathbf{1}_E$ and E has a non zero Lebesgue measure. The second point is proved in the same way.

2. Let us prove that $\gamma\tau_+(t, \sigma) > 0$ for $d\mu_v^-$-almost every $(t, \sigma) \in]0, T[\times\Gamma$. If we suppose that it is not true, then there exists $E \subset]0, T[\times\Gamma$ such that $d\mu_v^-(E) > 0$ and $\gamma\tau_+(t, \sigma) = 0$ for all $(t, \sigma) \in E$.

   Let ζ be the unique solution of the following problem

$$\begin{cases} \partial_t \zeta + v \cdot \nabla \zeta = 0, \\ \zeta(0) = 0, \\ \gamma\zeta = \mathbf{1}_E, \ \text{ on }]0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) < 0. \end{cases}$$

   Since, $\gamma\tau_+ = 0$ on the set E, we can see that the function $\tau_+\zeta$ is solution to the problem

$$\begin{cases} \partial_t(\tau_+\zeta) + v \cdot \nabla(\tau_+\zeta) = -\zeta, \\ \tau_+\zeta(0) = 0, \\ \tau_+\zeta(T) = 0, \\ \gamma(\tau_+\zeta) = 0, \ \text{ on }]0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) > 0, \\ \gamma(\tau_+\zeta) = 0, \ \text{ on }]0, T[\times\Gamma \text{ where } (v \cdot \boldsymbol{\nu}) < 0. \end{cases}$$

   If we integrate the previous equation on $]0, T[\times\Omega$, we find that

$$0 = \int_0^T \int_\Omega \zeta \, dx \, dt,$$

   which implies that $\zeta = 0$. Theorem 3.1 implies that the trace of ζ on $]0, T[\times\Gamma$ is zero $|d\mu_v|$-almost everywhere. This is impossible since by construction this trace is equal to 1 on E which has a positive $d\mu_v^-$-measure.

   Using the same method one can show that $\gamma\tau_-(t, \sigma) > 0$ for $d\mu_v^+$-almost every $(t, x) \in]0, T[\times\Gamma$.

■

5.2 Trace space for finite values of p

In the case $p = +\infty$, we have shown that the trace $\gamma\rho$ lies in the space $L^\infty(]0,T[\times\Gamma, |d\mu_v|)$. In the case $p < +\infty$, we only know at that time that $\gamma\rho$ is measurable. We can now define the space in which the traces $\gamma\rho$ are lying as follows.

Theorem 5.1
Suppose that $p < +\infty$ and that v satisfies the assumptions (2.5). For any $f \in L^1(]0,T[, L^p(\Omega))$ and any $\rho \in L^\infty(]0,T[, L^p(\Omega))$ solution to

$$\partial_t \rho + v \cdot \nabla \rho = f,$$

the trace $\gamma\rho$ lies in the space $L^p(]0,T[\times\Gamma, \gamma\tau|d\mu_v|)$ and we have

$$\|\gamma\rho\|_{L^p(]0,T[\times\Gamma,\gamma\tau|d\mu_v|)} \leq C(\|\rho\|_{L^\infty(]0,T[,L^p(\Omega))} + \|f\|_{L^1(]0,T[,L^p(\Omega))}),$$

where C depends only on T and p.

Notice that, thanks to Proposition 5.1, this theorem implies in particular that $\gamma\rho$ is finite $|d\mu_v|$-almost everywhere as claimed during the proof of Theorem 3.1.

Proof :
We choose $\varphi = \tau_- - \tau_+$ as a test function in (3.1). Notice that this function satisfies

$$\partial_t \varphi + v \cdot \nabla \varphi = 2.$$

Furthermore one can easily check that we have $\gamma\varphi\,(v \cdot \boldsymbol{\nu}) = \gamma\tau\,|v \cdot \boldsymbol{\nu}|$. Hence, we get

$$\int_0^T \int_\Gamma \beta(\gamma\rho)\gamma\tau\,|v \cdot \boldsymbol{\nu}|\,d\sigma\,dt = 2\int_0^T \int_\Omega \beta(\rho)\,dx\,dt - \int_\Omega \beta(\rho_0)\tau_+(0)\,dx - \int_\Omega \beta(\rho(T))\tau_-(T)\,dx + \int_0^T \int_\Omega \beta'(\rho) f \varphi\,dx\,dt.$$

For any $n \geq 0$ we choose $\beta_n(s) = \frac{|s|^p}{1+\frac{1}{n}|s|^p}$ which satisfies

$$|\beta_n(s)| \leq C|s|^p, \quad |\beta_n'(s)| \leq C|x|^{p-1}.$$

The previous estimate gives, since $|\varphi| \leq T$,

$$\begin{aligned}\int_0^T \int_\Gamma \beta_n(\gamma\rho)\gamma\tau\,|v \cdot \boldsymbol{\nu}|\,d\sigma\,dt &\leq C\int_0^T \int_\Omega |\rho|^p\,dx\,dt + C\int_0^T \int_\Omega |\rho|^{p-1} f\,dx\,dt \\ &\leq C\|\rho\|^p_{L^\infty(]0,T[,L^p(\Omega))} + C\|f\|^p_{L^1(]0,T[,L^p(\Omega))}.\end{aligned}$$

When n goes to infinity, $\beta_n(s)$ is non-decreasing and converges toward $|s|^p$. The claim follows using the monotone convergence theorem.

■

6 Initial and boundary value problem in L^p

We are now able to solve the initial and boundary value problem in the case $p < +\infty$. Notice that even though in general it is not true that $\gamma\rho \in L^p(]0,T[\times\Gamma, |d\mu_v|)$ (see the beginning of section 5), we show in the following result that if the inflow data ρ^i lies in $L^p(]0,T[\times\Gamma, d\mu_v^-)$ then the whole trace $\gamma\rho$ lies in $L^p(]0,T[\times\Gamma, |d\mu_v|)$. This fact was already pointed out in similar situations in [4, 5, 13].

Theorem 6.1
Let $p \in]1,+\infty[$, v a vector field satisfying (2.5) and $f \in L^1(]0,T[,L^p(\Omega))$. For any initial data $\rho_0 \in L^p(\Omega)$ and any boundary data $\rho^i \in L^p(]0,T[\times\Gamma, d\mu_v^-)$ there exists a unique couple $(\rho,\rho^o) \in L^\infty(]0,T[,L^p(\Omega)) \times L^p(]0,T[\times\Gamma,d\mu_v^+)$ such that for any $\varphi \in \mathcal{C}_c^1([0,T[\times\overline{\Omega})$ we have

$$\int_0^T \int_\Omega \rho(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho_0\varphi(0)\,dx \\ - \int_0^T \int_\Gamma \rho^o\varphi(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma + \int_0^T\int_\Gamma \rho^i\varphi(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + \int_0^T\int_\Omega f\varphi\,dt\,dx = 0. \quad (6.1)$$

Moreover, ρ is continuous in time with values in $L^p(\Omega)$ and the solution is a renormalized solution: for any regular function β such that $|\beta(s)| \leq C(1+|s|^{\bar{p}})$ and $|\beta'(s)| \leq C(1+|s|^{p-1})$, with $\bar{p} = \min(p, \frac{p}{p+d})$, the couple $(\beta(\rho),\beta(\rho^o))$ is a solution to (6.1) with data $(\beta(\rho_0),\beta(\rho^i))$ and source term $\beta'(\rho)f$.

Notice that, without any additional regularity assumption on the vector field v, we do not know in general whether the solution of (6.1) with data $(\beta(\rho_0),\beta(\rho^i))$ and source term $\beta'(\rho)f$ is unique.
Proof :

- **Existence :**

  For any $n \geq 1$, we consider the truncature function $T_n(s) = \min(\max(s,-n),n)$. Thanks to Theorem 4.1, there exists a unique solution $(\rho_n,\rho_n^o) \in L^\infty(]0,T[,L^\infty(\Omega)) \times L^\infty(]0,T[\times\Gamma,d\mu_v^+)$ to (4.1) with data $(T_n(\rho_0),T_n(\rho^i))$ and source term $T_n(f)$.

  Since this solution is a renormalized solution, we have the following estimate, for any $s \in [0,T]$

$$\begin{aligned}\int_\Omega |\rho_n(s)|^p\,dx + \int_0^s\int_\Gamma |\rho_n^o|^p(v\cdot\boldsymbol{\nu})^+\,d\sigma\,dt &= \int_\Omega |T_n(\rho_0)|^p\,dx + \int_0^s\int_\Gamma |T_n(\rho^i)|^p(v\cdot\boldsymbol{\nu})^-\,d\sigma\,dt \\ &\quad + p\int_0^s\int_\Omega |\rho_n|^{p-2}\rho_n T_n(f)\,dx\,dt \\ &\leq \int_\Omega |\rho_0|^p\,dx + \int_0^s\int_\Gamma |\rho^i|^p(v\cdot\boldsymbol{\nu})^-\,d\sigma\,dt \\ &\quad + p\int_0^s \|\rho_n(t)\|_{L^p(\Omega)}^{p-1}\|f(t)\|_{L^p(\Omega)}\,dt.\end{aligned}$$

  Using the Gronwall lemma, this estimate shows that $(\rho_n)_n$ is bounded in $L^\infty(]0,T[,L^p(\Omega))$ and that ρ_n^o is bounded in $L^p(]0,T[\times\Gamma,d\mu_v^+)$. Hence, we can find subsequences $(\rho_{n_k},\rho_{n_k}^o)_k$ which $*$-weakly converge towards a couple (ρ,ρ^o) in the spaces $L^\infty(]0,T[,L^p(\Omega)) \times L^p(]0,T[\times\Gamma,d\mu_v^+)$.

  Since $T_n(\rho_0), T_n(\rho^i)$ and $T_n(f)$ respectively converge towards ρ_0, ρ^i and f in the corresponding spaces, it is easily seen that we can perform the limit in the weak formulation satisfied by the solution ρ_n. Therefore, the couple (ρ,ρ^o) is a solution to the problem.

- **Identification of the trace :**

  Using Theorems 3.1 and 5.1, we know that there exists a unique trace $\gamma\rho \in L^p(]0,T[\times\Gamma,\gamma\tau|d\mu_v|)$ such that we have (3.1) for any test function φ and any function β in $\mathcal{C}_c^1(\mathbb{R})$.

  We want to find the relationship between (ρ^i,ρ^o) satisfying (6.1) on the one hand and $\gamma\rho$ in the other hand. This is not so obvious since we cannot choose $\beta(s) = s$ in (3.1) contrarily to the case $p = +\infty$ (see Remark 3.1).

  Let β be a smooth positive function such that $\beta(s) = \frac{1}{|s|}$ for all $|s| \geq 1$, and let us define $\beta_2(s) = s\beta(s)$. It is easily seen that β, β', β_2 and β_2' are bounded on $\mathbb{R}$.

Let $\psi \in \mathcal{C}^1([0,T[\times\overline{\Omega})$. We can take $\varphi = \beta(\rho_\varepsilon)\psi$ as a test function in (6.1), where $\rho_\varepsilon = \rho\star_{\boldsymbol{\nu}}\eta_\varepsilon$ is the regularization of ρ. Using the convergence properties of the sequence $(\rho_\varepsilon)_\varepsilon$ and of its traces established in the proof of Theorem 3.1, as well as Lemma 3.1, we can perform the limit when $\varepsilon \to 0$. We get:

$$\begin{aligned}\int_0^T\int_\Omega \rho\beta(\rho)(\partial_t\psi + v\cdot\nabla\psi)\,dt\,dx + \int_\Omega \rho_0\beta(\rho_0)\psi(0)\,dx - \int_0^T\int_\Gamma \rho^o\beta(\gamma\rho)\psi(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma \\ + \int_0^T\int_\Gamma \rho^i\beta(\gamma\rho)\psi(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + \int_0^T\int_\Omega f(\beta(\rho)+\rho\beta'(\rho))\psi\,dt\,dx = 0.\end{aligned} \tag{6.2}$$

Now let us take ψ as a test function in (3.1) used with the function β_2 instead of β. It follows:

$$\begin{aligned}\int_0^T\int_\Omega \rho\beta(\rho)(\partial_t\psi + v\cdot\nabla\psi)\,dt\,dx + \int_\Omega \rho_0\beta(\rho_0)\psi(0)\,dx - \int_0^T\int_\Gamma \gamma\rho\,\beta(\gamma\rho)\,\psi(v\cdot\boldsymbol{\nu})\,dt\,d\sigma \\ + \int_0^T\int_\Omega f(\beta(\rho)+\rho\beta'(\rho))\psi\,dt\,dx = 0.\end{aligned} \tag{6.3}$$

By comparison between (6.2) and (6.3), we obtain for any ψ

$$\int_0^T\int_\Gamma \rho^o\beta(\gamma\rho)\psi(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma - \int_0^T\int_\Gamma \rho^i\beta(\gamma\rho)\psi(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma = \int_0^T\int_\Gamma \gamma\rho\,\beta(\gamma\rho)\,\psi(v\cdot\boldsymbol{\nu})\,dt\,d\sigma.$$

This implies:

$$\gamma\rho\,\beta(\gamma\rho)\,(v\cdot\boldsymbol{\nu}) = \rho^o\beta(\gamma\rho)\psi(v\cdot\boldsymbol{\nu})^+ - \rho^i\beta(\gamma\rho)(v\cdot\boldsymbol{\nu})^-,$$

but as β does not vanish, we deduce that

$$\gamma\rho\,(v\cdot\boldsymbol{\nu}) = \rho^o(v\cdot\boldsymbol{\nu})^+ - \rho^i(v\cdot\boldsymbol{\nu})^-. \tag{6.4}$$

This proves that the trace $\gamma\rho$ in the sense of Theorem 3.1 is given by $\gamma\rho = \rho^o$ where $(v\cdot\boldsymbol{\nu}) > 0$ and $\gamma\rho = \rho^i$ where $(v\cdot\boldsymbol{\nu}) < 0$.

- **Renormalization :**

  Using (6.4), the formulation (3.1) gives, for any $\varphi \in \mathcal{C}^1([0,T[\times\overline{\Omega})$,

$$\begin{aligned}&\int_0^T\int_\Omega \beta(\rho)(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \beta(\rho_0)\varphi(0)\,dx \\ &- \int_0^T\int_\Gamma \beta(\rho^o)\varphi(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma + \int_0^T\int_\Gamma \beta(\rho^i)\varphi(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + \int_0^T\int_\Omega \beta'(\rho)f\varphi\,dt\,dx = 0,\end{aligned} \tag{6.5}$$

  for any $\beta \in \mathcal{C}_b^1(\mathbb{R})$. In fact, since $\rho \in L^\infty(]0,T[,L^p(\Omega))$, $\rho^i \in L^p(]0,T[\times\Gamma,d\mu_v^-)$, $\rho^o \in L^p(]0,T[\times\Gamma,d\mu_v^+)$, and $v \in L^1(]0,T[,(W^{1,p'}(\Omega))^d)$ it is easily seen that we can extend the formulation above using a truncature argument to functions $\beta \in \mathcal{C}^1(\mathbb{R})$ such that $|\beta(s)| \leq C(1+|s|^{\bar{p}})$ and $|\beta'(s)| \leq C(1+|s|^{p-1})$, with $\bar{p} = \min(p, \frac{p+d}{d})$. This value of $\bar{p}$ ensures that the term $\beta(\rho)v\cdot\nabla\varphi$ which appears in the weak formulation of the equation is integrable on $]0,T[\times\Omega$ for any smooth test function φ.

- **Uniqueness :**

  Since the equation is linear, it is enough to prove that, if $\rho_0 = 0$, $f = 0$ and $\rho^i = 0$ then $\rho = 0$ and $\rho^o = 0$. This claim is an straightforward consequence of (6.5) with $\beta(s) = \frac{s^2}{1+s^2}$.

- **Time continuity :**

  We already know that ρ lies in $\mathcal{C}^0([0,T],L^q(\Omega))$ for any $q < p$ and that ρ is continuous in time with values in $L^p(\Omega)$ for the weak topology (since $\rho \in L^\infty(]0,T[,L^p(\Omega))$).

Let $t_0, t_1 \in [0,T]$, with $t_0 < t_1$. Let $\varphi_\varepsilon(t,x)$ be defined by $\varphi_\varepsilon(t,x) = 0$ for $t \le t_0$ and $t \ge t_1$, $\varphi_\varepsilon(t,x) = 1$ for $t_0 + \varepsilon \le t \le t_1 - \varepsilon$ and which is linear on $[t_0, t_0+\varepsilon]$ and on $[t_1 - \varepsilon, t_1]$.

Let $\beta \in \mathcal{C}^1(\mathbb{R})$. If we take $\varphi_\varepsilon(t,x)$ as a test function in (6.5), we can perform the limit when ε goes to zero, since $\beta(\rho)$ is continuous in time with values in any $L^q(\Omega)$, $q < +\infty$ and we get

$$\int_\Omega \beta(\rho(t_0))\,dx - \int_\Omega \beta(\rho(t_1))\,dx - \int_{t_0}^{t_1}\int_\Gamma \beta(\rho^o)(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma \\ + \int_{t_0}^{t_1}\int_\Gamma \beta(\rho^i)(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma + \int_{t_0}^{t_1}\int_\Omega \beta'(\rho) f\,dt\,dx = 0,$$

Since $\rho(t_0), \rho(t_1) \in L^p(\Omega)$, we can easily extend this formula to the function $\beta(s) = |s|^p$.

Finally, since $\rho^i \in L^p(]0,T[\times\Gamma, d\mu_v^-)$, $\rho^o \in L^p(]0,T[\times\Gamma, d\mu_v^+)$, and $\rho \in L^\infty(]0,T[, L^p(\Omega))$, we deduce that

$$\lim_{t_1 \to t_0} \|\rho(t_1)\|_{L^p} = \|\rho(t_0)\|_{L^p}.$$

Hence, the continuity of ρ with values in $L^p(\Omega)$ for the strong topology follows.

■

7 Space continuity of solutions to the transport equation

We pointed out in section 1.2 of the introduction an example of the kind of spatial regularity property we can expect for the solutions to the transport equation. Let us now consider the following more general situation for which a simple proof is available.

Let $v = (v_1, v_2) \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^2)^2$ be a bounded smooth vector field with $\operatorname{div}(v) = 0$, and $f \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^2)$ be a smooth bounded source term. Let ρ be a bounded solution to the transport equation

$$\partial_t \rho + v_1 \partial_{x_1}\rho + v_2\partial_{x_2}\rho = f, \tag{7.1}$$

for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with an initial data $\rho(0) = \rho_0 \in L^\infty(\mathbb{R}^2)$. Suppose that we have $v_1(t,x) \ge \delta > 0$ for any (t,x), then (7.1) reads

$$\partial_t\left(\frac{\rho v_1}{v_1}\right) + \partial_{x_1}(\rho v_1) + \partial_{x_2}\left(\frac{v_2}{v_1}\rho v_1\right) = f.$$

Hence, this equation can be seen as a transport equation for the new unknown $R = \rho v_1$, in which x_1 is the "time" variable and (t, x_2) are the "space" variables. More precisely, we have

$$\partial_{x_1} R + w_0 \partial_t R + w_2 \partial_{x_2} R + cR = f, \tag{7.2}$$

where

$$w_0 = \frac{1}{v_1},\ \ w_2 = \frac{v_2}{v_1},\ \ c = \partial_t w_0 + \partial_{x_2} w_2.$$

The vector field (w_0, w_2) and the scalar function c are smooth and bounded. Hence the results of Di Perna and Lions [10] apply for this equation, at least locally, and we deduce that $R = \rho v_1$ is continuous with respect to the "time" variable (that is to say x_1) with values in any $L^q_{loc}(\mathbb{R}^2)$ in the variables (t, x_2), for any $q < +\infty$. Since v_1 is smooth and does not vanish, we deduce that ρ itself is continuous with respect to x_1 with values in some local Lebesgue space in the (t, x_2) variables.

If we want to make this proof rigorous for less regular vector fields v, we know that the study of (7.2) requires at least that (w_0, w_2) is integrable with respect to the "time" variable x_1 with values in a Sobolev space $W^{1,1}$ in the "space" variables (t, x_2). In particular, we see that it is needed to suppose some integrability condition on $\partial_t v$. Unfortunately, we have seen that in the applications $\partial_t v$ may not be integrable and we wish to avoid such an assumption.

The aim of the present section is to prove such continuity in space results for the weak solutions of the transport equation without any time regularity assumption on v. The proof we propose is very different from the one above in the case of smooth vector fields. It fundamentally relies on the properties of the traces that we established in the previous sections of this paper.

We consider here a vector field v satisfying (2.5) as well as the additional assumption

$$(v \cdot \boldsymbol{\nu}) \in \mathcal{C}^0_\xi([0, \xi_\Omega[, L^\alpha(]0, T[\times \Gamma)), \tag{7.3}$$

for a given $\alpha > 1$. As it was remarked in section 2.1, this last assumption is satisfied in particular by any v lying in $L^\alpha(]0, T[, (W^{\frac{1}{\alpha}+\varepsilon,\alpha}(\Omega))^d)$. In the applications to the study of incompressible viscous fluids for instance, the velocity field is known to lie in the energy space $L^2(]0, T[, (H^1(\Omega))^d)$ and then satisfies this additional assumption (see [3]).

7.1 Space continuity of bounded solutions

Notice that for any ξ small enough, Ω_ξ (defined in section 2.1) is a regular domain whose boundary is Γ_ξ. Thanks to (7.3), the velocity field v satisfies the assumptions (2.5) in each domain Ω_ξ. Hence, one can apply all the results of the previous sections for each domain Ω_ξ, and in particular we know that any function ρ_ξ solution to the transport equation in Ω_ξ has a trace denoted by $\gamma_\xi \rho_\xi$ on $]0, T[\times \Gamma_\xi$ defined almost everywhere for the measure $|v(t, \xi, \sigma) \cdot \boldsymbol{\nu}(\sigma)|\, dt\, d\sigma$.

The purpose of the first result below is to show that if we are given a family of solutions to the transport equation in each domain Ω_ξ and if this family of solutions depends continuously on ξ then, the family of the traces on Γ_ξ of these solutions also enjoy a continuity property with respect to ξ. The precise statement is the following.

Theorem 7.1
Let $f \in L^1(]0, T[, L^\infty(\Omega))$ and v satisfying (2.5) and (7.3). For any $\xi \in [0, \xi_\Omega[$, let ρ_ξ be a bounded solution to the transport equation (1.1) in the domain Ω_ξ. We extend ρ_ξ by zero in $\mathcal{O}_\xi = \Omega \backslash \overline{\Omega_\xi}$.

Suppose that the family $(\rho_\xi)_\xi$ is bounded in $L^\infty(]0, T[\times \Omega)$ and continuous with respect to ξ with values in the space $\mathcal{C}^0([0, T], L^q(\Omega))$ for any $q < +\infty$. Then, the family of functions $\gamma_\xi \rho_\xi (v(., \xi, .) \cdot \boldsymbol{\nu})$ is continuous with respect to ξ with values in $L^\alpha(]0, T[\times \Gamma)$.

Thanks to this theorem, we are able to prove that any bounded solution of the transport equation in Ω enjoys a continuity property with respect to the normal coordinate near the boundary Γ.

Theorem 7.2
Let $f \in L^1(]0, T[, L^\infty(\Omega))$ and v satisfying (2.5) and (7.3). Let ρ be any solution in $L^\infty(]0, T[\times \Omega)$ to the transport equation (1.1) in Ω.

The map

$$(t, \xi, \sigma) \in]0, T[\times [0, \xi_\Omega[\times \Gamma \mapsto \rho(t, \xi, \sigma)(v(t, \xi, \sigma) \cdot \boldsymbol{\nu}(\sigma)),$$

is continuous with respect to $\xi \in [0, \xi_\Omega[$ with values in $L^\alpha(]0, T[\times \Gamma)$.

Actually, the previous theorem let us deduce in a straightforward way a more general continuity in space result for any solution of the transport equation which is the main result of this section. Roughly speaking, the following result shows that the solution is continuous across any smooth hypersurface in Ω which is not characteristic for the vector field v.

Theorem 7.3
Let $f \in L^1(]0, T[, L^\infty(\Omega))$ and v satisfying (2.5) and such that $v \in L^\alpha(]0, T[, (W^{\frac{1}{\alpha}+\varepsilon,\alpha}(\Omega))^d)$ with $\varepsilon > 0$. Let ρ be any solution in $L^\infty(]0, T[\times \Omega)$ of the transport equation in Ω.

Consider any regular domain $\omega \Subset \Omega$ with boundary $S = \partial\omega$ and normal vector field $\boldsymbol{N}$ defined in a neighborhood of S. The function $\rho(v \cdot N)$ is continuous with respect to the normal coordinate relative to S and with values in $L^2(]0, T[\times S)$ in a neighborhood of S.

In particular, if there exist $\delta_1, \delta_2 > 0$ and $]t_0, t_1[\subset]0, T[$, such that

$$|v(t,x)\cdot N(x)| \geq \delta_1, \quad \text{for almost every } (t,x) \text{ in } U_{t_0,t_1,S,\delta_2} \equiv]t_0,t_1[\times\{x\in\Omega,\ \ d(x,S)\leq\delta_2\},$$

then ρ is continuous in U_{t_0,t_1,S,δ_2} with respect to the normal coordinate relative to S and with values in $L^2(]t_0,t_1[\times S)$.

Notice that this result also applies for bounded solutions of the transport equation in unbounded domain (with appropriate assumptions on the behavior of v and f at infinity, see [7, 10]) if we restrict ourselves to bounded subdomains in Ω.

Let us now prove Theorems 7.1 and 7.2.

Proof (of Theorem 7.1):

Since $(v\cdot\boldsymbol{\nu})$ lies in $\mathcal{C}^0_\xi([0,\xi_\Omega[, L^\alpha(]0,T[\times\Gamma))$, we can add any constant to each ρ_ξ so that we can suppose that $\rho_\xi \geq 0$ and $\gamma_\xi\rho_\xi \geq 0$. Let $\varphi \in \mathcal{C}^\infty([0,T]\times\overline{\Omega})$ be a smooth function and $0 \leq \xi_0 < \xi_1 < \xi_\Omega$ (the case $\xi_1 < \xi_0$ can be treated in the same way). For any $q \in [1,+\infty[$, let us take φ as a test function both for the equation satisfied by $\rho^q_{\xi_0}$ in Ω_{ξ_0} and for the equation satisfied by $\rho^q_{\xi_1}$ in Ω_{ξ_1}. We get for $i\in\{0,1\}$,

$$\begin{aligned}\int_0^T\int_{\Gamma_{\xi_i}}(\gamma_{\xi_i}\rho_{\xi_i})^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma &= \int_0^T\int_{\Omega_{\xi_i}}\big(\rho^q_{\xi_i}(\partial_t\varphi + v\cdot\nabla\varphi) + q\rho^{q-1}_{\xi_i}f\varphi\big)\,dt\,dx\\ &\quad + \int_{\Omega_{\xi_i}}\rho_{\xi_i}(0)^q\varphi(0)\,dx - \int_{\Omega_{\xi_i}}\rho_{\xi_i}(T)^q\varphi(T)\,dx.\end{aligned}$$

Using the fact that $\rho_{\xi_i} = 0$ outside Ω_{ξ_i}, it follows that

$$\begin{aligned}&\int_0^T\int_{\Gamma_{\xi_0}}(\gamma_{\xi_0}\rho_{\xi_0})^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma - \int_0^T\int_{\Gamma_{\xi_1}}(\gamma_{\xi_1}\rho_{\xi_1})^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma\\ &\qquad = \int_0^T\int_\Omega\big((\rho^q_{\xi_0}-\rho^q_{\xi_1})(\partial_t\varphi + v\cdot\nabla\varphi) + q(\rho^{q-1}_{\xi_0}-\rho^{q-1}_{\xi_1})f\varphi\big)\,dt\,dx\\ &\qquad + \int_\Omega(\rho_{\xi_0}(0)^q-\rho_{\xi_1}(0)^q)\varphi(0)\,dx - \int_\Omega(\rho_{\xi_0}(T)^q-\rho_{\xi_1}(T)^q)\varphi(T)\,dx.\end{aligned}$$

Thanks to the continuity assumption of the family $(\rho_\xi)_\xi$ with respect to ξ, we deduce that the right member of this identity tends to zero when ξ_1 converges towards ξ_0. Using the change of variables (2.3), we deduce:

$$\begin{aligned}&\int_0^T\int_\Gamma(\gamma_{\xi_0}\rho_{\xi_0})^q(t,\sigma)(v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma))\varphi(t,\xi_0,\sigma)J_{\xi_0}(\sigma)\,dt\,d\sigma\\ &-\int_0^T\int_\Gamma(\gamma_{\xi_1}\rho_{\xi_1})^q(t,\sigma)(v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma))\varphi(t,\xi_1,\sigma)J_{\xi_1}(\sigma)\,dt\,d\sigma \underset{\xi_1\to\xi_0}{\longrightarrow} 0.\end{aligned} \tag{7.4}$$

Since $(\xi,\sigma)\mapsto J_\xi(\sigma)$ is smooth and bounded from below by $\frac{1}{2}$ for $\xi\leq\xi_\Omega$ (see section 2.1), we can take

$$\varphi(t,\xi,\sigma) = \frac{\psi(t,\xi,\sigma)}{J_\xi(\sigma)},$$

extended smoothly to the whole domain Ω, where ψ is any function in $\mathcal{C}^\infty([0,T]\times\overline{\Omega})$. We get

$$\begin{aligned}&\int_0^T\int_\Gamma(\gamma_{\xi_0}\rho_{\xi_0})^q(t,\sigma)(v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma))\psi(t,\xi_0,\sigma)\,dt\,d\sigma\\ &-\int_0^T\int_\Gamma(\gamma_{\xi_1}\rho_{\xi_1})^q(t,\sigma)(v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma))\psi(t,\xi_1,\sigma)\,dt\,d\sigma \underset{\xi_1\to\xi_0}{\longrightarrow} 0.\end{aligned} \tag{7.5}$$

Since the family $(\rho_\xi)_\xi$ is uniformly bounded in $L^\infty(]0,T[\times\Gamma)$ and $(v\cdot\boldsymbol{\nu})\in\mathcal{C}^0_\xi(L^\alpha_{t,\sigma})$, the expression in the left-hand side of (7.5) is uniformly bounded by the norm of ψ in $\mathcal{C}^0_\xi(]0,\xi_\Omega[, L^{\alpha'}_{t,\sigma}(]0,T[\times\Gamma))$. Therefore, it is easily seen that the convergence (7.5) still holds for any $\psi\in\mathcal{C}^0_\xi(]0,\xi_\Omega[, L^{\alpha'}_{t,\sigma}(]0,T[\times\Gamma))$. Let us now consider two different choices for ψ.

1. We take $\psi(t,\xi,\sigma)=\psi(t,\sigma)\in L^{\alpha'}(]0,T[\times\Gamma)$ independent of ξ. The convergence (7.5) for $q=1$, proves that
$$\gamma_{\xi_1}\rho_{\xi_1}\left(v(.,\xi_1,.)\cdot\boldsymbol{\nu}(.)\right)\underset{\xi_1\to\xi_0}{\rightharpoonup}\gamma_{\xi_0}\rho_{\xi_0}\left(v(.,\xi_0,.)\cdot\boldsymbol{\nu}(.)\right),\ \text{ in } L^{\alpha}(]0,T[\times\Gamma).$$

2. Since $(v\cdot\boldsymbol{\nu})\in\mathcal{C}^0_\xi(L^\alpha_{t,\sigma})$ with $\alpha>1$, we can choose $\psi=|v\cdot\boldsymbol{\nu}|^{\alpha-2}(v\cdot\boldsymbol{\nu})\in\mathcal{C}^0_\xi(L^{\alpha'}_{t,\sigma})$ as a test function in (7.5) with $q=\alpha$. It follows
$$\int_0^T\int_\Gamma(\gamma_{\xi_1}\rho_{\xi_1})^\alpha(t,\sigma)|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^\alpha\,dt\,d\sigma\underset{\xi_1\to\xi_0}{\longrightarrow}\int_0^T\int_\Gamma(\gamma_{\xi_0}\rho_{\xi_0})^\alpha(t,\sigma)|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^\alpha\,dt\,d\sigma.$$

The weak convergence of the first point together with the convergence of the L^α norms obtained in the second point imply that the convergence is strong in $L^\alpha(]0,T[\times\Gamma)$ and the claim is proved.

■

Proof (of Theorem 7.2):

Thanks to assumption (7.3), the claim is clearly unchanged if we add a constant to ρ so that we can assume that $\rho\geq 0$, which implies that $\gamma\rho\geq 0$. For any $h>0$ small enough, let ψ_h be defined by $\psi_h(x)=1-\frac{d(x,\Gamma)}{h}$ in $\mathcal{O}_h$ and which is zero in Ω_h. Let $\varphi\in\mathcal{C}^\infty([0,T]\times\overline{\Omega})$, and let us take $\varphi(t,x)\,\psi_h(x)$ as a test function in the transport equation satisfied by ρ^q, for any $q\geq 1$
$$\begin{aligned}\int_0^T\int_\Gamma(\gamma\rho)^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma=&\frac{1}{h}\int_0^T\int_{\mathcal{O}_h}\rho^q(v\cdot\boldsymbol{\nu})\varphi\,dtdx+\int_0^T\int_{\mathcal{O}_h}\rho^q\psi_h(\partial_t\varphi+v\cdot\nabla\varphi)\,dt\,dx\\&+q\int_0^T\int_{\mathcal{O}_h}\rho^{q-1}f\varphi\psi_h\,dt\,dx+\int_{\mathcal{O}_h}\rho(0)^q\varphi(0)\psi_h\,dx-\int_{\mathcal{O}_h}\rho(T)^q\varphi(T)\psi_h\,dx.\end{aligned}$$
Since $|\psi_h|\leq 1$ and $|\mathcal{O}_h|\to 0$ when $h\to 0$, we deduce:
$$\int_0^T\int_\Gamma(\gamma\rho)^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma=\lim_{h\to 0}\frac{1}{h}\int_0^T\int_{\mathcal{O}_h}\rho^q(v\cdot\boldsymbol{\nu})\varphi\,dt\,dx.\tag{7.6}$$

Since for any $\xi\in[0,\xi_\Omega[$, Ω_ξ is a smooth domain and ρ is a solution to the transport equation in Ω_ξ, we can apply formula (7.6) with $q=1$ and replacing the initial domain Ω by each Ω_ξ. It follows that
$$\frac{d}{d\xi}\int_0^T\int_{\mathcal{O}_\xi}\rho(v\cdot\boldsymbol{\nu})\varphi\,dtdx=\int_0^T\int_{\Gamma_\xi}\gamma_\xi\rho\,(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma,\ \ \forall\xi\in[0,\xi_\Omega[.$$
Since the map
$$\xi\mapsto\int_0^T\int_{\mathcal{O}_\xi}\rho(v\cdot\boldsymbol{\nu})\varphi\,dtdx,$$
is absolutely continuous and vanishes for $\xi=0$, we deduce that, for any φ, we have
$$\int_0^T\int_{\mathcal{O}_\xi}\rho(v\cdot\boldsymbol{\nu})\varphi\,dt\,dx=\int_0^\xi\int_0^T\int_{\Gamma_\xi}\gamma_\xi\rho\,(v\cdot\boldsymbol{\nu})\varphi\,dt\,d\sigma\,d\xi,\ \ \forall\xi\in[0,\xi_\Omega[.\tag{7.7}$$
Hence, we get
$$\rho(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))=\gamma_\xi\rho(t,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)),\ \text{ for almost every } (t,\xi,\sigma)\in]0,T[\times]0,\xi_\Omega[\times\Gamma.\tag{7.8}$$

For any $\xi>0$ small enough, let ρ_ξ be the restriction of ρ to the domain Ω_ξ. Obviously, ρ_ξ is solution to the transport equation in Ω_ξ and is uniformly bounded in $L^\infty(]0,T[\times\Omega)$. It is easily seen that this family $(\rho_\xi)_\xi$ is continuous with respect to ξ in any space $\mathcal{C}^0([0,T],L^q(\Omega))$ with $q<+\infty$.

Using Theorem 7.1 with this particular family $(\rho_\xi)_\xi$, we deduce that the right-hand side of (7.8) is continuous with respect to ξ with values in $L^2(]0,T[\times\Gamma)$, which proves the claim.

■

7.2 An additional property of the trace of bounded solution

We show here that the trace $\gamma\rho$ of any bounded solution ρ to the transport equation can be expressed as the limit of the mean-values of ρ along the normal direction to Γ.

Proposition 7.1
Let $f \in L^1(]0,T[, L^\infty(\Omega))$ and v satisfying (2.5) with $p' = 1$ and (7.3). For any bounded solution ρ to the transport equation and any $1 \leq q < +\infty$, we have

$$\left((t,\sigma) \mapsto \frac{1}{h}\int_0^h \rho(t,\xi,\sigma)\, d\xi \right) \underset{h\to 0}{\longrightarrow} \gamma\rho, \quad \text{in } L^q(]0,T[\times\Gamma, d\mu_v^\alpha).$$

Notice that the convergence obviously still holds in $L^q(]0,T[\times\Gamma, |d\mu_v|)$.
Proof :
Without loss of generality, we assume that $\rho \geq 0$. Using the change of variables (2.4) in (7.6), we get:

$$\int_0^T \int_\Gamma (\gamma\rho)^q (v\cdot\boldsymbol{\nu})\varphi\, dt\, d\sigma = \lim_{h\to 0} \frac{1}{h}\int_0^T\int_0^h\int_\Gamma \rho^q(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))\varphi(t,\xi,\sigma)J_\xi(\sigma)\, dt\, d\xi\, d\sigma.$$

Since the Jacobian $J_\xi(\sigma)$ is smooth and satisfies $J_0(\sigma) = 1$ for any $\sigma \in \Gamma$, we can take $\varphi(t,\xi,\sigma) = \frac{\psi(t,\xi,\sigma)}{J_\xi(\sigma)}$ for any smooth ψ. It follows

$$\int_0^T \int_\Gamma (\gamma\rho)^q (v\cdot\boldsymbol{\nu})\psi\, dt\, d\sigma = \lim_{h\to 0} \frac{1}{h}\int_0^T\int_0^h\int_\Gamma \rho^q(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))\psi(t,\xi,\sigma)\, dt\, d\xi\, d\sigma. \tag{7.9}$$

Furthermore we have the following estimate, which is uniform with respect to $h > 0$,

$$\begin{aligned}
&\left| \frac{1}{h}\int_0^T\int_0^h\int_\Gamma \rho^q(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))\psi(t,\xi,\sigma)\, dt\, d\xi\, d\sigma \right| \\
&\qquad \leq \int_0^T\int_\Gamma \sup_\xi \Big(\rho^q(t,\xi,\sigma)|v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)||\psi(t,\xi,\sigma)| \Big)\, dt\, d\sigma \\
&\qquad \leq \|\rho^q\|_{L^\infty(]0,T[\times\Omega)} \|v\cdot\boldsymbol{\nu}\|_{\mathcal{C}^0_\xi(L^\alpha_{t,\sigma})} \|\psi\|_{\mathcal{C}^0_\xi(L^{\alpha'}_{t,\sigma})}.
\end{aligned}$$

This estimate shows that we can extend by density the formula (7.9) to any function $\psi \in \mathcal{C}^0_\xi([0,\xi_\Omega[, L^{\alpha'}(]0,T[\times\Gamma))$.

- We apply (7.9) with $q = 1$ and $\psi(t,\xi,\sigma) = \psi(t,\sigma) \in L^{\alpha'}(]0,T[\times\Gamma)$ which does not depend on ξ. We get

  $$\int_0^T\int_\Gamma \gamma\rho(t,\sigma)(v(t,\sigma)\cdot\boldsymbol{\nu}(\sigma))\psi(t,\sigma)\, dt\, d\sigma = \lim_{h\to 0}\int_0^T\int_\Gamma \psi(t,\sigma)\left(\frac{1}{h}\int_0^h \rho(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))\, d\xi\right) dt\, d\sigma.$$

  Hence, the family

  $$(t,\sigma) \mapsto f_h(t,\sigma) = \frac{1}{h}\int_0^h \rho(t,\xi,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))\, d\xi,$$

  weakly converges towards $\gamma\rho\,(v\cdot\boldsymbol{\nu})$ in $L^\alpha(]0,T[\times\Gamma)$.

- Thanks to the assumption (7.3), we can take $\varphi(t,\xi,\sigma) = |v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\alpha-2}(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma))$ as a test function in (7.9) with $q = \alpha$. It follows that

  $$\int_0^T\int_\Gamma (\gamma\rho)^\alpha |v.\boldsymbol{\nu}|^\alpha\, dt d\sigma = \lim_{h\to 0}\int_0^T\int_\Gamma \left(\frac{1}{h}\int_0^h \rho(t,\xi,\sigma)^\alpha |v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^\alpha\, d\xi\right) dt\, d\sigma.$$

Therefore we have

$$\limsup_{h\to 0} \int_0^T \int_\Gamma |f_h(t,\sigma)|^\alpha \, dt\, d\sigma \le \int_0^T \int_\Gamma (\gamma\rho)^\alpha |v\cdot \boldsymbol{\nu}|^\alpha \, dt\, d\sigma.$$

It follows that the family $(f_h)_h$ strongly converges when $h \to 0$ towards $\gamma\rho\,(v\cdot\boldsymbol{\nu})$ in $L^\alpha(]0,T[\times\Gamma)$.

- Finally we have the following estimate

$$\begin{aligned}\int_0^T \int_\Gamma \left|\gamma\rho(t,\sigma) - \frac{1}{h}\int_0^h \rho(t,\xi,\sigma)\, d\xi\right|^\alpha |v(t,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^\alpha \, dt\, d\sigma \\ \le C \int_0^T \int_\Gamma |\gamma\rho(t,\sigma)(v(t,\sigma)\cdot\boldsymbol{\nu}(\sigma)) - f_h(t,\sigma)|^\alpha \, dt\, d\sigma \\ + \frac{C}{h}\int_0^T \int_\Gamma \int_0^h |\rho(t,\xi,\sigma)|^\alpha |v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma) - v(t,0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^\alpha \, dt\, d\xi\, d\sigma.\end{aligned}$$

  As we have seen just before, the first term converges to zero when $h \to 0$ and the second one is bounded by $C\sup_{0\le\xi\le h} \|v(.,\xi,.)\cdot\boldsymbol{\nu} - v(.,0,.)\cdot\boldsymbol{\nu}\|^\alpha_{L^\alpha(]0,T[\times\Gamma)}$ which converges also to zero since we have assumed that $(v\cdot\boldsymbol{\nu})$ lies in $\mathcal{C}^0_\xi([0,\xi_\Omega[, L^\alpha(]0,T[\times\Gamma))$.

We proved the claim for $q=\alpha$ but since ρ and $\gamma\rho$ are bounded functions, the general case follows immediately. ■

7.3 Continuity of the life time with respect to the domain

Our goal now, is to investigate the spatial continuity properties of any solution ρ of the transport equation lying in $L^\infty(]0,T[, L^p(\Omega))$ for finite values of p. As we have seen before in the trace Theorem 5.1, the notion of life time with respect of the domain Ω is crucial to define the spaces in which the traces of such solutions are lying. Therefore, in order to use a similar strategy than in section 7.1 for the case of bounded solutions (*i.e.* by considering ρ as a solution of the transport equation on a family of domains Ω_ξ), it is crucial to investigate how the life time τ depends on the domain Ω. More precisely, in this section we will prove that the life time τ introduced in section 5 (and in fact both τ_+ and τ_-) is in some sense continuous with respect to the domain Ω we consider.

The proof of continuity properties of solutions of the transport equation in $L^\infty(]0,T[, L^p(\Omega))$ is then postponed to the next section.

Let us introduce some notations: as Ω_ξ is a regular bounded domain (for $\xi\in[0,\xi_\Omega[$) and the trace of $(v\cdot\boldsymbol{\nu})$ lies in $L^\alpha(]0,T[\times\Gamma_\xi)$ thanks to (7.3), we can define the life time τ_ξ (as well as $\tau_{+,\xi}$ and $\tau_{-,\xi}$) associated to the vector field v and to the domain Ω_ξ as we did in section 5 for the initial domain Ω. Furthermore, we extend each τ_ξ, $\tau_{+,\xi}$ and $\tau_{-,\xi}$ by the value 0 so that they are defined on the whole initial domain Ω.

Theorem 7.4

1. *The families $(\tau_\xi)_\xi$, $(\tau_{+,\xi})_\xi$ and $(\tau_{-,\xi})_\xi$ are continuous with respect to $\xi\in[0,\xi_\Omega[$ with values in $\mathcal{C}^0([0,T], L^q(\Omega))$ for any $1\le q<+\infty$.*
2. *The families $\big(\gamma_\xi\tau_{-,\xi}\,(v(.,\xi,.)\cdot\boldsymbol{\nu})\big)_\xi$, $\big(\gamma_\xi\tau_{+,\xi}\,(v(.,\xi,.)\cdot\boldsymbol{\nu})\big)_\xi$ and $\big(\gamma_\xi\tau_\xi\,(v(.,\xi,.)\cdot\boldsymbol{\nu})\big)_\xi$ are continuous with respect to ξ with values in $L^\alpha(]0,T[\times\Gamma)$.*

Proof :
We only give the proof of the claims concerning $\tau_{-,\xi}$ since the proofs for $\tau_{+,\xi}$ are similar and the results for τ_ξ will follow from the definition $\tau_\xi = \tau_{+,\xi}+\tau_{-,\xi}$.

1. Since all the functions $(\tau_{-,\xi})_\xi$ are uniformly bounded (by the final time T), it is enough to show the result for $q=1$. Let $0\le\xi_0<\xi_1<\xi_\Omega$. By definition of τ_{-,ξ_0} and τ_{-,ξ_1}, we deduce that $\tau_{-,\xi_0}-\tau_{-,\xi_1}$ is solution to the

problem

$$\begin{cases} \partial_t(\tau_{-,\xi_0} - \tau_{-,\xi_1}) + v \cdot \nabla(\tau_{-,\xi_0} - \tau_{-,\xi_1}) = 0, \text{ in } \Omega_{\xi_1}, \\ \tau_{-,\xi_0}(0) - \tau_{-,\xi_1}(0) = 0, \\ \gamma_{\xi_1}(\tau_{-,\xi_0} - \tau_{-,\xi_1}) = \gamma_{\xi_1}\tau_{-,\xi_0}, \text{ on }]0,T[\times\Gamma_{\xi_1} \text{ where } (v\cdot\boldsymbol{\nu}) < 0. \end{cases}$$

Notice that, thanks to Proposition 5.1 and to Remark 3.1, we know that $\gamma_{\xi_1}\tau_{-,\xi_0} \geq 0$ almost everywhere on $]0,T[\times\Gamma_{\xi_1}$ for the measure $|v\cdot\boldsymbol{\nu}|\,dt\,d\sigma$. Therefore, using the comparison principle (Proposition 4.1), we get that $\tau_{-,\xi_0} - \tau_{-,\xi_1} \geq 0$ almost everywhere in $]0,T[\times\Omega$, which is of course quite natural.

If we integrate the equation above on $]0,s[\times\Omega_{\xi_1}$ for any $s\in[0,T]$, we get

$$\int_{\Omega_{\xi_1}} (\tau_{-,\xi_0}(s) - \tau_{-,\xi_1}(s))\,dx + \int_0^s\int_{\Gamma_{\xi_1}} \gamma_{\xi_1}(\tau_{-,\xi_0} - \tau_{-,\xi_1})(v\cdot\boldsymbol{\nu})^+\,dt\,d\sigma \leq \int_0^s\int_{\Gamma_{\xi_1}} \gamma_{\xi_1}\tau_{-,\xi_0}(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma.$$

The boundary term in the left-hand size of this inequality is non-negative so that we deduce

$$\sup_{s\in[0,T]}\int_\Omega (\tau_{-,\xi_0}(s) - \tau_{-,\xi_1}(s))\,dx \leq T|\mathcal{O}_{\xi_0,\xi_1}| + \int_0^T\int_{\Gamma_{\xi_1}} \gamma_{\xi_1}\tau_{-,\xi_0}(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma. \tag{7.10}$$

Applying the continuity Theorem 7.2 to the bounded function τ_{-,ξ_0} on the domain Ω_{ξ_0} we easily get

$$\int_0^T\int_{\Gamma_{\xi_1}} \gamma_{\xi_1}\tau_{-,\xi_0}(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma \underset{\xi_1\to\xi_0}{\longrightarrow} \int_0^T\int_{\Gamma_{\xi_0}} \gamma_{\xi_0}\tau_{-,\xi_0}(v\cdot\boldsymbol{\nu})^-\,dt\,d\sigma = 0,$$

thanks to the boundary condition on τ_{-,ξ_0} in (5.1). Since $|\mathcal{O}_{\xi_0,\xi_1}| \to 0$ when $\xi_1 \to \xi_0$, the estimate (7.10) gives the first claim of the theorem.

2. The second claim is a consequence of Theorem 7.1 applied to the family $(\tau_{-,\xi})_\xi$ since we just have shown the continuity of this family of solutions with respect to ξ in every space $\mathcal{C}^0([0,T],L^q(\Omega))$, $q<+\infty$.

■

7.4 Space continuity for finite values of p

First of all, let us notice that if $p\in]1,+\infty[$ and $\rho\in L^\infty(]0,T[,L^p(\Omega))$ is a solution to the transport equation in Ω, the renormalization property (3.1) shows that for any $\beta\in\mathcal{C}^1_b(\mathbb{R})$, $\beta(\rho)$ is a bounded solution to the transport equation with source term $\beta'(\rho)f$. Hence, we can apply Theorem 7.2 to $\beta(\rho)$. In particular we deduce that, near Γ, the function $\beta(\rho)(v\cdot\boldsymbol{\nu})$ is continuous with respect to ξ with values in $L^2(]0,T[\times\Gamma)$.

Since we are restricted to bounded functions β, this first approach does not allow to prove a continuity result for the solution ρ itself. Of course, this obstruction follows from the fact that the trace space is not the L^p space on $]0,T[\times\Gamma$ for the measure $|d\mu_v|$ but only for the measure $\gamma\tau|d\mu_v|$, τ being the life time in Ω as we saw in Theorem 5.1.

Using the results of the previous sections, we can prove a more specific result for such solutions which reads as follows.

Theorem 7.5
For $p\in]1,+\infty[$, let us define $\bar{p}=\min(p,\frac{p+d}{d})$, where d is the space dimension. For any $\beta\in\mathcal{C}^1(\mathbb{R})$ such that

$$|\beta(s)|\leq C(1+|s|^{q_1}), \text{ and } |\beta'(s)|\leq C(1+|s|^{q_2-1}), \ \forall s\in\mathbb{R}, \tag{7.11}$$

with $q_1<\bar{p}$, $q_2<p$, and any $\rho\in L^\infty(]0,T[,L^p(\Omega))$ solution to the transport equation, the map

$$(t,\xi,\sigma)\in]0,T[\times[0,\xi_\Omega[\times\Gamma \mapsto \beta(\rho)(t,\xi,\sigma)\gamma_\xi\tau_\xi(t,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)),$$

is continuous with respect to ξ with values in $L^r(]0,T[\times\Gamma)$ for any $r<\frac{p\alpha}{p+(\alpha-1)q_1}$.

Notice that $\frac{p\alpha}{p+(\alpha-1)q_1} > 1$ since we have $q_1 < p$. Obviously, the identity function $\beta(s) = s$ satisfies the assumptions of the theorem so that we have proved, as a particular case, the continuity with respect to ξ of the function $\rho(.,\xi,.)\gamma_\xi\tau_\xi(v(.,\xi,.)\cdot\boldsymbol{\nu})$ in suitable spaces.

Furthermore, one can prove a result similar to Theorem 7.3 in the present case. In particular, if it happens that $|(v(.,\xi,.)\cdot N)\gamma_\xi\tau_\xi| \geq \delta_1 > 0$ on a neighborhood of $]t_0,t_1[\times S$ where S is a smooth hypersurface in Ω whose unitary normal is N, then the solution ρ itself is continuous with respect to the normal variable to S with values in a suitable local Lebesgue space in the transverse variables.

Let us give a simple example. We consider the domain $\Omega =]0,1[\times\mathbb{R}$ and $v(t,x,y) = (v_1(y), v_2(t,x))$ such that $v \in L^2(]0,T[, H^1_{loc}(\overline{\Omega})^2)$ for instance. The fact that Ω is unbounded is not a problem here since we are concerned with local properties. Suppose that there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < \alpha_1 \leq v_1(y) \leq \alpha_2, \forall y \in \mathbb{R}$. Then, any solution $\rho \in L^\infty(]0,T[, L^2(\Omega))$ to the transport equation is continuous with respect to the variable x and with values in $L^2(]0,T[, L^2_{loc}(\mathbb{R}))$ in the variables (t,y). Indeed, using the notations of the theorem, we have $v\cdot\boldsymbol{\nu} = v\cdot e_x = v_1 \geq \alpha_1 > 0$ and we can also prove (using Proposition 4.1 for instance) a bound from below for the life time

$$\gamma_\xi\tau_\xi \geq \frac{C(1-\xi)}{\alpha_2} > 0,$$

γ_ξ being the trace operator on the halfplane $\Gamma_\xi = \{(x,y)\in\Omega, x=\xi\}$ and τ_ξ the life time associated to v and to the domain $\Omega_\xi = \{(x,y)\in\Omega, x>\xi\}$.

Let us now give the proof of the previous theorem.

Proof :

- **Step 1 :**

Let $\beta_0 \in \mathcal{C}^1_b(\mathbb{R})$ be an injective function. We have seen that $\beta_0(\rho)$ is a bounded solution of a transport equation in Ω so that we can apply (7.8) to this solution. It follows that

$$\beta_0(\rho(t,\xi,\sigma))(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)) = \beta_0(\gamma_\xi\rho(t,\sigma))(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)), \text{ for a.e. } (t,\xi,\sigma)\in]0,T[\times]0,\xi_\Omega[\times\Gamma.$$

Since β_0 is injective, we deduce that, for any β satisfying (7.11), we have

$$\beta(\rho)(t,\xi,\sigma)\gamma_\xi\tau_\xi(t,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)) = \beta(\gamma_\xi\rho(t,\sigma))\gamma_\xi\tau_\xi(t,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)),$$

for almost every $(t,\xi,\sigma)\in]0,T[\times]0,\xi_\Omega[\times\Gamma$. Hence, it is enough to show that the right-hand side above is continuous with respect to ξ with values in $L^r(]0,T[\times\Gamma)$.

- **Step 2 :**

Let $F(s)$ be a smooth non-negative function such that $F(s) = 0$, for $s \leq 0$ and $F(s) = s$ for $s \geq 1$. For any $\beta \in \mathcal{C}^1(\mathbb{R})$ satisfying (7.11), we can write β as a sum

$$\beta = (F\circ\beta + 1) - (F\circ\beta - \beta + 2) + 1.$$

It is easily seen that the functions $F\circ\beta+1$ and $F\circ\beta-\beta+2$ are in $\mathcal{C}^1(\mathbb{R})$, satisfy (7.11) and are bounded from below by 1. Furthermore, when β is a constant function, the result is already proved in Theorem 7.4. Hence, the claim of the theorem being linear with respect to β, it is enough to restrict ourselves to the functions β which satisfy

$$\beta(s) \geq 1, \quad \forall s\in\mathbb{R}. \tag{7.12}$$

- **Step 3 :**

Using Theorem 5.1 in each domain Ω_ξ for $\xi\in[0,\xi_\Omega[$ and the assumption (7.11), we find that

$$(t,\xi,\sigma)\mapsto\beta(\gamma_\xi\rho(t,\sigma))\gamma_\xi\tau_\xi(t,\sigma)(v(t,\xi,\sigma)\cdot\boldsymbol{\nu}(\sigma)),$$

lies in $L^\infty_\xi(]0,\xi_\Omega[, L^1_{t,\sigma}(]0,T[\times\Gamma))$.

In particular the function $\beta(\gamma_\xi\rho)^{\frac{1}{d}}(\gamma_\xi\tau_\xi)^{\frac{1}{d}}|v(.,\xi,.)\cdot\boldsymbol{\nu}|^{\frac{1}{d}}$ lies in $L^\infty_\xi(]0,\xi_\Omega[,L^d_{t,\sigma}(]0,T[\times\Gamma))$. Hence, $\xi_0\in[0,\xi_\Omega[$ being fixed, there exists $L_{\xi_0}\in L^d(]0,T[\times\Gamma)$ such that, up to a subsequence, we have

$$\beta(\gamma_\xi\rho)^{\frac{1}{d}}(\gamma_\xi\tau_\xi)^{\frac{1}{d}}|v(.,\xi,.)\cdot\boldsymbol{\nu}|^{\frac{1}{d}} \underset{\xi\to\xi_0}{\rightharpoonup} L_{\xi_0} \quad \text{in } L^d(]0,T[\times\Gamma). \tag{7.13}$$

Let us introduce $\tilde\beta=\beta^{\frac{1}{d}}$. Using (7.11) and (7.12), we easily see that $\tilde\beta$ lies in $\mathcal{C}^1(\mathbb{R})$ and satisfies

$$|\tilde\beta(s)|\le C(1+|s|^{\frac{q}{d}}), \text{ and } |\tilde\beta'(s)|\le C(1+|s|^{p-1}),\ \ \forall s\in\mathbb{R},$$

so that, thanks to Theorem 6.1, we know that $\tilde\beta(\rho)$ is a solution to the transport equation with the corresponding data and source term. Therefore, for any $\xi\in[0,\xi_\Omega[$, the function $\tilde\beta(\rho)(\tau_{-,\xi}-\tau_{+,\xi})$ satisfies in Ω_ξ the equation

$$\partial_t\big(\tilde\beta(\rho)(\tau_{-,\xi}-\tau_{+,\xi})\big)+v\cdot\nabla\big(\tilde\beta(\rho)(\tau_{-,\xi}-\tau_{+,\xi})\big)=(\tau_{-,\xi}-\tau_{+,\xi})\tilde\beta'(\rho)f+2\tilde\beta(\rho), \tag{7.14}$$

with the corresponding initial data and boundary conditions. Let us choose the restriction to Ω_ξ of a smooth function $\varphi\in\mathcal{C}^\infty([0,T]\times\Omega)$ as a test function in (7.14). We get

$$\begin{aligned}\int_0^T\int_{\Gamma_\xi}\tilde\beta(\gamma_\xi\rho)\gamma_\xi\tau_\xi|v\cdot\boldsymbol{\nu}|\varphi\,dt\,d\sigma &= \int_0^T\int_{\Omega_\xi}\tilde\beta(\rho)(\tau_{-,\xi}-\tau_{+,\xi})(\partial_t\varphi+v\cdot\nabla\varphi)\,dt\,dx\\ &\quad+\int_0^T\int_{\Omega_\xi}\varphi\big[(\tau_{-,\xi}-\tau_{+,\xi})\tilde\beta'(\rho)f+2\tilde\beta(\rho)\big]\,dt\,dx\\ &\quad-\int_{\Omega_\xi}\tilde\beta(\rho(0))\varphi(0)\tau_{+,\xi}(0)\,dx-\int_{\Omega_\xi}\tilde\beta(\rho(T))\varphi(T)\tau_{-,\xi}(T)\,dx,\end{aligned}$$

We deduce that for any $0<\xi_0\le\xi_1<\xi_\Omega$ we can write

$$\begin{aligned}&\int_0^T\int_{\Gamma_{\xi_0}}\tilde\beta(\gamma_{\xi_0}\rho)\gamma_{\xi_0}\tau_{\xi_0}|v\cdot\boldsymbol{\nu}|\varphi\,dt\,d\sigma-\int_0^T\int_{\Gamma_{\xi_1}}\tilde\beta(\gamma_{\xi_1}\rho)\gamma_{\xi_1}\tau_{\xi_1}|v\cdot\boldsymbol{\nu}|\varphi\,dt\,d\sigma\\ &=\int_0^T\int_\Omega F_1\Big((\tau_{-,\xi_0}-\tau_{-,\xi_1})-(\tau_{+,\xi_0}-\tau_{+,\xi_1})\Big)\,dt\,dx+\int_0^T\int_{\mathcal{O}_{\xi_0,\xi_1}}F_2\,dt\,dx\\ &\quad-\int_\Omega\Big(G_1(\tau_{+,\xi_0}(0)-\tau_{+,\xi_1}(0))+G_2(\tau_{-,\xi_0}(T)-\tau_{-,\xi_1}(T))\Big)\,dx,\end{aligned} \tag{7.15}$$

where $G_1,G_2\in L^{1+\varepsilon}(\Omega)$, $F_1,F_2\in L^1(]0,T[,L^{1+\varepsilon}(\Omega))$ for some suitable $\varepsilon>0$, are functions depending only on φ,f and ρ. The fact that these functions are in a slightly better space than L^1 is due to the fact that in (7.11), we have supposed $q_1<\bar p$ and $q_2<p$. Thanks to Theorem 7.4, we know that τ_{-,ξ_1} and τ_{+,ξ_1} converge respectively towards τ_{-,ξ_0} and τ_{-,ξ_0} in $\mathcal{C}^0([0,T],L^{\frac{1+\varepsilon}{\varepsilon}}(\Omega))$ when ξ_1 goes to ξ_0. Therefore, we can pass to the limit in the right-hand side member of (7.15) and deduce that, φ being fixed, we have

$$\int_0^T\int_{\Gamma_{\xi_0}}\tilde\beta(\gamma_{\xi_0}\rho)\gamma_{\xi_0}\tau_{\xi_0}|v\cdot\boldsymbol{\nu}|\varphi\,dt\,d\sigma-\int_0^T\int_{\Gamma_{\xi_1}}\tilde\beta(\gamma_{\xi_1}\rho)\gamma_{\xi_1}\tau_{\xi_1}|v\cdot\boldsymbol{\nu}|\varphi\,dt\,d\sigma\underset{\xi_1\to\xi_0}{\longrightarrow}0.$$

Using (2.3), we write both terms above as integrals over Γ, and choose $\varphi(t,\xi,\sigma)=\frac{\psi(t,\xi,\sigma)}{J_\xi(\sigma)}$ for any smooth ψ since the Jacobian $J_\xi(\sigma)$ is smooth and does not vanish. It follows

$$\begin{aligned}&\int_0^T\int_\Gamma\tilde\beta(\gamma_{\xi_1}\rho)\gamma_{\xi_1}\tau_{\xi_1}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\psi(t,\xi_1,\sigma)\,dt\,d\sigma\underset{\xi_1\to\xi_0}{\longrightarrow}\\ &\qquad\int_0^T\int_\Gamma\tilde\beta(\gamma_{\xi_0}\rho)\gamma_{\xi_0}\tau_{\xi_0}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\psi(t,\xi_0,\sigma)\,dt\,d\sigma.\end{aligned} \tag{7.16}$$

Let us introduce $d' = \frac{d}{d-1}$. Using Theorem 7.4 and the Lebesgue theorem, we know that $(\gamma_{\xi_1}\tau_{\xi_1})^{\frac{1}{d'}}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d'}}$ converges, up to an extraction of a subsequence, towards $(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d'}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d'}}$ almost everywhere but also for the strong topology of $L^{\alpha d'}(]0,T[\times\Gamma)$. Using the weak convergence (7.13) we deduce that

$$\tilde{\beta}(\gamma_{\xi_1}\rho)\gamma_{\xi_1}\tau_{\xi_1}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)| \underset{\xi_1\to\xi_0}{\rightharpoonup} L_{\xi_0}(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d'}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d'}},$$

weakly in $L^{\alpha' d'}(]0,T[\times\Gamma)$. If we compare this result to (7.16), we deduce that

$$L_{\xi_0}(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d'}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d'}} = \tilde{\beta}(\gamma_{\xi_0}\rho)\gamma_{\xi_0}\tau_{\xi_0}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|.$$

Hence, the weak limit L_{ξ_0} introduced in (7.13) has necessarily the following form

$$L_{\xi_0} = \tilde{\beta}(\gamma_{\xi_0}\rho)(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}} + \tilde{L}_{\xi_0}, \tag{7.17}$$

where $\tilde{L}_{\xi_0}$ is in $L^d(]0,T[\times\Gamma)$ and has its support in the set $\{(t,\sigma)\in]0,T[\times\Gamma,\ \ v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)=0\}$. Notice that we used here the fact that $\gamma_{\xi_0}\tau_{\xi_0}\neq 0$ for almost every (t,σ) such that $v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)\neq 0$ (see Proposition 5.1)

We now remark that (7.16) also holds if we replace $\tilde{\beta}$ by β, thanks to the assumptions (7.11). Choosing $\psi = 1$ in this case, we deduce that

$$\int_0^T\int_\Gamma \beta(\gamma_{\xi_1}\rho)\gamma_{\xi_1}\tau_{\xi_1}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\,dt\,d\sigma \underset{\xi_1\to\xi_0}{\longrightarrow} \int_0^T\int_\Gamma \beta(\gamma_{\xi_0}\rho)\gamma_{\xi_0}\tau_{\xi_0}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|\,dt\,d\sigma.$$

Since $\tilde{\beta} = \beta^{\frac{1}{d}}$, the convergence above reads

$$\begin{aligned}\big\|\tilde{\beta}(\gamma_{\xi_1}\rho)(\gamma_{\xi_1}\tau_{\xi_1})^{\frac{1}{d}}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}}\big\|^d_{L^d(]0,T[\times\Gamma)} &\underset{\xi_1\to\xi_0}{\longrightarrow} \big\|\tilde{\beta}(\gamma_{\xi_0}\rho)(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}}\big\|^d_{L^d(]0,T[\times\Gamma)}\\ &= \|L_{\xi_0}\|^d_{L^d(]0,T[\times\Gamma)} - \|\tilde{L}_{\xi_0}\|^d_{L^d(]0,T[\times\Gamma)} \leq \|L_{\xi_0}\|^d_{L^d(]0,T[\times\Gamma)},\end{aligned}$$

where we used the decomposition of L_{ξ_0} as a sum of two functions with disjoint supports obtained in (7.17).

Therefore, we proved that the convergence of $\tilde{\beta}(\gamma_{\xi_1}\rho)(\gamma_{\xi_1}\tau_{\xi_1})^{\frac{1}{d}}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}}$ towards L_{ξ_0} is strong in $L^d(]0,T[\times\Gamma)$, and that $\tilde{L}_{\xi_0}=0$. As a consequence, we have shown that $\tilde{\beta}(\gamma_{\xi_1}\rho)(\gamma_{\xi_1}\tau_{\xi_1})^{\frac{1}{d}}|v(t,\xi_1,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}}$ converges strongly in $L^d(]0,T[\times\Gamma)$ towards $\tilde{\beta}(\gamma_{\xi_0}\rho)(\gamma_{\xi_0}\tau_{\xi_0})^{\frac{1}{d}}|v(t,\xi_0,\sigma)\cdot\boldsymbol{\nu}(\sigma)|^{\frac{1}{d}}$ when $\xi_1\to\xi_0$. Since $\beta=\tilde{\beta}^d$, it follows that $\beta(\gamma_\xi\rho)\gamma_\xi\tau_\xi|v(.,\xi,.)\cdot\boldsymbol{\nu}|$ is continuous with respect to ξ with values in $L^1(]0,T[\times\Gamma)$.

Using the fact that $(v\cdot\boldsymbol{\nu})\in\mathcal{C}^0_\xi(L^\alpha_{t,\sigma})$ and Theorem 5.1, we deduce that $\beta(\gamma_\xi\rho)\gamma_\xi\tau_\xi|v(.,\xi,.)\cdot\boldsymbol{\nu}|$ is in fact bounded in $L^\infty_\xi(L_{t,\sigma}^{\frac{p\alpha}{p+(\alpha-1)q_1}})$. Therefore, we proved that this function is continuous with respect to ξ and with values in $L^r(]0,T[\times\Gamma)$ for any $r<\frac{p\alpha}{p+(\alpha-1)q_1}$.

Finally, it is easily seen that the result still holds for the function $\beta(\gamma_\xi\rho)\gamma_\xi\tau_\xi(v(.,\xi,.)\cdot\boldsymbol{\nu})$, that is to say without absolute values. Indeed, one can prove the continuity for the weak topology of $L^r(]0,T[\times\Gamma)$ by the same kind of arguments than above and the continuity of the norms in $L^r(]0,T[\times\Gamma)$ follows from the previous results concerning $\beta(\gamma_\xi\rho)\gamma_\xi\tau_\xi|v(.,\xi,.)\cdot\boldsymbol{\nu}|$.

■

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