Mathematical study of multiphase flow under shear through order parameter formulation

Franck Boyer[1],

Mathématiques Appliquées de Bordeaux, Université Bordeaux 1, 351 cours de la libération, 33405 Talence cedex, France.

Abstract - In this paper we study the coupling of the Navier-Stokes equations and the Cahn-Hilliard equation which stands for a model of a multiphase fluid under shear. We first study existence and uniqueness of solutions of the system in dimension 2 and 3 even if the diffusion coefficient is allowed to degenerate. In the last part, an asymptotic stability result is shown.

Key Words - Shear flow, Navier-Stokes, Cahn-Hilliard, Order parameter.

AMS Subject Classification - 35K55, 35K65, 76D05, 76E05, 82C26

Physical background

The problem of building mathematical models for multi-phase fluids have been extensively studied in the literature ([6], [8], [11], [17]). Among all those results, we choose to study the order parameter formulation. If we only consider the diffusion phenomenon, the evolution of the order parameter is given by the Cahn-Hilliard equation

$$\frac{\partial \varphi}{\partial t} - \text{ div } (B(\varphi)\nabla \mu) = 0,$$

where $\mu = -\alpha\Delta\varphi + F'(\varphi)$ is a chemical potential derived from a coarse-grained study of the free energy of the fluid ([11]) and $B(\varphi)$ is a non-negative mobility coefficient which depends on the local composition of the alloy.

Furthermore, this equation is provided with the conditions

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0, \tag{$*$}$$

where ν is the outward normal on the boundary of the domain, which can be read as no-flux conditions. For a mathematical study of this equation we refer to [4], [7], [9], [19].

Hereafter, we are interested in the evolution of a binary alloy in a channel under shear. This situation arises for example in viscometric experiments. Numerous physical papers ([6], [8] for example), propose to consider in this case, a coupling of Cahn-Hilliard and Navier-Stokes equations given by

$$\begin{aligned}
&\frac{\partial \varphi}{\partial t} + v.\nabla\varphi - \text{ div } (B(\varphi)\nabla \mu) = 0\\
&\mu = -\alpha\Delta\varphi + F'(\varphi)\\
&\frac{\partial v}{\partial t} - 2\,\text{div } (\eta(\varphi)D(v)) + (v.\nabla)v + \nabla p = \mu\nabla\varphi\\
&\text{div } (v) = 0
\end{aligned}$$

with the boundary conditions $(*)$, and the shear boundary conditions for v.

In this system, μ is the chemical potential, $B(\varphi)$ the mobility, $\eta(\varphi)$ is the viscosity of the fluid depending on the composition of the alloy and $D(v)$ is the tensor $\frac{1}{2}(\nabla v + \nabla v^t)$. The addition of a transport term in

[1] e-mail: fboyer@math.u-bordeaux.fr

the Cahn-Hilliard equation is quite natural and the forces appearing in the right member of the Navier-Stokes equation must be considered as capillary forces acting only near the interface.

The physical-relevant assumption on the function F is that it must have a double-well structure, each of them representing the two phases of the fluid. Physicians ([6], [8], [11]) often propose to consider either functions like

$$F(x) = (1-x^2)^2,$$

either functions like

$$F(x) = 1 - x^2 + c\Big((1+x)\log(1+x) + (1-x)\log(1-x)\Big),$$

where we suppose that φ is normalized in such a way that the two pure phases of the fluid are respectively represented by the values 1 and -1 of the order parameter.

In the sequel of this paper we will concentrate on making assumptions on the function F which match these two choices.

As far as the mobility is concerned, a mathematical difficulty is that B may or may not degenerate that is to say that B can eventually vanish at the points -1 and 1; a possible choice for B is for example

$$B(x) = (1-x^2)^r, \ r \geq 0.$$

Contents

1 Introduction

In this paper we investigate first the question of existence, uniqueness, regularity and globality of solutions. Then, we are interested in qualitative results on the solutions, in particular in their asymptotic behavior.

1.1 Mathematical setting of the problem

We will consider both dimensions $d = 2$ and $d = 3$ for this problem. The system of spatial coordinates chosen is (x, z) if $d = 2$, and (x, y, z) if $d = 3$. The channel we consider is

$$\{-1 \leq z \leq 1\} \subset \mathbb{R}^d$$

and the shear velocity is chosen to be $U.e_x$ on the upper boundary of the channel $\{z=1\}$ and $-U.e_x$ on the lower boundary $\{z=-1\}$, where U is a positive constant.

The channel is supposed to be infinite in the direction x and y (if $d=3$) but for the mathematical study of the problem, we will represent it by a bounded domain $\Omega =]-L, L[^{d-1} \times]-1, 1[$ with L large, provided with periodicity conditions in the variable x and y. The system we deal with reads

$$\frac{\partial \varphi}{\partial t} + v.\nabla\varphi - \text{ div } (B(\varphi)\nabla\mu) = 0 \tag{1.1}$$

$$\mu = -\alpha\Delta\varphi + F'(\varphi) \tag{1.2}$$

$$\frac{\partial v}{\partial t} - 2 \,\text{div } (\eta(\varphi)D(v)) + (v.\nabla)v + \nabla p = \mu\nabla\varphi \tag{1.3}$$

$$\text{div } (v) = 0 \text{ in } \Omega \tag{1.4}$$

$$v(0) = v_0 \text{ and } \ \varphi(0) = \varphi_0 \tag{1.5}$$

provided with the boundary conditions

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \{z=1\} \cup \{z=-1\} \tag{1.6}$$

$$\varphi \text{ is 2L-periodic in the variables } x,\, y \tag{1.7}$$

$$v = Ue_x \quad \text{on} \quad \{z=1\} \text{ and } v = -Ue_x \text{ on } \ \{z=-1\} \tag{1.8}$$

$$v \text{ is 2L-periodic in the variables } x,\, y \tag{1.9}$$

1.2 Functional spaces and notations

In this paper let $|.|_p$ denote the natural norm on $L^p(\Omega)$, for $p=2$ the inner product is $(.,.)$, and $\|.\|_s$ is the norm on the space $H^s(\Omega)$ induced by the inner product $((.,.))_s$. Let $\mathbb{L}^p(\Omega) = (L^p(\Omega))^d$ and $\mathbb{H}^s(\Omega) = (H^s(\Omega))^d$.

The inner product of two tensors σ and σ' is defined by

$$\sigma : \sigma' = \sum_{i,j=1}^{d} \sigma_{i,j}\sigma'_{i,j}.$$

1.2.1 Velocity

For the mathematical study of the problem, we introduce the following homogeneous conditions

$$u = 0 \text{ on } \{z = \pm 1\}, \tag{1.10}$$

$$u \text{ is } 2L\text{-periodic in the variables } x,\, y, \tag{1.11}$$

and the spaces

$$\mathcal{V} = \{u \in \mathcal{D}(\overline{\Omega})^d, \ \text{div } (u) = 0, \text{ satisfying (1.10) and (1.11)}\}$$

$$V_s = \overline{\mathcal{V}}^{\mathbb{H}^s(\Omega)}.$$

As usual the spaces V_0 and V_1 will be noted H and V. Let us recall the following results.

We introduce the Stokes operator, which is a non-bounded operator in H of domain $D(A) = V_2$ defined by

$$Au = -\Delta u + \nabla\pi \in H, \ \forall u \in V_2.$$

Lemma 1.1 (Poincaré's inequality [1])
The application $u \mapsto |\nabla u|_2$ is a norm on V equivalent to the $\|.\|_1$-norm that is to say there exists $C(\Omega) > 0$ such that for all $u \in V$ we have

$$\|u\|_1 \leq C(\Omega)|\nabla u|_2. \tag{1.12}$$

Lemma 1.2 (Korn's inequality [13])
Let Ω an open and regular set in $\mathbb{R}^d$ and $u \in \mathbb{H}_0^1(\Omega)$ a vector field on Ω, then we have

$$|\nabla u|_2 \leq \sqrt{2}|D(u)|_2,$$

with the equality if we suppose $div(u) = 0$.

Lemma 1.3 (Regularity of the Stokes problem [20])
There exists a constant $C > 0$ such that for any $u \in V_2$ we have

$$\|u\|_2 \leq C|Au|_2,$$

$$\|\pi\|_{H^1\backslash\mathbb{R}} \leq C|Au|_2,$$
$$\|\pi\|_{L^2\backslash\mathbb{R}} \leq C\|u\|_1.$$

Lemma 1.4
For $u, v, w \in \mathbb{H}^1(\Omega)$ we define

$$b(u, v, w) = \sum_{i,j=1}^{d} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j,$$

then b is endowed by the following properties

$$b(u, v, w) + b(u, w, v) = 0, \ \forall u, v, w \in V, \tag{1.13}$$

and

$$\begin{aligned} &|b(u, v, w)| \leq C\|u\|_1\|v\|_1\|w\|_1 \ \forall u, v, w \in \mathbb{H}^1(\Omega), \\ &|b(u, v, w)| \leq C|u|_2^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}|v|_2^{\frac{1}{2}}\|v\|_1^{\frac{1}{2}}\|w\|_{\frac{d}{2}}, \ \forall u, v \in \mathbb{H}^1(\Omega), \forall w \in V_{\frac{d}{2}}. \end{aligned} \tag{1.14}$$

Later on, we denote by $B(u, v)$ the continuous linear form on V defined by $B(u, v).w = b(u, v, w)$. The previous result shows that B is bilinear and continuous on $V \times V$.

Finally, if $U > 0$ is given we introduce the stationary solution of (1.3)-(1.4) under conditions (1.8)-(1.9) when φ is constant which is given by

$$v_\infty^U = Uze_x. \tag{1.15}$$

1.2.2 Order parameter

Here we introduce the homogeneous boundary conditions given by

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \Delta\varphi}{\partial \nu} = 0 \text{ on } \{z = \pm 1\}, \tag{1.16}$$
$$\varphi \text{ is } 2L\text{-periodic in the variables } x,\ y, \tag{1.17}$$

and we will use the following functional spaces

$$\Phi = \{\psi \in \mathcal{D}(\overline{\Omega}), \text{ satisfying (1.16) and (1.17)}\},$$

and for any $s \geq 0$,

$$\Phi_s = \overline{\Phi}^{H^s(\Omega)},$$

endowed with the H^s norm. For any $f \in L^1(\Omega)$, let $m(f) = \frac{1}{|\Omega|}\int_\Omega f$ its average.

One can find in [19] the following lemma.

Lemma 1.5
For any $\varphi \in \Phi_1$ we have

$$\|\varphi - m(\varphi)\|_1 \leq C|\nabla\varphi|_2, \tag{1.18}$$

and for any $\varphi \in \Phi_{s+2}$ we have

$$\|\varphi - m(\varphi)\|_{s+2} \leq C\|\Delta\varphi\|_s. \tag{1.19}$$

Remark 1.1
From this lemma one deduce

$$\|\nabla\varphi\|_1 = \|\nabla(\varphi - m(\varphi))\|_1 \leq \|\varphi - m(\varphi)\|_2 \leq C|\Delta\varphi|_2, \ \forall\varphi \in \Phi_2.$$

We will often use those kind of estimates.

Later on, we will use some basic Hilbertian interpolation results which can be found in [15]. We also recall the following results ([15] and [18]).

Lemma 1.6
Let $X \subset Y \subset Z$ three Hilbert spaces, and suppose that the embedding of X into Y is compact.

i) For any $p_1, p_2 \in]1, +\infty[$ the embedding

$$\{f \in L^{p_1}(0,T;X),\ \frac{df}{dt} \in L^{p_2}(0,T;Z)\} \hookrightarrow L^{p_1}(0,T;Y)$$

is compact.

ii) For any $p > 1$ the embedding

$$\{f \in L^{\infty}(0,T;X),\ \frac{df}{dt} \in L^{p}(0,T;Z)\} \hookrightarrow C([0,T],Y)$$

is compact.

iii) The following continuous embedding holds

$$\{f \in L^2(0,T;X),\ \frac{df}{dt} \in L^2(0,T;Y)\} \hookrightarrow C(0,T;[X,Y]_{\frac{1}{2}})$$

2 Main results

In this section we specify what we mean by weak and strong solutions of the problem (1.1)-(1.9) and we state precisely the different results of this paper.

2.1 Definitions

We define a first notion of weak solutions.

Definition 2.1
Let $U > 0$, $v_0 \in v_\infty^U + H$, $\varphi_0 \in \Phi_1$ given. We say that (φ, v) is a weak solution of (1.1)-(1.9) on $[0,T[$ $(0 < T \leq +\infty)$, if

- *v and φ satisfy*

$$v - v_\infty^U \in L^\infty(0,T;H) \cap L^2_{loc}(0,T;V) \cap C^0([0,T[;V'_{\frac{d-2}{4}}),$$
$$\varphi \in L^\infty(0,T;\Phi_1) \cap L^2_{loc}(0,T;\Phi_3) \cap C^0([0,T[;\Phi_1).$$

- *If we let*

$$\mu = -\alpha\Delta\varphi + F'(\varphi) \quad \text{in } \mathcal{D}'(]0,T[\times\Omega), \tag{2.1}$$

then μ satisfies

$$\mu \in L^2_{loc}(0,T;\Phi_1).$$

- *We have the initial conditions*

$$v(0) = v_0, \quad \text{and} \quad \varphi(0) = \varphi_0.$$

- *For any w in V,*

$$\frac{d}{dt}(v,w) + b(v,v,w) + 2\int_\Omega \eta(\varphi)D(v) : D(w) = -\int_\Omega (w.\nabla\mu)\varphi \text{ in } \mathcal{D}'(]0,T[). \tag{2.2}$$

- *For any ψ in Φ_1,*

$$\frac{d}{dt}(\varphi,\psi) + \int_\Omega B(\varphi)\nabla\mu.\nabla\psi - \int_\Omega (v.\nabla\psi)\varphi = 0 \text{ in } \mathcal{D}'(]0,T[). \tag{2.3}$$

We also define the notion of strong solutions.

Definition 2.2
Given $U > 0$, $v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_2$ we say that (v,φ) is a strong solution of (1.1)-(1.9) on $[0,T[$, if (v,φ) is a weak solution on $[0,T[$ $(0 < T \leq +\infty)$ and if furthermore we have

$$v - v_\infty^U \in L^\infty(0,T;V) \cap L^2_{loc}(0,T;V_2) \cap C^0([0,T[;V),$$
$$\varphi \in L^\infty(0,T;\Phi_2) \cap L^2_{loc}(0,T;\Phi_4) \cap C^0([0,T[;\Phi_2).$$

Remark 2.1
The functions of $H^2(\Omega)$ being continuous on $\overline{\Omega}$, any strong solution φ is continuous on $[0,T[\times\overline{\Omega}$.

In the degenerate case, we allow F to be singular at the points where B vanishes. That's the reason why the definition 2.1 can be irrelevant and so, if we only impose that BF'' stay bounded, we have another definition of weak solutions, weaker than definition 2.1.

Definition 2.3
Let $U > 0$, $v_0 \in v_\infty^U + H$, $\varphi_0 \in \Phi_1$ given. We say that (φ,v) is a weak solution, in a weaker sense than the one of definition 2.1, of (1.1)-(1.9) on $[0,T[$ $(0 < T \leq +\infty)$, if

- *v and φ satisfy*

$$v - v_\infty^U \in L^\infty(0,T;H) \cap L^2_{loc}(0,T;V) \cap C^0([0,T[;V'_{\frac{d-2}{4}}),$$
$$\varphi \in L^\infty(0,T;\Phi_1) \cap L^2_{loc}(0,T;\Phi_2) \cap C^0([0,T[;\Phi_{\frac{1}{2}}).$$

- *We have the initial conditions*

$$v(0) = v_0 \quad \text{and} \quad \varphi(0) = \varphi_0.$$

- *For any w in V,*

$$\frac{d}{dt}(v,w) + b(v,v,w) + 2\int_\Omega \eta(\varphi)D(v) : D(w) = -\alpha\int_\Omega (w.\nabla\varphi)\Delta\varphi \text{ in } \mathcal{D}'(]0,T[), \tag{2.4}$$

- *For any ψ in V_3,*

$$\frac{d}{dt}(\varphi,\psi) + \alpha \int_\Omega \Delta\varphi \,\text{div}\,(B(\varphi)\nabla\psi) + \int_\Omega B(\varphi)F''(\varphi)\nabla\varphi.\nabla\psi - \int_\Omega (v.\nabla\psi)\varphi = 0 \text{ in } \mathcal{D}'(]0,T[). \quad (2.5)$$

Remark 2.2
In this weak formulation the chemical potential μ does not appear any more. In particular, in (2.4) we replaced $\mu\nabla\varphi$ by $-\alpha\Delta\varphi\nabla\varphi$ in the right member, because the term $F'(\varphi)\nabla\varphi$ is the gradient of $F(\varphi)$ and can be considered as a part of the pressure gradient.

2.2 Statement of the results

Later on, the functions B,η are supposed to be, at least, locally Lipschitz on $\mathbb{R}$. We also suppose that there exists $\eta_1,\eta_2 > 0$ such that

$$\eta_1 \le \eta(x) \le \eta_2, \ \forall x \in \mathbb{R}.$$

2.2.1 Non-degenerate case

We suppose here that

$$\exists\, B_1, B_2 > 0, \quad B_1 \le B(x) \le B_2, \ \forall x \in \mathbb{R}. \quad (2.6)$$

Moreover we suppose that the function F satisfies

$$F \text{ is of } C^2 \text{ class, and } \ F \ge 0, \quad (2.7)$$

$$\begin{gathered}\exists\, F_1, F_2 > 0 \text{ such that } \ |F'(x)| \le F_1|x|^p + F_2, \ |F''(x)| \le F_1|x|^{p-1} + F_2, \ \forall x \in \mathbb{R}, \\ \text{where } 1 \le p \le 3 \text{ if } d = 3 \text{ and } 1 \le p < +\infty \text{ if } d = 2\end{gathered} \quad (2.8)$$

$$\begin{gathered}\forall\gamma \in \mathbb{R}, \exists\, F_3(\gamma) > 0, \ F_4(\gamma) \ge 0 \text{ such that,} \\ (x-\gamma)F'(x) \ge F_3(\gamma)F(x) - F_4(\gamma), \ \forall x \in \mathbb{R},\end{gathered} \quad (2.9)$$

$$\exists\, F_5 \ge 0 \text{ such that } F''(x) \ge -F_5, \ \forall x \in \mathbb{R}. \quad (2.10)$$

Remark 2.3

- *Let us emphasize that, the physical-relevant functions F are always bounded from below and so we can suppose $F \ge 0$ because adding a constant to F does not change the equations.*
- *The assumption (2.9) is satisfied for example by any convex function with $F_3(\gamma) = 1$ and $F_4(\gamma) = F(\gamma)$, but we remind that we are interested in functions with double-well structure which are not convex. Hence, this assumption must be seen as a generalization of a convexity property.*
- *As an example, polynomial functions of even order for $d = 2$, or of order 4 for $d = 3$, with strictly positive dominant coefficient can be used as a function F. The Cahn-Hilliard equation is often studied with such polynomial nonlinearities ([19]).*

Theorem 2.1 (Existence of global weak solutions)
Given $U > 0, v_0 \in v_\infty^U + H, \varphi_0 \in \Phi_1$, if B satisfies (2.6) and F satisfies (2.7)-(2.10) then there exists a global weak solution (v,φ) to (1.1)-(1.9) in the sense of definition 2.1.

Furthermore, in order to prove a result concerning strong solutions we must suppose

$$B \text{ and } \eta \text{ are of } C^1\text{-class and } B',\ \eta' \text{ are bounded on } \mathbb{R}, \tag{2.11}$$

$$F \text{ is of } C^3\text{-class and } \exists F_6 > 0, \quad |F'''(x)| \le F_6(1+|x|^q),\ \forall x \in \mathbb{R}, \tag{2.12}$$

where $q < 3$ if $d = 3$ and $q < +\infty$ if $d = 2$.

Theorem 2.2 (Strong solutions)
Given $U > 0, v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_2$, if B satisfies (2.6), (2.11), and F satisfies (2.7)-(2.10) and (2.12), then

- *If $d = 2$, there exists a unique global strong solution of (1.1)-(1.9) on $\mathbb{R}^+$.*
- *If $d = 3$, there exists $T_0(v_0, U, \varphi_0) > 0$ and a unique strong solution of (1.1)-(1.9) on $[0, T_0[$.*

2.2.2 Degenerate case

The situation is slightly different and we will follow [9]. As we will see, we are going to show that if the initial data φ_0 takes its values in $[-1,1]$ then for almost every time, $\varphi(t)$ has the same property. That's the reason why we suppose that B is a positive function of C^1-class defined on $[-1,1]$ and such that

$$x \in [-1,1] \text{ and } B(x) = 0 \Leftrightarrow x \in \{-1,1\}. \tag{2.13}$$

For technical reasons we extend B to $\mathbb{R}$ letting $B(x) = 0$ if $|x| > 1$. In the same way we suppose that

$$F = F_1 + F_2,$$

where F_1 is a convex function defined on $]-1,1[$, of C^2-class such that the product BF_1'' is continuous on $[-1,1]$, and F_2 a function of C^2-class on $[-1,1]$. We extend F_2 to be a function of C^2-class on $\mathbb{R}$ with $\|F_2''\|_{L^\infty(\mathbb{R})} \le F_0$. We also suppose that

$$F_1'' \text{ is non-decreasing near } x = 1 \text{ and non-increasing near } x = -1. \tag{2.14}$$

Finally we will be able to derive some estimates, using the function G defined on $]-1,1[$ by

$$G(0) = 0,\ G'(0) = 0,\ G''(x) = B(x)^{-1}, \quad \forall x \in]-1,1[.$$

We can now state the following result on the existence of weak solutions in the degenerate case.

Theorem 2.3
Under assumptions (2.6),(2.13)-(2.14), for any given $0 < T < +\infty$, $U > 0$, $v_0 \in v_\infty^U + H$ and $\varphi_0 \in \Phi_1$, such that $|\varphi_0|_\infty \le 1$ and

$$\int_\Omega \Big(F(\varphi_0) + G(\varphi_0)\Big) < +\infty, \tag{2.15}$$

there exists a weak solution of (1.1)-(1.9) on $[0,T[$ in the sense of definition 2.3 which satisfies

$$|\varphi(t,x)| \le 1 \text{ for almost every } (t,x) \in]0,T[\times\Omega.$$

Moreover, if we suppose $B'(1) = 0$ and $B'(-1) = 0$, then for almost every $t \in [0,T[$ the set

$$\{x \in \Omega,\ |\varphi(t,x)| = 1\}$$

has zero measure.

Remark 2.4

1) *If $B'(-1) \neq 0$ and $B'(1) \neq 0$ then one can see that G is bounded on $[-1,1]$ and that, as BF'' is supposed to be bounded, the function F is also bounded. In that case, (2.15) is satisfied for every initial data. That is for example the case if we take $B(x) = 1-x^2$ and $F(x) = -r.x^2 + (1+x)\log(1+x) + (1-x)\log(1-x)$. In all other cases, the condition (2.15) is not trivial and imposes that φ_0 must be far enough from the critical values -1 and 1. An important case where this condition is fulfilled is the one where we have $|\varphi_0|_\infty < 1$.*

2) *We want to point out the importance of the L^∞-estimate obtained in this theorem because, in a physical point of view, only the values of φ between -1 and 1 are meaningful.*

3) *If B is degenerating enough, then (2.15) implies that $\{x,\ |\varphi(t,x)| = 1\}$ has zero measure for $t = 0$, and then the last point of the theorem ensures that it is still true for almost every time. In a physical point of view we can say that if the initial state does not contain any pure phase area, then the alloy will never contain any pure phase area.*

2.2.3 Qualitative behavior

If we only consider the Cahn-Hilliard equation, some results are known upon the asymptotic behavior of the solutions. The existence of a Lyapunov function and of a global attractor is shown in [7] and [19].

In space dimension $d = 1$, several studies ([4], [5]) concern the stationary solutions which are minima of the free energy given by

$$E(\varphi) = \int_\Omega \left(\frac{\alpha}{2} |\nabla\varphi|^2 + F(\varphi) \right).$$

In this case, the linear stability of those solutions is investigated in [4]. They show that for a mean ω fixed, these solutions are

i) the constant solution which is locally linearly stable,

ii) a boundary layer solution which is linearly unstable,

iii) a transition layer solution which is globally stable.

The next theorem deals with the point i). The solutions ii) and iii) are specific to the one-dimensional case and the study above do not allow any conclusion about this kind of non constant stationary solutions. Even if some stationary solutions are known in higher dimensions ([21], [22], [23]), we do not know any stability result about them.

Moreover, one can see that if we consider the Cahn-Hilliard equation with a supplementary transport term $v.\nabla\varphi$ with $v(t,x)$ regular and fixed, the previous results in dimension $d = 1$ does not subsist.

With our complete model (1.1)-(1.9), we are able to show the stability of the stationary solution where φ is a constant ω, and $v = v_\infty^U$ under the conditions that U is small enough and that the mean ω lies in a metastable region of F ([4], [11]) that is to say if F is convex near ω.

Theorem 2.4
Let I an open interval of $\mathbb{R}$ and $\omega \in I$ given. We suppose that B is a non-negative function of C^1-class defined on I, F is a function of C^3-class defined on I. Then, for $U > 0$ small enough, the stationary solution of (1.1)-(1.9) given by $\varphi_\infty = \omega$ and v_∞^U defined by (1.15) is asymptotically stable under the conditions

$$\begin{aligned} B(\omega) &> 0, && (2.16)\\ F''(x) &\geq 0 \text{ for any } x \text{ in a neighborhood of } \omega. && (2.17) \end{aligned}$$

More precisely, for any $\varepsilon > 0$ there exists $\beta > 0$ such that for any $U > 0$, $v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_2$ satisfying

$$m(\varphi_0) = \omega,\ |U| \leq \beta,\ \|v_0\|_1 \leq \beta,\ \|\varphi_0 - m(\varphi_0)\|_2 \leq \beta,$$

there exists a unique global strong solution on $\mathbb{R}^+$ of the problem (1.1)-(1.9) *in the sense of definition 2.2 such that*

$$\|v - v_0\|_{L^\infty(\mathbb{R}^+;V)} + \|\varphi - \varphi_0\|_{L^\infty(\mathbb{R}^+;\Phi_2)} \leq \varepsilon,$$

and if ε is small enough, we have as $t \to +\infty$

$$\varphi(t) \longrightarrow \varphi_\infty \text{ in } \Phi_s \text{ for any } 0 \leq s < 2,$$
$$v(t) - v_\infty^U \longrightarrow 0 \text{ in } V_s \text{ for any } 0 \leq s < 1.$$

Remark 2.5

1) *We point out that this theorem is proved in both cases $d = 2$ and $d = 3$ and that we allow B to degenerate far from ω and F to be only defined in a neighborhood of ω. Hence, this theorem improves theorems 2.2 and 2.3 because we were not able under general assumptions to prove existence of global strong solutions in the tridimensional case or in the degenerate case.*

2) *This theorem is for example applicable if*

$$B(x) = (1 - x^2)^r, \; r \geq 0, \quad \text{and } F(x) = 1 - x^2 + c\Big((1+x)\log(1+x) + (1-x)\log(1-x)\Big),$$

for ω in $]-1, -1+\delta[$ or in $]1-\delta, 1[$ for a $\delta > 0$ small enough. That is to say that the homogeneous alloy is stable if one of the component is predominant.

3) *Let us suppose that B is constant ([6], [7]) and that the theoretical thickness of the interface is not too small, that is to say ([6], [8]) that for a constant C depending only on Ω, we have*

$$-F''(x) \leq \alpha C, \text{ for any } x \in I. \tag{2.18}$$

Then, with straightforward computations, one can deduce from the estimates given in the proof of this theorem (section 3.5) that for $U > 0$, φ_0 and v_0 given not necessarily small, if a global strong solution (φ, v) of the problem exists (if $d = 2$ for example), we have

$$\varphi(t) - m(\varphi_0) \to 0 \text{ in } \Phi_s \text{ for any } 0 \leq s < 2.$$

We want to emphasize that we do not suppose that $m(\varphi_0)$ lies in a metastable region of F.

Furthermore, if we suppose that

$$U \leq C\eta_2,$$

with C depending only on Ω, that is to say that the Reynolds number of the flow is small, then we also have

$$v(t) - v_\infty^U \to 0 \text{ in } V_s \text{ for any } 0 \leq s < 1.$$

In that case, we have a quantitative estimate on the smallness of the shear velocity we must impose to control the behavior of the flow.

3 Proofs of the main results

3.1 The non-homogeneous boundary conditions

One of the key-points in the following proofs is to get rid of the non-homogeneous boundary condition (1.7). We will use for that the following lemma ([16]) both in dimension $d = 2$ or $d = 3$.

Lemma 3.1
For any $\lambda > 0$, there exists a smooth function $v_\lambda = f(z)e_x$ such that $f(1) = 1$, $f(-1) = -1$ and

$$|v_\lambda|_\infty = 1,\ |v_\lambda|_4 \le C_\Omega \lambda, \tag{3.1}$$
$$|b(u, v_\lambda, u)| \le C_\Omega \lambda \|u\|_1^2 \text{ for any } u \text{ in } V, \tag{3.2}$$
$$|\nabla v_\lambda|_2^2 = 2|D(v_\lambda)|_2^2 \le C_\Omega \frac{1}{\lambda^4}(1+\lambda^8),\ |\Delta v_\lambda|_2^2 \le C_\Omega \frac{1}{\lambda^{12}}(1+\lambda^{24}), \tag{3.3}$$

where the constant C_Ω depends only on the open set Ω.

Remark 3.1
Such a function v_λ is divergence-free and satisfies

$$(v_\lambda.\nabla)v_\lambda = 0.$$

3.2 Non-degenerate case

In this subsection we give the proof of the theorem 2.1.

Theorem
Given $U > 0, v_0 \in v_\infty^U + H, \varphi_0 \in \Phi_1$, if B satisfies (2.6) and F satisfies (2.7)-(2.10) then there exists a global weak solution (v, φ) to (1.1)-(1.9) in the sense of definition 2.1.

Proof :
The idea of the proof is to use the lemma 3.1 and to let, for a given λ,

$$v = u + U v_\lambda,$$

so that the boundary conditions for u are homogeneous. Hence, as $Uv_\lambda - u_\infty^U \in V$, the theorem will be proved if we show that, for a suitable λ depending on U, there exists (u, φ) such that

$$u \in L^\infty(\mathbb{R}^+; H) \cap L^2_{loc}(\mathbb{R}^+; V),$$
$$\varphi \in L^\infty(\mathbb{R}^+; \Phi_1) \cap L^2_{loc}(\mathbb{R}^+; \Phi_2),$$

which satisfy the initial condition, $u(0) = v_0 - Uv_\lambda \equiv u_0,\quad \varphi(0) = \varphi_0$, and

- for any w in V

$$\begin{aligned}\frac{d}{dt}(u, w) + b(u, u, w) + 2\int_\Omega \eta(\varphi)D(u) : D(w) + Ub(u, v_\lambda, w) + Ub(v_\lambda, u, w)\\ + 2U\int_\Omega \eta(\varphi)D(v_\lambda) : D(w) = -\int_\Omega (w.\nabla\mu)\varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^{+*}),\end{aligned} \tag{3.4}$$

- for any ψ in Φ_1

$$\frac{d}{dt}(\varphi, \psi) + \int_\Omega B(\varphi)\nabla\mu.\nabla\psi - \int_\Omega (u.\nabla\psi)\varphi - U\int_\Omega (v_\lambda.\nabla\psi)\varphi = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^{+*}), \tag{3.5}$$

where μ is always given by (2.1).

In order to show this last result we use a Galerkin approximation. We consider $(w_i)_{i\ge1}$ the family of the eigen functions of the Stokes operator A as a Galerkin base in the space V and $(\psi_i)_{i\ge1}$ the family of the eigen functions of the operator $-\Delta$ with the boundary conditions (1.16) and (1.17) as a Galerkin base in Φ_1. We can impose ψ_1 to be the constant function equals to 1 and we remark, as usual, that the (ψ_i) are orthogonal both in Φ_0 and in Φ_1. We define the n-dimensional spaces $\Psi_n = Span(\psi_1, ..., \psi_n)$ and $\mathcal{W}_n = Span(w_1, ..., w_n)$, and P_{Ψ_n} and $P_{\mathcal{W}_n}$ the orthogonal projectors on these spaces in Φ_0 (resp. in H). We remark that the choice of the (ψ_i) implies that Ψ_n is stable under the operator $-\Delta$.

Let us remark that, even if Ω is not regular enough, one can see that because of the periodic boundary conditions, the eigenfunctions of the Stokes operator are regular.

• ***Step 1:*** At this point, we seek three functions of the form

$$u_n(t) = \sum_{i=1}^n \alpha_i(t) w_i \in \mathcal{W}_n,\ \varphi_n(t) = \sum_{i=1}^n \beta_i(t)\psi_i \in \Psi_n,\ \mu_n(t) = \sum_{i=1}^n \gamma_i(t)\psi_i \in \Psi_n,$$

where α_i, β_i and γ_i are real-valued functions of C^1 class, such that $u_n(0) = P_{\mathcal{W}_n}(u_0)$, $\varphi_n(0) = P_{\Psi_n}(\varphi_0)$ and

- for any w in $\mathcal{W}_n$

$$\begin{aligned}(\frac{du_n}{dt}, w) + b(u_n, u_n, w) + 2\int_\Omega \eta(\varphi_n)D(u_n) : D(w) + Ub(u_n, v_\lambda, w) + Ub(v_\lambda, u_n, w)\\ + 2U\int_\Omega \eta(\varphi_n)D(v_\lambda) : D(w) = -\int_\Omega (w.\nabla\mu_n)\varphi_n,\end{aligned} \tag{3.6}$$

- for any ψ in Ψ_n,

$$(\frac{d\varphi_n}{dt}, \psi) + \int_\Omega B(\varphi_n)\nabla\mu_n.\nabla\psi - \int_\Omega (u_n.\nabla\psi)\varphi_n - U\int_\Omega (v_\lambda.\nabla\psi)\varphi_n = 0, \tag{3.7}$$

where

$$\mu_n = -\alpha\Delta\varphi_n + P_{\Psi_n}(F'(\varphi_n)). \tag{3.8}$$

The functions B, η and F' being locally Lipschitz, one can easily see that this system of equations is equivalent to a Cauchy problem for an ordinary differential equations system in the unknowns $\alpha_i, \beta_i, \gamma_i$. The Cauchy-Lipschitz theorem ensures that this system has a unique solution into an interval $[0, t_n[$, $t_n > 0$.

• ***Step 2:*** Putting $\psi_1 = 1$ as a test function in (3.6), we find that, for any $n \geq 1$ and any $t \in [0, t_n[$, we have

$$\frac{d}{dt}m(\varphi_n) = \frac{1}{|\Omega|}\int_\Omega \frac{d\varphi_n}{dt}\psi_1 = 0,$$

so that

$$m(\varphi_n(t)) = m(\varphi_n(0)) = m(P_{\Psi_n}(\varphi_0)) = m(\varphi_0). \tag{3.9}$$

That is to say that $m(\varphi_n)$ is independent of t and n and its value depends only on the initial data φ_0.

• ***Step 3:*** We must now derive some *a priori* estimates in order to show that $t_n = +\infty$ for every $n \geq 0$ and that the sequences (u_n), (φ_n) and (μ_n) are bounded in the appropriate functional spaces.

- First of all, we use $\mu_n(t) \in \Psi_n$ as a test function in (3.7) and we get

$$(\frac{d\varphi_n}{dt}, \mu_n(t)) + \int_\Omega B(\varphi_n)|\nabla\mu_n|^2 - \int_\Omega (u_n.\nabla\mu_n)\varphi_n = -U\int_\Omega (v_\lambda.\nabla\mu_n)\varphi_n,$$

and so, using (3.8), we have

$$\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_n|_2^2 + \int_\Omega F(\varphi_n)\right) + \int_\Omega B(\varphi_n)|\nabla\mu_n|^2 - \int_\Omega (u_n.\nabla\mu_n)\varphi_n = -U\int_\Omega (v_\lambda.\nabla\mu_n)\varphi_n.$$

- If we use $u_n(t)$ as a test function in (3.6), we get

$$\frac{d}{dt}\left(\frac{1}{2}|u_n|_2^2\right) + 2\int_\Omega \eta(\varphi_n)D(u_n) : D(u_n) = -Ub(u_n, v_\lambda, u_n)$$
$$- 2U\int_\Omega \eta(\varphi_n)D(v_\lambda) : D(u_n) - \int_\Omega (u_n.\nabla\mu_n)(\varphi_n - m(\varphi_n)),$$

because u_n is divergence free and so, with (1.16)-(1.11), one can easily see that

$$\int_\Omega u_n.\nabla\mu_n = \int_\Omega \text{div } (\mu_n u_n) = \int_{\partial\Omega} \mu_n.(u_n.\nu) = 0.$$

Now we use (1.13) and Korn's inequality, and then, by summing those two last estimates, we get

$$\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_n|_2^2 + \frac{1}{2}|u_n|_2^2 + \int_\Omega F(\varphi_n)\right) + B_1|\nabla\mu_n|_2^2 + \eta_1|\nabla u_n|_2^2$$
$$\leq U|b(u_n, v_\lambda, u_n)| + \sqrt{2}\eta_2 U|D(v_\lambda)|_2|\nabla u_n|_2 + U|v_\lambda|_4|\nabla\mu_n|_2|\varphi_n - m(\varphi)|_4.$$

Using now (1.14), (3.1), (3.2), the Sobolev embedding $H^1 \subset L^4$ ([1]) and Young's inequality we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_n|_2^2 + \frac{1}{2}|u_n|_2^2 + \int_\Omega F(\varphi_n)\right) + B_1|\nabla\mu_n|_2^2 + \eta_1|\nabla u_n|_2^2\\ &\leq C\lambda U|\nabla u_n|_2^2 + \frac{\eta_1}{2}|\nabla u_n|_2^2 + CU^2|D(v_\lambda)|_2^2 + \frac{B_1}{2}|\nabla\mu_n|_2^2 + CU^2\lambda^2|\nabla\varphi_n|_2^2.\end{aligned} \tag{3.10}$$

- ***Step 4:*** We use $\varphi_n - m(\varphi_0) = \varphi_n - m(\varphi_n)\psi_1$ as a test function in (3.8) so that we obtain, using the boundary conditions for φ_n,

$$(\mu_n, \varphi_n - m(\varphi_n)) = \alpha|\nabla\varphi_n|_2^2 + \int_\Omega F'(\varphi_n)(\varphi_n - m(\varphi_0)).$$

But, as $m(\varphi_n - m(\varphi_n)) = 0$, this expression is equal to $(\mu_n - m(\mu_n), \varphi_n - m(\varphi_n))$. Using now (1.18), Young's inequality and (2.9), one has

$$\frac{1}{2\alpha}|\nabla\mu_n|_2^2 \geq \frac{\alpha}{2}|\nabla\varphi_n|_2^2 + F_3(m(\varphi_0))\left(\int_\Omega F(\varphi_n)\right) - F_4(m(\varphi_0))|\Omega|. \tag{3.11}$$

Moreover, if we multiply (3.8) by $-\Delta\varphi_n$, and integrate by parts, we get

$$(\nabla\mu_n, \nabla\varphi_n) = \alpha|\Delta\varphi_n|_2^2 + \int_\Omega F''(\varphi_n)|\nabla\varphi_n|^2.$$

Young's inequality and (2.10) lead to

$$\frac{1}{2}|\nabla\mu_n|_2^2 \geq \alpha|\Delta\varphi_n|^2 - (F_5 + \frac{1}{2})|\nabla\varphi_n|_2^2. \tag{3.12}$$

Finally, estimates (3.11) and (3.12) show that there exists $C_1, C_2 > 0$, depending only on α, B_1 and F_5 such that

$$\begin{aligned}\frac{B_1}{2}|\nabla\mu_n|_2^2 \geq& \frac{B_1}{4}|\nabla\mu_n|_2^2 + C_1|\nabla\varphi_n|^2 + C_1|\Delta\varphi_n|_2^2\\ &+ C_1F_3(m(\varphi_0))\left(\int_\Omega F(\varphi_n)\right) - C_2F_4(m(\varphi_0)).\end{aligned} \tag{3.13}$$

Later on, we choose

$$\lambda = \lambda(U) = \min(k/U, \sqrt[4]{U}). \tag{3.14}$$

where k depends only on the functions B, η and will be fixed in the sequel. Using (3.10) with (3.13) and (3.3) leads to

$$\begin{aligned} \frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_n|_2^2+\frac{1}{2}|u_n|_2^2+\int_\Omega F(\varphi_n)\right)&+\frac{B_1}{4}|\nabla\mu_n|_2^2+\frac{\eta_1}{2}|\nabla u_n|_2^2\\ &+C_1|\nabla\varphi_n|_2^2+C_1|\Delta\varphi_n|_2^2+C_1F_3(m(\varphi_0))\left(\int_\Omega F(\varphi_n)\right)\\ &\le C\lambda U|\nabla u_n|_2^2+CU^2\frac{1}{\lambda}(1+\lambda^2)+CU^2\lambda^2|\nabla\varphi_n|_2^2+C_2F_4(m(\varphi_0))\\ &\le Ck|\nabla u_n|_2^2+f_1(U)+Ck^2|\nabla\varphi_n|_2^2+C_2F_4(m(\varphi_0)), \end{aligned}$$

where $f_1(U)=CU^2(1+\lambda(U)^4)/\lambda(U)^8$ depends only on U and satisfies when U is small enough $f_1(U)=CU(1+U^2)$ which tends to zero when U tends to zero. If now we let

$$k=\min\left(\frac{\eta_1}{4C},\sqrt{\frac{C_1}{2C}}\right), \tag{3.15}$$

we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_n|_2^2+\frac{1}{2}|u_n|_2^2+\int_\Omega F(\varphi_n)\right)&+\frac{B_1}{4}|\nabla\mu_n|_2^2+\frac{\eta_1}{4}|\nabla u_n|_2^2+\frac{C_1}{2}|\nabla\varphi_n|_2^2+C_1|\Delta\varphi_n|^2\\ &+CF_3(m(\varphi_0))\int_\Omega F(\varphi_n)\\ &\le C'F_4(m(\varphi_0))+f_1(U). \end{aligned} \tag{3.16}$$

Let us remark that Poincaré's inequality (1.12) implies that, if we introduce

$$y_n(t)=\frac{\alpha}{2}|\nabla\varphi_n(t)|^2+\frac{1}{2}|u_n|^2+\int_\Omega F(\varphi_n),$$

we have for $\gamma>0$ small enough,

$$y_n'(t)+\gamma y_n(t)\le C'F_4(m(\varphi_0))+f_1(U).$$

As we suppose that F is positive, a Gronwall argument shows that $t_n=+\infty$ for any $n\ge 1$ and that, with (2.7)

$$y_n(t)\le y_n(0)+\gamma^{-1}(C'F_4(m(\varphi_0))+f_1(U)),\ \forall t\ge 0.$$

If we use the fact that P_{Ψ_n} and $P_{\mathcal{W}_n}$ are orthogonal projectors in the spaces H^1 and L^2, we have with (2.7)-(2.12), and the embedding $H^1\subset L^p$ valid with the choice of p in (2.8),

$$\begin{aligned} y_n(0) &= \frac{\alpha}{2}|\nabla P_{\Psi_n}(\varphi_0)|_2^2+\frac{1}{2}|P_{\mathcal{W}_n}(u_0)|_2^2+\int_\Omega F(P_{\Psi_n}(\varphi_0))\\ &\le \frac{\alpha}{2}|\nabla\varphi_0|_2^2+\frac{1}{2}|u_0|_2^2+\left(F_4(m(\varphi_0))|\Omega|+\int_\Omega|P_{\Psi_n}(\varphi_0)-m(\varphi_0))|\ |F'(P_{\Psi_n}(\varphi_0))|\right)F_3(m(\varphi_0))^{-1}\\ &\le C|\nabla\varphi_0|_2^2+C|u_0|_2^2+\Big(|\Omega|F_4(m(\varphi_0))+|P_{\Psi_n}(\varphi_0-m(\varphi_0))|_2|F_1+F_2|P_{\Psi_n}(\varphi_0)|^p|_2\Big)F_3(m(\varphi_0))^{-1}\\ &\le C|\nabla\varphi_0|_2^2+C|v_0|_2^2+CU^2+\Big(|\Omega|F_4(m(\varphi_0))+C|\varphi_0-m(\varphi_0)|_2(1+\|\varphi_0\|_1^p)\Big)F_3(m(\varphi_0))^{-1}. \end{aligned}$$

We let

$$\begin{aligned} k_1(v_0,\varphi_0,U) \equiv & CF_4(m(\varphi_0)) + C|\nabla\varphi_0|_2^2 + C|v_0|_2^2 + Cf_1(U) + CU^2 \\ & + \Big(CF_4(m(\varphi_0)) + C|\varphi_0 - m(\varphi_0)|_2(1+\|\varphi_0\|_1^p)\Big)F_3(m(\varphi_0))^{-1}, \end{aligned} \tag{3.17}$$

so that, using Gronwall's lemma, we have

$$\|u_n\|_{L^\infty(\mathbb{R}+;H)} + \|\varphi_n - m(\varphi_0)\|_{L^\infty(\mathbb{R}+;\Phi_1)} \leq k_1(v_0,\varphi_0,U), \tag{3.18}$$

Integrating in time (3.16) and using (3.18), one easily gets for every $t_0 \geq 0, \tau > 0$,

$$\|u_n\|_{L^2(t_0,t_0+\tau;V)} + \|\varphi_n - m(\varphi_0)\|_{L^2(t_0,t_0+\tau;\Phi_2)} + \|\nabla\mu_n\|_{L^2(t_0,t_0+\tau;L^2(\Omega))} \leq k_2(v_0,\varphi_0,U,\tau), \tag{3.19}$$

where

$$k_2(v_0,\varphi_0,U,\tau) \equiv C(1+\tau)k_1(v_0,\varphi_0,U). \tag{3.20}$$

- ***Step 5:*** We can derive from (3.8) that

$$m(\mu_n) = (\mu_n,\psi_1) = \alpha(-\Delta\varphi_n,\psi_1) + (P_{\Psi_n}F'(\varphi_n),\psi_1) = \int_\Omega F'(\varphi_n),$$

as the first term vanishes thanks to the boundary conditions (1.16) and (1.11).

So we have by (2.8) and the embedding $H^1 \subset L^p$, for almost every $t \geq 0$

$$|m(\mu_n)| \leq F_1\int_\Omega |\varphi_n|^p + F_2|\Omega| \leq F_1|\varphi_n|_p^p + F_2|\Omega| \leq F_1\|\varphi_n(t)\|_1^p + F_2 \leq C(u_0,\varphi_0,U).$$

Finally, we get from (1.18) and (3.19)

$$\|\mu_n\|_{L^2(t_0,t_0+\tau;\Phi_1)} \leq C(v_0,\varphi_0,U,\tau). \tag{3.21}$$

Moreover we have

$$\begin{aligned} \alpha^2|\nabla\Delta\varphi_n|_2^2 &\leq 2|\nabla\mu_n|_2^2 + 2|\nabla P_{\Psi_n}F'(\varphi_n)|_2^2 \\ &\leq 2|\nabla\mu_n|_2^2 + 2|\nabla F'(\varphi_n)|_2^2, \end{aligned}$$

because P_{Ψ_n} is built on the eigen functions of $-\Delta$. Using now (2.8) we derive

$$\begin{aligned} |\nabla F'(\varphi_n)|_2^2 &\leq C\int_\Omega (1+|\varphi_n|^{2p-2})|\nabla\varphi_n|^2 \leq C|\nabla\varphi_n|_2^2 + C||\varphi_n|^{2p-2}|_{\frac{3}{2}}||\nabla\varphi_n|^2|_3 \\ &\leq C|\nabla\varphi_n|_2^2 + C|\varphi_n|_{3p-3}^{2p-2}|\nabla\varphi_n|_6^2 \leq C|\nabla\varphi_n|_2^2 + C|\nabla\varphi_n|_2^{2p-2}|\Delta\varphi_n|_2^2, \end{aligned}$$

thanks to the choice of p in (2.8) which implies the Sobolev embeddings $H^1 \subset L^6$ and $H^1 \subset L^{3p-3}$ hold in both dimensions $d=2$ or $d=3$.

Finally, with (3.18), (3.19) and (3.21) we get

$$\|\varphi_n - m(\varphi_0)\|_{L^2(t_0,t_0+\tau;\Phi_3)} \leq C(v_0,\varphi_0,U,\tau). \tag{3.22}$$

- ***Step 6:***

- Equation (3.6) can also be written

$$\frac{du_n}{dt} + P^*_{\mathcal{W}_n}\Big(B(u_n,u_n) + UB(u_n,v_\lambda) + UB(v_\lambda,u_n) + A(\varphi_n,u_n) + UA(\varphi_n,v_\lambda) + \varphi_n\nabla\mu_n\Big) = 0, \tag{3.23}$$

where, for $\varphi \in \Phi_1, u \in V$, we define $A(\varphi,u) \in V'$ by

$$A(\varphi,u).w = 2\int_\Omega \eta(\varphi)D(u) : D(w), \ \ \forall w \in V.$$

We easily get from Korn's inequality

$$|A(\varphi,u)|_{V'} \leq C\|u\|_1.$$

The fact that $P_{\mathcal{W}_n}$ is an orthogonal projector in V_s for any $s \geq 0$ implies

$$\|P_{\mathcal{W}_n}\|_{L(V_s,V_s)} \leq 1 \text{ and } \ \|P^*_{\mathcal{W}_n}\|_{L(V'_s,V'_s)} \leq 1.$$

And finally, using the Sobolev embedding $H^1 \subset L^4$ (in dimension $d=2$ or 3) one gets

$$|\varphi\nabla\mu|_{V'} \leq C|\varphi|_4|\nabla\mu|_2 \leq C\|\varphi\|_1|\nabla\mu|_2.$$

Using these properties, the embedding $V' \subset V'_{\frac{d}{2}}$ and (1.14) we deduce from (3.23) that

$$\left|\frac{du_n}{dt}\right|_{V'_{\frac{d}{2}}} \leq C + C|u_n|_2\|u_n\|_1 + C\|u_n\|_1 + C\|\varphi_n\|_1|\nabla\mu_n|_2.$$

We deduce from this estimate and from (3.18)-(3.19),(3.21) that for any $n \geq 1$, $t_0 \geq 0$ and $\tau > 0$,

$$\left\|\frac{du_n}{dt}\right\|_{L^2(t_0,t_0+\tau;V'_{\frac{d}{2}})} \leq C(u_0,\varphi_0,U,\tau). \tag{3.24}$$

- We work in the same way to control $\dfrac{d\varphi_n}{dt}$. With (3.7) we can write

$$\frac{d\varphi_n}{dt} + P^*_{\Psi_n}\Big(\text{div }(B(\varphi_n)\nabla\mu_n) + \text{ div }(\varphi_n u_n) + U\,\text{div }(\varphi_n v_\lambda)\Big) = 0, \tag{3.25}$$

and we clearly have

$$|\text{ div }(B(\varphi)\nabla\mu)|_{\Phi'_1} \leq B_2|\nabla\mu|_2, \text{ and } |\text{ div }(\varphi u)|_{\Phi'_1} \leq |\varphi|_4|u|_4 \leq C\|\varphi\|_1\|u\|_1.$$

Finally, the fact that P_{Ψ_n} is an orthogonal projector in Φ_1 leads to

$$\left|\frac{d\varphi_n}{dt}\right|_{\Phi'_1} \leq C|\nabla\mu_n|_2 + C\|\varphi_n\|_1 + C\|\varphi_n\|_1\|u_n\|_1,$$

which implies, using (3.18)-(3.19) that we have for any $t_0 \geq 0$ and $\tau > 0$

$$\left\|\frac{d\varphi_n}{dt}\right\|_{L^2(t_0,t_0+\tau;\Phi'_1)} \leq C(u_0,\varphi_0,U,\tau). \tag{3.26}$$

- ***Step 7:*** We use the estimates (3.18)-(3.19), (3.21), (3.24), (3.26) and the lemma 1.6 so that we can extract subsequences of (u_n), (φ_n) and (μ_n) which satisfy

$$\begin{gathered}
u_n \rightharpoonup u \text{ in } L^\infty(\mathbb{R}^+;H) \text{ weak-}*,\\
u_n \rightharpoonup u \text{ in } L^2_{loc}(\mathbb{R}^+;V) \text{ weak},\\
u_n \to u \text{ in } L^2_{loc}(\mathbb{R}^+;H) \text{ strong},\\
\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2_{loc}(\mathbb{R}^+;V'_{\frac{d}{2}}) \text{ weak},\\
\varphi_n \rightharpoonup \varphi \text{ in } L^2_{loc}(\mathbb{R}^+;\Phi_3) \text{ weak}\\
\varphi_n \rightharpoonup \varphi \text{ in } L^\infty(\mathbb{R}^+;\Phi_1) \text{ weak-}*,\\
\varphi_n \to \varphi \text{ in } L^2_{loc}(\mathbb{R}^+;\Phi_0) \text{ strong and a.e.},\\
\frac{d\varphi_n}{dt} \rightharpoonup \frac{d\varphi}{dt} \text{ in } L^2_{loc}(\mathbb{R}^+;\Phi'_1) \text{ weak},\\
\mu_n \rightharpoonup \mu \text{ in } L^2_{loc}(\mathbb{R}^+;\Phi_1) \text{ weak}.
\end{gathered}$$

Moreover, using the interpolations $[V_1, V'_{\frac{d}{2}}]_{\frac{1}{2}} = V'_{\frac{d-2}{4}}$, $[\Phi_3, \Phi'_1]_{\frac{1}{2}} = \Phi_1$ and the last point of lemma 1.6, we infer that

$$u_n \rightharpoonup u \in C^0([0,T[, V'_{\frac{d-2}{4}}) \text{ weak}, \quad \varphi_n \rightharpoonup \varphi \in C^0([0,T[, \Phi_1) \text{ weak}.$$

In particular, $u_n(0)$ weakly converges to $u(0)$ in $V'_{\frac{d-2}{4}}$ and so $u(0) = u_0$ because $P_{\mathcal{W}_n}$ converges to the identity for the strong topology of operators. In the same way we prove $\varphi(0) = \varphi_0$

Finally, we must prove that the functions u, φ and μ satisfy (3.4)-(3.5).

Consider $\rho \in \mathcal{D}(\mathbb{R}^{+*})$, and $N \geq 1$. For any $n \geq N$, u_n satisfies (3.6) with $w = w_N$; we multiply this equation by $\rho(t)$ and then we integrate by parts. One can easily see that the convergence properties of the sequences (u_n), (φ_n) and (μ_n) let us pass to the limit in this equation: we recall that the convergence in the non linear term $b(u_n, u_n, \rho w_N)$ is valid because we have $u_n \to u$ in $L^2(\mathbb{R}^+, H)$ strong. The limit equation obtained is fulfilled for any N and any $\rho \in \mathcal{D}(\mathbb{R}^{+*})$ so that the density of $Span(w_i)_{i\geq 1}$ in V let us conclude that u, φ and μ satisfy (3.4). In the same way we show that (3.5) is fulfilled.

As far as (2.1) is concerned, the result comes from the fact that P_{Ψ_n} converges to the identity for the strong topology of operators and from the dominated convergence theorem.

■

3.3 Strong solutions in the non-degenerate case

We are going to prove the theorem 2.2.

Theorem

Given $U > 0, v_0 \in v^U_\infty + V$, $\varphi_0 \in \Phi_2$, if B satisfies (2.6), (2.11), *and F satisfies* (2.7)-(2.10) *and* (2.12), *then*

- *If $d = 2$, there exists a unique global strong solution of* (1.1)-(1.9) *on* $\mathbb{R}^+$.
- *If $d = 3$, there exists $T_0(v_0, U, \varphi_0) > 0$ and a unique strong solution of* (1.1)-(1.9) *on* $[0, T_0[$.

Proof :

In the first part of the proof we will show the existence of strong solutions; the uniqueness will be investigated in a second part.

- ***Step 1:*** First of all, let us remark that if we have a couple of functions defined on $[0, T[$ with $0 < T \leq +\infty$, such that

$$v - v^U_\infty \in L^\infty(0,T;V) \cap L^2_{loc}(0,T;V_2), \quad \varphi \in L^\infty(0,T;V_2) \cap L^2_{loc}(0,T;\Phi_4), \tag{3.27}$$

which satisfy (2.2)-(2.3) then we have similar formulas than (3.23) and (3.25) from which we easily infer that $\frac{du}{dt}$ lies in $L^2_{loc}(0,T;H)$ and $\frac{d\varphi}{dt}$ lies in $L^2_{loc}(0,T;\Phi_0)$. Hence, we deduce from the third point of the lemma 1.6 that

$$v - v^U_\infty \in C^0([0,T[,V), \quad \text{and} \quad \varphi \in C^0([0,T[,\Phi_2),$$

and finally (v,φ) is a strong solution of the problem. Therefore in the following we only have to prove the existence of a solution which fulfills (3.27).

• ***Step 2:*** As we did for weak solutions, we seek a strong solution of the form $v = u + Uv_\lambda$ and we are going to derive some other *a priori* estimates on the solution of the Galerkin approximation (3.6)-(3.8) we used in the proof of theorem 2.1.

In the following, instead of (3.14), we need to let

$$\lambda = \min(\sqrt[12]{U}, k/U), \tag{3.28}$$

where k is always given by (3.15). One can easily verify that the estimates in the proof of theorem 2.1 are the same with this choice, which only change the value of the function f_1 in (3.16), which becomes for U small enough $f_1(U) = CU^{\frac{5}{3}}(1+U^{\frac{2}{3}})$. The key point here is to remark that $f_1(U)$ tends to zero when U tends to zero.

Moreover, with this new choice of λ we have with (3.3), for U small enough

$$f_2(U) \equiv U^2|\Delta v_\lambda|_2^2 \leq CU^2\frac{1}{\lambda^{12}}(1+\lambda^{24}) \leq CU(1+U^2).$$

Later on, f_i will always denote a positive continuous function of U, which satisfies $f_i(0)=0$.

The following estimates are made on the Galerkin approximation of the weak solution that we built in the proof of theorem 2.1, but in order to simplify the notations we will drop the subscript n in u_n, φ_n, or μ_n.

We use Au as a test function in (3.6), and after integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla u|_2^2 &- 2\int_\Omega \eta'(\varphi)\nabla\varphi.(D(u+Uv_\lambda).Au) - 2\int_\Omega \eta(\varphi)\Delta(u+Uv_\lambda).Au \\ &+ b(u,u,Au) + Ub(v_\lambda,u,Au) + Ub(u,v_\lambda,Au) \\ &= -\alpha\int_\Omega (Au.\nabla\varphi)\Delta\varphi.\end{aligned} \tag{3.29}$$

We remark that, if the Stokes operator is written $Au = -\Delta u + \nabla\pi$, we have

$$-\int_\Omega \eta(\varphi)\Delta u.Au = \int_\Omega \eta(\varphi)|Au|^2 - \int_\Omega \eta(\varphi)\nabla\pi.Au. \tag{3.30}$$

Now, we use $\Delta^2\varphi$ as a test function in (3.7), it leads to

$$\begin{aligned}\frac{1}{2}\frac{d|\Delta\varphi|_2^2}{dt} &+ \int_\Omega u.\nabla\varphi\Delta^2\varphi + U\int_\Omega v_\lambda.\nabla\varphi\Delta^2\varphi \\ &- \int_\Omega B'(\varphi)\nabla\varphi.\nabla\mu\Delta^2\varphi - \int_\Omega B(\varphi)\Delta\mu\Delta^2\varphi = 0.\end{aligned} \tag{3.31}$$

Using the three estimates (3.29)-(3.31) one gets

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|\Delta\varphi|_2^2+\frac{1}{2}|\nabla u|_2^2\right)+\alpha B_1|\Delta^2\varphi|_2^2+2\eta_1|Au|_2^2\\
&\qquad\le U\int_\Omega |v_\lambda|\ |\nabla\varphi|\ |\Delta^2\varphi|+B_2|\Delta F'(\varphi)|_2|\Delta^2\varphi|_2+2\eta_2 U|\Delta v_\lambda|_2|Au|_2\\
&\qquad+2\|\eta'\|_\infty\int_\Omega|\nabla\varphi|\ |D(u+Uv_\lambda)|\ |Au|+\|B'\|_\infty\int_\Omega|\nabla\varphi|\ |\nabla\mu|\ |\Delta^2\varphi| \qquad (3.32)\\
&\qquad+|b(u,u,Au)|+U|b(v_\lambda,u,Au)|+U|b(u,v_\lambda,Au)|\\
&\qquad+\alpha\int_\Omega|Au||\nabla\varphi||\Delta\varphi|+\int_\Omega|u||\nabla\varphi||\Delta^2\varphi|+\left|\int_\Omega\nabla\pi.Au\right|.
\end{aligned}
$$

• ***Step 3:*** We estimate each of the ten terms of the right member of (3.32), denoted by $I_1,...,I_{11}$. As $|v_\lambda|_\infty=1$, we get

$$
I_1\le U|v_\lambda|_\infty|\nabla\varphi|_2|\Delta^2\varphi|_2\le\frac{\alpha B_1}{10}|\Delta^2\varphi|_2^2+CU^2|\nabla\varphi|_2^2. \qquad (3.33)
$$

Following [19], we have

$$
\begin{aligned}
|\Delta F'(\varphi)|_2^2&\le|F'''(\varphi)|_\infty^2|\nabla\varphi|_4^4+|F''(\varphi)|_\infty^2|\Delta\varphi|_2^2\\
&\le C(1+|\varphi-m(\varphi)|_\infty^{2q})|\nabla\varphi|_4^4+C(1+|\varphi-m(\varphi)|_\infty^{2q+2})|\Delta\varphi|_2^2.
\end{aligned}
$$

- If $d=2$: we choose $\varepsilon>0$ such that

$$
\varepsilon<\frac{2}{q+1}, \qquad (3.34)
$$

and then using the Sobolev embedding $H^{1+\varepsilon}\subset L^\infty$, the interpolation $H^{1+\varepsilon}=[H^1,H^4]_{\frac{\varepsilon}{3}}$ and (1.19), we get

$$
|\varphi-m(\varphi)|_\infty\le C|\nabla\varphi|_2^{1-\frac{\varepsilon}{3}}|\Delta^2\varphi|_2^{\frac{\varepsilon}{3}}.
$$

Moreover, by the embedding $H^{\frac12}\subset L^4$ and the interpolation $H^{\frac12}=[L^2,H^3]_{\frac16}$, we have

$$
|\nabla\varphi|_4=|\nabla(\varphi-m(\varphi))|_4\le C|\nabla\varphi|_2^{\frac56}\|\varphi-m(\varphi)\|_4^{\frac16}\le C|\nabla\varphi|_2^{\frac56}|\Delta^2\varphi|_2^{\frac16}.
$$

In the same way we get

$$
|\Delta\varphi|_2\le C\|\varphi-m(\varphi)\|_2\le C\|\varphi-m(\varphi)\|_1^{\frac23}\|\varphi-m(\varphi)\|_4^{\frac13}\le C|\nabla\varphi|_2^{\frac23}|\Delta^2\varphi|_2^{\frac13}. \qquad (3.35)
$$

Finally, we have

$$
\begin{aligned}
|\Delta F'(\varphi)|_2^2&\le C\left(1+|\nabla\varphi|_2^{2q(1-\frac{\varepsilon}{3})}|\Delta^2\varphi|_2^{2q\frac{\varepsilon}{3}}\right)|\nabla\varphi|_2^{\frac{10}{3}}|\Delta^2\varphi|_2^{\frac23}\\
&+C\left(1+|\nabla\varphi|_2^{(2q+2)(1-\frac{\varepsilon}{3})}|\Delta^2\varphi|_2^{\frac{\varepsilon}{3}(2q+2)}\right)|\nabla\varphi|_2^{\frac43}|\Delta^2\varphi|_2^{\frac23}.
\end{aligned}
$$

One easily see that (3.34) implies that all the powers of $|\Delta^2\varphi|_2$ in this estimate are strictly lesser than 2, and so by Young's inequality and (3.18) we have

$$
I_2\le C|\nabla\varphi|_2^{\frac43}(1+|\nabla\varphi|_2^{\alpha_1})+\frac{\alpha B_1}{10}|\Delta^2\varphi|_2^2, \qquad (3.36)
$$

where $\alpha_1>0$ depends on q and ε.

- If $d = 3$: we use Agmon's inequality ([3]) and interpolation $H^2 = [H^1, H^4]_{\frac{1}{3}}$ to obtain

$$|\varphi - m(\varphi)|_\infty \leq C|\nabla\varphi|^{\frac{1}{2}} \|\varphi - m(\varphi)\|_2^{\frac{1}{2}} \leq C|\nabla\varphi|_2^{\frac{5}{6}} |\Delta^2\varphi|_2^{\frac{1}{6}}.$$

Using now the embedding $H^{\frac{3}{4}} \subset L^4$ and $H^{\frac{3}{4}} = [L^2, H^3]_{\frac{1}{4}}$, we have

$$|\nabla\varphi|_4 \leq C|\nabla\varphi|_2^{\frac{3}{4}} |\Delta^2\varphi|_2^{\frac{1}{4}}.$$

Hence, with (3.35) which is also valid in dimension 3, we get

$$\begin{aligned} |\Delta F'(\varphi)|_2^2 &\leq C\left(1 + |\nabla\varphi|_2^{\frac{5}{3}q} |\Delta^2\varphi|_2^{\frac{q}{3}}\right) |\nabla\varphi|_2^3 |\Delta^2\varphi|_2 \\ &\quad + C\left(1 + |\nabla\varphi|_2^{\frac{5}{3}(q+1)} |\Delta^2\varphi|_2^{\frac{1}{3}(q+1)}\right) |\nabla\varphi|_2^{\frac{4}{3}} |\Delta^2\varphi|_2^{\frac{2}{3}}. \end{aligned}$$

Thanks to the assumption on q, one see that the power of $|\Delta^2\varphi|_2$ are lesser than 2 and so, with Young's inequality

$$I_2 \leq C|\nabla\varphi|_2^{\frac{4}{3}} (1 + |\nabla\varphi|_2^{\alpha_2}) + \frac{\alpha B_1}{10} |\Delta^2\varphi|_2^2, \tag{3.37}$$

where $\alpha_2 > 0$.

As far as the third term is concerned, we can write

$$I_3 \leq CU^2 |\Delta v_\lambda|_2^2 + \frac{\eta_1}{5} |Au|_2^2 \leq C f_2(U) + \frac{\eta_1}{5} |Au|_2^2. \tag{3.38}$$

If now we use $H^{\frac{3}{2}} \subset L^\infty$ for $d = 2$, or Agmon's inequality if $d = 3$, and the interpolation between L^2 and H^3, we have in both cases

$$|\nabla\varphi|_\infty \leq C|\nabla\varphi|_2^{\frac{1}{2}} |\Delta^2\varphi|_2^{\frac{1}{2}}, \tag{3.39}$$

so that

$$\begin{aligned} I_4 &\leq 2\|\eta'\|_\infty U |D(v_\lambda)|_2 |\nabla\varphi|_2^{\frac{1}{2}} |\Delta^2\varphi|_2^{\frac{1}{2}} |Au|_2 + 2\|\eta'\|_\infty |\nabla\varphi|_2^{\frac{1}{2}} |\Delta^2\varphi|_2^{\frac{1}{2}} |\nabla u|_2 |Au|_2 \\ &\leq \frac{\eta_1}{5} |Au|_2^2 + \frac{\alpha B_1}{10} |\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2 |\nabla u|_2^4 + f_3(U) |\nabla\varphi|_2^2. \end{aligned} \tag{3.40}$$

We deal with the fifth term as follows

- if d=2: we have seen, when we estimate the term I_2 that

$$|\Delta\mu|_2^2 \leq 2\alpha |\Delta^2\varphi|_2^2 + 2|\Delta F'(\varphi)|_2^2 \leq C(1 + |\Delta^2\varphi|_2^2).$$

We use another time the embedding $H^1 \subset L^4$ and interpolation results to derive

$$\begin{aligned} I_5 &\leq C|\nabla\varphi|_4 |\nabla\mu|_4 |\Delta^2\varphi|_2 \leq C|\nabla\varphi|_2^{\frac{1}{2}} |\Delta\varphi|_2^{\frac{1}{2}} |\nabla\mu|_2^{\frac{1}{2}} |\Delta\mu|_2^{\frac{1}{2}} |\Delta^2\varphi|_2 \\ &\leq \frac{\alpha B_1}{10} |\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2 + C|\nabla\mu|_2^2 |\Delta\varphi|_2^2 + C|\nabla\varphi|_2^2 |\nabla\mu|_2^2 |\Delta\varphi|_2^2. \end{aligned} \tag{3.41}$$

- if d=3: we need to remark that
$$|\nabla\mu| \leq \alpha|\nabla\Delta\varphi| + |F''(\varphi)||\nabla\varphi|.$$
Using Agmon's inequality together with (1.18) and (1.19) to derive
$$\begin{aligned}|\nabla\varphi|_\infty &= |\nabla(\varphi - m(\varphi))|_\infty \leq \|\varphi - m(\varphi)\|_2^{\frac{1}{2}}\|\varphi - m(\varphi)\|_3^{\frac{1}{2}} \\ &\leq C|\Delta\varphi|_2^{\frac{1}{2}}|\nabla\Delta\varphi|_2^{\frac{1}{2}}.\end{aligned}$$
Finally we will use the following interpolation result
$$|\nabla\Delta\varphi|_2 \leq \|\Delta\varphi\|_1 \leq |\Delta\varphi|_2^{\frac{1}{2}}\|\Delta\varphi\|_2^{\frac{1}{2}} \leq |\Delta\varphi|_2^{\frac{1}{2}}|\Delta^2\varphi|_2^{\frac{1}{2}}.$$
If we notice that $H^1 \subset L^4$ and $H^2 \subset L^\infty$, we can now estimate I_5 by
$$\begin{aligned}I_5 &\leq C|\nabla\varphi|_\infty|\nabla\Delta\varphi|_2|\Delta^2\varphi|_2 + C|F''(\varphi)|_\infty|\nabla\varphi|_4^2|\Delta^2\varphi|_2 \\ &\leq C|\Delta\varphi|_2^{\frac{5}{4}}|\Delta^2\varphi|_2^{\frac{7}{4}} + C(1+|\varphi|_\infty^{q+1})|\Delta\varphi|_2^2|\Delta^2\varphi|_2 \\ &\leq C|\Delta\varphi|_2^{\frac{5}{4}}|\Delta^2\varphi|_2^{\frac{7}{4}} + C(1+|\Delta\varphi|_2^{q+1})|\Delta\varphi|_2^2|\Delta^2\varphi|_2 \\ &\leq \frac{\alpha B_1}{10}|\Delta^2\varphi|_2^2 + C|\Delta\varphi|_2^{10} + C(1+|\Delta\varphi|_2^{2q+2})|\Delta\varphi|_2^2.\end{aligned} \tag{3.42}$$

The term I_6 is estimated as follows.

- if d=2: we use the Sobolev embedding $H^{\frac{1}{2}} \subset L^4$ and the interpolation $H^{\frac{1}{2}} = [L^2, H^1]_{\frac{1}{2}}$ to obtain
$$\begin{aligned}I_6 = |b(u,u,Au)| &\leq C|u|_4|\nabla u|_4|Au|_2 \leq C|u|_2^{\frac{1}{2}}|\nabla u|_2|Au|_2^{\frac{3}{2}} \\ &\leq \frac{\eta_1}{5}|Au|_2^2 + C|u|_2^2|\nabla u|_2^4.\end{aligned} \tag{3.43}$$

- if d=3: we use here the embeddings $H^1 \subset L^6$ and $H^{\frac{1}{2}} \subset L^3$ together with interpolation results to get
$$\begin{aligned}I_6 = |b(u,u,Au)| &\leq C|u|_6|\nabla u|_3|Au|_2 \leq C|\nabla u|_2^{\frac{3}{2}}|Au|_2^{\frac{3}{2}} \\ &\leq \frac{\eta_1}{5}|Au|_2^2 + C|\nabla u|_2^6.\end{aligned} \tag{3.44}$$

Moreover, we easily get
$$I_7 \leq U|v_\lambda|_\infty|\nabla u|_2|\Delta u|_2 \leq \frac{\eta_1}{5}|Au|_2^2 + CU^2|\nabla u|_2^2, \tag{3.45}$$
and with Agmon's inequality,
$$\begin{aligned}I_8 &\leq U|u|_\infty|\nabla v_\lambda|_2|Au|_2 \leq U|\nabla u|_2^{\frac{1}{2}}|\nabla v_\lambda|_2|Au|_2^{\frac{3}{2}} \\ &\leq \frac{\eta_1}{5}|Au|_2^2 + f_4(U)|\nabla u|_2^2.\end{aligned} \tag{3.46}$$
Another time we use Agmon's inequality to deduce that
$$\begin{aligned}I_9 &\leq \alpha|Au|_2|\nabla\varphi|_\infty|\Delta\varphi|_2 \leq C|Au|_2|\Delta\varphi|_2^{\frac{3}{2}}|\nabla\Delta\varphi|_2^{\frac{1}{2}} \\ &\leq \frac{\eta_1}{5}|Au|_2^2 + C|\Delta\varphi|_2^3|\nabla\Delta\varphi|_2.\end{aligned} \tag{3.47}$$

With (3.39) we get

$$\begin{aligned} I_{10} &\leq |u|_2|\nabla\varphi|_\infty|\Delta^2\varphi|_2 \leq |u|_2|\nabla\varphi|_2^{\frac{1}{2}}|\Delta^2\varphi|_2^{\frac{3}{2}} \\ &\leq \frac{\alpha B_1}{10}|\Delta^2\varphi|_2^2 + C|u|_2^4|\nabla\varphi|_2^2. \end{aligned} \tag{3.48}$$

Finally, we use the lemma 1.3 to deduce after integration by parts

$$\begin{aligned} I_{11} &= \left|\int_\Omega \eta'(\varphi)\pi\nabla\varphi.Au\right| \\ &\leq \|\eta'\|_{L^\infty}|\nabla\varphi|_\infty|\pi|_2|Au|_2 \leq C|\nabla\Delta\varphi|_2\|u\|_1|Au|_2 \leq \frac{\eta_1}{5}|Au|_2^2 + C|\nabla\Delta\varphi|_2^2|\nabla u|_2^2. \end{aligned} \tag{3.49}$$

• ***Step 4 - Study of the two-dimensional case:***
We let

$$y(t) = \frac{\alpha}{2}|\Delta\varphi|_2^2 + \frac{1}{2}|\nabla u|_2^2,$$

so that, using (3.33)-(3.48), estimate (3.32) reads

$$\begin{aligned} y'(t) &+ \frac{\alpha B_1}{10}|\Delta^2\varphi|_2^2 + \frac{\eta_1}{5}|Au|_2^2 \\ &\leq C|\nabla\varphi|_2 + C|\nabla\varphi|_2^2 + C|u|_2^4|\nabla\varphi|_2^2 + (C + f_3(U))|\nabla\varphi|_2^{\frac{4}{3}}(1 + |\nabla\varphi|_2^{\alpha_1}) + Cf_2(U) \\ &+ C\Big(U^2 + f_4(U) + |\nabla\varphi|_2^2|\nabla u|_2^2 + |\nabla\mu|_2^2 + |\nabla\varphi|_2^2|\nabla\mu|_2^2 \\ &+ |u|_2^2|\nabla u|_2^2 + |\Delta\varphi|_2|\nabla\Delta\varphi|_2 + |\nabla\Delta\varphi|_2^2\Big)y(t) \\ &\equiv k_3(v_0,\varphi_0,U) + g(t)y(t). \end{aligned} \tag{3.50}$$

where, using the estimates on the weak solutions (3.18)-(3.19) we define

$$k_3(v_0,\varphi_0,U) = C(k_1 + k_1^2 + k_1^6) + (C + f_3(U))k_1^{\frac{4}{3}}(1 + k_1^{\alpha_1}) + Cf_2(U). \tag{3.51}$$

In the same way, we define

$$k_4(v_0,\varphi_0,U,\tau) = C(U^2 + f_4(U))\tau + C(k_1^2k_2^2 + k_2^2), \tag{3.52}$$

so that we have, for any $t_0 \geq 0, \tau > 0$

$$\int_{t_0}^{t_0+\tau} g(s)ds \leq k_4(v_0,\varphi_0,U,\tau).$$

The key-point now is to use the uniform Gronwall lemma ([19]) and (3.19) to deduce that for any $t \geq 0$,

$$y(t) \leq (\max(y(0),k_2^2) + k_3)e^{k_4}.$$

We want to point out the fact that in all the previous estimates, u and φ are the Galerkin approximations of the weak solution and so we have not $u(0) = u_0$ and $\varphi(0) = \varphi_0$ but only $u(0) = Pu_0$ and $\varphi(0) = P'\varphi_0$ where P and P' are projectors on Galerkin spaces. Nevertheless, as we made it for weak solutions, one can see that we have with (3.3) and (3.28)

$$y(0) \leq \frac{\alpha}{2}|\Delta\varphi_0|_2^2 + \frac{1}{2}|\nabla u_0|_2^2 \leq \frac{\alpha}{2}|\Delta\varphi_0|_2^2 + \frac{1}{2}|\nabla v_0|_2^2 + f_5(U).$$

Finally if we let

$$k_5(v_0, \varphi_0, U) \equiv \left(\max \left(\frac{\alpha}{2} |\Delta \varphi_0|_2^2 + \frac{1}{2} |\nabla v_0|_2^2 + f_5(U), k_2^2 \right) + k_3 \right) e^{k_4}, \tag{3.53}$$

we have

$$\|\Delta \varphi\|_{L^\infty(\mathbb{R}^+;\Phi_0)} + \|\nabla u\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} \leq C.k_5(v_0, \varphi_0, U),$$

and going back to (3.50) we see that φ and u are bounded in $L^2_{loc}(\mathbb{R}^+;\Phi_4)$ and $L^2_{loc}(\mathbb{R}^+;V_2)$ respectively. It's now straightforward to deduce the global existence of strong solutions in the two-dimensional case.

• ***Step 5 - Study of the three dimensional case:***

In the same way we introduce $y(t)$ and , with (3.33)-(3.48), we see that (3.32) reads

$$\begin{aligned} y'(t) &+ \frac{\alpha B_1}{10} |\Delta^2 \varphi|_2^2 + \frac{\eta_1}{5} |Au|_2^2 \\ &\leq C|\nabla \varphi|_2^2 + C|u|_2^4 |\nabla \varphi|_2^2 + (C + f_3(U)) |\nabla \varphi|_2^{\frac{4}{3}} (1 + |\nabla \varphi|_2^{\alpha_2}) + C f_2(U) \\ &+ C \Big(U^2 + f_4(U) + |\nabla \varphi|_2^2 |\nabla u|_2^2 + |u|_2^2 |\nabla u|_2^2 + |\Delta \varphi|_2 |\nabla \Delta \varphi|_2 + |\nabla \Delta \varphi|_2^2 \Big) y(t) \\ &+ C_1 (y^3(t) + y^5(t)) \\ &\equiv \tilde{k}_3(v_0, \varphi_0, U) + \tilde{g}(t) y(t) + C_1 (y^3(t) + y^5(t)). \end{aligned} \tag{3.54}$$

A standard argument of the theory of ordinary differential equations shows that, there exists a time $T = T(v_0, \varphi_0, U)$ such that y is bounded on any compact interval of $[0, T_0[$ and so we can prove that there exists a local strong solution of our problem in the time interval $[0, T_0[$.

• ***Step 6 - Uniqueness :***

As usual we suppose that we have two strong solutions v_1, φ_1, μ_1 and v_2, φ_2, μ_2 of (2.1)-(2.3) defined on the interval $[0, T[$.

By the remark 2.1 we have $\varphi_1, \varphi_2 \in C^0([0, T] \times \overline{\Omega})$ and so there exists $R > 0$ such that

$$|\varphi_1(t, x)| \leq R, \quad |\varphi_2(t, x)| \leq R, \ \forall x \in \Omega, \ \forall t \in [0, T],$$

$$\|\varphi_1(t)\|_2 \leq R, \quad \|\varphi_2(t)\|_2 \leq R, \text{ for a.e. } \ t \in [0, T[,$$

$$\|v_1(t)\|_1 \leq R, \quad \|v_2(t)\|_1 \leq R, \text{ for a.e. } \ t \in [0, T[.$$

We let $\varphi = \varphi_1 - \varphi_2$ and $v = v_1 - v_2$, we can derive from (2.4) and (2.5) that

• For any w in$V$,

$$\begin{aligned} &\frac{d}{dt}(v, w) + b(v, v_1, w) + b(v_2, v, w) + 2 \int_\Omega (\eta(\varphi_1) - \eta(\varphi_2)) D(v_1) : D(w) + 2 \int_\Omega \eta(\varphi_2) D(v) : D(w) \\ &\quad = -\alpha \int_\Omega (w.\nabla \varphi_1) \Delta \varphi - \alpha \int_\Omega (w.\nabla \varphi) \Delta \varphi_2. \end{aligned} \tag{3.55}$$

• For any ψ in V_3,

$$\begin{aligned} &\frac{d}{dt}(\varphi, \psi) - \int_\Omega (v.\nabla \psi) \varphi_1 - \int_\Omega (v_2.\nabla \psi) \varphi \\ &\quad - \alpha \int_\Omega (B(\varphi_1) - B(\varphi_2)) \nabla \Delta \varphi_1 . \nabla \psi - \alpha \int_\Omega B(\varphi_2) \nabla \Delta \varphi . \nabla \psi \\ &\quad + \int_\Omega (BF''(\varphi_1) - BF''(\varphi_2)) \nabla \varphi_1 . \nabla \psi + \int_\Omega BF''(\varphi_2) \nabla \varphi . \nabla \psi = 0. \end{aligned} \tag{3.56}$$

As we deal with strong solutions, φ and v are regular enough to take $w = v$ as a test function in (3.55) and $\psi = -\alpha\Delta\varphi$ as a test function in (3.56) (remark that $v \in V$). By summing the results, integrating by parts and using mean's value formula, one gets

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi|_2^2 + \frac{1}{2}|v|_2^2\right) + \eta_1|\nabla v|_2^2 + \alpha^2 B_1|\nabla\Delta\varphi|_2^2 \\
&\qquad \le |b(v, v_1, v)| + 2\|\eta'\|_\infty \int_\Omega |\varphi||D(v_1)||D(v)| \\
&\qquad + \alpha \int_\Omega |v_2||\nabla\varphi||\Delta\varphi| + \alpha \int_\Omega |v||\nabla\varphi||\Delta\varphi_2| \\
&\qquad + \alpha \int_\Omega |B(\varphi_1)F''(\varphi_1) - B(\varphi_2)F''(\varphi_2)||\nabla\varphi_1||\nabla\Delta\varphi| \\
&\qquad + \alpha \int_\Omega |B(\varphi_2)F''(\varphi_2)||\nabla\varphi||\nabla\Delta\varphi| \\
&\qquad + \alpha^2\|B'\|_\infty \int_\Omega |\varphi||\nabla\Delta\varphi_1||\nabla\Delta\varphi|.
\end{aligned} \tag{3.57}$$

Once again, we are going to estimate each of the seven terms $J_1, ..., J_7$ of the right member in this last inequality. Notice that the following estimates are valid in both cases $d = 2$ and $d = 3$.

We first use the embedding $H^{\frac{3}{4}} \subset L^4$ and the interpolation $H^{\frac{3}{4}} = [L^2, H^1]_{\frac{3}{4}}$ to get

$$\begin{aligned}
J_1 &\le |\nabla(v_1)|_2 |v|_4^2 \\
&\le C(R)|v|_2^{\frac{1}{2}}|\nabla v|_2^{\frac{3}{2}} \\
&\le \frac{\eta_1}{8}|\nabla v|_2^2 + C(R)|v|_2^2.
\end{aligned} \tag{3.58}$$

For the second term, we use the embeddings $H^{\frac{1}{2}} \subset L^3$, $H^1 \subset L^6$ and interpolation results to get

$$\begin{aligned}
J_2 &\le \|\eta'\|_\infty |\varphi|_6 |D(v_1)|_3 |D(v)|_2 \\
&\le C|\nabla\varphi|_2|\nabla v|_2|\nabla v_1|_2^{\frac{1}{2}}\|v_1\|_2^{\frac{1}{2}} \\
&\le \frac{\eta_1}{8}|\nabla v|_2^2 + C(R)|\nabla\varphi|_2^2.
\end{aligned} \tag{3.59}$$

With the embedding $H^2 \subset L^\infty$, one easily obtains

$$\begin{aligned}
J_3 &\le \alpha|\nabla\varphi|_\infty|\Delta\varphi_2||v|_2 \\
&\le \alpha|\nabla\Delta\varphi|_2|\Delta\varphi_2|_2|v|_2 \\
&\le \frac{\alpha^2 B_1}{8}|\nabla\Delta\varphi|_2^2 + C(R)|v|_2^2,
\end{aligned} \tag{3.60}$$

and

$$\begin{aligned}
J_4 &\le \alpha|v_2|_2|\nabla\varphi|_\infty|\Delta\varphi|_2 \\
&\le C(R)|\nabla\Delta\varphi|_2|\nabla\varphi|_2^{\frac{1}{2}}|\nabla\Delta\varphi|_2^{\frac{1}{2}} \\
&\le \frac{\alpha^2 B_1}{8}|\nabla\Delta\varphi|_2^2 + C(R)|\nabla\varphi|_2^2.
\end{aligned} \tag{3.61}$$

To estimate the fifth term we use mean's value formula to get

$$
\begin{aligned}
J_5 &\leq \left(\sup_{|x|\leq R} |(BF'')'(x)| \right) |\varphi|_3 |\nabla\varphi_1|_6 |\nabla\Delta\varphi|_2 \\
&\leq C(R)|\nabla\varphi|_2|\Delta\varphi_1|_2|\nabla\Delta\varphi|_2 \\
&\leq \frac{\alpha^2 B_1}{8}|\nabla\Delta\varphi|_2^2 + C(R)|\nabla\varphi|_2^2.
\end{aligned} \tag{3.62}
$$

In a similar way we have

$$
\begin{aligned}
J_6 &\leq \left(\sup_{|x|\leq R} |BF''(x)| \right) |\nabla\varphi|_2 |\nabla\Delta\varphi|_2 \\
&\leq \frac{\alpha^2 B_1}{8}|\nabla\Delta\varphi|_2^2 + C(R)|\nabla\varphi|_2^2.
\end{aligned} \tag{3.63}
$$

Once again we will use the inequality

$$
|f|_\infty \leq C\|f\|_1^{\frac{3}{4}}\|f\|_3^{\frac{1}{4}},
$$

satisfied by any function $f \in H^3(\Omega)$ which comes from Agmon's inequality and the interpolation $H^2 = [H^1, H^3]_{\frac{1}{2}}$. If now we remark that $m(\varphi) = m(\varphi_1) - m(\varphi_2) = 0$, we get with (1.18)-(1.19)

$$
\begin{aligned}
J_7 &\leq \alpha^2\|B'\|_\infty|\varphi|_\infty|\nabla\Delta\varphi_1|_2|\nabla\Delta\varphi|_2 \\
&\leq C|\nabla\varphi|_2^{\frac{3}{4}}|\Delta\varphi_1|_2^{\frac{1}{2}}|\Delta^2\varphi_1|_2^{\frac{1}{2}}|\nabla\Delta\varphi|_2^{\frac{5}{4}} \\
&\leq \frac{\alpha^2 B_1}{8}|\nabla\Delta\varphi|_2^2 + C|\Delta\varphi_1|_2^{\frac{4}{3}}|\Delta^2\varphi_1|_2^{\frac{4}{3}}|\Delta\varphi|_2^2.
\end{aligned} \tag{3.64}
$$

Finally, using (3.58)-(3.64), the estimate (3.57) reads, after integration with respect to t,

$$
z(t) \leq z(0) + \int_0^t h(s)z(s)ds, \ \forall 0 \leq t < T,
$$

where we let

$$
z(t) = \frac{\alpha}{2}|\nabla\varphi|_2^2 + \frac{1}{2}|v|^2,
$$

and $h(t) = C(R) + C(R)|\Delta^2\varphi_1|_2^{\frac{4}{3}} \in L^1(0,T)$.
Using Gronwall lemma, we obtain

$$
z(t) \leq z(0)e^{\int_0^t h(u)du} \quad \forall 0 \leq t < T.
$$

As φ_1, v_1 and φ_2, v_2 have the same initial data φ_0, v_0, we have $z(0) = 0$ and so $z(t) = 0$ for any $t \in [0, T[$ which implies the uniqueness of the strong solution of the problem.

■

3.4 Weak solutions in the degenerate case

In this part we prove the theorem 2.3

Theorem
Under assumptions (2.6),(2.13)-(2.14), *for any given* $0 < T < +\infty$, $U > 0$, $v_0 \in v_\infty^U + H$ *and* $\varphi_0 \in \Phi_1$, *such that* $|\varphi_0|_\infty \leq 1$ *and*

$$
\int_\Omega \Big(F(\varphi_0) + G(\varphi_0) \Big) < +\infty,
$$

there exists a weak solution of (1.1)-(1.9) *on* $[0,T[$ *in the sense of definition 2.3 which satisfies*

$$|\varphi(t,x)| \le 1 \text{ for almost every } (t,x) \in]0,T[\times\Omega.$$

Moreover, if we suppose $B'(1) = 0$ *and* $B'(-1) = 0$, *then for almost every* $t \in [0,T[$ *the set*

$$\{x \in \Omega,\ |\varphi(t,x)| = 1\}$$

has zero measure.

Proof : Following [9], we introduce a non-degenerate approximation of this problem. For any $0 < \varepsilon < 1$, let

$$\begin{array}{rcl} B_\varepsilon(x) = & B(-1+\varepsilon) & \text{if } x \le -1+\varepsilon \\ B_\varepsilon(x) = & B(x) & \text{if } -1+\varepsilon \le x \le 1-\varepsilon \\ B_\varepsilon(x) = & B(1-\varepsilon) & \text{if } x \ge 1-\varepsilon. \end{array}$$

We suppose that F_2 is extended to be a function on all $\mathbb{R}$ of C^2 class with $\|F_2\|_{C^2(\mathbb{R})} \le C$ and we introduce an approximation of F_1 as

$$\begin{array}{rcl} F_1^\varepsilon(0) = & F_1(0) & \text{and } F_1^{\varepsilon\prime}(0) = F_1'(0) \\ F_1^{\varepsilon\prime\prime}(x) = & F_1''(-1+\varepsilon) & \text{if } x \le -1+\varepsilon \\ F_1^{\varepsilon\prime\prime}(x) = & F_1''(x) & \text{if } -1+\varepsilon \le x \le 1-\varepsilon \\ F_1^{\varepsilon\prime\prime}(x) = & F_1''(1-\varepsilon) & \text{if } x \ge 1-\varepsilon. \end{array}$$

Notice that we have $F_1(x) = F_1^\varepsilon(x)$ if $|x| \le 1-\varepsilon$. The function G is also approached by G_ε defined on $\mathbb{R}$ by

$$G_\varepsilon(0) = 0,\ G_\varepsilon'(0) = 0,\ G_\varepsilon''(x) = B_\varepsilon(x)^{-1}.$$

One can easily verify that B_ε satisfies (2.6) and $F_\varepsilon = F_1^\varepsilon + F_2$ satisfies (2.7)-(2.10) with $p = 2$. Hence, theorem 2.1 can be applied here to obtain a solution φ_ε, μ_ε and v_ε of the approached problem with $B = B_\varepsilon$ and $F = F_\varepsilon$ in the sense of definition 2.1.

• ***Step 1:*** We are going to derive some new uniform estimates on φ_ε, μ_ε and v_ε. In fact, we point out that, as in the proof of theorem 2.1, the following computations have to be performed on the Galerkin approximations of $\varphi_\varepsilon, \mu_\varepsilon$ and v_ε to be fully justified.

We use $\mu_\varepsilon(t) \in \Phi_1$ as a test function in (2.3) and $u_\varepsilon = v_\varepsilon(t) - Uv_\lambda \in V$ in (2.2), λ being chosen later. By summing the results we obtain in a similar way than in the proof of theorem 2.1,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_\varepsilon|_2^2 + \frac{1}{2}|u_\varepsilon|_2^2 + \int_\Omega F_\varepsilon(\varphi_\varepsilon)\right) &+ \int_\Omega B_\varepsilon|\nabla\mu_\varepsilon|^2 + \eta_1|\nabla u_\varepsilon|_2^2 \\ &\le \alpha U\int_\Omega |v_\lambda||\nabla\varphi_\varepsilon||\Delta\varphi_\varepsilon| + U|b(u_\varepsilon, v_\lambda, u_\varepsilon)| \\ &+ 2U\eta_2\int_\Omega |D(v_\lambda)||D(u_\varepsilon)|. \end{aligned}$$

As G_ε'' is bounded, we can use $G_\varepsilon'(\varphi_\varepsilon)$ as a test function in (2.3) so that we have, using $B_\varepsilon G_\varepsilon'' = 1$,

$$\frac{d}{dt}\left(\int_\Omega G_\varepsilon(\varphi_\varepsilon)\right) + \int_\Omega \nabla\mu_\varepsilon.\nabla\varphi_\varepsilon = 0,$$

where we use the divergence theorem (because div $(v_\varepsilon) = 0$) and the boundary conditions on v_ε which imply that $v_\varepsilon.\nu = 0$ on $\partial\Omega$. Integrating by parts, this last estimate can be written

$$\frac{d}{dt}\left(\int_\Omega G_\varepsilon(\varphi_\varepsilon)\right) + \alpha|\Delta\varphi_\varepsilon|_2^2 + \int_\Omega (F_\varepsilon^1)''(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2 \le \|F_2''\|_\infty|\nabla\varphi_\varepsilon|_2^2.$$

We choose $\lambda = \min(k/U, \sqrt[4]{U})$, where k is a constant being fixed later, and using the convexity of F^1_ε and (3.2)-(3.3) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_\varepsilon|_2^2 + \frac{1}{2}|u_\varepsilon|_2^2 + \int_\Omega F_\varepsilon(\varphi_\varepsilon) + \int_\Omega G_\varepsilon(\varphi_\varepsilon)\right) + \int_\Omega B_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2 + \alpha|\Delta\varphi_\varepsilon|_2^2 + \frac{\eta_1}{2}|\nabla u_\varepsilon|_2^2 \\ \le \|F_2''\|_\infty|\nabla\varphi_\varepsilon|_2^2 + \alpha k|\Delta\varphi_\varepsilon|_2^2 \\ + f_1(U) + k|\nabla u_\varepsilon|_2^2,\end{aligned}$$

where $f_1(U)$ which tends to zero when U tends to zero. Hence, if we let $k = \min(1/2, \eta_1/4)$, we have obtained

$$\begin{aligned}\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi_\varepsilon|_2^2 + \frac{1}{2}|u_\varepsilon|_2^2 + \int_\Omega F_\varepsilon(\varphi_\varepsilon) + \int_\Omega G_\varepsilon(\varphi_\varepsilon)\right) + \frac{\alpha}{2}|\Delta\varphi_\varepsilon|_2^2 + \frac{\eta_1}{2}|\nabla u_\varepsilon|_2^2 + \int_\Omega B_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2 \\ \le \|F_2''\|_\infty|\nabla\varphi_\varepsilon|_2^2 + f_1(U).\end{aligned} \tag{3.65}$$

One can easily see that the technical assumption (2.14) implies that for $\varepsilon > 0$ small enough, we have $F^1_\varepsilon(x) \le F^1(x)$ and $G_\varepsilon(x) \le G(x)$ for any $x \in]-1,1[$. By Gronwall lemma and (2.15) we get from (3.65) that there exists $C > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H)} + \|\varphi_\varepsilon\|_{L^\infty(0,T;\Phi_1)} \le C, \tag{3.66}$$

$$\|u_\varepsilon\|_{L^2(0,T;V)} + \|\varphi_\varepsilon\|_{L^2(0,T;\Phi_2)} \le C, \tag{3.67}$$

$$\left\|\int_\Omega G_\varepsilon(\varphi_\varepsilon)\right\|_{L^\infty(0,T)} \le C, \tag{3.68}$$

$$\|B_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon\|_{L^2(]0,T[\times\Omega)} \le C. \tag{3.69}$$

As in the proof of theorem 2.1 and using (3.69), one can derive easily the following estimates

$$\left\|\frac{d\varphi_\varepsilon}{dt}\right\|_{L^2(0,T;\Phi_1')} \le C, \tag{3.70}$$

$$\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2(0,T;V_{\frac{d}{2}}')} \le C. \tag{3.71}$$

We can then extract a subsequence of (u_ε) and (φ_ε) which will be always denoted (u_ε) and (φ_ε) such that, using lemma 1.6,

$$\varphi_\varepsilon \longrightarrow \varphi \text{ in } L^2(0,T;\Phi_1) \text{ strong and a.e.,}$$

$$\Delta\varphi_\varepsilon \rightharpoonup \Delta\varphi \text{ in } L^2(]0,T[\times\Omega) \text{ weak,}$$

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ in } C^0([0,T[,\Phi_{\frac{1}{2}}) \text{ weak,}$$

$$u_\varepsilon \longrightarrow u \text{ in } L^2(0,T;H) \text{ strong and a.e,}$$

$$u_\varepsilon \rightharpoonup u \text{ in } L^2(0,T,V) \text{ weak,}$$

$$u_\varepsilon \rightharpoonup u \text{ in } C^0([O,T[,V_{\frac{d-2}{4}}) \text{ weak.}$$

Finally, λ being fixed independently of ε, we see that if we let $v = u + Uv_\lambda$ we have

$$v_\varepsilon \longrightarrow v \text{ in } L^2(]0,T[\times\Omega) \text{ strong and a.e.,}$$

$$v_\varepsilon - v_\infty^U \rightharpoonup v - v_\infty^U \text{ in } L^2(0,T;V) \text{ weak.}$$

- ***Step 2 - L^∞ estimate :***

Let us remark that, by definition, $G(x) \geq 0$, $G'(x) \geq 0$ if $x \geq 0$ and $G(x) \geq 0$, $G'(x) \leq 0$ if $x \leq 0$. Following [9], if $x > 1$ we get

$$\begin{aligned} G_\varepsilon(x) &= G_\varepsilon(1-\varepsilon) + G'_\varepsilon(1-\varepsilon)(x-(1-\varepsilon)) + \frac{1}{2} G''_\varepsilon(1-\varepsilon)(x-(1-\varepsilon))^2 \\ &= G(1-\varepsilon) + G'(1-\varepsilon)(x-(1-\varepsilon)) + \frac{1}{2} G''(1-\varepsilon)(x-(1-\varepsilon))^2 \\ &\geq \frac{1}{2} \frac{1}{B(1-\varepsilon)} (x-1)^2, \end{aligned}$$

and in the same way, if $x < -1$ we have

$$\begin{aligned} G_\varepsilon(x) &= G_\varepsilon(-1+\varepsilon) + G'_\varepsilon(-1+\varepsilon)(x-(-1+\varepsilon)) + \frac{1}{2} G''_\varepsilon(-1+\varepsilon)(x-(-1+\varepsilon))^2 \\ &= G(-1+\varepsilon) + G'(-1+\varepsilon)(x-(-1+\varepsilon)) + \frac{1}{2} G''(-1+\varepsilon)(x-(-1+\varepsilon))^2 \\ &\geq \frac{1}{2} \frac{1}{B(-1+\varepsilon)} (x+1)^2. \end{aligned}$$

This implies

$$\int_\Omega (|\varphi_\varepsilon| - 1)_+^2 \leq 2 \max(B(1-\varepsilon), B(-1+\varepsilon)) \int_\Omega G_\varepsilon(\varphi_\varepsilon),$$

and so, using (3.68), and the fact that $B(1-\varepsilon)$ and $B(-1+\varepsilon)$ tend to zero when ε tend to zero, we deduce that

$$\int_\Omega (|\varphi_\varepsilon| - 1)_+^2 \longrightarrow 0.$$

Using, for example, Vitali's theorem, we can perform the limit in this last integral, so that we get

$$\int_\Omega (|\varphi| - 1)_+^2 = 0,$$

which implies that $|\varphi| \leq 1$ for almost every time t and almost every $x \in \Omega$.

• ***Step 3 - Passing to the limit in the equations :***

To conclude the proof of the theorem, we have to verify that the functions φ and v are solutions of the problem.

As equation (2.4) is satisfied by v_ε we have for any $w \in V$, $\rho \in \mathcal{D}(]0,T[)$,

$$-\int_0^T (v_\varepsilon, \rho'(t)w)dt + \int_0^T b(v_\varepsilon, v_\varepsilon, \rho(t)w) + 2\int_{]0,T[\times\Omega} \eta(\varphi_\varepsilon) D(v_\varepsilon) : D(\rho w) = -\alpha \int_{]0,T[\times\Omega} ((\rho w).\nabla\varphi_\varepsilon)\Delta\varphi_\varepsilon.$$

Passing to the limit is classical in the first two terms. In the third term we remark that, by Lebesgue's theorem $\eta(\varphi_\varepsilon)D(\rho w)$ converges to $\eta(\varphi)D(\rho w)$ in $L^2(]0,T[\times\Omega)$ strongly and so the weak convergence of $D(v_\varepsilon)$ to $D(v)$ in $L^2(]0,T[\times\Omega)$ let us pass to the limit. Finally, we remark that lemma 1.6 together with (3.67),(3.70) implies that (up to another extraction of a subsequence) φ_ε converges to φ in $L^2(0,T,H^{\frac{3}{2}}(\Omega))$ strongly and so by Sobolev's embedding $\nabla\varphi_\varepsilon$ converges to $\nabla\varphi$ in $L^2(0,T,\mathbb{L}^3(\Omega))$ strongly. Using the fact that $\rho.w \in L^\infty(0,T;\mathbb{L}^6(\Omega))$ and the weak convergence of $\Delta\varphi_\varepsilon$ to $\Delta\varphi$ in $L^2(]0,T[\times\Omega)$, one can easily pass to the limit in the last term so that φ and v satisfy (2.4).

As far as equation (2.5) is concerned, we have for any $\psi \in \Phi_2$, $\rho \in \mathcal{D}(]0,T[)$

$$
\begin{aligned}
-\int_0^T (\varphi_\varepsilon, \rho(t)\psi)dt + \alpha \int_{]0,T[\times\Omega} B_\varepsilon(\varphi_\varepsilon)\Delta\varphi_\varepsilon(\rho\Delta\psi) + \alpha \int_{]0,T[\times\Omega} B'_\varepsilon(\varphi_\varepsilon)\Delta\varphi_\varepsilon\nabla\varphi_\varepsilon.(\rho\nabla\psi) \\
+ \int_{]0,T[\times\Omega} (B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon)\nabla\varphi_\varepsilon.(\rho\nabla\psi) - \int_{]0,T[\times\Omega} (v_\varepsilon.\nabla(\rho\psi))\varphi_\varepsilon = 0.
\end{aligned}
\tag{3.72}
$$

Passing to the limit in the first term is straightforward. Let us remark that B_ε is uniformly bounded by $\|B\|_\infty$ and that B_ε converges uniformly to B on $\mathbb{R}$. This implies, with Lebesgue's theorem, that $B_\varepsilon(\varphi_\varepsilon)(\rho.\Delta\psi)$ converges to $B(\varphi)(\rho.\Delta\psi)$ in $L^2(]0,T[\times\Omega)$ strongly and so using the weak convergence of $\Delta\varphi_\varepsilon$ in $L^2(]0,T[\times\Omega)$, we can pass to the limit in the second term.

In the third term we have $\rho.\nabla\psi \in L^\infty(]0,T[\times\Omega)$ because $\psi \in V_3$ and $\Delta\varphi_\varepsilon$ converges weakly to $\Delta\varphi$ in $L^2(]0,T[\times\Omega)$. We can pass to the limit in this term if we show that $B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon$ converges strongly to $B'(\varphi).\nabla\varphi$ in $L^2(]0,T[\times\Omega)$. Using the L^∞-estimate we have established on φ in the previous step, we have

$$
\begin{aligned}
\int_{]0,T[\times\Omega} |B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - B'(\varphi)\nabla\varphi|^2 &= \int_{]0,T[\times\Omega\cap\{|\varphi|<1\}} |B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - B'(\varphi)\nabla\varphi|^2 \\
&+ \int_{]0,T[\times\Omega\cap\{|\varphi|=1\}} |B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - B'(\varphi)\nabla\varphi|^2.
\end{aligned}
$$

In the first integral, we remark that, as $|\varphi| < 1$ we have $B'_\varepsilon(\varphi_\varepsilon) \to B'(\varphi)$ and $\nabla\varphi_\varepsilon \to \nabla\varphi$ almost everywhere and

$$
|B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - B'(\varphi)\nabla\varphi|^2 \leq 2\|B'\|_\infty(|\nabla\varphi_\varepsilon|^2 + |\nabla\varphi|^2),
$$

so that Vitali's theorem let us pass to the limit in this integral. A classical lemma ([10]) says that $\nabla\varphi = 0$ almost everywhere in $\{|\varphi| = 1\}$, hence the second integral can be written

$$
\begin{aligned}
\int_{]0,T[\times\Omega\cap\{|\varphi|=1\}} |B'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon|^2 &\leq \|B'\|^2_\infty \int_{]0,T[\times\Omega\cap\{|\varphi|=1\}} |\nabla\varphi_\varepsilon|^2 \\
&\to \|B'\|^2_\infty \int_{]0,T[\times\Omega\cap\{|\varphi|=1\}} |\nabla\varphi|^2 = 0.
\end{aligned}
$$

Therefore, we have proved that we can pass to the limit in the third term of (3.72).

As we supposed that BF'' is continuous on $[-1,1]$, one can easily see that $B_\varepsilon F''_\varepsilon$ is uniformly bounded and so, in order to pass to the limit in the fourth term of (3.72), we just have to show that $(B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon)$ converges to $(BF'')(\varphi)$ almost everywhere. This last point is obvious if $|\varphi| < 1$ because for ε small enough we have the equality $(B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon) = (BF'')(\varphi_\varepsilon)$ and we conclude by continuity. Let us show that it remains true if, for example, $\varphi(t,x) = 1$ (the case $\varphi = -1$ is treated in the same way).

- If ε is such that $0 \leq \varphi_\varepsilon(t,x) \leq 1 - \varepsilon$ then we have the equality

$$
(B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon(t,x)) = (BF'')(\varphi_\varepsilon(t,x)).
$$

- If ε is such that $\varphi_\varepsilon > 1 - \varepsilon$ then we have

$$
(B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon(t,x)) = (BF''_1)(1-\varepsilon) + B(1-\varepsilon)F''_2(\varphi_\varepsilon(t,x)).
$$

In both cases we can conclude that $(B_\varepsilon F''_\varepsilon)(\varphi_\varepsilon)$ converges to $(BF'')(\varphi)$ almost everywhere.

Finally, passing to the limit in the fifth term of (3.72) is straightforward because $\rho.\nabla\psi \in L^\infty(]0,T[\times\Omega)$ and $v_\varepsilon, \varphi_\varepsilon$ both converge strongly in $L^2(]0,T[\times\Omega)$.

Hence, we proved that the functions v, φ are weak solutions of the problem in the sense of definition 2.3.

- ***Step 4:*** In this last step we suppose that $B'(1) = B'(-1) = 0$. It is easily seen that this implies that $G(x)$ converges to $+\infty$ when x converges to 1 or -1. Using (3.68) and Fatou's lemma we have for almost every $t \in [0,T[$,

$$\int_\Omega \liminf_{\varepsilon \to 0} G_\varepsilon(\varphi_\varepsilon) \leq C. \tag{3.73}$$

- If $|\varphi(t,x)| < 1$ then for ε small enough we have $G_\varepsilon(\varphi_\varepsilon(t,x)) = G(\varphi_\varepsilon(t,x))$ and so by continuity

$$\lim_{\varepsilon \to 0} G_\varepsilon(\varphi_\varepsilon(t,x)) = G(\varphi(t,x)).$$

- If $\varphi(t,x) = 1$ for example, then for any $\varepsilon > 0$ we have

$$G_\varepsilon(\varphi_\varepsilon(t,x)) \geq \min(G(1-\varepsilon), G(\varphi_\varepsilon(t,x))).$$

  And, as $G(x) \to +\infty$ when $x \to 1$, we see that

$$G_\varepsilon(\varphi_\varepsilon(t,x)) \to +\infty, \tag{3.74}$$

- In the same way, when $\varphi(t,x) = -1$, we have also (3.74).

Finally, using (3.73) and (3.74), one sees that

$$\{x \in \Omega,\ |\varphi(t,x)| = 1\}$$

has zero measure. ■

3.5 A qualitative result

We finally give the proof of theorem 2.4.

Theorem
Let I an open interval of $\mathbb{R}$ and $\omega \in I$ given. We suppose that B is a non-negative function of C^1-class defined on I, F is a function of C^3-class defined on I. Then, for $U > 0$ small enough, the stationary solution of (1.1)-(1.9) given by $\varphi_\infty = \omega$ and v_∞^U defined by (1.15) is asymptotically stable under the conditions

$$\begin{aligned} B(\omega) &> 0, & (3.75)\\ F''(x) &\geq 0 \text{ for any } x \text{ in a neighborhood of } \omega. & (3.76) \end{aligned}$$

More precisely, for any $\varepsilon > 0$ there exists $\beta > 0$ such that for any $U > 0$, $v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_2$ satisfying

$$m(\varphi_0) = \omega,\ |U| \leq \beta,\ \|v_0\|_1 \leq \beta,\ \|\varphi_0 - m(\varphi_0)\|_2 \leq \beta,$$

there exists a unique global strong solution on $\mathbb{R}^+$ of the problem (1.1)-(1.9) in the sense of definition 2.2 such that

$$\|v - v_0\|_{L^\infty(\mathbb{R}^+;V)} + \|\varphi - \varphi_0\|_{L^\infty(\mathbb{R}^+;\Phi_2)} \leq \varepsilon,$$

and if ε is small enough, we have as $t \to +\infty$

$$\begin{aligned} &\varphi(t) \longrightarrow \varphi_\infty \text{ in } \Phi_s \text{ for any } 0 \leq s < 2,\\ &v(t) - v_\infty^U \longrightarrow 0 \text{ in } V_s \text{ for any } 0 \leq s < 1. \end{aligned}$$

Proof :

- ***Step 1 - Truncation :***

Let $\delta > 0$ such that

$$I_\delta = [\omega - \delta, \omega + \delta] \subset I,$$

$$F'' \geq 0, B > 0 \text{ on a neighborhood of } I_\delta.$$

One can easily build a function F_ω of C^3-class such that

$$F_\omega(\omega) = 0,\ F'_\omega(\omega) = 0,\ F''_\omega(x) = F''(x) \text{ on } I_\delta,\ F''_\omega(x) \geq 0 \text{ if } x \notin I_\delta,$$

and such that F'''_ω is bounded on $\mathbb{R}$. This function satisfies (2.7)-(2.10) and (2.12) and moreover, the convexity of F_ω leads to

$$0 = F_\omega(\omega) \geq F(x) + (\omega - x)F'(x),\ \forall x \in \mathbb{R},$$

so that in (2.9) we can take the constants equal to

$$F_3(\omega) = 1, \quad F_4(\omega) = 0. \tag{3.77}$$

We can also easily build a function B_ω of C^1-class satisfying (2.6) and (2.11) such that

$$B_\omega(x) = B(x),\ \forall x \in I_\omega.$$

Therefore we can apply theorem 2.2 where we replaced F by F_ω and B by B_ω to get the existence of a strong solution $(\varphi_\omega, v_\omega)$ of this modified problem.

- ***Step 2 - Two-dimensional case estimates :***

The key-point in this proof is (3.77). Indeed, in (3.17) the terms in $F_4(m(\varphi_0)) = F_4(\omega)$ vanish and so we see that $k_1(v_0, \varphi_0, U)$ tends to zero when U, $|v_0|_2$ and $\|\varphi_0 - \omega\|_1$ tends to zero. Using now (3.20), (3.51), (3.52) and (3.53) we see finally that $k_5(v_0, \varphi_0, U)$ tends to zero when U, $\|v_0\|_1$ and $\|\varphi_0 - \omega\|_2$ tends to zero.

This implies that there exists $\beta > 0$ small enough such that if we suppose

$$|U| \leq \beta,\ \|v_0\|_1 \leq \beta,\ \|\varphi_0 - \omega\|_2 \leq \beta,$$

we have

$$\|v_\omega - v_0\|_{L^\infty(\mathbb{R}^+;V)} + \|\varphi_\omega - \varphi_0\|_{L^\infty(\mathbb{R}^+;\Phi_2)} \leq \varepsilon,$$

We can also choose β small enough to have $|\varphi_\omega(t,x) - \omega| \leq \delta$ for any t, x (see remark 2.1) so that for any t, x we have

$$B_\omega(\varphi_\omega(t,x)) = B(\varphi_\omega(t,x)),$$

and

$$F_\omega(\varphi_\omega(t,x)) = F(\varphi_\omega(t,x)) - F(\omega) - (\varphi_\omega(t,x) - \omega)F'(\omega).$$

We see that in (2.2) and (2.3), it is only $\nabla\mu$ who appears and so, we can add an affine part to F without changing the equations. We conclude that φ_ω and v_ω are strong solutions of the original problem with F and B.

- ***Step 3 - Existence of global strong solutions in dimension $d = 3$:***

We have obtained the estimate (3.54) which can be written

$$y'(t) + \gamma y(t) \leq \tilde{k}_3(v_0, \varphi_0, U) + \tilde{g}(t)y(t) + C(y^2(t) + y^4(t))y(t), \tag{3.78}$$

where γ is a constant depending only on B_1, η_1 and Ω and where g satisfies

$$\int_{t_0}^{t_0+\tau} g(s)ds \leq \tilde{k}_4(v_0, \varphi_0, U, \tau),$$

and we have, as in the case $d=2$ above, k_1, k_2, $\tilde{k}_3$ and $\tilde{k}_4$ tending to zero when $\|v_0\|_1$, $\|\varphi_0-\omega\|_2$ and U tend to zero.

Now we choose $\beta>0$ such that for any $U\le\beta$, $\|v_0\|_1\le\beta$ and $\|\varphi_0-\omega\|_2\le\beta$ we have

$$C(y(0)^2+y(0)^4)\le\frac{\gamma}{2}.$$

By a continuity argument we know that there exists a time $T>0$ such that for any $0\le t<T$ we have

$$C(y(t)^2+y(t)^4)<\gamma, \tag{3.79}$$

suppose that T is the maximum time satisfying this property. On the interval $[0,T[$, (3.78) reads

$$y'(t)\le\tilde{k}_3+\tilde{g}(t)y(t),$$

and so using uniform gronwall's lemma as we did in the study of the two-dimensional case in the proof of theorem 2.2 we have

$$y(t)\le\tilde{k}_5(v_0,\varphi_0,U),\quad\forall\,0\le t<T,$$

where

$$\tilde{k}_5(v_0,\varphi_0,U)=\left(\max\left(\frac{\alpha}{2}|\Delta\varphi_0|_2^2+\frac{1}{2}|\nabla v_0|_2^2+f_5(U),k_2^2\right)+\tilde{k}_3\right)e^{\tilde{k}_4}.$$

As in the two-dimensional case, we infer from (3.77) and the estimates on the weak and strong solutions, that $\tilde{k}_5$ tends to zero when U, $\|v_0\|_1$ and $\|\varphi_0-\omega\|_2$ tend to zero. Hence, if we impose β to be small enough such that

$$C(\tilde{k}_5^2+\tilde{k}_5^4)<\gamma,$$

we see that the maximal time where (3.79) is true is $T=+\infty$. Therefore we have proved the existence of a global strong solution for the modified problem and this solution satisfies

$$y(t)\le\tilde{k}_5(v_0,\varphi_0,U),\quad\forall t\ge0.$$

We can now conclude as we did in the end of step 2: φ_ω and v_ω are in fact strong solutions of the initial problem.

Therefore, we have proved the first point of the theorem.

• ***Step 4 - Asymptotic behavior :***

The following proof is valid in both dimensions $d=2$ or $d=3$. We recall that we have shown that for any (t,x), $\varphi(t,x)$ lies in the interval $[\omega-\delta,\omega+\delta]$ where B and B_ω coincide and F'' is positive.

We deduce from equation (2.5) that for any $\psi\in\Phi_2$ we have

$$\begin{aligned}\frac{d}{dt}(\varphi-\omega,\psi)&+\alpha\int_\Omega B_\omega'(\varphi)\Delta\varphi\nabla\varphi.\nabla\psi+\alpha\int_\Omega B_\omega(\varphi)\Delta\varphi\Delta\psi\\&+\int_\Omega B_\omega(\varphi)F''(\varphi)\nabla\varphi.\nabla\psi-\int_\Omega(v.\nabla\psi)(\varphi-\omega)=0,\end{aligned}$$

and so taking $\psi=\varphi-\omega$ we have using divergence theorem that

$$\frac{1}{2}\frac{d}{dt}|\varphi-\omega|_2^2+\alpha B_1|\Delta\varphi|_2^2\le\alpha\int_\Omega|B_\omega'(\varphi)||\Delta\varphi||\nabla\varphi|^2,$$

where we recall that B_ω is bounded from below by a constant B_1. Finally, we have

$$\frac{1}{2}\frac{d}{dt}|\varphi-\omega|_2^2+B_1|\Delta\varphi|_2^2\le\alpha\|B_\omega'\|_\infty|\Delta\varphi|_2|\nabla\varphi|_4^2\ \le C\|\varphi-\omega\|_2|\Delta\varphi|_2^2\ \le C\varepsilon|\Delta\varphi|_2^2.$$

If we choose ε small enough we have

$$\frac{1}{2}\frac{d}{dt}|\varphi-\omega|_2^2+\frac{B_1}{2}|\Delta\varphi|_2^2\leq 0,$$

and so using (1.19), there exists a constant $\gamma>0$ such that

$$|\varphi-\omega|_2\leq Ce^{-\gamma t},$$

and therefore $\varphi(t)\to\varphi_\infty$ in Φ_0. But we recall that $\varphi(t)-\varphi_\infty$ lies in $L^\infty(\mathbb{R}^+;\Phi_2)$. So, by interpolation results, we have $\varphi(t)\to\varphi_\infty$ in Φ_s for any $0\leq s<2$.

If we notice that $(v_\infty^U.\nabla)v_\infty^U=0$ and $\Delta v_\infty^U=0$, equation (2.2) can be written,

$$\frac{d}{dt}(v-v_\infty^U,w)+b(v,v-v_\infty^U,w)+b(v-v_\infty^U,v_\infty^U,w)+2\int_\Omega\eta(\varphi)D(v-v_\infty^U):D(w)$$
$$+\int_\Omega(\eta(\varphi)-\eta(\omega))D(v_\infty^U):D(w)=-\alpha\int_\Omega(w.\nabla\varphi)\Delta\varphi \text{ in } \mathcal{D}'(\mathbb{R}+),\ \forall w\in V$$

so that if we take $w=v-v_\infty^U\in V$ we have

$$\frac{1}{2}\frac{d}{dt}|v-v_\infty^U|_2^2+2\eta_1|\nabla(v-v_\infty^U)|_2^2\leq|b(v-v_\infty^U,v_\infty^U,v-v_\infty^U)|+\int_\Omega|\eta(\varphi)-\eta(\omega)||D(v-v_\infty^U)||D(v_\infty^U)|$$
$$+\alpha\int_\Omega|\nabla\varphi||\Delta\varphi||v-v_\infty^U|.$$

Hence, using $|\nabla v_\infty^U|_1=|\Omega|U$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v-v_\infty^U|_2^2+2\eta_1|\nabla(v-v_\infty^U)|_2^2&\leq C\|v_\infty^U\|_1\|v-v_\infty^U\|_1^2+\|\eta'\|_\infty|\varphi-\varphi_\infty|_\infty|D(v-v_\infty^U)|_2|D(v_\infty^U)|_2\\&\quad+|\nabla\varphi|_3|\Delta\varphi|_2\|v-v_\infty^U\|_1\\&\leq CU|\nabla(v-v_\infty^U)|_2^2+CU^2|\nabla(v-v_\infty^U)|_2^2+C|\varphi-\varphi_\infty|_\infty^2\\&\quad+\frac{1}{2}|\Delta\varphi|_2^2||\nabla(v-v_\infty^U)|_2^2+|\nabla(\varphi-\varphi_\infty)|_3^2.\end{aligned}$$

We recall that we have $|\Delta\varphi|_2\leq\varepsilon$ and $U\leq\beta$, so that if we impose ε and β to be small enough, we get by Sobolev embedding $H^{\frac{1}{2}}\subset L^3$,

$$\frac{1}{2}\frac{d}{dt}|v-v_\infty^U|_2^2+\frac{\eta_1}{2}|\nabla(v-v_\infty^U)|_2^2\leq C|\varphi-\varphi_\infty|_\infty^2+C\|\varphi-\varphi_\infty\|_{\frac{3}{2}}^2. \tag{3.80}$$

Using (1.12), and the convergence of $\varphi(t)$ to φ_∞ in Φ_s for any $s<2$ we derive from (3.80) that

$$\frac{d}{dt}|v-v_\infty^U|_2^2+\gamma|v-v_\infty^U|_2^2\leq f(t),$$

where $\gamma>0$ is a constant, and $f(t)$ tends to zero when $t\to+\infty$. A classical argument of ordinary differential equations leads to

$$|v-v_\infty^U|_2^2\leq|v_0-v_\infty^U|_2^2e^{-\gamma t}+e^{-\gamma t}.\int_0^t f(s)e^{\gamma s}ds,$$

and so, as $f(t)\to 0$, we easily deduce that $v(t)-v_\infty^U$ converges to zero in H.

As $v(t)-v_\infty^U$ is uniformly bounded in V_1 with respect to t, we finally infer that the convergence of $v(t)-v_\infty^U$ to zero takes place in V_s for any $0\leq s<1$.

■

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