OUTFLOW BOUNDARY CONDITIONS FOR THE INCOMPRESSIBLE NON-HOMOGENEOUS NAVIER-STOKES EQUATIONS

FRANCK BOYER AND PIERRE FABRIE

Abstract. In this paper we propose the analysis of the incompressible non-homogeneous Navier-Stokes equations with nonlinear outflow boundary condition. This kind of boundary condition appears to be, in some situations, a useful way to perform numerical computations of the solution to the unsteady Navier-Stokes equations when the Dirichlet data are not given explicitly by the physical context on a part of the boundary of the computational domain.

The boundary condition we propose, following previous works in the homogeneous case, is a relationship between the normal component of the stress and the outflow momentum flux taking into account inertial effects. We prove the global existence of a weak solution to this model both in 2D and 3D. In particular, we show that the nonlinear boundary condition under study holds for such a solution in a weak sense, even though the normal component of the stress and the density may not have traces in the usual sense.

1. Introduction

1.1. **Statement of the problem.** We are given a smooth and bounded (say $\mathcal{C}^2$) domain Ω in $\mathbb{R}^d$ ($d=2$ or $d=3$) and we denote by $\boldsymbol{\nu}$ the outward unit normal on the boundary $\Gamma = \partial\Omega$. For any real number x we define its positive and negative part respectively by $x^+ = \max(x,0)$ and $x^- = \max(0,-x)$.

This work is concerned with the study of the non-homogeneous Navier-Stokes equations :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \operatorname{div} v = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div}(\sigma) = \rho f, \\ \rho(0) = \rho_0, \ v(0) = v_0, \end{cases} \tag{1}$$

where the stress tensor, with density-dependent viscosity, is defined by

$$\sigma = 2\mu(\rho) D(v) - p\,\mathrm{Id},$$

ρ and p being respectively the density and the pressure of the fluid and $D(v) = \frac{1}{2}(\nabla v + {}^t\nabla v)$ the symmetric part of the gradient of the velocity field v. This system models the flow of an incompressible non-homogeneous viscous fluid.

It is necessary to prescribe boundary conditions for problem (1) on $\partial\Omega$. One part of the boundary of Ω (denoted by Γ_{in} in the sequel) is said to be the inflow boundary (see Figure 1). On this part the Dirichlet data ρ_{in} and v_{in} are supposed to be given. Then, on the remaining part of boundary of Ω (denoted by Γ_{out} in the sequel), we assume that no physical boundary data is available. This typically

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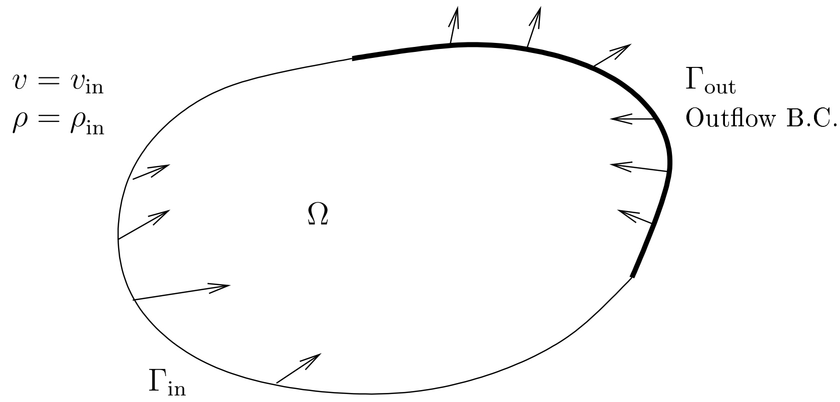

FIGURE 1. Inflow and outflow parts of the boundary of Ω

happens when, for computational reasons, the domain Ω is strictly included in the physical domain so that some part of the boundary is purely artificial. We are interested here in the analysis of a nonlinear outflow boundary condition model on Γ_{out} given by

$$\begin{cases} \rho = \rho_{\text{in}}, \quad \text{on } \Gamma \text{ where } (v \cdot \boldsymbol{\nu}) < 0, \\ v = v_{\text{in}}, \quad \text{on } \Gamma_{\text{in}}, \\ \sigma.\boldsymbol{\nu} = \sigma_{\text{ref}}.\boldsymbol{\nu} - \dfrac{1}{2}\rho_{\text{in}}(v \cdot \boldsymbol{\nu})^{-}(v - v_{\text{ref}}), \quad \text{on } \Gamma_{\text{out}}. \end{cases} \tag{2}$$

Here, v_{ref} and $\sigma_{\text{ref}}.\boldsymbol{\nu}$ are given reference boundary data for the velocity field and the normal component of the stress tensor such that $\displaystyle\int_{\Gamma_{\text{out}}} v_{\text{ref}} \cdot \boldsymbol{\nu} d\omega + \int_{\Gamma_{\text{in}}} v_{\text{in}} \cdot \boldsymbol{\nu} d\omega = 0$ at any time t. This nonlinear term will let us obtain an energy inequality for system (1) preventing finite time blow up of the solution. Notice that, even though Γ_{in} and Γ_{out} are respectively called *inflow* and *outflow* part of the boundary, it is not assumed in the following analysis that $v_{\text{in}} \cdot \boldsymbol{\nu} \leq 0$ on Γ_{in} and of course it can occur than $v \cdot \boldsymbol{\nu} < 0$ even on Γ_{out} (it is the situation where the nonlinear term has a role to play). In particular, the Dirichlet condition on the density takes place on the whole part of the boundary Γ where $v \cdot \boldsymbol{\nu} < 0$.

The analysis of such a model is motivated by previous studies of similar boundary conditions for homogeneous fluids. Indeed, in that case, it is shown in [7, 8] that such a model is well-posed and can be successfully used to compute flows in artificial domains without too much vortexes reflexions on Γ_{out} and good agreement with the expected solution.

Of course, the numerical efficiency of this approach highly depends on the choice of the reference boundary data v_{ref} and $\sigma_{\text{ref}}.\boldsymbol{\nu}$. To our knowledge there is no universal strategy to make such a choice. Nevertheless in many cases the physical intuition of the behavior of the flow may help us to do so. As an example, for the classical computation of a flow past obstacles in an open channel, the Poiseuille reference flow is used in [6, 7, 8] and gives results that do not depend too much on the distance between the obstacles and the artificial open boundary of the computational domain. Furthermore, in the same references, numerical comparisons with the usual *imposed normal stress condition* are given showing that, for high Reynolds numbers, the nonlinear term in (2) is crucial to avoid non physical reflexions and blows up of the solution. Unfortunately, no rigorous results are available yet to give a precise justification of the method and to understand the influence of the reference flow on the solution.

Remark 1. Let us notice that the outflow boundary condition we propose is consistent with the Navier-Stokes equations in the physical domain in the following sense.

Indeed, consider the 2D homogeneous case ($\rho = \mathrm{cst} = 1$), which is the only one for which uniqueness of weak solutions is known and for which the present remark can be made rigorous. Suppose that the physical domain $\Omega_{\rm phys}$ contains Ω and that $\overline{\Gamma_{\rm in}} = \partial\Omega \cap \partial\Omega_{\rm phys}$, so that $\Gamma_{\rm out} \subset \Omega_{\rm phys} \setminus \Omega$. Let us consider now an initial data v_0 for the NS equation on $\Omega_{\rm phys}$ and Dirichlet boundary data $v_{\rm bound}$ on $\partial\Omega_{\rm phys}$. Denote by $(v_{\rm phys}, p_{\rm phys})$ the unique solution to the incompressible homogeneous NS equations associated to those data and by $\sigma_{\rm phys}$ the corresponding stress tensor. Assume finally that $(v_{\rm phys}, p_{\rm phys})$ is smooth enough, that is for instance a so-called *strong* solution (see [22]).

Let $(v_{\rm ref}, \sigma_{\rm ref})$ given and consider now (v, p) the unique solution to problem (1)-(2) with $\rho = 1$ (see [8]), the initial data v_0 restricted to Ω and the inflow data $v_{\rm in} = v_{\rm bound}$ on $\Gamma_{\rm in}$. Then, the following estimate can be shown

$$\begin{aligned} \|v - v_{\rm phys}\|_{L^\infty(]0,T[,L^2(\Omega))} &+ \|v - v_{\rm phys}\|_{L^2(]0,T[,H^1(\Omega))} \\ &\leq C\Bigg(\left\|\sigma_{\rm ref}.\boldsymbol{\nu} - \sigma_{\rm phys}.\boldsymbol{\nu}\right\|_{L^2(]0,T[,H^{-\frac{1}{2}}(\Gamma_{\rm out}))} \\ &\qquad + \|(v \cdot \boldsymbol{\nu})^-\|_{L^3(]0,T[\times\Gamma_{\rm out})} \|v_{\rm ref} - v_{\rm phys}\|_{L^3(]0,T[\times\Gamma_{\rm out})} \Bigg), \end{aligned}$$

the constant C depending only on the data T, v_0, and $v_{\rm bound}$. This result shows in particular that, if we are able to choose $v_{\rm ref}$ and $\sigma_{\rm ref}$ close enough to $v_{\rm phys}$ and $\sigma_{\rm phys}$ (which of course we do not know exactly) then the solution v computed in Ω with our outflow boundary condition will be close to the exact physical solution $v_{\rm phys}$.

Such an estimate is not achievable in the present non-homogeneous framework due to the lack of regularity of the density ρ (that is the same difficulty which prevents us to prove uniqueness of weak solutions).

In the last years, many authors have considered the problem of outflow (or artificial) boundary conditions for various kinds of equations. In the particular framework of fluid mechanics, the case of Oseen equations were considered for instance in [14] while the case of stationnary homogeneous Navier-Stokes equations is treated e.g. in [12, 17, 18] (see also the references therein). In each of these works the original physical domain is supposed to have a particular shape (typically an exterior domain or an half space). This is used in a fundamental way by the authors to derive their method and to prove convergence results. On the contrary, our boundary condition (2) does not rely on particular geometric assumptions on the domain, provided that suitable reference flow can be chosen.

Boundary conditions only involving the pressure were considered for instance in [2, 9, 15] where well-posedness results are given but no convergence results. In these references, the validity of the chosen approach is discussed through numerical simulations. Some other authors have proposed to build artificial boundary conditions in the velocity-vorticity formulation (see for instance [3]). Unfortunately, this formulation is not available for non-homogeneous flows and in that case we are needed to work with the primitive variables: density, velocity and pressure. To our knowledge, artificial boundary conditions in this particular framework were only studied for compressible viscous or inviscid flows (see [19, 23, 24]).

1.2. **Outline.** The present analysis is devoted to the proof of existence of weak solutions for the above problem, that is the transient non-homogeneous incompressible Navier-Stokes equation (1) with boundary conditions (2). As usual in the theory of weak solutions of the incompressible Navier-Stokes equations (see e.g. [22]), uniqueness of such solutions is only established for homogeneous fluids (that is when the density ρ is a constant) in the two dimension case (see [8, 5]). In comparison with the homogeneous situation, many new difficulties appear in this analysis.

- We need to give a precise sense to the Cauchy/Dirichlet problem for the transport equation with non tangential and non smooth vector field v. Most of this material is already available from a previous work by the first author [4] and is recalled in Section 2. Nevertheless, we needed in the present paper to complete this framework with a stability result of the solution ρ as a function of v (see Theorem 4), which is an important result by itself. This result will let us pass to the limit in an approximate problem, in particular in boundary and viscosity terms.
- Suitable weak formulations of the problem, with divergence free test functions, formally obtained by integrating by parts the equation an using (2) are discussed in Section 3. Our main results are then stated in Section 4.
- In Section 5, we introduce (in the same spirit than [16, 21]) an approximate problem for the weak formulation under study (equations (21) and (26)) and we show that it admits a global solution, through a fixed point procedure. Then, in Section 6, we provide estimates on the approximate solution and we perform the limit in the approximate problem. This concludes the proof of Theorem 5.
- Section 7 is devoted to the proof of Theorem 6 which states that the non-linear boundary condition under study is satisfied, in a weak sense, by the solution to the weak formulation of the problem. This is done using once more the fine properties of the traces of solutions to the transport equation obtained in [4]. Finally, we conclude the paper by studying in Section 8 a slight modification of our outflow boundary condition which let us prove the same results under weaker assumptions on the data.

1.3. **Notations and assumptions.** We suppose that $\Gamma = \overline{\Gamma_{\text{in}}} \cup \overline{\Gamma_{\text{out}}}$ where Γ_{in} and Γ_{out} are disjoint open sets in Γ with positive measures. We introduce the space

$$\mathcal{V} = \left\{ v \in (\mathcal{C}^\infty(\overline{\Omega}))^d, \text{ s.t. } \operatorname{div} v = 0, \text{ and } v = 0 \text{ on } \Gamma_{\text{in}} \right\},$$

and let H (resp. V) be the closure of $\mathcal{V}$ in $(L^2(\Omega))^d$ (resp. $(H^1(\Omega))^d$). We also define the space $H^1_{\text{in}}(\Omega)$ of functions in $H^1(\Omega)$ whose trace vanishes on Γ_{in} and $H^{\frac{1}{2}}_{\text{in}}(\Gamma)$ the space of functions in $H^{\frac{1}{2}}(\Gamma)$ which vanishes on Γ_{in}.

Let $T > 0$ be fixed. We assume that $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \geq 0$ and $\rho_{\text{in}} \in L^\infty(]0,T[\times\Gamma)$, $\rho_{\text{in}} \geq 0$. In this paper, excepted in Section 8, we assume that there exists $\alpha > 0$ such that

$$\frac{1}{\rho_0^\alpha} \in L^1(\Omega), \tag{3}$$

$$\frac{1}{\rho_{\text{in}}^\alpha} \in L^1(]0,T[, L^1(\Gamma)). \tag{4}$$

These conditions allow the given initial and inflow boundary densities to vanish on zero measure sets. They are obviously satisfied if we assume that $\inf \rho_0 > 0$ and $\inf \rho_{\text{in}} > 0$. Notice that, condition (3) was considered in [16, 21] in order to improve some of the results concerning the problem (1) with homogeneous Dirichlet of periodic boundary conditions. In Section 8 we will study a slightly modified boundary condition (see (56)) for which we are able to perform the analysis under the weaker assumptions that ρ_0 and ρ_{in} are positive almost everywhere.

Notice that the boundary condition (2) only depends on the value of the reference flow v_{ref} on the outflow boundary Γ_{out}. Nevertheless, in order to perform the analysis, we need to consider a divergence free extension of this boundary data and of v_{in}, still denoted by v_{ref} in the sequel and satisfying

$$\begin{cases} v_{\text{ref}} \in L^\infty(]0,T[,(L^2(\Omega))^d) \cap L^r(]0,T[,(H^1(\Omega))^d), \\ \qquad \text{with } r=2 \text{ if } d=2 \text{ and } r=4 \text{ if } d=3, \\ \operatorname{div} v_{\text{ref}} = 0, \\ \partial_t v_{\text{ref}} \in L^2(]0,T[,(L^2(\Omega))^d), \\ v_{\text{ref}} = v_{\text{in}}, \text{ on } \Gamma_{\text{in}}. \end{cases} \tag{5}$$

We also suppose given σ_{ref} such that

$$\sigma_{\text{ref}}.\boldsymbol{\nu} \in L^2(]0,T[,(H^{-\frac{1}{2}}(\Gamma))^d). \tag{6}$$

We take $v_0 \in H$ and $f \in L^2(]0,T[,(L^2(\Omega))^d)$. Finally, the viscosity μ is supposed to be a continuous function such that there exists $\mu_{\min}, \mu_{\max} \in \mathbb{R}$ satisfying

$$0 < \mu_{\min} \leq \mu(s) \leq \mu_{\max}, \quad \forall s \in \mathbb{R}. \tag{7}$$

2. The transport equation

2.1. Notations. For any $x \in \overline{\Omega}$ we denote by $d(x,\Gamma)$ the distance between x and the compact set Γ. For any $\xi \geq 0$, we define the following two open sets

$$\mathcal{O}_\xi = \{x \in \Omega,\ d(x,\Gamma) < \xi\}, \quad \text{and} \quad \Omega_\xi = \{x \in \Omega,\ d(x,\Gamma) > \xi\}.$$

Since Ω is bounded and regular, there exists $\xi_\Omega > 0$ such that the maps $d(\cdot,\Gamma)$ (distance to Γ) and P_Γ (projection on Γ) are well defined and smooth in $\overline{\mathcal{O}_{\xi_\Omega}}$.

As a consequence it is possible to use $(d(x,\Gamma), P_\Gamma(x)) \in [0,\xi_\Omega]\times\Gamma$ as a coordinate system in $\overline{\mathcal{O}_\xi}$ (see for instance [4, 5]). For any function $F :]0,T[\times\overline{\Omega_{\xi_\Omega}} \mapsto \mathbb{R}$ we will use the notation:

$$F(t,\xi,\omega) \equiv F(t,\omega - \xi\boldsymbol{\nu}(\omega)), \quad \forall (t,\xi,\omega) \in]0,T[\times[0,\xi_\Omega]\times\Gamma.$$

Notice that for any $\omega \in \Gamma$, we have $F(t,0,\omega) = F(t,\omega)$. The reverse formula obviously reads

$$F(t,x) = F(t,d(x,\Gamma),P_\Gamma(x)), \quad \text{for almost every } (t,x) \in]0,T[\times\overline{\mathcal{O}_{\xi_\Omega}}.$$

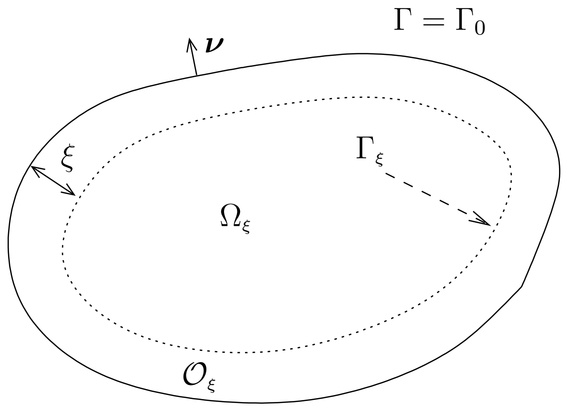

FIGURE 2. Notations near the boundary of Ω

We can now introduce the space $\mathcal{C}^0([0,\xi_\Omega], L^q(]0,T[\times\Gamma))$ of measurable functions $F(t,x)$ such that $\xi \mapsto F(\cdot,\xi,\cdot)$ is continuous with respect to ξ with values in $L^q(]0,T[\times\Gamma)$ in the variables (t,ω).

For any $\xi \in [0,\xi_\Omega[$, we note $\Gamma_\xi = \{x \in \overline{\Omega},\ d(x,\Gamma) = \xi\}$. Notice that for any $\xi \in [0,\xi_\Omega]$ we have

$$\partial\mathcal{O}_\xi = \Gamma \cup \Gamma_\xi, \quad \partial\Omega_\xi = \Gamma_\xi.$$

For any $0 \leq \xi \leq \xi_\Omega$, the manifolds Γ and Γ_ξ are isomorphic through the parallel transport with respect to the vector field $-\xi\boldsymbol{\nu}$. Let $J_\xi(\omega), \omega \in \Gamma$ be the Jacobian determinant of the isomorphism between the manifolds Γ and Γ_ξ. For any $G \in L^1(\Gamma_\xi)$, we have

$$\int_{\Gamma_\xi} G(\omega')\,d\omega' = \int_\Gamma G(\omega - \xi\boldsymbol{\nu}(\omega))J_\xi(\omega)\,d\omega = \int_\Gamma G(\xi,\omega)J_\xi(\omega)\,d\omega.$$

Therefore, for any $F \in L^1(]0,T[\times\mathcal{O}_{\xi_\Omega})$ the following change of variables formula holds:

$$\int_0^T \int_{\mathcal{O}_\eta} F(t,x)\,dt\,dx = \int_0^T \int_0^\eta \int_\Gamma F(t,\xi,\omega)J_\xi(\omega)\,dt\,d\xi\,d\omega, \quad \forall \eta \in]0,\xi_\Omega]. \tag{8}$$

Notice that $J_0(\omega) = 1$ for any $\omega \in \Gamma$. Furthermore, we choose ξ_Ω small enough so that $(\xi,\omega) \mapsto J_\xi(\omega)$ is smooth and satisfy $\frac{1}{2} \leq J_\xi(\omega) \leq \frac{3}{2}$ for any $(\xi,\sigma) \in [0,\xi_\Omega]\times\Gamma$.

Finally, notice that there exists a smooth function $\widetilde{d} : \overline{\Omega} \mapsto \mathbb{R}$ such that $\widetilde{d} = d(\cdot,\Gamma)$ into $\mathcal{O}_{\xi_\Omega}$ so that we can define for any $x \in \overline{\Omega}$, the vector field $\boldsymbol{\nu}(x) = -\nabla\widetilde{d}(x)$ which is regular bounded and which coincides with the outward normal vector field near the boundary of the domain. More precisely, we have

$$\boldsymbol{\nu}(x) = \boldsymbol{\nu}(P_\Gamma(x)), \quad \text{for any } x \in \mathcal{O}_{\xi_\Omega}.$$

2.2. **Trace theorem - Initial and boundary value problem.** We recall here some of the results proved by the first author in [4] concerning the transport equation for non tangential vector field with Sobolev regularity. These results are among the main tools in the study of our problem. The proofs of these results use, in particular, the fundamental concept of renormalized solutions as introduced in [13] for the study of the transport equation for velocity fields tangent to the boundary of the domain. This concept was used to study problem (1) with usual boundary conditions (periodic or homogeneous Dirichlet) in [10, 11, 16] for instance.

Theorem 1 (Trace theorem)**.** *Let $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ such that* $\operatorname{div} v = 0$ *and $(v \cdot \boldsymbol{\nu}) \in L^\delta(]0,T[\times\Gamma)$ for some $\delta > 1$. Then, any weak solution (in the distribution sense) $\rho \in L^\infty(]0,T[\times\Omega)$ of the transport equation*

$$\partial_t \rho + v \cdot \nabla \rho = 0, \tag{9}$$

lies in $\mathcal{C}^0([0,T], L^q(\Omega))$ for any $q \in [1,+\infty[$. Furthermore, there exists a unique $\gamma(\rho)$ in $L^\infty(]0,T[\times\Gamma, |v\cdot\boldsymbol{\nu}|\,dt\,d\omega)$ (called the trace of ρ) such that for any $\varphi \in \mathcal{C}^1([0,T]\times\overline{\Omega})$ and any $[t_1,t_2] \subset [0,T]$ we have

$$\int_{t_1}^{t_2}\int_\Omega \rho(\partial_t\varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho(t_1)\varphi(t_1)\,dx - \int_\Omega \rho(t_2)\varphi(t_2)\,dx \\ - \int_{t_1}^{t_2}\int_\Gamma \gamma(\rho)\varphi\,(v\cdot\boldsymbol{\nu})\,dt\,d\omega = 0.$$

Theorem 2 (Initial and boundary value problem)**.** *Let $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ such that* $\operatorname{div} v = 0$ *and $(v\cdot\boldsymbol{\nu}) \in L^\delta(]0,T[\times\Gamma)$ for some $\delta > 1$. For any initial data $\rho_0 \in L^\infty(\Omega)$ and any inflow data $\rho_{\text{in}} \in L^\infty(]0,T[\times\Gamma,(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega)$ there exists a unique couple $(\rho,\rho_{\text{out}}) \in L^\infty(]0,T[\times\Omega)\times L^\infty(]0,T[\times\Gamma,(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega)$ such that:*

- *ρ is a weak solution to the transport equation* (9) *with $\rho(0) = \rho_0$.*
- *The trace of ρ is characterized by $\gamma(\rho)(v\cdot\boldsymbol{\nu}) = \rho_{\text{out}}(v\cdot\boldsymbol{\nu})^+ - \rho_{\text{in}}(v\cdot\boldsymbol{\nu})^-$.*

Furthermore, we have the renormalization property*:*

> *For any $\beta \in \mathcal{C}^1(\mathbb{R})$, the couple $(\beta(\rho),\beta(\rho_{\text{out}}))$ is the unique weak solution to the transport equation with initial data $\beta(\rho_0)$ and inflow boundary data $\beta(\rho_{\text{in}})$.*

Notice that the initial condition $\rho(0) = \rho_0$ makes sense since, by Theorem 1, we know that any weak solution of the transport equation is continuous in time with values in any $L^q(\Omega)$, $q \in [1,+\infty[$.

Theorem 3 (L^∞ estimate)**.** *Using the notations of Theorem 2, we introduce*

$$\rho_{\min} = \min\left(\inf_\Omega \rho_0, \inf_{]0,T[\times\Gamma} \rho_{\text{in}}\right), \tag{10}$$

$$\rho_{\max} = \max\left(\sup_\Omega \rho_0, \sup_{]0,T[\times\Gamma} \rho_{\text{in}}\right), \tag{11}$$

where the infimum and supremum of ρ_{in} on $]0,T[\times\Gamma$ are taken with respect to the measure $(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega$. Then, we have

$$\rho_{\min} \le \rho(t,x) \le \rho_{\max}, \quad \forall t \in [0,T],\ \textit{for a.e. } x \in \Omega,$$

and

$$\rho_{\min} \le \rho_{\text{out}}(t,\omega) \le \rho_{\max}, \quad \textit{for a.e. } (t,\omega) \in]0,T[\times\Gamma,$$

with respect to the measure $(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega$.

2.3. **Stability with respect to the velocity field.** Using the results recalled above concerning the initial and boundary value problem for the transport equation, we can prove the stability of the solution with respect to the data v, ρ_0 and ρ_{in}. Since this result is interesting for itself we give here a quite general statement applying to more general situations than the particular one addressed in this paper.

Theorem 4 (Stability with respect to v)**.** *For any $k \geq 1$, let $v_k \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ such that $\operatorname{div} v_k = 0$ and $(v_k \cdot \boldsymbol{\nu}) \in L^\delta(]0,T[\times\Gamma)$ for some $\delta > 1$. We suppose given, for any $k \geq 1$, an initial data $\rho_{0,k} \in L^\infty(\Omega)$, and an inflow boundary data $\rho_{\text{in},k} \in L^\infty(]0,T[\times\Gamma)$. We denote by*

$$(\rho_k, \rho_{\text{out},k}) \in L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, (v_k \cdot \boldsymbol{\nu})^+ \, dt \, d\omega),$$

the unique solution to the problem

$$\begin{cases} \partial_t \rho_k + v_k \cdot \nabla \rho_k = 0, \\ \rho_k(0) = \rho_{0,k}, \\ \gamma(\rho_k) = \rho_{\text{in},k}, \quad \text{where} \quad (v_k \cdot \boldsymbol{\nu}) < 0. \end{cases} \tag{12}$$

We assume that

- *$(\rho_{0,k})_k$ is bounded in $L^\infty(\Omega)$ and strongly converges towards $\rho_0 \in L^\infty(\Omega)$ for the $L^1(\Omega)$ topology.*
- *$(\rho_{\text{in},k})_k$ is bounded in $L^\infty(]0,T[\times\Gamma)$ and strongly converges towards $\rho_{\text{in}} \in L^\infty(]0,T[\times\Gamma)$ for the $L^1(]0,T[\times\Gamma)$ topology.*
- *$(v_k)_k$ converges towards v in $L^1(]0,T[,(L^1(\Omega))^d)$, where v is supposed to belong to $L^1(]0,T[,(W^{1,1}(\Omega))^d)$.*
- *$(v_k \cdot \boldsymbol{\nu})_k$ strongly converges towards $v \cdot \boldsymbol{\nu}$ in $L^\delta(]0,T[\times\Gamma)$.*

Then, if we denote by $(\rho, \rho_{\text{out}})$ the solution to the transport problem associated to the vector field v, the initial data ρ_0 and the boundary data ρ_{in}, we have

- *$(\rho_k)_k$ strongly converges towards ρ in all the spaces $L^q(]0,T[\times\Omega)$, $q \in [1,+\infty[$. And more precisely we have*

$$\rho_k(t) \xrightarrow[k\to+\infty]{} \rho(t), \quad \text{in } L^q(\Omega), \ \forall t \in [0,T], \forall q \in [1,+\infty[. \tag{13}$$

- *$\big(\gamma(\rho_k)(v_k \cdot \boldsymbol{\nu})\big)_k$ strongly converges towards $\gamma(\rho)(v \cdot \boldsymbol{\nu})$ in $L^\delta(]0,T[\times\Gamma)$.*

Remark 2. Notice that we do not need the strong convergence of v_k towards v in the space $L^1(]0,T[,(W^{1,1}(\Omega))^d)$ but only in $L^1(]0,T[,(L^1(\Omega))^d)$ supplemented by the strong convergence of the normal traces $(v_k \cdot \boldsymbol{\nu})$.

In sections 5 and 6 we will use this stability result in the case where $(v_k)_k$ converges towards v weakly in $L^2(]0,T[,(H^1(\Omega))^d)$ and strongly in $(L^\gamma(]0,T[\times\Omega))^d$ for some $\gamma \in]1,2[$. These convergences imply in particular the strong convergence of the traces $(v_k \cdot \boldsymbol{\nu})_k$ in some space $L^\delta(]0,T[\times\Gamma)$ as required in the assumptions of the theorem.

Proof. Let us assume that $\rho_{0,k}$ and $\rho_{\text{in},k}$ are non-negative for any k. This is not restrictive since it is always possible to add a constant to all the data without changing the convergence properties in the statement of the Theorem.

- We first recall that $\rho_{\text{out},k}$ is uniquely determined only on the part of the boundary where $v_k \cdot \boldsymbol{\nu} > 0$. Hence, for simplicity, we impose in the sequel $\rho_{\text{out},k} = 0$ on the part of $]0,T[\times\Gamma$ where $v_k \cdot \boldsymbol{\nu} \leq 0$.

The sequences $(\rho_{0,k})_k$ and $(\rho_{\text{in},k})_k$ being bounded in L^∞ we know by Theorem 3 that $(\rho_k)_k$ is bounded in $L^\infty(]0,T[\times\Omega)$ and that $(\rho_{\text{out},k})_k$ is bounded in $L^\infty(]0,T[\times\Gamma)$. Therefore, we can extract subsequences, always denoted by $(\rho_k)_k$ and $(\rho_{\text{out},k})_k$ which $\star$-weakly converge respectively in $L^\infty(]0,T[\times\Omega)$ and $L^\infty(]0,T[\times\Gamma)$.

Using these weak convergences and the assumptions concerning the convergence of the sequences $(\rho_{0,k})_k$, $(\rho_{\text{in},k})_k$ and $(v_k)_k$ we see that for any test function $\varphi \in \mathcal{C}^1([0,T]\times\Omega)$ we can perform the limit in the weak formulation of the problem (12). We find that the weak limits of $(\rho_k)_k$ and $(\rho_{\text{out},k})_k$ satisfy the weak formulation for the transport problem associated to the velocity field v, the initial data ρ_0 and the inflow boundary data ρ_{in}. By Theorem 2, $(\rho, \rho_{\text{out}})$ is the unique couple satisfying this formulation. Hence, we proved that

$$\rho_k \xrightarrow[k\to+\infty]{} \rho, \quad \text{in } L^\infty(]0,T[\times\Omega) \ \star\text{-weak,}, \tag{14}$$

$$\rho_{\text{out},k} \xrightarrow[k\to+\infty]{} \rho_{\text{out}}, \quad \text{in } L^\infty(]0,T[\times\Gamma, (v\cdot\boldsymbol{\nu})^+ dt\, d\omega) \ \star\text{-weak.} \tag{15}$$

Since ρ is unique, the convergence (14) holds in fact for the whole sequence $(\rho_k)_k$ and not only for a subsequence. As far as the outflow boundary term is concerned the situation is slightly different since ρ_{out} is only uniquely defined on the set where $v\cdot\boldsymbol{\nu} > 0$. Nevertheless, we obtain that the whole sequence of the traces $(\gamma(\rho_k))_k$ satisfies the weak convergence

$$\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu}) \xrightarrow[k\to+\infty]{} \gamma(\rho)(v\cdot\boldsymbol{\nu}), \quad \text{in } L^\delta(]0,T[\times\Gamma) \text{ weak.} \tag{16}$$

Finally, performing the limit in the weak formulation satisfied by ρ_k, it is easily seen that

$$\rho_k(t) \xrightarrow[k\to+\infty]{} \rho(t), \quad \text{in } L^\delta(\Omega) \text{ weak, for any } t\in[0,T]. \tag{17}$$

- Our goal is now to prove that the above convergences hold in fact for the strong topologies.

We use here the renormalization property given by Theorem 2. It implies in particular that, for any k, $(\rho_k^\delta, \rho_{\text{out},k}^\delta)$ is the unique solution to the transport problem (12) with initial data $\rho_{0,k}^\delta$ and inflow boundary data $\rho_{\text{in},k}^\delta$. Using the convergence assumptions on the data, we easily see that the sequences $(\rho_{0,k}^\delta)_k$ and $(\rho_{\text{in},k}^\delta)_k$ are bounded in L^∞ and converge strongly towards ρ_0^δ in $L^1(\Omega)$ and towards ρ_{in}^δ in $L^1(]0,T[\times\Gamma)$ respectively.

Furthermore, using once more the renormalization property, we know that the solution to the transport problem associated to the limit vector field v and to the data ρ_0^δ and ρ_{in}^δ is unique and given by $(\rho^\delta, \rho_{\text{out}}^\delta)$. Hence, we can apply the argument of the first point of the proof to obtain the weak convergences

$$\rho_k^\delta \xrightarrow[k\to+\infty]{} \rho^\delta, \quad \text{in } L^\infty(]0,T[\times\Omega) \ \star\text{-weak,}, \tag{18}$$

$$\gamma(\rho_k)^\delta(v_k\cdot\boldsymbol{\nu}) \xrightarrow[k\to+\infty]{} \gamma(\rho)^\delta(v\cdot\boldsymbol{\nu}), \quad \text{in } L^\delta(]0,T[\times\Gamma) \text{ weak.} \tag{19}$$

The convergence (18) imply in particular that

$$\|\rho_k\|^\delta_{L^\delta(]0,T[\times\Omega)} = \int_0^T \int_\Omega \rho_k^\delta \, dt\, dx \xrightarrow[k\to+\infty]{} \int_0^T \int_\Omega \rho^\delta \, dt\, dx = \|\rho\|^\delta_{L^\delta(]0,T[\times\Omega)},$$

and we deduce that the convergence of $(\rho_k)_k$ towards ρ is strong in the space $L^\delta(]0,T[\times\Omega)$. Using (14) it follows that this strong convergence holds in fact in any $L^q(]0,T[\times\Omega)$, $q \in [1,+\infty[$.

By assumption, $(v_k\cdot\boldsymbol{\nu})_k$ strongly converges towards $v\cdot\boldsymbol{\nu}$ in $L^\delta(]0,T[\times\Gamma)$. It follows that

$$|v_k\cdot\boldsymbol{\nu}|^{\delta-2}(v_k\cdot\boldsymbol{\nu}) \xrightarrow[k\to+\infty]{} |v\cdot\boldsymbol{\nu}|^{\delta-2}(v\cdot\boldsymbol{\nu}), \quad \text{in } L^{\frac{\delta}{\delta-1}}(]0,T[\times\Gamma). \tag{20}$$

Hence, using (19) and (20), we have

$$\begin{aligned}\|\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu})\|^\delta_{L^\delta(]0,T[\times\Gamma)} &= \int_0^T \int_\Gamma \gamma(\rho_k)^\delta(v_k\cdot\boldsymbol{\nu})\Big(|v_k\cdot\boldsymbol{\nu}|^{\delta-2}(v_k\cdot\boldsymbol{\nu})\Big)\, dt\, d\omega \\ &\xrightarrow[k\to+\infty]{} \int_0^T \int_\Gamma \gamma(\rho)^\delta(v\cdot\boldsymbol{\nu})\Big(|v\cdot\boldsymbol{\nu}|^{\delta-2}(v\cdot\boldsymbol{\nu})\Big)\, dt\, d\omega \\ &= \|\gamma(\rho)(v\cdot\boldsymbol{\nu})\|^\delta_{L^\delta(]0,T[\times\Gamma)}.\end{aligned}$$

Therefore, we have shown the convergence of the L^δ norm of $(\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu}))_k$ towards the one of $\gamma(\rho)(v\cdot\boldsymbol{\nu})$ which, using the weak convergence (16), implies the strong convergence announced.

- It remains to prove (13). Let $t\in[0,T]$ be fixed. Using the renormalization property with $\beta(s) = s^\delta$ for any k and taking $\varphi = 1$ as a test function in the weak formulation we get

$$\int_\Omega \rho_k^\delta(t)\, dx = \int_\Omega \rho_{0,k}^\delta \, dx - \int_0^t \int_\Gamma \gamma(\rho_k)^\delta(v_k\cdot\boldsymbol{\nu})\, dt\, d\omega.$$

By using the strong convergences proved above, we can perform the limit in the right-hand side and we get that

$$\int_\Omega \rho_k^\delta(t)\, dx \xrightarrow[k\to+\infty]{} \int_\Omega \rho_0^\delta\, dx - \int_0^t \int_\Gamma \gamma(\rho)^\delta(v\cdot\boldsymbol{\nu})\, dt\, d\omega = \int_\Omega \rho^\delta(t)\, dx.$$

Notice that the last equality comes from the renormalization property applied to the limit transport problem satisfied by ρ. Hence, this proves that $\|\rho_k(t)\|_{L^\delta(\Omega)}$ converges towards $\|\rho(t)\|_{L^\delta(\Omega)}$ and then, using the weak convergence (17) the claim is proved.

□

3. Weak formulations of the Navier-Stokes problem

We describe here the weak formulations of the problem (1)-(2) we deal with in this paper. In this section we only give formal computations that will be justified in Section 5 on the approximate problem.

3.1. **The continuity equation.** Following Theorem 2 (see the details in [4]), the natural weak formulation of the transport equation is

$$\int_0^T \int_\Omega \rho(\partial_t \varphi + v\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho_0\varphi(0,.)\,dx \\ - \int_0^T \int_\Gamma \rho_{\text{out}}\varphi(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega + \int_0^T \int_\Gamma \rho_{\text{in}}\varphi(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega = 0, \quad (21)$$

for any $\varphi \in \mathcal{C}_c^1([0,T[,H^1(\Omega))$.

3.2. **The momentum balance equation.** As usual it is possible, at least at a formal level, to consider the equivalent non-conservative or conservative weak formulation for the momentum balance equation. In this section we present these two formulations but also a third one, called *intermediate*. As we will see later, the non-conservative formulation is useful to express the problem (more precisely the approximate problem that we will introduce) as a finite dimensional ordinary differential equation. The intermediate one is used to obtain the energy estimate and the conservative formulation is the one which does not involve the time derivative of the velocity, so that it will be easier to perform the limit in the approximate problem.

3.2.1. *Non-conservative formulation.* Let us introduce $\widetilde{v} = v - v_{\text{ref}}$. The non-conservative formulation reads

$$\int_\Omega \rho\big(\partial_t v + ((v\cdot\nabla)v)\big)\cdot\psi\,dx + \int_\Omega 2\mu(\rho)D(v) : D(\psi)\,dx - \langle \sigma_{\text{ref}}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac12},H^{\frac12}} \\ + \frac12\int_\Gamma \rho_{\text{in}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^-\,d\omega = \int_\Omega \rho f\cdot\psi\,dx,\ \forall t\in[0,T], \quad (22)$$

for any $\psi \in V$ which does not depend on t, with the initial data $\widetilde{v}(0) = \widetilde{v}_0 = v_0 - v_{\text{ref}}(0)$.

3.2.2. *Intermediate formulation.* Consider $\psi \in \mathcal{C}_c^1([0,T[,V)$ and let us take $\varphi = \frac12(\widetilde{v}\cdot\psi)$ in (21). We get

$$\int_0^T \int_\Omega \rho\left(\frac12\partial_t\widetilde{v}\cdot\psi + \frac12\widetilde{v}\cdot\partial_t\psi + \frac12(v\cdot\nabla)(\widetilde{v}\cdot\psi)\right)dt\,dx + \frac12\int_\Omega \rho_0\widetilde{v}_0\cdot\psi(0)\,dx \\ - \frac12\int_0^T\int_\Gamma \rho_{\text{out}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega + \frac12\int_0^T\int_\Gamma \rho_{\text{in}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega = 0. \quad (23)$$

We now take ψ depending on t in (22) and we integrate with respect to t, then we subtract (23) and we get:

$$\int_0^T\int_\Omega \rho\left(\frac12\partial_t\widetilde{v}\cdot\psi - \frac12\partial_t\psi\cdot\widetilde{v} + \frac12((v\cdot\nabla)\widetilde{v})\cdot\psi - \frac12((v\cdot\nabla)\psi)\cdot\widetilde{v}\right)dt\,dx \\ - \frac12\int_\Omega \rho_0\widetilde{v}_0\cdot\psi(0)\,dx + \int_0^T\int_\Omega 2\mu(\rho)D(v):D(\psi)\,dt\,dx - \int_0^T \langle\sigma_{\text{ref}}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac12},H^{\frac12}}\,dt \\ + \frac12\int_0^T\int_\Gamma \rho_{\text{out}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega = \int_0^T\int_\Omega \rho\Big(f - \partial_t v_{\text{ref}} - ((v\cdot\nabla)v_{\text{ref}})\Big)\cdot\psi\,dt\,dx. \quad (24)$$

This formulation will be useful to obtain the energy estimate (formally by taking $\psi = \widetilde{v}$) since the first term is antisymmetric and vanishes when $\psi = \widetilde{v}$.

3.2.3. *Conservative formulation.* We can now obtain a third weak formulation of the momentum balance equation which is the "conservative" form of the problem, and which is in fact the one that will be solved. For any $\psi \in \mathcal{C}^1_c([0,T[,V)$ we take $\varphi = (v\cdot\psi)$ in (21) to obtain

$$\int_0^T\int_\Omega \rho\left(\partial_t v\cdot\psi + v\cdot\partial_t\psi + (v\cdot\nabla)(v\cdot\psi)\right)\,dt\,dx + \int_\Omega \rho_0 v_0\cdot\psi(0)\,dx \\ - \int_0^T\int_\Gamma \rho_{\text{out}}(v\cdot\psi)(v\cdot\boldsymbol{\nu})^+\,dt\,d\omega + \int_0^T\int_\Gamma \rho_{\text{in}}(v\cdot\psi)(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega = 0. \quad (25)$$

We integrate (22) on the time interval $[0,T]$ and we subtract (25). It follows

$$-\int_0^T\int_\Omega \rho v\cdot\Big(\partial_t\psi + ((v\cdot\nabla)\psi)\Big)\,dt\,dx - \int_\Omega \rho_0 v_0\cdot\psi(0)\,dx \\ + \int_0^T\int_\Omega 2\mu(\rho)D(v):D(\psi)\,dt\,dx - \int_0^T \langle\sigma_{\text{ref}}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac12},H^{\frac12}}\,dt \\ + \frac{1}{2}\int_0^T\int_\Gamma \rho_{\text{in}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega + \int_0^T\int_\Gamma \gamma(\rho)(v\cdot\psi)(v\cdot\boldsymbol{\nu})\,dt\,d\omega \\ = \int_0^T\int_\Omega \rho f\cdot\psi\,dt\,dx. \quad (26)$$

4. Main results

The main results of this paper are described in this section. First of all, we show the existence of the density ρ and velocity v satisfying the conservative formulation of the problem introduced above.

Theorem 5. *Under the assumptions stated in Section 1.3, there exists a density ρ in $L^\infty(]0,T[\times\Omega)$, and a velocity field v in $L^2(]0,T[,(H^1(\Omega))^d)$ such that $\operatorname{div} v = 0$ and $v = v_{\text{ref}}$ on Γ_{in} and satisfying (21) and (26).*

Then, we deduce the existence of the pressure term *via* the de Rham theorem. We obtain a triple (ρ, v, p) satisfying the Navier-Stokes equation in the distribution sense.

Then, we are able to prove that the outflow boundary condition on Γ_{out} in (2) is satisfied. This is not obvious since ρ is not smooth and does not have traces in the usual Sobolev sense. The precise result is given by Theorem 7 in Section 7 but at this point we only state the following formal result. We use here the notations of Section 2.1.

Theorem 6. *Let ρ and v given by Theorem 5. There exists a unique pressure field $p \in W^{-1,\infty}(]0,T[,L^2(\Omega))$ such that the total stress tensor $\sigma = 2\mu(\rho)D(v) - p\,\mathrm{Id}$ satisfies*

$$\partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) - \operatorname{div}(\sigma) = \rho f,$$

in the distribution sense, and such that furthermore the outflow boundary condition in (2) is satisfied in the sense

$$\frac{1}{\eta}\int_0^\eta \sigma(\cdot,\xi,\cdot).\boldsymbol{\nu}\,d\xi \underset{\eta\to 0}{\rightharpoonup} \sigma_{\text{ref}}.\boldsymbol{\nu} - \frac{1}{2}\rho_{\text{in}}(v\cdot\boldsymbol{\nu})^-(v - v_{\text{ref}}), \quad \text{on } \Gamma_{\text{out}}.$$

That is to say that the mean values along the normal coordinate of the normal component of the stress $\sigma.\boldsymbol{\nu}$ near the boundary converges towards the imposed stress $\sigma_{\mathrm{ref}}.\boldsymbol{\nu} - \frac{1}{2}\rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(v - v_{\mathrm{ref}})$ on Γ_{out} in a suitable weak topology.

5. Approximate problem

5.1. **Definition.** For any integer $k \geq 1$, let V_k be a k-dimensional Galerkin approximation space in V and $(\eta_k)_k$ a smooth approximation of the identity in the time variable. We introduce approximations of the data defined by:

$$\begin{aligned} &\rho_{\mathrm{in},k} = \rho_{\mathrm{in}} \star \eta_k + \frac{1}{k}, \rho_{0,k} = \rho_0 + \frac{1}{k}, \\ &v_{\mathrm{ref},k} = v_{\mathrm{ref}} \star \eta_k, \sigma_{\mathrm{ref},k} = \sigma_{\mathrm{ref}} \star \eta_k, f_k = f \star \eta_k. \end{aligned} \tag{27}$$

Hence, $\rho_{\mathrm{in},k}, v_{\mathrm{ref},k}, \sigma_{\mathrm{ref},k}$ and f_k are continuous in time and furthermore $\rho_{0,k}$ and $\rho_{\mathrm{in},k}$ are bounded from below by $\frac{1}{k}$, since ρ_0 and ρ_{in} are non-negative.

Let us consider the following approximate problem :

Find $\widetilde{v}_k \in \mathcal{C}^1([0,T], V_k)$, $\rho_k \in \mathcal{C}^0([0,T], L^1(\Omega)) \cap L^\infty(]0,T[\times\Omega)$ and $\rho_{\mathrm{out},k} \in L^\infty(]0,T[\times\Gamma)$ such that if we introduce $v_k = v_{\mathrm{ref},k} + \widetilde{v}_k$:

1. The couple (ρ_k, v_k) is solution to (21) for any $\varphi \in \mathcal{C}^1_c([0,T[, H^1(\Omega))$ with initial data $\rho_{0,k}$ and inflow boundary condition $\rho_{\mathrm{in},k}$.
2. The couple (ρ_k, v_k) is solution to (22) with regularized data for any $\psi \in V_k$.

If (ρ_k, v_k) is such an approximate solution then, it also satisfies the equivalent formulations (24) and (26) for any $\psi \in \mathcal{C}^1([0,T], V_k)$, with regularized data. Indeed, the formal computations of Section 3 are now justified since the approximate solutions are smooth enough.

Notice that the approximate density ρ_k does not lie in a finite dimension space and then the resolution of the approximate problem is not a straightforward consequence of the ordinary differential equations theory.

5.2. **Resolution of the approximate problem.** The resolution of the approximate problem is performed using a fixed point method (see for instance [5, 16]).

Let $w_k \in v_{\mathrm{ref},k} + \mathcal{C}^0([0,T], V_k)$ be given and consider the equations (21)-(22) with regularized data and where the advection field is taken to be w_k instead of v, that is

$$\begin{aligned} \int_0^T \int_\Omega \rho_k(\partial_t\varphi + w_k\cdot\nabla\varphi)\,dt\,dx + \int_\Omega \rho_{0,k}\varphi(0,\cdot)\,dx - \int_\Omega \rho_k(T)\varphi(T)\,dx \\ - \int_0^T \int_\Gamma \rho_{\mathrm{out},k}\varphi(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega + \int_0^T\int_\Gamma \rho_{\mathrm{in},k}\varphi(w_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega = 0, \end{aligned} \tag{28}$$

for any $\varphi \in \mathcal{C}^1([0,T], H^1(\Omega))$ and

$$\begin{aligned} &\int_\Omega \rho_k(\partial_t v_k + (w_k\cdot\nabla)v_k)\cdot\psi\,dx + \int_\Omega 2\mu(\rho_k)D(v_k) : D(\psi)\,dx \\ &-\langle\sigma_{\mathrm{ref},k}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} + \frac{1}{2}\int_\Gamma \rho_{\mathrm{in},k}(\widetilde{v}_k\cdot\psi)(w_k\cdot\boldsymbol{\nu})^-\,d\omega = \int_\Omega \rho_k f_k\cdot\psi\,dx,\ \forall 0\leq t\leq T, \end{aligned} \tag{29}$$

for any $\psi \in V_k$ and with the initial data $\widetilde{v}_k(0) = \mathcal{P}_{V_k}\widetilde{v}_0 = \mathcal{P}_{V_k}(v_0 - v_{\mathrm{ref}})$, $\mathcal{P}_{V_k}$ being the orthogonal projector in H onto V_k.

The vector field w_k being fixed in $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T],V_k) \subset L^2(]0,T[,(H^1(\Omega))^d)$, we know by Theorem 2 that (28) has a unique weak solution

$$(\rho_k, \rho_{\mathrm{out},k}) \in L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, (w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega).$$

Furthermore, this solution enjoys the renormalization property and, in particular, ρ_k is continuous with respect to t and with values in $L^q(\Omega)$ for any $q\in[1,+\infty[$. Even though $\rho_{\mathrm{out},k}$ is only uniquely defined on the part of boundary where $(w_k\cdot\boldsymbol{\nu}) > 0$, it is convenient for the analysis to extend it to the whole boundary by letting $\rho_{\mathrm{out},k} = 0$ where $(w_k\cdot\boldsymbol{\nu}) \leq 0$.

By using Theorem 3 we deduce the first useful estimate on $(\rho_k, \rho_{\mathrm{out},k})$.

Lemma 1. *We define $\rho_{\min}$ and $\rho_{\max}$ by* (10)-(11). *For any k we have*

$$\rho_{\min} + \frac{1}{k} \leq \rho_k(t,x) \leq \rho_{\max} + \frac{1}{k}, \quad \textit{for a.e. } (t,x) \in]0,T[\times\Omega,$$

and

$$\rho_{\min} + \frac{1}{k} \leq \rho_{\mathrm{out},k}(t,\omega) \leq \rho_{\max} + \frac{1}{k}, \quad \textit{for a.e. } (t,\omega) \in]0,T[\times\Gamma,$$

with respect to the measure $(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega$.

In particular, for any k, $\rho_k \geq \frac{1}{k} > 0$ and then, $(\rho_k, \rho_{\mathrm{out},k})$ being fixed, the equation (29) for v_k is now a classical finite dimensional ordinary differential equation since the regularized data are continuous with respect to the time variable. Furthermore, the advection velocity field w_k being fixed, the system is linear. Using the Cauchy theorem, there exists a unique (global) solution $v_k \in v_{\mathrm{ref},k} + \mathcal{C}^1([0,T],V_k)$ to this problem for the given approximate initial data. We denote this solution by $v_k = \Theta_k(w_k)$. We are now going to show that the map Θ_k has a fixed point in a suitable space.

Before this, let us observe that, the solutions ρ_k, v_k of (28) and (29) being smooth enough, we can justify the algebraic manipulations of section 3 so that we have for any $\psi \in \mathcal{C}^1([0,T],V_k)$

$$\begin{aligned}
&\int_0^T\int_\Omega \rho_k\left(\frac{1}{2}\partial_t\widetilde{v}_k\cdot\psi - \frac{1}{2}\partial_t\psi\cdot\widetilde{v}_k + \frac{1}{2}((w_k\cdot\nabla)\widetilde{v}_k)\cdot\psi - \frac{1}{2}((w_k\cdot\nabla)\psi)\cdot\widetilde{v}_k\right)dt\,dx\\
&+\frac{1}{2}\int_\Omega \rho_k(T)\widetilde{v}_k(T)\cdot\psi(T)\,dx - \frac{1}{2}\int_\Omega \rho_{0,k}\widetilde{v}_{0,k}\cdot\psi(0)\,dx - \int_0^T \langle\sigma_{\mathrm{ref},k}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\,dt\\
&+\int_0^T\int_\Omega 2\mu(\rho_k)D(v_k):D(\psi)\,dt\,dx + \frac{1}{2}\int_0^T\int_\Gamma \rho_{\mathrm{out},k}(\psi\cdot\widetilde{v}_k)(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega\\
&=\int_0^T\int_\Omega \rho_k\big(f_k - \partial_t v_{\mathrm{ref},k} - ((w_k\cdot\nabla)v_{\mathrm{ref},k})\big)\cdot\psi\,dt\,dx.
\end{aligned} \tag{30}$$

and

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho_k v_k\cdot\Big(\partial_t\psi+((w_k\cdot\nabla)\psi)\Big)\,dt\,dx-\int_\Omega \rho_{0,k}v_{0,k}\cdot\psi(0)\,dx+\int_\Omega \rho_k(T)v_k(T)\cdot\psi(T)\,dx\\
&+\int_0^T\int_\Omega 2\mu(\rho_k)D(v_k):D(\psi)\,dt\,dx-\int_0^T\langle\sigma_{\mathrm{ref},k}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac12},H^{\frac12}}\,dt\\
&+\frac12\int_0^T\int_\Gamma \rho_{\mathrm{in},k}(\widetilde{v}_k\cdot\psi)(w_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega+\int_0^T\int_\Gamma\gamma(\rho_k)(v_k\cdot\psi)(w_k\cdot\boldsymbol{\nu})\,dt\,d\omega\\
&=\int_0^T\int_\Omega\rho_k f_k\cdot\psi\,dt\,dx.
\end{aligned}
\tag{31}
$$

By taking $\varphi=(\psi\cdot v_{\mathrm{ref},k})$ as a test function in (28) we also get the following equation satisfied by $\widetilde{v}_k$

$$
\begin{aligned}
&-\int_0^T\int_\Omega\rho_k\Big(\widetilde{v}_k\cdot\partial_t\psi+((w_k\cdot\nabla)\psi)\cdot\widetilde{v}_k\Big)\,dt\,dx\\
&+\int_\Omega\rho_k(T)(\widetilde{v}_k(T)\cdot\psi(T))\,dx-\int_\Omega\rho(0)(\widetilde{v}_{0,k}\cdot\psi(0))\,dx\\
&+\int_0^T\int_\Omega 2\mu(\rho_k)D(v_k):D(\psi)\,dt\,dx-\int_0^T\langle\sigma_{\mathrm{ref},k}.\boldsymbol{\nu},\psi\rangle_{H^{-\frac12},H^{\frac12}}\,dt\\
&-\frac12\int_0^T\int_\Gamma\rho_{\mathrm{in},k}(\widetilde{v}_k\cdot\psi)(w_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega+\int_0^T\int_\Gamma\rho_{\mathrm{out},k}(\widetilde{v}_k\cdot\psi)(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega\\
&=\int_0^T\int_\Omega\rho_k\big(f_k-\partial_t v_{\mathrm{ref},k}-((w_k\cdot\nabla)v_{\mathrm{ref},k})\big)\cdot\psi\,dt\,dx.
\end{aligned}
\tag{32}
$$

This equation will be useful to obtain time translation estimates in the sequel.

5.2.1. *Energy estimate.* Consider $w_k\in v_{\mathrm{ref},k}+\mathcal{C}^0([0,T],V_k)$ and $v_k=\Theta_k(w_k)$ as defined previously. Let us choose $\psi=\widetilde{v}_k=v_k-v_{\mathrm{ref},k}$ as a test function in (30), it follows

$$
\begin{aligned}
\frac12\int_\Omega\rho_k(T)|\widetilde{v}_k(T)|^2\,dx+\int_0^T\int_\Omega 2\mu(\rho_k)|D(\widetilde{v}_k)|^2\,dt\,dx+\frac12\int_0^T\int_\Gamma\rho_{\mathrm{out},k}|\widetilde{v}_k|^2(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega&\\
=\frac12\int_\Omega\rho_{0,k}|\widetilde{v}_{0,k}|^2\,dx-\int_0^T\int_\Omega 2\mu(\rho_k)D(\widetilde{v}_k):D(v_{\mathrm{ref},k})\,dt\,dx&\\
+\int_0^T\langle\sigma_{\mathrm{ref},k}.\boldsymbol{\nu},\widetilde{v}_k\rangle_{H^{-\frac12},H^{\frac12}}\,dt-\int_0^T\int_\Omega\rho_k((w_k\cdot\nabla)v_{\mathrm{ref},k})\cdot\widetilde{v}_k\,dt\,dx&\\
-\int_0^T\int_\Omega\rho_k\partial_t v_{\mathrm{ref},k}\cdot\widetilde{v}_k\,dt\,dx+\int_0^T\int_\Omega\rho_k f_k\cdot\widetilde{v}_k\,dt\,dx.&
\end{aligned}
$$

Using the L^∞ bound on ρ_k given by Lemma 1, assumption (7) and Hölder and Young's inequalities, we classically deduce the estimate

$$\begin{aligned}\int_\Omega \rho_k(T)|\widetilde{v}_k(T)|^2\,dx + \int_0^T\int_\Omega 2\mu(\rho_k)|D(\widetilde{v}_k)|^2\,dt\,dx + \int_0^T\int_\Gamma \rho_{\mathrm{out},k}|\widetilde{v}_k|^2(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega \\ \leq C(\rho_0, v_0, \mu, \|v_{\mathrm{ref}}\|_{L^2(H^1)}, \|f\|_{L^2(L^2)}, \|\sigma_{\mathrm{ref}}\|_{L^2(H^{-\frac{1}{2}})}, \|\partial_t v_{\mathrm{ref}}\|_{L^2(L^2)}) \\ + C(\rho_{max},\mu)\left(\int_0^T \|v_{\mathrm{ref},k}\|^4_{H^1(\Omega)}\|\sqrt{\rho_k}w_k\|^2_{L^2}\,dt\right)^{\frac{1}{2}}\left(\int_0^T\int_\Omega |\nabla w_k|^2\,dt\,dx\right)^{\frac{1}{2}},\end{aligned}$$

where we emphasize that the constants C above do not depend on k. Hence, since $v_k = v_{\mathrm{ref},k} + \widetilde{v}_k$ and using (5) we deduce

$$\begin{aligned}\int_\Omega \rho_k(T)|v_k(T)|^2\,dx + \int_0^T\int_\Omega \mu(\rho_k)|D(v_k)|^2\,dt\,dx + \int_0^T\int_\Gamma \rho_{\mathrm{out},k}|\widetilde{v}_k|^2(w_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega \\ \leq C(\rho_0, v_0, \mu, \|v_{\mathrm{ref}}\|_{L^2(H^1)}, \|f\|_{L^2(L^2)}, \|\sigma_{\mathrm{ref}}\|_{L^2(H^{-\frac{1}{2}})}, \|\partial_t v_{\mathrm{ref}}\|_{L^2(L^2)}) \\ + C(\rho_{max},\mu)\left(\int_0^T \|v_{\mathrm{ref},k}\|^4_{H^1(\Omega)}\|\sqrt{\rho_k}\,w_k\|^2_{L^2}\,dt\right)^{\frac{1}{2}}\left(\int_0^T\int_\Omega |\nabla w_k|^2\,dt\,dx\right)^{\frac{1}{2}},\end{aligned} \tag{33}$$

We now use the fact that, k being fixed, $v_{\mathrm{ref},k}$ belongs to $L^\infty(]0,T[,(H^1(\Omega))^d)$. Furthermore on the finite dimensional space V_k, the $H^1(\Omega)$ norm is equivalent to the $L^2(\Omega)$ norm so that it follows from (33)

$$\|v_k(T)\|^2_{L^2} \leq C_k + D_k\int_0^T \|w_k(t)\|^2_{L^2}\,dt,$$

where the constants C_k and D_k depend on k.

The above estimate applies for any final time $T>0$ so that we have in fact

$$\|v_k(t)\|^2_{L^2} \leq C_k + D_k\int_0^t \|w_k(s)\|^2_{L^2}\,ds, \quad \forall t\in[0,T]. \tag{34}$$

Let us introduce

$$M_k(t) = C_k e^{D_k t}, \quad \forall t \geq 0.$$

Suppose that we have

$$\|w_k(t)\|^2_{L^2} \leq M_k(t), \quad \forall t\in[0,T],$$

then, using (34) we deduce

$$\|v_k(t)\|^2_{L^2} \leq M_k(t), \quad \forall t\in[0,T].$$

Hence, we proved that Θ_k maps the convex set

$$\mathcal{K}_0 = \left\{ v \in v_{\mathrm{ref},k} + \mathcal{C}^1([0,T],V_k),\ \text{such that } \|v(t)\|^2_{L^2} \leq M_k(t),\ \forall t\in[0,T]\right\},$$

into itself. Notice in particular that the elements of $\mathcal{K}_0$ are uniformly bounded in $L^\infty(]0,T[,(L^2(\Omega))^d)$ by a constant depending only on k, on the data and on the final time T. Moreover, since V_k is a finite dimensional subspace of $(H^1(\Omega))^d$, the L^2-norm and the H^1-norm are equivalent on V_k and then the set $\mathcal{K}_0$ is also bounded in $L^\infty(]0,T[,(H^1(\Omega))^d)$ since $v_{\mathrm{ref},k} \in L^\infty(]0,T[,(H^1(\Omega))^d)$.

5.2.2. *Compactness.* Let $w_k \in \mathcal{K}_0$ and $v_k = \Theta_k(w_k)$. We take $\psi = \partial_t \widetilde{v}_k$ in (29). Since all the norms in V_k are equivalent and using the fact that $\partial_t v_{\mathrm{ref},k}$ lies in $L^2(]0,T[\times\Omega)$, we easily get a bound

$$\sup_{0\le t\le T} \|\partial_t \widetilde{v}_k\|_{V_k} \le C'_k,$$

where C'_k depends only on T, k and on the data. Of course, we used here that the approximate density ρ_k is bounded from below by $\frac{1}{k}$.

As a consequence, the convex set

$$\mathcal{K}_1 = \left\{ v \in \mathcal{K}_0, \quad \sup_{0\le t\le T} \|\partial_t \widetilde{v}\|_{V_k} \le C'_k \right\},$$

is invariant through the map Θ_k. Using the Ascoli theorem, we know that the set $\mathcal{K}_1$ is relatively compact in $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T], V_k)$. In order to apply the Schauder fixed point theorem to the map Θ_k on the compact convex set $\overline{\mathcal{K}_1}$, it remains to show that Θ_k is continuous for the topology of $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T], V_k)$. In fact it is enough to show that Θ_k is sequentially continuous.

5.2.3. *Continuity of* Θ_k. Recall that k is a fixed integer. Let $(w_k^n)_n$ we a sequence in $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T], V_k)$ which converges towards w_k in this space. For any n let $(\rho_{k,n}, \rho_{\mathrm{out},k,n})$ be the solution to the transport problem (28) with $w_k = w_k^n$ and let $(\rho_k, \rho_{\mathrm{out},k})$ the solution to (28) for the limit velocity field w_k.

Since V_k is embedded in $(H^1(\Omega))^d$, the sequence $(w_k^n)_n$ strongly converges in $\mathcal{C}^0([0,T], (H^1(\Omega))^d)$. In particular, the traces $(w_k^n \cdot \boldsymbol{\nu})_n$ converge towards $(w_k \cdot \boldsymbol{\nu})$ in $L^2(]0,T[\times\Gamma)$. Hence, by the stability Theorem 4 we deduce that $(\rho_{k,n})_n$ strongly converges towards ρ_k in all the spaces $L^q(]0,T[\times\Omega)$ and that $(\rho_{\mathrm{out},k,n}(w_k^n \cdot \boldsymbol{\nu})^+)_n$ strongly converges towards $\rho_{\mathrm{out},k}(w_k \cdot \boldsymbol{\nu})^+$ in $L^2(]0,T[\times\Gamma)$.

Finally, since μ is a bounded continuous function, we deduce that $(\mu(\rho_{k,n}))_n$ converges towards $\mu(\rho_k)$ in all the spaces $L^q(]0,T[\times\Omega)$, $q < +\infty$.

Let us now consider the solution v_k^n to (29) for the advection vector field w_k^n and the density $\rho_{k,n}$ constructed above. Since $(w_k^n)_n$ is bounded in $\mathcal{C}^0([0,T], (H^1(\Omega))^d)$, the energy estimate (33) leads to

$$\|v_k^n\|_{\mathcal{C}^0([0,T],(L^2(\Omega))^d)} \le C_k, \ \forall n \ge 0,$$

and we also easily get that

$$\|\partial_t v_k^n\|_{\mathcal{C}^0([0,T],(L^2(\Omega))^d)} \le C'_k, \ \forall n \ge 0.$$

Using the Ascoli theorem, there exists a subsequence always denoted by $(v_k^n)_n$ which strongly converges towards a limit v_k in the space $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T], V_k)$, and in particular strongly in $\mathcal{C}^0([0,T], (H^1(\Omega))^d)$. Furthermore, up to another extraction of a subsequence, $(\partial_t v_k^n)_n$ $\star$-weakly converges towards $\partial_t v_k$ in $L^\infty(]0,T[, V_k)$.

Thanks to the convergences obtained above, we can perform the limit in the equation satisfied by v_k^n. We obtain that ρ_k, v_k, w_k and $\rho_{\mathrm{out},k}$ satisfy (29) and also (31). Since the solution to (31) is unique as soon as ρ_k, $\rho_{\mathrm{out},k}$ and w_k are fixed we deduce that the whole sequence $(v_k^n)_n$ converges in $\mathcal{C}^0([0,T], (H^1(\Omega))^d)$ towards $v_k = \Theta_k(w_k)$.

This concludes the proof of the continuity of the map Θ_k.

5.2.4. *Conclusion.* We just proved that Θ_k is a continuous map from $v_{\mathrm{ref},k} + \mathcal{C}^0([0,T],V_k)$ into itself and that the convex compact set $\overline{\mathcal{K}_1}$ is invariant by Θ_k. Thanks to the Schauder fixed point theorem, we find that there exists at least one fixed point v_k of Θ_k into $\overline{\mathcal{K}_1}$.

Hence, there exists at least one solution to the approximate problem under study. Furthermore, this solution (ρ_k, v_k) being continuous in time, it also satisfies all the equivalent weak formulations of the momentum balance equation.

In the following section, we are going to provide uniform estimates with respect to k for this approximate solution which let us perform the limit when k goes to infinity.

6. Estimates for the approximate solution and proof of Theorem 5

6.1. **Energy estimate.** Since we have $v_k = \Theta_k(v_k)$, the inequality (33) provides a first useful inequality which implies, using the Gronwall lemma and assumption (5), the estimate

$$\|\sqrt{\rho_k} v_k\|^2_{L^\infty(]0,T[,L^2)} + \|v_k\|^2_{L^2(]0,T[,H^1)} + \int_0^T \int_\Gamma \rho_{\mathrm{out},k} |v_k|^2 (v_k\cdot\boldsymbol{\nu})^+ \, dt\, d\omega \leq C_0, \quad (35)$$

where C_0 depends only on the data and the final time T, and in particular is uniform with respect to k. Notice that we used here the Korn inequality which says that there exists $C>0$ such that

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|D(u)\|_{L^2(\Omega)}, \quad \forall u \in (H^1_{\mathrm{in}}(\Omega))^d, \quad \operatorname{div} u = 0.$$

6.2. **Additional estimates on ρ_k and v_k.** Since ρ_k may vanish we do not have a classical $L^\infty(]0,T[,(L^2(\Omega))^d)$ estimate on the velocity field v_k from the energy estimate (35). Nevertheless, using (3) and (4), we can obtain estimates on v_k and on its trace in spaces with time integrability index greater than 2 which will be very useful in the sequel.

Lemma 2. *Under the assumptions* (3)-(4)*, there exist* $C_1 > 0$*,* $\beta_1, \beta_2 \in]2,+\infty]$*, such that for any* k

$$\sup_{t\in[0,T]} \int_\Omega \frac{1}{\rho_k^{\frac{\alpha}{2}}(t)}\, dx \leq C_1. \quad (36)$$

$$\|v_k\|_{L^{\beta_1}(]0,T[,L^1(\Omega))} \leq C_1, \quad (37)$$

$$\|v_k\|_{L^{\beta_2}(]0,T[,L^2(\Gamma))} \leq C_1. \quad (38)$$

Proof. Let k being fixed.

- Using Lemma 1 we know that ρ_k is bounded from below by $\frac{1}{k}$. Let $\beta_k \in \mathcal{C}^1(\mathbb{R})$ such that $\beta_k(s) = s^{-\frac{\alpha}{2}}$ for any $s \geq \frac{1}{k}$. Using the renormalization property given in Lemma 2 we know that $(\beta_k(\rho_k), \beta_k(\rho_{\mathrm{out},k}))$ is solution to the transport equation with initial data $\beta_k(\rho_{0,k})$ and inflow data $\beta_k(\rho_{\mathrm{in},k})$. It follows that for any $t\in[0,T]$

$$\int_\Omega \beta_k(\rho_k(t))\, dx + \int_0^t \int_\Gamma \beta_k(\rho_{\mathrm{out},k})(v_k\cdot\boldsymbol{\nu})^+ \, dt\, d\omega$$
$$= \int_\Omega \beta_k(\rho_{0,k})\, dx + \int_0^t \int_\Gamma \beta_k(\rho_{\mathrm{in},k})(v_k\cdot\boldsymbol{\nu})^- \, dt\, d\omega,$$

and then we have

$$\begin{aligned}\int_\Omega \frac{1}{\rho_k(t)^{\frac{\alpha}{2}}}\,dx &\le \int_\Omega \frac{1}{\rho_{0,k}{}^{\frac{\alpha}{2}}}\,dx + \int_0^t \int_\Gamma (\rho_{\text{in}} \star \eta_k)^{-\frac{\alpha}{2}} (v_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega \\ &\le \|\rho_0\|_{L^\infty}^{\frac{\alpha}{2}} \int_\Omega \frac{1}{{\rho_0}^{\alpha}}\,dx \\ &\quad + \|(\rho_{\text{in}}\star\eta_k)^{-\frac{\alpha}{2}}\|_{L^2(]0,T[,L^2(\Gamma))}\|(v_k\cdot\boldsymbol{\nu})^-\|_{L^2(]0,T[,L^2(\Gamma))}.\end{aligned} \tag{39}$$

Since $(v_k)_k$ is bounded in $L^2(]0,T[,(H^1(\Omega))^d)$, we know that $(v_k\cdot\boldsymbol{\nu})^-$ is bounded in $L^2(]0,T[,L^2(\Gamma))$. Furthermore, since $s\mapsto s^{-\frac{\alpha}{2}}$ is convex on $\mathbb{R}^+$, Jensen's inequality gives

$$(\rho_{\text{in}}\star\eta_k)^{-\frac{\alpha}{2}} \le ({\rho_{\text{in}}}^{-\frac{\alpha}{2}})\star\eta_k.$$

Then, by assumptions (3) and (4) and using Young's inequality, we see that the right-hand side of (39) is bounded. This proves the first point of the lemma.

- We first assume that $\alpha\ge 2$, then we write $|v_k| = (\sqrt{\rho_k}|v_k|)\frac{1}{\sqrt{\rho_k}}$. Using (35), the first factor is bounded in $L^\infty(]0,T[,L^2(\Omega))$ and using (36) the second factor is bounded in $L^\infty(]0,T[,L^\alpha(\Omega))$. By using Hölder's inequality, we find that (37) holds with $\beta_1=+\infty$ in that case.
- We assume now that $\alpha<2$ and let $\beta=\frac{5\alpha}{2\alpha+6}\in]0,1[$. We write

$$|v_k| = \left({\rho_k}^{\frac{\beta}{2}}|v_k|^\beta\right)|v_k|^{1-\beta}{\rho_k}^{-\frac{\beta}{2}}. \tag{40}$$

Using (35), one can see that the first factor is bounded in $L^\infty(]0,T[,L^{\frac{2}{\beta}}(\Omega))$ and that $(v_k)_k$ is bounded in $L^2(]0,T[,(H^1(\Omega))^d))\subset L^2(]0,T[,(L^6(\Omega))^d)$. Hence, the factor $|v_k|^{1-\beta}$ in (40) is bounded in $L^{\frac{2}{1-\beta}}(]0,T[,L^{\frac{6}{1-\beta}}(\Omega))$. Finally, using (36), we see that the third factor in (40) is bounded in the space $L^\infty(]0,T[,L^{\frac{\alpha}{\beta}}(\Omega))$. Hence, we find that (37) holds with $\beta_1=\frac{4(\alpha+3)}{3(2-\alpha)}$ in that case.
- We recall that there exists $C>0$ such that

$$\|u\|_{L^2(\Gamma)} \le C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{H^1(\Omega)}^{\frac{1}{2}},\ \forall u\in H^1(\Omega).$$

Using interpolation properties between Lebesgue spaces on bounded domains and the embedding of $H^1(\Omega)$ into $L^6(\Omega)$ as soon as $d\le 3$, we have

$$\|u\|_{L^2(\Gamma)} \le C\|u\|_{L^1(\Omega)}^{\frac{1}{5}}\|u\|_{H^1(\Omega)}^{\frac{4}{5}},\ \forall u\in H^1(\Omega). \tag{41}$$

From (35) and (37) we know that $(v_k)_k$ is bounded in $L^2(]0,T[,(H^1(\Omega))^d)$ and in $L^{\beta_1}(]0,T[,(L^1(\Omega))^d)$. Hence, applying (41) with $u=v_k$, we obtain the bound (38) with $\beta_2=\frac{5\beta_1}{2\beta_1+1}>2$.

□

6.3. **Time translations estimates.** In order to perform the limit in the approximate problem, it is necessary to prove some compactness property for the sequence $(v_k)_k$. As in [21] for instance, this compactness property will follow from fractional time derivatives and more precisely from time translations estimates.

Let us denote by τ_h, $h>0$, the time translation operator defined for any Banach space X and any $u\in L^1_{loc}(]0,T[,X)$ by

$$\tau_h u(t) = u(t+h),\quad \forall t\in]0,T-h[.$$

Lemma 3. *There exist $\delta>0$ and $C_2>0$ depending only on the data and the final time T, such that for any $k\geq 0$ and any $h>0$, we have*

$$\|\sqrt{\tau_h\rho_k}(\tau_h\widetilde{v}_k-\widetilde{v}_k)\|_{L^2(]0,T-h[,(L^2(\Omega))^d)} \leq C_2 h^\delta. \tag{42}$$

Proof. Let us first write the following identity

$$\begin{aligned}\tau_h\rho_k(\tau_h\widetilde{v}_k-\widetilde{v}_k)\cdot(\tau_h\widetilde{v}_k-\widetilde{v}_k) =&(\tau_h(\rho_k\widetilde{v}_k)-\rho_k\widetilde{v}_k)\cdot(\tau_h\widetilde{v}_k-\widetilde{v}_k)\\ &-\Big((\tau_h\rho_k-\rho_k)\widetilde{v}_k\Big)\cdot(\tau_h\widetilde{v}_k-\widetilde{v}_k) = A+B.\end{aligned} \tag{43}$$

• Estimate of the term A :

Consider $\Psi(t,x)\in\mathcal{C}^1([0,T],V_k)$ and let us introduce $\psi(t,x)=1_{[s,s+h]}(t)\Psi(s,x)$. We use $\psi(t,x)$ as a test function in the conservative formulation (32) where we recall that $w_k=v_k$. This computation is allowed since ρ_k and $\widetilde{v}_k$ are continuous with respect to the time variable. We get

$$\begin{aligned}&\int_\Omega(\tau_h(\rho_k\widetilde{v}_k)(s)-\rho_k\widetilde{v}_k(s))\cdot\Psi(s)\,dx\\ &=\int_s^{s+h}\int_\Omega\rho_k((v_k\cdot\nabla)\Psi(s))\cdot\widetilde{v}_k\,dt\,dx-\int_s^{s+h}\int_\Omega 2\mu(\rho_k)D(v_k):D(\Psi(s))\,dt\,dx\\ &\quad+\int_s^{s+h}\langle\sigma_{\text{ref},k}.\boldsymbol{\nu},\Psi(s)\rangle_{H^{-\frac12},H^{\frac12}}\,dt+\frac12\int_s^{s+h}\int_\Gamma\rho_{\text{in},k}(\widetilde{v}_k\cdot\Psi(s))(v_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega\\ &\quad-\int_s^{s+h}\int_\Gamma\rho_{\text{out},k}(\widetilde{v}_k\cdot\Psi(s))(v_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega\\ &\quad+\int_s^{s+h}\int_\Omega\rho_k\big(f_k-\partial_t v_{\text{ref},k}-((v_k\cdot\nabla)v_{\text{ref},k})\big)\cdot\Psi(s)\,dt\,dx.\end{aligned}$$

Using the bounds on ρ_k given by Lemma 1, we get

$$\begin{aligned}&\left|\int_\Omega(\tau_h(\rho_k\widetilde{v}_k)(s)-\rho_k\widetilde{v}_k(s))\cdot\Psi(s)\,dx\right|\\ &\leq C\left(\int_s^{s+h}\|\sqrt{\rho_k}v_k\|_{L^4}(\|\sqrt{\rho_k}\|\widetilde{v}_k\|_{L^4}+\|\nabla v_{\text{ref},k}\|_{L^2})\,dt\right)\|\nabla\Psi(s)\|_{L^2}\\ &+C\left(\int_s^{s+h}\|\nabla v_k\|_{L^2}\,dt\right)\|\nabla\Psi(s)\|_{L^2}+C\left(\int_s^{s+h}\|\sigma_{\text{ref},k}.\boldsymbol{\nu}\|_{H^{-\frac12}}\,dt\right)\|\Psi(s)\|_{H^{\frac12}(\Gamma)}\\ &+\frac12\left|\int_s^{s+h}\int_\Gamma\rho_{\text{in},k}(\widetilde{v}_k\cdot\Psi(s))(v_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega\right|+\left|\int_s^{s+h}\int_\Gamma\rho_{\text{out},k}(\widetilde{v}_k\cdot\Psi(s))(v_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega\right|\\ &+C\left(\int_s^{s+h}\|f_k\|_{L^2}+\|\partial_t v_{\text{ref},k}\|_{L^2}\,dt\right)\|\nabla\Psi(s)\|_{L^2}.\end{aligned}$$

Using the energy estimate (35) and assumptions (5)-(6) we can easily bound all the interior terms by $Ch^\delta\|\nabla\Psi(s)\|_{L^2}$ (see [5, 21]). It remains to bound the boundary terms.

To this end we use the trace estimate (38), where we recall that $\beta_2 > 2$. Hence, using the Hölder inequality, the two boundary terms above can be bounded by $Ch^{\frac{\beta_2-2}{2\beta_2}}\|\nabla\Psi(s)\|_{L^2}$. Hence, for $\delta > 0$ small enough, we finally proved the following estimate

$$\left|\int_\Omega (\tau_h(\rho_k\widetilde{v}_k)(s) - \rho_k\widetilde{v}_k(s)) \cdot \Psi(s)\,dx\right| \le Ch^\delta\|\nabla\Psi(s)\|_{L^2}.$$

Now let us take $\Psi(s) = \tau_h(\widetilde{v}_k)(s) - \widetilde{v}_k(s)$. It follows

$$\left|\int_\Omega (\tau_h(\rho_k\widetilde{v}_k)(s) - \rho_k\widetilde{v}_k(s)) \cdot (\tau_h(\widetilde{v}_k)(s) - \widetilde{v}_k(s))\,dx\right| \le Ch^\delta\|\tau_h\widetilde{v}_k(s) - \widetilde{v}_k(s)\|_{H^1}.$$

By integrating this inequality with respect to s and using (35), we get

$$\left|\int_0^{T-h}\int_\Omega (\tau_h(\rho_k\widetilde{v}_k)(s) - \rho_k\widetilde{v}_k(s)) \cdot (\tau_h(\widetilde{v}_k)(s) - \widetilde{v}_k(s))\,ds\,dx\right| \\ \le Ch^\delta\|\widetilde{v}_k\|_{L^2(0,T-h,H^1)} \le K_1h^\delta. \quad (44)$$

This is the estimate of the term A in (43).

• Estimate for the term B :

Consider a given time $s \in]0, T-h[$, and choose a time-independent function $\varphi \in H^1(\Omega)$. We take $(t,x) \mapsto 1_{[s,s+h]}(t)\varphi(x)$ as a test function in (28) (this is possible since ρ_k is continuous in time), it follows

$$\int_\Omega (\tau_h\rho_k(s,x) - \rho_k(s,x))\varphi(x)\,dx = \\ \int_\Omega\left(\int_s^{s+h}\rho_k v_k\,dt\right)\cdot\nabla\varphi(x)\,dx + \int_s^{s+h}\int_\Gamma \gamma(\rho_k)\varphi(v_k\cdot\boldsymbol{\nu})\,dt\,d\omega.$$

Let $\psi \in \mathcal{C}^0([0,T],(H^1(\Omega))^d)$. We choose $\varphi(x) = \widetilde{v}_k(s,x)\cdot\psi(s,x)$ in the above identity. It follows, using Sobolev embeddings, trace theorems and Lemma 1 that :

$$\begin{aligned} &\left|\int_\Omega (\tau_h\rho_k(s) - \rho_k(s))\widetilde{v}_k(s)\cdot\psi(s)\,dx\right| \\ &\qquad \le C\left(\int_s^{s+h}\|v_k(t)\|_{H^1}\,dt\right)\|\widetilde{v}_k(s)\|_{H^1}\|\psi(s)\|_{H^1_0} \\ &\qquad \le Ch^{\frac12}\|v_k\|_{L^2(]0,T[,H^1)}\|\widetilde{v}_k(s)\|_{H^1}\|\psi(s)\|_{H^1}. \end{aligned} \quad (45)$$

We now take $\psi(s) = \tau_h\widetilde{v}_k(s) - \widetilde{v}_k(s)$ in (45), so that integrating with respect to s we get

$$\left|\int_0^{T-h}\int_\Omega (\tau_h\rho_k - \rho_k)\widetilde{v}_k\cdot(\tau_h\widetilde{v}_k - \widetilde{v}_k)\,ds\,dx\right| \\ \le Ch^{\frac12}\|v_k\|_{L^2(]0,T[,(H^1(\Omega))^d)}\|\widetilde{v}_k\|^2_{L^2(]0,T[,(H^1(\Omega))^d)} \le K_1'h^{\frac12}. \quad (46)$$

Combining estimates (44) and (46) gives the claim thanks to (43). □

Since ρ_k is not necessarily bounded from below away from 0 uniformly with respect to k, it is needed to use the estimates of Lemma 2 in order to deduce translation estimates on $\widetilde{v}_k$ from the previous Lemma.

Lemma 4. *For any $\gamma < 2$, there exists $\delta > 0$ such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ satisfying*

$$\|\tau_h \widetilde{v}_k - \widetilde{v}_k\|_{L^\gamma(]0,T-h[\times\Omega)} \leq \varepsilon + C_\varepsilon h^\delta, \quad \forall k \geq 0, \ \forall h > 0.$$

Proof. For any $\eta > 0$ we have

$$\eta^{\frac{\gamma}{2}} \int_0^{T-h} \int_\Omega |\tau_h \widetilde{v}_k - \widetilde{v}_k|^\gamma \, dt \, dx \leq \eta^{\frac{\gamma}{2}} \iint_{\{\tau_h \rho_k < \eta\}} |\tau_h \widetilde{v}_k - \widetilde{v}_k|^\gamma \, dt \, dx + \iint_{\{\tau_h \rho_k \geq \eta\}} |\tau_h \rho_k|^{\frac{\gamma}{2}} |\tau_h \widetilde{v}_k - \widetilde{v}_k|^\gamma \, dt \, dx.$$

It follows by using the Hölder inequality that

$$\int_0^{T-h} \int_\Omega |\tau_h \widetilde{v}_k - \widetilde{v}_k|^\gamma \, dt \, dx \leq \Big|\{\tau_h \rho_k < \eta\}\Big|^{\frac{2-\gamma}{2}} \|\tau_h \widetilde{v}_k - \widetilde{v}_k\|^\gamma_{L^2(]0,T-h[\times\Omega)} + \frac{|\Omega|^{\frac{2-\gamma}{2}}}{\eta^{\frac{\gamma}{2}}} \|\sqrt{\tau_h \rho_k}(\tau_h \widetilde{v}_k - \widetilde{v}_k)\|^\gamma_{L^2(]0,T-h[\times\Omega)},$$

where $|A|$ denotes the Lebesgue measure of any measurable set $A \subset]0,T[\times\Omega$. Using Poincaré's inequality we deduce from (35) a bound in $L^2(]0,T[,(L^2(\Omega))^d)$ for $\widetilde{v}_k$ which let us bound the first term above. We use (42) to treat the second term. It follows

$$\int_0^{T-h} \int_\Omega |\tau_h \widetilde{v}_k - \widetilde{v}_k|^\gamma \, dt \, dx \leq C \Big|\{\tau_h \rho_k < \eta\}\Big|^{\frac{2-\gamma}{2}} + C \frac{1}{\eta^{\frac{\gamma}{2}}} h^{\delta\gamma}.$$

The claim will be proved if we show that for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\Big|\{\tau_h \rho_k < \eta\}\Big| \leq \varepsilon$ for any k and any h. To this end, we use (36) which gives

$$\Big|\{\tau_h \rho_k < \eta\}\Big| \leq \eta^{\frac{\alpha}{2}} \int_0^T \int_\Omega \frac{1}{\rho_k^{\frac{\alpha}{2}}} \, dt \, dx \leq \eta^{\frac{\alpha}{2}} T C_0,$$

and the claim is proved. □

6.4. **Performing the limit in the equations.** From Lemma 3 and the compactness results by J. Simon (see [20]) we deduce that the sequences $(\widetilde{v}_k)_k$ and $(v_k = \widetilde{v}_k + v_{\mathrm{ref},k})_k$ are relatively compact in $(L^\gamma(]0,T[\times\Omega))^d$ for any $\gamma < 2$. Since $(v_k)_k$ is bounded in $L^2(]0,T[,(H^1(\Omega))^d)$ thanks to (35), we can extract a subsequence always denoted by $(v_k)_k$ which converges weakly in $L^2(]0,T[,(H^1(\Omega))^d)$ and strongly in all the intermediate spaces $L^\gamma(]0,T[,(H^{1-\varepsilon}(\Omega))^d)$ for any $\gamma < 2$ and any $\varepsilon > 0$. In particular, the normal traces $(v_k \cdot \boldsymbol{\nu})_k$ converge towards $(v \cdot \boldsymbol{\nu})$ in $L^\gamma(]0,T[\times\Gamma)$ for any $\gamma < 2$.

Hence, we can apply the stability Theorem 4 which gives us the strong convergence of $(\rho_k)_k$ towards ρ in all the spaces $L^q(]0,T[\times\Omega)$, $q < +\infty$. Since μ is a bounded continuous function we get the convergence of $(\mu(\rho_k))_k$ towards $\mu(\rho)$ is the same spaces. Furthermore, the stability theorem also gives the convergence of the traces, that is

$$\gamma(\rho_k)(v_k \cdot \boldsymbol{\nu}) \xrightarrow[k \to +\infty]{} \gamma(\rho)(v \cdot \boldsymbol{\nu}), \quad \text{in } L^\gamma(]0,T[\times\Gamma), \text{ for any } \gamma < 2. \tag{47}$$

All the convergences above (and the convergence properties of the regularized data defined in (27)) let us perform the limit in all the interior terms in (31) (with w_k replaced by v_k) without any difficulties.

Let us now treat the boundary terms. We proved that $(v_k)_k$ strongly converges towards v in $L^\gamma(]0,T[,(H^{1-\varepsilon}(\Omega))^d)$, for any $\gamma < 2$ and $\varepsilon > 0$. We deduce, in particular, that the trace of v_k strongly converges towards the trace of v in $L^\gamma(]0,T[,(L^3(\Gamma))^d)$ for instance. Thanks to the estimate (38) (recall that $\beta_2 > 2$) and to a classical interpolation argument, we deduce that there exists $\beta > 2$ such that the trace of v_k strongly converges in $L^\beta(]0,T[,(L^\beta(\Gamma))^d)$. Since we also have (47), it is now straightforward to perform the limit in all the boundary terms in (31).

This concludes the proof of Theorem 5 since the limits ρ and v obtained above have been proved to satisfy the weak formulations (21) and (26).

7. Interpretation of the outflow boundary condition

Let us now prove that the artificial nonlinear outflow boundary condition that we proposed in (2) is satisfied in a suitable weak sense by the solution (ρ, v) of (21) and (26) that we obtained in the previous section.

First of all, if we restrict (26) to test functions $\psi \in W_0^{1,1}(]0,T[,(H_0^1(\Omega))^d)$ such that $\operatorname{div}\psi = 0$ then we can use the classical de Rham argument (see for instance [21]) to get the existence of the pressure. More precisely, there exists $p \in W^{-1,\infty}(]0,T[,L_0^2(\Omega))$ such that, introducing the stress tensor $\sigma = 2\mu(\rho)D(v) - p\,\mathrm{Id}$, we have

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div}(\sigma) = \rho f, \tag{48}$$

in the distribution sense on $]0,T[\times\Omega$. Notice that all the terms in this equation belong to $W^{-1,\infty}(]0,T[,(H^{-1}(\Omega))^d)$ so that any element of $W_0^{1,1}(]0,T[,(H_0^1(\Omega))^d)$ can be chosen as a test function in (48).

From now on, we denote by $\langle\cdot,\cdot\rangle_{W^{-1,\infty},W_0^{1,1}}$ the duality bracket between the spaces $W^{-1,\infty}(]0,T[)$ and $W_0^{1,1}(]0,T[)$.

Let us first prove that, even though the term $\rho v \otimes v$ is not smooth enough, we can give a weak sense to its normal trace on the boundary of the domain by using the fact that ρ is the solution of the transport equation associated to the velocity field v. The precise result is the following.

Lemma 5. *For any $\psi \in \mathcal{C}^0([0,T],(H^1(\Omega))^d)$ and any $\alpha \in L^\infty(]0,T[)$, we have*

$$\frac{1}{\xi}\int_0^T \alpha(t)\int_{\mathcal{O}_\xi} \rho(v\cdot\psi)(v\cdot\boldsymbol{\nu})\,dt\,dx \xrightarrow[\xi\to 0]{} \int_0^T \alpha(t)\int_\Gamma \gamma(\rho)(v\cdot\psi)(v\cdot\boldsymbol{\nu})\,dt\,d\omega.$$

Proof. We refer to Section 2.1 for the definition of Ω_ξ.

First of all, since $v \in L^2(]0,T[,(H^1(\Omega))^d)$, the claim follows from the results in Section 7 of [4] in the case where ψ is assumed to be smooth enough.

Then, since $v \in L^2(]0,T[,(H^1(\Omega))^d)$, $\psi \in \mathcal{C}^0([0,T],(H^1(\Omega))^d)$ and $d \le 3$, we easily see that $(v\cdot\psi)(v\cdot\boldsymbol{\nu}) \in L^1(]0,T[,W^{1,1}(\Omega))$ and that

$$\|(v\cdot\psi)(v\cdot\boldsymbol{\nu})\|_{L^1(W^{1,1})} \le C\|v\|_{L^2(H^1)}^2\|\psi\|_{\mathcal{C}^0(H^1)}.$$

We deduce that $(v\cdot\psi)(v\cdot\boldsymbol{\nu})$ lies in $\mathcal{C}^0([0,\xi_\Omega],L^1(]0,T[\times\Gamma))$ and its norm in this space is controlled by $\|v\|^2_{L^2(H^1)}\|\psi\|_{\mathcal{C}^0(H^1)}$. Hence, using (8), for any $\xi\in[0,\xi_\Omega]$ we have

$$\left|\frac{1}{\xi}\int_0^T \alpha(t)\int_{\mathcal{O}_\xi}\rho(v\cdot\psi)(v\cdot\boldsymbol{\nu})\,dt\,dx\right| \leq C\|\alpha\|_{L^\infty}\|\rho\|_{L^\infty}\|v\|^2_{L^2(H^1)}\|\psi\|_{\mathcal{C}^0(H^1)}.$$

The claim follows by density of smooth functions in $\mathcal{C}^0([0,T],(H^1(\Omega))^d)$. □

We now prove that, against a test function vanishing on Γ, the mean-value along the normal direction of the normal component of the stress tensor tends to zero when approaching the boundary. This result is very natural but important since it implies that, for any $\psi\in W^{1,1}_0(]0,T[,(H^1(\Omega))^d)$ the limit (if it exists !) when $\xi\to 0$ of the quantity

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi}\langle\sigma.\boldsymbol{\nu},\psi\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx,$$

only depends on the trace of ψ on the boundary Γ.

Lemma 6. *For any $\psi\in W^{1,1}_0(]0,T[,(H^1_0(\Omega))^d)$ we have*

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi}\langle\sigma.\boldsymbol{\nu},\psi\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx \xrightarrow[\xi\to 0]{} 0.$$

Proof. Notice first that the claim makes sense since $\sigma=2\mu(\rho)D(v)+p\,\mathrm{Id}$ belongs to $W^{-1,\infty}(]0,T[,(L^2(\Omega))^d)$.

For any $\xi\in[0,\xi_\Omega]$, we introduce the function θ_ξ defined by $\theta_\xi=1$ on Ω_ξ and $\theta_\xi=\frac{d(x,\Gamma)}{\xi}$ on $\mathcal{O}_\xi$. Hence, $\theta_\xi\in W^{1,\infty}(\Omega)$ and $\nabla\theta_\xi=0$ in Ω_ξ, $\nabla\theta_\xi=-\frac{1}{\xi}\boldsymbol{\nu}$ in $\mathcal{O}_\xi$.

Consider now $\psi\in W^{1,1}_0(]0,T[,(H^1_0(\Omega))^d)$ and, for ξ small enough, let us take $\theta_\xi\psi$ as a test function in (48). We get

$$\begin{aligned}-\int_0^T\int_\Omega\theta_\xi\rho v\cdot(\partial_t\psi+(v\cdot\nabla)\psi)\,dt\,dx+\int_0^T\frac{1}{\xi}\int_{\mathcal{O}_\xi}\rho(v\cdot\psi)(v\cdot\boldsymbol{\nu})\,dt\,dx\\ +\int_\Omega\theta_\xi\,\langle\sigma,D(\psi)\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx-\int_0^T\int_\Omega\theta_\xi\rho f\cdot\psi\,dt\,dx\\ =\frac{1}{\xi}\int_{\mathcal{O}_\xi}\langle\sigma.\boldsymbol{\nu},\psi\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx.\end{aligned}\tag{49}$$

By Lemma 5, the second term tends to zero as $\xi\to 0$ because ψ vanishes on the boundary. Since $\theta_\xi\to 1$ when $\xi\to 0$ we can perform the limit in the other terms of the left-hand side. It follows

$$\begin{aligned}\frac{1}{\xi}\int_{\mathcal{O}_\xi}\langle\sigma.\boldsymbol{\nu},\psi\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx \xrightarrow[\xi\to 0]{} -\int_0^T\int_\Omega\rho v\cdot(\partial_t\psi+(v\cdot\nabla)\psi)\,dt\,dx\\ +\int_\Omega\langle\sigma,D(\psi)\rangle_{W^{-1,\infty},W^{1,1}_0}\,dx-\int_0^T\int_\Omega\rho f\cdot\psi\,dt\,dx.\end{aligned}$$

One can finally see that this limit is zero by taking ψ as a test function in (48). □

We are now able to express the limit of the mean-value along the normal direction of the normal component of the stress tested against any smooth divergence free test function vanishing on Γ_{in}.

Lemma 7. *For any* $\psi \in W_0^{1,1}(]0,T[,V)$ *we have*

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi} \langle \sigma.\boldsymbol{\nu}, \psi\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx \xrightarrow[\xi\to 0]{} \int_0^T \langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu}, \psi\rangle_{H^{-\frac12},H^{\frac12}}\,dt - \frac{1}{2}\int_0^T\int_\Gamma \rho_{\mathrm{in}}(\widetilde{v}\cdot\psi)(v\cdot\boldsymbol{\nu})^-\,dt\,d\omega.$$

Proof. Notice first that for any $\xi > 0$ small enough, the equality (49) also holds for any $\psi \in W_0^{1,1}(]0,T[,V)$ since $\theta_\xi\psi$ is a suitable test function for (48). Furthermore, for this particular choice of ψ, we have

$$\int_\Omega \theta_\xi \langle \sigma, D(\psi)\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx = \int_0^T\int_\Omega 2\mu(\rho)\theta_\xi D(v):D(\psi)\,dt\,dx,$$

since $\operatorname{div}\psi = 0$ and then the pressure term (which is the only one which is not integrable in time) is cancelled. Using Lemma 5 we can perform the limit in the second term in (49). It follows that

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi} \langle \sigma.\boldsymbol{\nu}, \psi\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx \xrightarrow[\xi\to 0]{} -\int_0^T\int_\Omega \rho v\cdot(\partial_t\psi + (v\cdot\nabla)\psi)\,dt\,dx$$
$$+\int_0^T\int_\Gamma \gamma(\rho)(v\cdot\boldsymbol{\nu})(v\cdot\psi)\,dt\,d\omega + \int_0^T\int_\Omega 2\mu(\rho)D(v):D(\psi)\,dt\,dx - \int_0^T\int_\Omega \rho f\,dt\,dx.$$

Using now the weak formulation (26) satisfied by ρ and v, the claim follows. $\square$

The nonlinear boundary condition under consideration in this paper is a condition on the normal component of the stress tensor at the boundary. In this kind of situations, the pressure is uniquely determined (see [5]), contrarily to the case of Dirichlet boundary conditions where the pressure is only defined modulo a space independent term (this degree of freedom is often fixed by imposing a zero mean-value condition on p).

Since the trace of the test functions ψ allowed in Lemma 7 are constraint to satisfy $\int_\Gamma(\psi\cdot\boldsymbol{\nu})d\omega = 0$, we will determine uniquely the pressure by removing this constraint. To this end we will need the following result.

Lemma 8. *For any* $\Phi \in (H^1(\Omega))^d$ *there exists* $g_\Phi \in W^{-1,\infty}(]0,T[)$ *such that, for any* $\alpha \in W_0^{1,1}(]0,T[)$ *we have*

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi} \langle \sigma.\boldsymbol{\nu}, \alpha\Phi\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx \xrightarrow[\xi\to 0]{} \langle g_\Phi, \alpha\rangle_{W^{-1,\infty},W_0^{1,1}}.$$

Proof. We take $\psi = \theta_\xi\alpha(t)\Phi$ as a test function in (48). It follows

$$-\int_0^T(\partial_t\alpha)\left(\int_\Omega \theta_\xi\rho v\cdot\Phi\,dx\right)dt - \int_0^T\alpha(t)\left(\int_\Omega\theta_\xi\rho v\cdot((v\cdot\nabla)\Phi)\,dx\right)dt$$
$$+\int_0^T\alpha(t)\left(\frac{1}{\xi}\int_{\mathcal{O}_\xi}\rho(v\cdot\Phi)(v\cdot\boldsymbol{\nu})\,dx\right)dt + \int_\Omega\theta_\xi\langle\sigma,\alpha D(\Phi)\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx$$
$$-\int_0^T\alpha(t)\left(\int_\Omega\theta_\xi\rho f\cdot\Phi\,dx\right)dt = \frac{1}{\xi}\int_{\mathcal{O}_\xi}\langle\sigma.\boldsymbol{\nu},\alpha\Phi\rangle_{W^{-1,\infty},W_0^{1,1}}\,dx.$$

By Lemma 5 we can perform the limit in the third term. The other terms in the left-hand side are treated by using the Lebesgue theorem. It follows

$$\frac{1}{\xi}\int_{\mathcal{O}_\xi} \langle \sigma.\boldsymbol{\nu}, \alpha\Phi\rangle_{W^{-1,\infty},W_0^{1,1}}\, dx \xrightarrow[\xi\to 0]{} -\int_0^T (\partial_t\alpha)\left(\int_\Omega \rho v\cdot\Phi\, dx\right) dt$$
$$-\int_0^T \alpha(t)\left(\int_\Omega \rho v\cdot((v\cdot\nabla)\Phi)\, dx\right) dt + \int_0^T \alpha(t)\left(\int_\Gamma \gamma(\rho)(v\cdot\Phi)(v\cdot\boldsymbol{\nu})\, d\omega\right) dt$$
$$+\left\langle \int_\Omega \sigma : D(\Phi)\, dx, \alpha\right\rangle_{W^{-1,\infty},W_0^{1,1}} - \int_0^T \alpha(t)\left(\int_\Omega \rho f\cdot\Phi\, dx\right) dt.$$

Since $L^1(]0,T[)\subset W^{-1,\infty}(]0,T[)$, the limit obtained above can clearly be expressed as a duality bracket $\langle g_\Phi,\alpha\rangle_{W^{-1,\infty},W_0^{1,1}}$, with $g_\Phi\in W^{-1,\infty}(]0,T[)$. □

We can now prove Theorem 6 which gives existence and uniqueness of the pressure and the interpretation of the outflow boundary condition. More precisely, we prove the following result.

Theorem 7. *Let ρ and v given by Theorem 5. There exists a unique pressure field $p\in W^{-1,\infty}(]0,T[,L^2(\Omega))$ such that the total stress tensor $\sigma = 2\mu(\rho)D(v) - p\,\mathrm{Id}$ satisfies*

$$\partial_t(\rho v) + \mathrm{div}\,(\rho v\otimes v) - \mathrm{div}\,(\sigma) = \rho f,$$

in the distribution sense, and such that furthermore the outflow boundary condition is satisfied on Γ_{out} in the following sense:

For any $\varphi\in W_0^{1,1}(]0,T[,(H^{\frac12}_{\mathrm{in}}(\Gamma))^d)$, we have

$$\left\langle \frac{1}{\eta}\int_0^\eta \sigma(\xi,\cdot).\boldsymbol{\nu}\, d\xi, \varphi\right\rangle_{W^{-1,\infty}(H^{-\frac12}),W_0^{1,1}(H^{\frac12})} \xrightarrow[\eta\to 0]{}$$
$$\int_0^T \langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu},\varphi\rangle_{H^{-\frac12},H^{\frac12}}\, dt - \frac12\int_0^T\int_\Gamma \rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\varphi)\, dt\, d\omega. \quad (50)$$

Notice that, since φ vanishes on Γ_{in}, formula (50) gives a sense to the outflow boundary condition only on Γ_{out} as expected.

Proof. We know that the trace operator from $H^1(\Omega)$ onto $H^{\frac12}(\Gamma)$ admits a continuous right inverse denoted by R. Furthermore, the divergence operator admits a continuous right inverse from $L^2_0(\Omega)$ into $(H^1_0(\Omega))^d$ denoted by Π. For any $\varphi\in W_0^{1,1}(]0,T[,(H^{\frac12}_{\mathrm{in}}(\Gamma))^d)$ we define its spatial mean-value $m(\varphi\cdot\boldsymbol{\nu})(t) = \int_\Gamma \varphi\cdot\boldsymbol{\nu}\, d\omega$, for any $t\in[0,T]$.

For any φ such that $m(\varphi\cdot\boldsymbol{\nu})(t) = 0$ for any t, the function $G(\varphi)$ defined by

$$G(\varphi) = R(\varphi) - \Pi(\mathrm{div}\,(R(\varphi))),$$

lies in $W_0^{1,1}(]0,T[,V)$ and its trace on Γ is φ. Consider now the map $F(\varphi)$ defined by

$$F(\varphi)(t,\xi,\omega) = \frac{\varphi(t,\omega)}{J_\xi(\omega)},\quad \forall t\in]0,T[,\forall\xi\in[0,\xi_\Omega],\forall\omega\in\Gamma,$$

and extended in a regular way to the whole domain Ω. In this formula, we recall that $J_\xi(\omega)$ is the smooth Jacobian determinant appearing in (8). By definition, we have

$$\left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} = \frac{1}{\eta}\int_{\mathcal{O}_\eta} \langle \sigma.\boldsymbol{\nu}, F(\varphi)\rangle_{W^{-1,\infty},W_0^{1,1}}\ dx. \quad (51)$$

Furthermore, by construction the trace of $F(\varphi)$ on Γ is ψ. Hence, the difference $F(\varphi) - G(\varphi)$ lies in $W_0^{1,1}(]0,T[,(H_0^1(\Omega))^d)$ and then, by Lemma 6 we get

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\mathcal{O}_\eta} \langle \sigma.\boldsymbol{\nu}, F(\varphi) - G(\varphi)\rangle_{W^{-1,\infty},W_0^{1,1}}\ dx. \quad (52)$$

Hence, applying Lemma 7 to $\psi = G(\varphi)$, and using (51)-(52) we find

$$\begin{aligned}\lim_{\eta\to 0}&\left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \\ &= \int_0^T \langle \sigma_{\rm ref}.\boldsymbol{\nu}, \varphi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\ dt - \frac{1}{2}\int_0^T\int_\Gamma \rho_{\rm in}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\varphi)\, dt\, d\omega. \quad (53)\end{aligned}$$

Hence, we proved (53) for any φ such that $m(\varphi\cdot\boldsymbol{\nu}) = 0$ for any $t\in[0,T]$. Let us now choose $\Phi_0 \in (H^1_{\rm in}(\Omega))^d$, independent of t, such that $m(\Phi_0\cdot\boldsymbol{\nu}) = 1$. This is possible since we assume that $\Gamma_{\rm out}$ has a positive measure. For any φ like in the statement of the theorem we introduce $\widetilde{\varphi} = \varphi - m(\varphi\cdot\boldsymbol{\nu})\Phi_0$. By construction, $m(\widetilde{\varphi}\cdot\boldsymbol{\nu}) = 0$ and then (53) holds with φ replaced by $\widetilde{\varphi}$. Let us now write by using formula (51)

$$\begin{aligned}&\left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \\ &\qquad = \left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, \widetilde{\varphi} \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \\ &\qquad\quad + \left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, m(\varphi\cdot\boldsymbol{\nu})\Phi_0 \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \\ &\qquad = \left\langle \frac{1}{\eta}\int_0^\eta \sigma.\boldsymbol{\nu}\, d\xi, \widetilde{\varphi} \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \\ &\qquad\quad + \frac{1}{\eta}\int_{\mathcal{O}_\eta} \langle \sigma.\boldsymbol{\nu}, m(\varphi\cdot\boldsymbol{\nu})F(\Phi_0)\rangle_{W^{-1,\infty},W_0^{1,1}}\ dx. \quad (54)\end{aligned}$$

The first term in the right-hand side converges when $\eta\to 0$ as we have seen in (53). By Lemma 8, the second one converges towards $\langle g_{F(\Phi_0)}, m(\varphi\cdot\boldsymbol{\nu})\rangle_{W^{-1,\infty},W_0^{1,1}}$ since $m(\varphi\cdot\boldsymbol{\nu}) \in W_0^{1,1}(]0,T[)$. We remark that $g_{F(\Phi_0)}$ depends only on the time variable so that we can also write this term as follows

$$\langle g_{F(\Phi_0)}, m(\varphi\cdot\boldsymbol{\nu})\rangle_{W^{-1,\infty},W_0^{1,1}} = \left\langle \frac{1}{\eta}\int_0^\eta g_{F(\Phi_0)}\,{\rm Id}.\boldsymbol{\nu}\, d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})}. \quad (55)$$

It is then natural to introduce the new stress tensor $\widetilde{\sigma} = \sigma - g_{F(\Phi_0)}\,{\rm Id}$, which amounts to add a term depending only on the time variable to the pressure. Of course, (48) is also satisfied when we replace σ by $\widetilde{\sigma}$. Furthermore, from (53), (54) and (55) it follows that

$$\left\langle \frac{1}{\eta}\int_0^\eta \widetilde{\sigma}.\boldsymbol{\nu}\,d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} \xrightarrow[\xi\to 0]{} \int_0^T \langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu}, \widetilde{\varphi}\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\,dt - \frac{1}{2}\int_0^T\int_\Gamma \rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\widetilde{\varphi})\,dt\,d\omega.$$

By definition, we have $\widetilde{\varphi} = \varphi - m(\varphi\cdot\boldsymbol{\nu})\Phi_0$ so that we get

$$\begin{aligned}\left\langle \frac{1}{\eta}\int_0^\eta \widetilde{\sigma}.\boldsymbol{\nu}\,d\xi, \varphi \right\rangle_{W^{-1,\infty}(H^{-\frac{1}{2}}),W_0^{1,1}(H^{\frac{1}{2}})} &\xrightarrow[\eta\to 0]{} \int_0^T \langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu}, \varphi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\,dt - \frac{1}{2}\int_0^T\int_\Gamma \rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\varphi)\,dt\,d\omega \\ &\quad - \int_0^T m(\varphi\cdot\boldsymbol{\nu})\,\langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu}, \Phi_0\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\,dt \\ &\quad + \frac{1}{2}\int_0^T m(\varphi\cdot\boldsymbol{\nu})\int_\Gamma \rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\Phi_0)\,dt\,d\omega.\end{aligned}$$

If we add to the pressure the quantity

$$\pi(t) = -\langle \sigma_{\mathrm{ref}}.\boldsymbol{\nu}, \Phi_0\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} + \frac{1}{2}\int_\Gamma \rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(\widetilde{v}\cdot\Phi_0)\,d\omega \in W^{-1,\infty}(]0,T[),$$

that is letting

$$\widetilde{\widetilde{\sigma}} = \widetilde{\sigma} - \pi(t)\,\mathrm{Id} = \sigma - (g_{F(\Phi_0)}(t) + \pi(t))\,\mathrm{Id} = 2\mu(\rho)D(v) - (p + \pi(t) + g_{F(\Phi_0)}(t))\,\mathrm{Id},$$

and replacing the pressure p by $p + \pi(t) + g_{F(\Phi_0)}(t)$, the claim is proved. □

8. A possible variant for the outflow boundary condition

In this section we propose to prove that similar results than the one of Theorems 5 and 7 still hold in the case where the initial density ρ_0 and the inflow data ρ_{in} are only supposed to be non-negative almost everywhere provided that we consider a slightly modified outflow boundary condition. Hence, in this section we do not assume that (3) and (4) hold.

We propose to consider here the following outflow boundary condition

$$\sigma.\boldsymbol{\nu} = \sigma_{\mathrm{ref}}.\boldsymbol{\nu} - \theta\rho_{\mathrm{in}}(v\cdot\boldsymbol{\nu})^-(v - v_{\mathrm{ref}}), \tag{56}$$

instead of the one in (2), where θ is any real number such that $\theta > \frac{1}{2}$. This choice corresponds, roughly speaking, to add a small dissipation at the outflow boundary for our problem.

Theorem 8. *We only assume that $\rho_0 > 0$ and $\rho_{\mathrm{in}} > 0$ almost everywhere instead of* (3) *and* (4).

Then, the results of Theorems 5 and 7 hold if we replace the last boundary condition in (2) *by* (56) *with $\theta > \frac{1}{2}$ (and replacing the coefficient $\frac{1}{2}$ by θ in* (50)*).*

Proof. The main lines of the proof of Theorem 5 are the same, we only give the details of the points that need a particular attention.

We introduce in the same way an approximate problem for which existence of a solution is proved with the same fixed point technique. It is now easy to see that the energy estimate (35) now becomes

$$\begin{aligned}&\|\sqrt{\rho_k}v_k\|^2_{L^\infty(]0,T[,L^2)} + \|v_k\|^2_{L^2(]0,T[,H^1)}\\ &+\int_0^T\int_\Gamma \rho_{\mathrm{out},k}|v_k|^2(v_k\cdot\boldsymbol{\nu})^+\,dt\,d\omega + \left(\theta-\frac{1}{2}\right)\int_0^T\int_\Gamma \rho_{\mathrm{in},k}|v_k|^2(v_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega \le C_0,\end{aligned}$$

which imply a bound, uniform with respect to k

$$\int_0^T\int_\Gamma \gamma(\rho_k)|v_k|^2|v_k\cdot\boldsymbol{\nu}|\,dt\,d\omega \le C_1,$$

from which we deduce an $L^3(]0,T[\times\Gamma)$ estimate which reads

$$\int_0^T\int_\Gamma (\gamma(\rho_k)|v_k\cdot\boldsymbol{\nu}|)^3\,dt\,d\omega \le C_1. \tag{57}$$

Since we do not assume that (3) and (4) hold, we clearly see that estimates (36)-(38) are not valid anymore. Nevertheless we are going to show that Lemmas 3 and 4 are still valid. Indeed, in the proof of Lemma 3 the only thing which change is the estimate of the boundary term since we do not have (38). Let us now bound these boundary terms by using (57):

$$\begin{aligned}&\left|\int_s^{s+h}\int_\Gamma \gamma(\rho_k)(v_k\cdot\boldsymbol{\nu})(\widetilde{v}_k\cdot\Psi(s))\,dt\,d\omega\right|\\ &\qquad\le\left|\int_s^{s+h}\int_\Gamma \gamma(\rho_k)(v_k\cdot\boldsymbol{\nu})(\widetilde{v}_k\cdot\Psi(s))\,dt\,d\omega\right|\\ &\qquad\le Ch^{\frac16}\|\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu})\|_{L^3(]0,T[\times\Gamma)}\|\widetilde{v}_k\|_{L^2(]0,T[,H^1)}\|\Psi(s)\|_{H^1}\\ &\qquad\le C'h^{\frac16}\|\Psi\|_{H^1}.\end{aligned}$$

This estimate let us conclude the proof of Lemma 3 as before.

Let us now turn to Lemma 4. Here again most of the proof still holds. We just have to provide an alternative argument for the last point. More precisely we are going to show, without using (36), that for any $\varepsilon>0$, there exists $\eta>0$ and $k_0\ge 0$, such that

$$|\{\tau_h\rho_k<\eta\}|\le\varepsilon,\quad\forall k\ge k_0,\quad\forall h>0. \tag{58}$$

To this end, let us show

$$|\{x\in\Omega,\rho_k(t,x)<\eta\}|\le\varepsilon,\quad\forall k\ge k_0,\quad\forall t\in[0,T], \tag{59}$$

which imply (58) by integration with respect to the time variable.

Let $\beta\in\mathcal{C}^1(\mathbb{R})$ be a non-negative non-increasing function such that $\beta(s)=1$ for any $s\le 1$ and $\beta(s)=0$ for any $s\ge 2$. For any $\eta>0$, we let $\beta_\eta(s)=\beta\left(\frac{s}{\eta}\right)$. We now use the renormalization property for (28) (with $w_k=v_k$) applied to the function β_η and we finally take $\varphi=1$ as a test function in the equation satisfied by

$\beta_\eta(\rho_k)$. It follows that for any $t \in [0,T]$ we have

$$\begin{aligned}\int_\Omega \beta_\eta(\rho_k(t))\,dx &\leq \int_\Omega \beta_\eta(\rho_{0,k})\,dx + \int_0^T\int_\Gamma \beta_\eta(\rho_{\text{in},k})(v_k\cdot\boldsymbol{\nu})^-\,dt\,d\omega \\ &\leq \int_\Omega \beta_\eta(\rho_0)\,dx + \|v_k\|_{L^2(]0,T[\times\Gamma)}\left(\int_0^T\int_\Gamma \beta_\eta^2(\rho_{\text{in},k})\,dt\,d\omega\right)^{\frac12} \\ &\leq \int_\Omega \beta_\eta(\rho_0)\,dx + C\left(\int_0^T\int_\Gamma \beta_\eta^2(\rho_{\text{in}})\,dt\,d\omega\right)^{\frac12} \\ &\quad + C\left(\int_0^T\int_\Gamma |\beta_\eta^2(\rho_{\text{in}})-\beta_\eta^2(\rho_{\text{in},k})|\,dt\,d\omega\right)^{\frac12}\end{aligned}$$

Since $\beta_\eta(s)=0$ as soon as $s\geq 2\eta$ the first two terms above are bounded respectively by $|\{\rho_0<2\eta\}|$ and by $C|\{\rho_{\text{in}}<2\eta\}|^{\frac12}$. The last term, called S, can be controlled as follows

$$\begin{aligned}S &\leq C\|\beta_\eta(\rho_{\text{in}})-\beta_\eta(\rho_{\text{in},k})\|^{\frac12}_{L^1(]0,T[\times\Gamma)} \\ &\leq C\frac{1}{\eta^{\frac12}}\|\rho_{\text{in}}-\rho_{\text{in},k}\|^{\frac12}_{L^1(]0,T[\times\Gamma)}.\end{aligned}$$

Since, by construction, $\beta_\eta(s)=1$ any for $s<\eta$, we finally proved for any $t\in[0,T]$, and any $k\geq 0$

$$\big|\{\rho_k(t)<\eta\}\big| \leq \big|\{\rho_0<2\eta\}\big| + C\big|\{\rho_{\text{in}}<2\eta\}\big|^{\frac12} + C\frac{1}{\eta^{\frac12}}\|\rho_{\text{in}}-\rho_{\text{in},k}\|^{\frac12}_{L^1(]0,T[\times\Gamma)}.$$

Since ρ_0 and ρ_{in} are positive almost everywhere, there exists $\eta>0$ small enough such that each of the first two terms are smaller than ε. This η being fixed, we can now find k_0 such that the last term is smaller than ε as soon as $k\geq k_0$. This proves (59) and then (58).

Finally, following the proof of Lemma 4, we proved that, for any $\gamma<2$, there exists $\delta>0$ such that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ and k_ε such that

$$\|\tau_h\widetilde{v}_k-\widetilde{v}_k\|_{L^\gamma(]0,T-h[\times\Omega)} \leq \varepsilon + C_\varepsilon h^\delta,\quad \forall k\geq k_\varepsilon,\ \forall h>0.$$

This translation estimate is sufficient to apply the compactness results in [20] and we deduce in that case the same compactness properties than in Section 6.4.

Finally, it remains to perform the limit in the approximate problem. As far as the transport equation is concerned the proof is the same than in Section 6.4 by using the weak-$\star$ convergence of ρ_k and $\rho_{\text{out},k}$ in L^∞ and the strong convergence of v_k obtained above. In the same way we can prove the strong convergence of ρ_k towards ρ in all the spaces $L^q(]0,T[\times\Omega)$ with $q<+\infty$.

Remark now that $(\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu}))_k$ weakly converges towards $\gamma(\rho)(v\cdot\boldsymbol{\nu})$ in the space $L^\gamma(]0,T[,L^2(\Gamma))$ for any $\gamma<2$. But, from (57) we know that $(\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu}))_k$ is bounded in $L^3(]0,T[\times\Gamma)$. Hence, up to a subsequence, we deduce that $(\gamma(\rho_k)(v_k\cdot\boldsymbol{\nu}))_k$ weakly converges in $L^3(]0,T[\times\Gamma)$ towards $\gamma(\rho)(v\cdot\boldsymbol{\nu})$. Since the trace of v_k strongly converges in $L^\gamma(]0,T[,(L^3(\Gamma))^d)$ we can now perform the limit in the boundary terms in (31) (with $w_k=v_k$, $\psi(T)=0$ and θ instead of $\frac12$).

Finally, existence and uniqueness of the pressure satisfying (48) and the weak convergence (50) with θ instead of $\frac{1}{2}$ are proved in exactly the same way than in Theorem 7. □

9. Conclusions

In this paper, we perform the analysis of the non-homogeneous incompressible Navier-Stokes equations with non standard nonlinear outflow boundary conditions. In the homogeneous case, this boundary condition has already been shown to lead to a well-posed problem and to be efficient, provided a suitable reference flow can be chosen.

In the present work we establish an existence result of weak solutions for this model in the non-homogeneous framework. Particular attention is paid to the interpretation of the outflow boundary condition for weak solutions. This point is not obvious and relies upon the analysis of the properties of the traces of weak solutions of the Cauchy/Dirichlet problem for the transport equation.

Many questions remain open like the problems of regularity and uniqueness of solutions. It would also be interesting to have a better understanding of how the solution depend on the reference data v_{ref} and σ_{ref} and how to choose those data depending on the physics of the flow we are interested in.

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Franck Boyer
Université de Provence,
Laboratoire d'Analyse Topologie et Probabilités, UMR 6632,
39 rue Joliot-Curie, 13453 Marseille Cedex 13 FRANCE
E-mail address: fboyer@cmi.univ-mrs.fr

Pierre Fabrie
Université Bordeaux 1,
Mathématiques Appliquées de Bordeaux, UMR 5466,
351 cours de la Libération, 33405 Talence Cedex FRANCE
E-mail address: fabrie@math.u-bordeaux1.fr