1 About Theorem VI.1.6 and Lemma VI.1.7

The analysis of the parabolic approximation introduced page 424 during the proof of Theorem VI.1.6 (in particular Lemma VI.1.7) is not precise enough (and actually, what is written is not completely correct).

On top of page 425, it should be precised that the solution $\tilde{\rho}_\varepsilon$ we are looking for need to belong to $L^\infty(]0,T[\times\Omega)\cap L^2(]0,T[,H^1(\Omega))$. If not, all the integrals in the weak formulation (VI.23) may not be defined due to the lack of integrability in space of c and v.

Anyway, let us show how to make all this work.

- First of all, we observe that the proof which is given in the book is correct as soon as v and c are smooth enough.
- If v and c simply possess the regularity which is assumed here, that is (VI.4)-(VI.6), we must use a regularisation argument.
  Let us consdider a sequence $(v_n)_n$ of smooth vector fields which converges to v in $L^1(]0,T[,(W^{1,1}(\Omega))^d)$ and a sequence of smooth scalar fields $(\psi_n)_n$ which converges in $L^1(]0,T[\times\Omega)$ towards $c+\operatorname{div} v$.
  Notice that this can be done (using the smoothing operators defined in Chapter III) in such a way that $\psi_n^- \in L^1(]0,T[,L^\infty(\Omega))$ and

$$\|\psi_n^-\|_{L^1(L^\infty)} \leq C\|(c+\operatorname{div} v)^-\|_{L^1(L^\infty)}, \quad \forall n \geq 0.$$

  We set $c_n = \psi_n - \operatorname{div} v_n \in L^1(]0,T[\times\Omega)$ and we consider now, $\varepsilon > 0$ being fixed, the parabolic problem

$$\begin{aligned}
&-\int_0^T\int_\Omega \tilde{\rho}_{\varepsilon,n}\left(\frac{\partial\varphi}{\partial t} + v_n\cdot\nabla\varphi - c_n\varphi\right)dx\,dt + \varepsilon\int_0^T\int_\Omega \nabla\tilde{\rho}_{\varepsilon,n}\cdot\nabla\varphi\,dx\,dt\\
&+\int_0^T\int_\Gamma (\tilde{\rho}_{\varepsilon,n}(v_n\cdot\nu)^+\varphi - \rho^{in}(v_n\cdot\nu)^-\varphi)\,d\sigma\,dt\\
&+\int_\Omega \tilde{\rho}_{\varepsilon_n}(T)\varphi(T)\,dx - \int_\Omega \rho_0\varphi(0)\,dx = 0.
\end{aligned} \tag{I.1}$$

  For any n, this problem has a unique solution (since v_n and c_n are smooth) which satisfies all the properties given in the book.
  Let us check that the main estimates hold uniformly in n.
  - The first important point is that the L^∞ bound on $\tilde{\rho}_{\varepsilon,n}$ does not depend on n. Indeed, with Lemma VI.1.7, the estimate reads

$$\begin{aligned}
\|\tilde{\rho}_{\varepsilon,n}\|_{L^\infty(]0,T[\times\Omega)} &\leq M\exp\left(\|(c_n+\operatorname{div} v_n)^-\|_{L^1(L^\infty)}\right)\\
&= M\exp\left(\|\psi_n^-\|_{L^1(L^\infty)}\right)\\
&\leq M\exp\left(C\|(c+\operatorname{div} v)^-\|_{L^1(L^\infty)}\right).
\end{aligned}$$

- Second, we have to prove estimates (VI.24) and (VI.25) uniformly in n. It suffices to write (VI.26) with v_n and c_n in place of v and c and to change a little bit the final argument. The term that we need to take care of is

$$I_{n,\varepsilon} = \int_0^T \int_\Omega (2c_n + \operatorname{div} v_n)(\tilde{\rho}_{\varepsilon,n})^2 \, dx \, dt.$$

  Since we do not know, due to the regularisation process, if $(\operatorname{div} v_n)^+$ is uniformly bounded in $L^1(]0,T[, L^\infty(\Omega))$, we cannot use the Gronwall lemma. Instead, we simply use the L^∞ bound $\|\tilde{\rho}_{\varepsilon,n}\|_{L^\infty} \leq C_1$ to get

$$|I_{n,\varepsilon}| \leq C_1^2(2\|c_n\|_{L^1} + \|\operatorname{div} v_n\|_{L^1}) \leq C(\|c\|_{L^1} + \|\operatorname{div} v\|_{L^1}).$$

Considering a suitable weakly converging subsequence of $(\tilde{\rho}_{\varepsilon,n})_n$, we can pass to the limit with respect to n in (I.1) and in the estimates (VI.24) and (VI.25).
Note that for any $\alpha > 0$, $(\tilde{\rho}_{\varepsilon,n})_n$ is compact in $L^2(]0,T[, H^{1-\alpha}(\Omega))$ and therefore the traces (which belong to L^∞) strongly converge in any $L^p(]0,T[\times\partial\Omega)$, $p < +\infty$.