Selected topics in statistics
Spatial Statistics
Homework 2

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Exercise 1

Let A, and B be two random variables with means m_A and m_B, variances v_A and v_B and covariance $c_{A,B}$. Consider the stochastic process Y, defined on $\mathcal{D} = \mathbb{R}$ by $Y(t) = At + B$. Calculate the mean and covariance functions of Y. What are the conditions on $m_A, m_B, v_A, v_B, c_{A,B}$ for the covariance function of Y to be stationary?

Exercise 2

Let B_1 and B_2 be two Brownian motions on $\mathbb{R}^+$. Assume that for any t_1, t_2, $B(t_1)$ and $B(t_2)$ are independent. Let Y be the stochastic process on $\mathcal{D} = [0,1]$ defined $Y(t) = B_1(t) + B_2(1-t)$. Show that Y is stationary (hint: a multidimensional Gaussian distribution is characterized by its mean vector and its covariance matrix).

Exercise 3

Let U be a uniform random variable on $[0,1]$. Let Y be the stochastic process on $\mathcal{D} = [0,1]$ defined by $Y(t) = \mathbf{1}_{U \leq t}$. Calculate the mean and covariance functions of Y. Is Y mean square continuous on $[0,1]$? Is Y almost-surely continuous on $[0,1]$?

Exercise 4

Let $\mathcal{D} = l^1$, be the space of all the real sequences $(s_i)_{i \in \mathbb{N}^*}$, indexed by $\mathbb{N}^*$, verifying $\sum_{i=1}^{+\infty} |s_i| < +\infty$. Notice that, as a consequence, $\sum_{i=1}^{+\infty} s_i^2 < +\infty$. Let $(U_i)_{i \in \mathbb{N}^*}$ be a sequence of *iid* random variables with uniform distribution on $[0,1]$.

1) Show that there exists a measurable subset $\tilde{\Omega}$ of the probability space Ω of the $(U_i)_{i \in \mathbb{N}^*}$, with $P(\tilde{\Omega}) = 1$, so that for any $\omega \in \tilde{\Omega}$: for any $s \in \mathcal{D}$, the formula $Y(\omega, s) = \sum_{i=1}^{+\infty} s_i U_i(\omega)$ is valid. Then (do not show it), $Y(\omega, s)$ can be extended to a real random variable defined on Ω and hence, we have just defined a stochastic process Y on $\mathcal{D}$.

2) Show that, for any $s \in \mathcal{D}$, the sequence of random variables $Y_n(s) = \sum_{i=1}^{n} s_i U_i$ is a Cauchy sequence with the norm $||X - X'||_2^2 = \mathbb{E}((X - X')^2)$. We admit that, as a consequence, there exists a random variable $\tilde{Y}(s)$ so that $\mathbb{E}((Y_n(s) - \tilde{Y}(s))^2) \to_{n \to +\infty} 0$ and $\mathbb{E}(\tilde{Y}^2(s)) < +\infty$ (notice that as a consequence, $\mathbb{E}(|\tilde{Y}(s)|) < +\infty$).

3) Show that, for any $s \in \mathcal{D}$, $Y(s) = \tilde{Y}(s)$ almost surely.

4) Show that $\mathbb{E}(Y(s)) = \lim_{n \to +\infty} \mathbb{E}(Y_n(s))$ and $cov(Y(s), Y(s')) = \lim_{n \to +\infty} cov(Y_n(s), Y_n(s'))$.

5) Show that the mean function of Y is $m(s) = \frac{1}{2} \sum_{i=1}^{+\infty} s_i$ and that the covariance function of Y is $K(s, s') = \frac{1}{12} \sum_{i=1}^{+\infty} s_i s_i'$.