Thèse

Présentée pour obtenir le titre de

Docteur en Sciences

de l'Université Côte d'Azur

Spécialité Mathématiques

Effets dispersifs et asymptotique en temps long d'équations d'ondes dans des domaines extérieurs

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Soutenue le 25 septembre 2018

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Résumé : L'objet de cette thèse est l'étude des équations de Schrödinger et des ondes, à la fois linéaires et non linéaires, dans des domaines extérieurs.

Nous nous intéressons en particulier aux inégalités dites de Strichartz, qui sont une famille d'estimations dispersives mesurant la décroissance du flot linéaire, particulièrement utiles à l'étude des problèmes non linéaires correspondants. Dans des géométries dites non-captantes, c'est à dire où tous les rayons de l'optique géométrique partent à l'infini, de nombreux résultats montrent que de telles estimations sont aussi bonnes que dans l'espace libre. D'autre part, la présence d'une trajectoire captive induit nécessairement une perte au niveau d'une autre famille d'estimations à priori, les estimations d'effet régularisant et de décroissance locale de l'énergie, respectivement pour Schrödinger et pour les ondes. En contraste de quoi, nous montrons des estimations de Strichartz sans perte dans une géométrie captante instable : l'extérieur de plusieurs obstacles strictement convexes vérifiant la condition d'Ikawa.

La seconde partie de cette thèse est dédiée à l'étude du comportement en temps long des équations non-linéaires sous jacentes. Lorsque le domaine dans lequel elles vivent n'induit pas trop de concentration de l'énergie, on s'attend à ce qu'elles diffusent, c'est à dire se comportent de manière linéaire asymptotiquement en temps. Nous montrons un tel résultat pour les ondes non linéaires critiques à l'extérieur d'une classe d'obstacles généralisant la notion d'étoilé. A l'extérieur de deux obstacles strictement convexes, nous obtenons un résultat de rigidité concernant les solutions à flot compact, premier pas vers un résultat général. Enfin, nous nous intéressons à l'équation de Schrödinger non linéaire, dans l'espace libre, mais avec un potentiel. Nous montrons que les solutions diffusent si l'on prend un potentiel répulsif, ainsi qu'une somme de deux potentiels répulsifs ayant des surfaces de niveau convexes, ce qui fournit un exemple de diffusion dans une géométrie captante analogue à l'extérieur de deux convexes stricts.

Mots clés : Equations des ondes et de Schrödinger, estimations de Strichartz, ensemble capté, équations non linéaires, diffusion.

Dispersive effects and long-time asymptotics for wave equations in exterior domains

Abstract: We are concerned with Schrödinger and wave equations, both linear and non linear, in exterior domains.

In particular, we are interested in the so-called Strichartz estimates, which are a family of dispersive estimates measuring decay for the linear flow. They turn out to be particularly useful in order to study the corresponding non linear equations. In non-captive geometries, where all the rays of geometrical optics go to infinity, many results show that Strichartz estimates hold with no loss with respect to the flat case. Moreover, the local smoothing estimates for the Schrödinger equation, respectively the local energy decay for the wave equation, which are another family of dispersive estimates, are known to fail in any captive geometry. In contrast, we show Strichartz estimates without loss in an unstable captive geometry: the exterior of many strictly convex obstacles verifying Ikawa's condition.

The second part of this thesis is dedicated to the study of the long time asymptotics of the corresponding non linear equations. We expect that they behave linearly in large times, or scatter, when the domain they live in does not induce too much concentration effect. We show such a result for the non linear critical wave equation in the exterior of a class of obstacles generalizing star-shaped bodies. In the exterior of two strictly convex obstacles, we obtain a rigidity result concerning compact flow solutions, which is a first step toward a general result. Finally, we consider the non linear Schrödinger equation in the free space but with a potential. We prove that solutions scatter for a repulsive potential, and for a sum of two repulsive potentials with strictly convex level surfaces. This provides a scattering result in a framework similar to the exterior of two strictly convex obstacles.

Keywords: Schrödinger and wave equations, Strichartz estimates, trapped set, non linear equations, scattering.

Remerciements

Je tiens bien sûr à remercier en premier lieu mon directeur de thèse, Fabrice Planchon. Je lui suis profondément reconnaissant pour m'avoir fait découvrir un domaine de recherche passionnant, pour sa constante disponibilité, ses conseils toujours avisés, et la grande confiance qu'il m'a témoignée. Sa vision mathématique riche et pleine d'intuition m'a beaucoup apportée. Au-delà de son encadrement scientifique, ses qualités humaines m'ont touchées, et ce fut une grande chance et un grand plaisir, sous tous ces aspects, que d'effectuer une thèse sous sa direction.

Un grand merci à Nicolas Burq et Jared Wunsch pour m'avoir fait l'honneur d'être mes rapporteurs de thèse. Je suis particulièrement reconnaissant à Nicolas Burq pour avoir inspiré en partie le sujet de cette thèse, et pour les quelques discussions mathématiques que j'ai eues avec lui durant ces trois années, qui ont toujours été à la fois enthousiasmantes et d'un grand secours. Je le remercie également pour faire partie du jury, et Jared Wunsch, en plus de l'attention et l'interêt avec lesquels il a lu ce manuscrit et les commentaires enrichissants qui en ont découlés, pour son accueil à Northwestern durant les deux semaines qui ont précédées ma soutenance. Echanger avec lui y fut un grand plaisir.

Je suis également très honoré que Jean-Marc Delort, Thomas Duyckaerts, Isabelle Gallagher, Oana Ivanovici, et Pierre Raphaël aient accepté de faire partie du jury. Je suis touché de l'intérêt qu'ils portent à mon travail de recherche et je les en remercie chaleureusement.

Parmi eux, j'aimerais exprimer ma gratitude à Pierre Raphaël pour avoir en plus encadré mon premier stage de recherche, alors que j'étais un jeune étudiant de M1. Son enthousiasme et les sujets qu'il m'a alors fait entrevoir ont comptés dans le choix de mon domaine de recherche. Merci à Thomas Duyckaerts pour son invitation à Paris 13 et pour les discussions que nous y avons eues et que j'espère poursuivre. Un grand merci également à Oana Ivanovici pour l'intérêt qu'elle a porté à mon travail durant ces quelques années, sa gentillesse et ses encouragements.

Nicola Visciglia m'a accueilli quelques mois à Pise au début de ma thèse, où il a participé à l'encadrement de mes tous premiers travaux. Ce séjour a été important pour moi et je lui en suis très reconnaissant. Merci également à Rémi Carles, avec qui il a été un plaisir de discuter, pour son invitation à Montpellier, et à Camille Laurent pour une semaine très enrichissante au laboratoire Jacques-Louis Lions et un début de collaboration enthousiasmant.

C'est au cours de cette thèse que j'ai donné mes premiers cours. Erwann Aubry, Joachim Yameogo et Mohamed Elkadi ont encadré ces premières expériences d'enseignement, ce fut un réel plaisir de donner des TD pour leurs cours et je voudrais leur exprimer ma gratitude.

Je souhaiterais remercier également ici tous les professeurs que j'ai eu et qui m'ont donné envie de faire des mathématiques. Je pense en particulier à mon professeur de classes préparatoires Michel Alessandri, à la passion si communicative, à Laurent Desvillettes pour un cours d'analyse fonctionnelle qui m'est toujours utile, à Thomas Alazard pour un premier contact passionnant avec les EDP, et à Patrick Gérard pour un cours de M2 lumineux.

Parmi les chercheurs que je n'ai pas encore eus la chance de citer, j'aimerais remercier Gilles Lebeau pour avoir fait partie de mon comité de suivi de thèse, Maxime Ingremeau avec qui il fut un plaisir d'échanger, Anne-Sophie de Suzzoni, pour ses conseils et pour m'avoir secouru alors que sur le point de terminer cet écrit, mon ordinateur a décidé de rendre l'âme, et Euan Spence pour la confiance qu'il me témoigne pour la suite.

Ce travail n'aurait pu voir le jour sans le personnel du laboratoire. Je le remercie profondément. Une pensée toute particulière pour Jean-Marc Lacroix, Roland Ruelle, Angélique Guitard et Chiara Soresi, grâce auxquels ma thèse s'est déroulée dans de si bonnes conditions, et dont la patience et la gentillesse est à toute épreuve. A l'école doctorale, j'ai été très heureux de toujours pouvoir compter sur Catherine Briet et je l'en remercie.

Ce fut un très grand plaisir de partager mon bureau avec Julie et Reine. Un énorme merci à elles pour avoir rendu ce lieu si agréable, et pour tout ce que nous y avons partagé. Plus généralement, j'aimerais remercier tous les doctorants et post-doctorants que j'ai croisés et avec qui j'ai eu le plaisir d'échanger. En particulier, merci à Bjorn pour la compagnie sportive et des discussions passionnantes, à Charles pour des échanges mathématiques motivants, à Arthur pour la vie nocturne, et à Amine pour m'avoir le premier accueilli au laboratoire. Je pense également, au laboratoire, à Jean-Baptiste, Jiqiang, Chenmin, Lucrezia, Stefania, Alexis, Medhi, Laurence, Léo, Giulia, Rinel, Eduard, Liana, et en dehors à Annalaura, Baptiste, Victor, Olivier, Clémentine, et bien d'autres.

Un grand merci à mes camarades cachanais : Augustin, Bathilde, Zoé, Boris, Romain, Christophe, Sarah, Léo, Julie, ainsi qu'à Alessio pour son acceuil à Pise, à Laurent pour des échanges inestimables, et à Constantin, dont l'amitié m'est très précieuse. Leur présence a beaucoup compté lors de ma scolarité et au-delà.

Mon séjour à Nice n'aurait pas été le même sans mes amis comédiens : Emmanuelle, Benjamin, Benoît, Roberta, Brigitte, Claude, Ghyslaine, Maxime, Pierre, Priscillia, Marc, Marine, et tous les autres, qui m'ont permis de m'évader un peu des mathématiques et m'ont tant apporté. Qu'il me soit ici permis de les remercier.

Je ne peux achever ce manuscrit sans penser à Lucie, Nora, Eugène, Chantal, Coco, avec qui j'aurais tant aimé partager ce moment. Un grand merci à Michaël et Haroumi, dont la présence à la soutenance me touche beaucoup. Enfin, je ne pourrais jamais assez remercier mes parents, qui m'ont toujours fait confiance et soutenu dans tous mes projets.

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Part I.

Introduction (Français)

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1 Estimations de Strichartz à l'extérieur de plusieurs obstacles convexes

Soit $\Omega \subset \mathbb{R}^d$ un domaine à bord. Considérons l'équation de Schrödinger avec conditions au bord de Dirichlet

$$\begin{cases} i\partial_t u - \Delta u = 0, \ u = 0 \text{ sur } \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{LS}$$

ainsi que l'équation des ondes

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \ u = 0 \text{ sur } \partial\Omega, \\ (u, \partial_t u)(0) = (u_0, u_1). \end{cases} \tag{LW}$$

Afin de comprendre la théorie perturbative, ainsi que les équations non linéaires associées à de telles équations, il est crucial de comprendre comment les solutions des équations *linéaires* décroissent. En particulier, la mesure des normes de Lebesgue espace-temps $L^p_t L^q_x$, ou estimations de Strichartz, s'avère particulièrement pertinente. L'objet de la première partie de cette thèse est l'obtention de telles estimations, aussi bonnes que dans l'espace libre, dans une géométrie captante instable : l'extérieur de plusieurs obstacles strictement convexes.

1.1. Estimations de Strichartz dans l'espace libre

Dans le cas de l'espace libre $\Omega = \mathbb{R}^d$ pour l'équation de Schrödinger, prendre la transformée de Fourier en espace dans (LS) et intégrer en temps fournit la représentation explicite

$$u(x,t) = \frac{1}{4i\pi|t|^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy,$$

d'où on peut tirer l'estimation de dispersion ponctuelle

$$\|e^{it\Delta}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{t^{d/2}},$$

qui est une première mesure de la décroissance du flot linéaire. Elle permet en particulier, combinée à des arguments de dualité, d'obtenir des estimations dites de Strichartz :

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2},$$

où le couple (q,r), doit respecter la condition d'admissibilité donnée par l'invariance d'échelle de l'équation, soit

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2},\ (q,r,d)\neq(2,\infty,2).$$

Localement en temps, une telle estimation reflète un gain d'intégrabilité : si la donnée est L^2, la solution est dans L^r, $r>2$ pour presque tout temps. Globalement, elle décrit un effet de décroissance : la norme L^r de la solution doit tendre vers zéro, au moins dans un sens en moyenne L^q en temps.

Pour l'équation des ondes dans $\mathbb{R}^d$, les estimations de Strichartz peuvent s'écrirent

$$\|u\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \lesssim \|u_0\|_{\dot H^s}+\|u_1\|_{\dot H^{s-1}}$$

où $2\leq q\leq\infty$ et $2\leq r<\infty$ respectent l'invariance d'échelle

$$\frac{1}{q}+\frac{d}{r}=\frac{d}{2}-s,$$

et la condition d'admissibilité pour les ondes

$$\frac{1}{q}+\frac{d-1}{2r}\leq\frac{d-1}{4}.$$

Elles peuvent être obtenues par sommation via la théorie de Littlewood-Paley d'estimations localisées en fréquence, elles mêmes par exemple conséquences de l'estimation de dispersion pour les ondes

$$\|e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0\|_{L^\infty} \lesssim h^{-\frac{d+1}{2}}\frac{1}{t^{\frac{d-1}{2}}}\|\Psi(h\sqrt{-\Delta})u_0\|_{L^1},$$

$\Psi(h\sqrt{-\Delta})$ localisant en fréquences $\sim h^{-1}$.

Pour Schrödinger comme pour les ondes, de telles estimations reflètent la nature dispersive de l'équation : chaque fréquence se propage selon des directions différentes, et sont moins bonnes pour les ondes, pour laquelle la dispersion est plus faible, chaque fréquence se déplacant à la même vitesse. En plus de leur interêt intrinsèque, elles fournissent un outils très robuste pour l'étude des équations non-linéaires correspondantes, en donnant, par exemple, de bons espaces dans lesquels travailler pour obtenir des résultats d'existence locale via des méthodes de point fixe, ainsi qu'un cadre utile à l'étude de l'asymptotique en temps long de telles équations.

Leur longue histoire remonte aux travaux fondateurs de Strichartz [Str77] pour le cas $p=q$ pour les ondes dans $\mathbb{R}^d$, étendu à tous les exposants $q\neq2$ pour Schrödinger par

Ginibre et Velo [GV85b], et à tous les exposants $r \neq \infty$ pour les ondes par les mêmes auteurs [GV85a, GV87, GV95] ainsi que Lindblad et Sogge [LS95]. Le cas limite (ou *endpoint*) $q = 2$ pour Schrödinger, respectivement $r = \infty$ pour les ondes, est plus délicat. Il a été obtenu dans un cadre général par [KT98], et est en particulier faux pour Schrödinger en dimension deux $(p, q, r) = (2, \infty, 2)$ – voir par exemple [MS98], et pour les ondes en dimension trois [KM93].

1.2. Estimations de Strichartz et géométrie

Nous donnons, dans ce paragraphe qui n'a pas vocation à être exhaustif, un bref historique de quelques-uns des principaux résultats obtenus dans des géométries non euclidiennes qui nous semblent pertinents pour la suite de l'exposition.

1.2.1. L'équation des ondes

Pour l'équation des ondes dans une variété sans bord, les singularités se propagent selon les géodésiques à vitesse constante et la vitesse finie de propagation montre qu'il suffit de se placer en coordonnées locales pour obtenir des estimations locales. C'est le chemin suivit par exemple par [Kap89], [MSS93], [Smi98], et [Tat02]. Le cas d'un domaine à bord, où l'on doit traiter la propagation des singularités au bord, est plus délicat. L'étude de la propagation des singularités dans un tel cadre a été menée par Melrose et Sjöstrand [MS78, MS82] qui définissent la notion de rayon bicaractéristique généralisé, généralisant la notion de rayon optique, selon lesquels ont lieu la propagation.

A l'extérieur d'un convexe strict, la paramétrix de Melrose et Taylor [MT87] fournit une description explicite de la solution près des points diffractifs. Elle est notamment utilisée par Smith et Sogge [SS95] pour montrer que les estimations de l'espace libre sont vraies en dehors d'un obstacle strictement convexe.

Dans le cadre d'une géométrie quelconque, une telle paramétrix n'est plus utilisable à cause de la présence de rayons réfléchis plusieurs fois et de leur limites, les rayons dits *glissants*. Si on a longtemps pensé que les estimations de l'espace libre devaient être vraies à l'intérieur d'un convexe strict puisque les modes de galerie, accumulant l'énergie près du bord, ne sont responsables d'aucune perte, Ivanovici [Iva12] donne un contre exemple fourni par les caustiques qui peuvent apparaître près du bord en temps petit.

Les premières estimations locales pour l'équation des ondes dans un domaine général on été obtenues par Burq, Lebeau et Planchon [BLP08]. Elles sont la conséquence d'estimations de projecteurs spectraux dus à Smith et Sogge [SS07a], et sont généralisées ensuite par Blair, Smith et Sogge [BSS09a]. Elles se font sans perte de dérivées (la perte venant des directions tangentes étant compensée par le fait que ces directions vivent dans un petit cône de l'espace des fréquences), mais sur des plages d'indices restreintes par rapport à l'espace libre.

Récemment, Ivanovici, Lebeau et Planchon [ILP14a] dans un cas modèle, et Ivanovici, Lascar, Lebeau et Planchon [ILLP] améliorent tous les résultats connus pour $d \geq 3$ à l'intérieur d'un convexe strict quelconque. Ils obtiennent une estimation de dispersion ponctuelle avec une perte d'1/4 de dérivées, et montrent que cette perte, due à l'apparition de caustiques, est inévitable dans l'estimation dispersive. De plus, en montrant qu'elles apparaissent de manière suffisamment exceptionnelle pour que cet effet puisse être atténué par une moyenne en temps, ils obtiennent dans [ILP14b] des estimations de Strichartz sans perte de dérivée, avec des exposants sur une plage réduite qui correspondrait à une perte d'1/6 dans l'estimation dispersive.

1.2.2. L'équation de Schrödinger

Vitesse finie de propagation pour l'équation semi-classique

Dans le cas de l'équation de Schrödinger, la vitesse infinie de propagation rend les choses beaucoup plus délicates. Cependant, le changement d'échelle semi-classique permet grâce à la méthode de Lebeau [Leb92] de se ramener à une vitesse finie (mais non constante) de propagation. On considère une donnée localisée en fréquences d'ordre h^{-1}, et on utilise le changement de variable en temps $t \to ht$. On considérera l'équation dite *semi-classique* :

$$\begin{cases} i\partial_t u - h\Delta u = 0, \ u = 0 \text{ sur } \partial\Omega, \\ u(0) = \Psi(-h^2\Delta)u_0. \end{cases} \tag{LS$_{sc}$}$$

Maintenant, soit $(e_n)_{n\geq 0}$ un système orthonormal dans $L^2(\Omega)$ de fonctions propres de $-\Delta_D$ sur Ω, de valeurs propres $(\lambda_n)_{n\geq 0}$. On identifie les données

$$u_{k,0} \in \text{Vect}\left\{e_n, \ 2^k\alpha \leq \sqrt{\lambda_n} \leq 2^k\beta\right\}$$

avec les solutions du système, pour $h_k = 2^{-k}$

$$\begin{cases} i\partial_t u_k - h_k\Delta u_k = 0, \ u_k = 0 \text{ sur } \partial\Omega, \\ u_k(0) = u_{k,0}. \end{cases} \tag{LS$_{sc}$}$$

On dira que $\rho_0 = (y_0, \eta_0) \in T^\star(\Omega \times \mathbb{R}) \cup T^\star(\partial\Omega \times \mathbb{R})$ n'est *pas* dans le front d'onde au bord $WF_b(U)$ de $U = (u_{k,0})_{k\geq 0}$, s'il existe un opérateur h_k-pseudodifférentiel tangentiel au bord, de symbole p compactement supporté, elliptique en ρ_0, et $\psi \in C_c^\infty$ égale à 1 près de y_0 tels que

$$\forall\sigma, \forall N, \ \|\text{Op}_{h_k}(p)\psi u_k\|_{H^\sigma(\mathbb{R}\times\Omega)} \leq C_{N,\sigma}h_k^N.$$

Alors, d'après [Leb92],

$$WF_b(u_k) \subset \Sigma_b \cap \{\tau \in [\alpha, \beta], \ |\xi| \leq D\},$$

où Σ_b est l'ensemble caractéristique $\Sigma_b = \left\{\tau - |\xi|^2 = 0\right\}$. De plus, en posant $\Theta(U)(x,t,s) = \sum_k e^{-ih_k^{-1}s}u_k(x,t,s)$, $\Theta(U)$ vérifie

$$\partial_{t,s}^2\Theta(U) - \Delta(U) = 0, \ u = 0 \text{ sur } \partial\Omega \times \mathbb{R} \times \mathbb{R},$$

pour lequel on dispose donc du théorème de Melrose et Sjöstrand. D'après [Leb92], on a

Lemme 1.1. *Pour tout $s_0 \in \mathbb{R}$, pour tout $\rho_0 \in T^\star(\Omega \times \mathbb{R}) \cup T^\star(\Omega \times \mathbb{R})$,*

$$\rho_0 \in WF_b(U) \iff \theta(\rho_0, s_0) \in WF_b(\Theta(U))$$

où $\theta(\rho, s) := (\rho; s, 1)$.

Le front d'onde au bord de la suite U est donc invariant par le flot bicaractérisque généralisé sur $T^\star(\Omega \times \mathbb{R}) \cup T^\star(\partial\Omega \times \mathbb{R})$.

Estimations de Strichartz

En travaillant sur l'équation semi-classique avec des données spectralement localisées, on peut donc tirer parti de la vitesse finie de propagation et, dans le cas d'une variété sans bord, travailler en coordonnées locales. C'est par exemple la stratégie suivie par Burq, Gérard et Tzetkov [BGT04b], qui, s'intéressent à une variété compacte sans bord. En travaillant sur l'équation semi-classique avec donnée localisée en fréquence $\sim h^{-1}$, ils construisent une paramétrix en temps h dont ils déduisent des estimations de Strichartz avec perte d'$1/p$ dérivées :

$$\|e^{it\Delta}u_0\|_{L^p([0,T],L^q(M))} \leq C(T)\|u_0\|_{H^{1/p}(M)}.$$

Notons qu'une telle perte est naturelle au sens où l'énergie se propageant à la vitesse h^{-1}, on s'attend à pouvoir montrer des estimations sans perte en temps h, qui après sommation, donnent une perte $1/p$. Mentionnons également, dans le cas d'un opérateur de Schrödinger à coefficients variables, les travaux de Staffilani et Tataru [ST02].

Le cas d'une variété avec bord, où il faut tenir compte des effets de reflexion, est, encore une fois, plus délicat. Une stratégie, suivie par Anton [Ant08] puis Blair, Smith et Sogge [BSS08], consiste, en redressant le bord et en faisant une symétrie par rapport à celui-ci, à se ramener au cas d'une variété sans bords mais avec coefficients Lipshitziens. Ils obtiennent ainsi des estimations avec perte de $\frac{4}{3p}$ dérivées, soit

$$\|e^{it\Delta}u_0\|_{L^p([0,T],L^q(M))} \leq C(T)\|u_0\|_{H^{4/3p}(M)}.$$

Cependant, une telle méthode ne tient pas compte de la géométrie particulière du bord et n'a pas d'espoir d'être utilisées dans des géométries plus particulières où des estimations meilleures sont espérées. Dans le cas d'une variété compacte avec bord strictement concave, dont un exemple est donné par le billard de Sinaï, Ivanovici tire parti dans [Iva10] de la vitesse finie de propagation pour l'équation semi-classique et de la paramétrix de Melrose et Taylor, pour montrer des estimations en temps h sur les données localisées en fréquence, qui une nouvelle fois fournissent après sommation de Littlewood-Paley, des estimations locales en temps avec perte d'$1/p$ dérivées sur l'équation classique. A l'aide d'un argument dont nous discuterons au paragraphe suivant, elle en déduit des estimations globales aussi bonnes que dans l'espace libre à l'extérieur dans convexe strict. Par ailleurs, elle montre dans [Iva12] qu'à l'intérieur d'un convexe strict, une perte d'$1/6$ de dérivées est inévitable. Son contre exemple est donné par les modes de galerie, qui, rappelons-le, n'induisent aucune perte dans le cas de l'équation des ondes.

Enfin, notons que généraliser les travaux [ILP14a] et [ILLP] à l'équation de Schrödinger afin d'obtenir des estimations fines à l'intérieur d'un convexe strict semble pour le moment hors de portée : la vitesse non-constante de propagation rend les réflexions multiples au bord extrêmement difficiles à gérer.

Remarque. Des estimations de Strichartz sans perte n'impliquent pas nécessairement une estimation de dispersion ponctuelle aussi bonne que dans l'espace libre. Un exemple remarquable de ce fait est fournit par Ivanovici et Lebeau, qui montrent en particulier dans [IL17] que l'estimation de dispersion ponctuelle est *fausse* à l'extérieur d'une boule en dimensions $d \geq 4$. Comme il n'y a pas d'effet concentration explicite de l'énergie dans un tel problème, ce résultat est surprenant. Cependant, d'après le principe de Huygens, chaque point du bord atteint par l'onde se comporte comme une nouvelle source de lumière, qui peut donc refocaliser dans l'ombre de l'obstacle en un point expérimentalement observé dit de Poisson-Arago, qui est à l'origine du contre exemple donné par les auteurs.

1.3. Effet régularisant, estimations de Strichartz et non-captivité

Une obstruction intuitive à l'obtention d'estimations de Strichartz globales et sans pertes est la présence de *rayons captés,* qui devraient, au moins en un certain sens, induire des effets de concentration de l'énergie. Pour cette raison, il est naturel de se placer sous l'hypothèse de *non-captivité :*

Définition. On dit qu'un obstacle $\Theta \subset \mathbb{R}^d$ est non captant si tous les rayons de l'optique géométrique de $\mathbb{R}^d \backslash \Theta$ sortent de tout compact en temps fini. De la même manière, on dit qu'une variété sans bord (M, g) est non-captante si toutes les géodésiques sortent de tout compact en temps fini.

Plusieurs auteurs obtiennent des résultats en se placant sous cette hypothèse : citons notamment les travaux de Smith et Sogge [SS00], Staffilani et Tataru [ST02], Burq [Bur03], Metcalfe [Met04], Hassell, Tao et Wunsch [HTW06], Bouclet et Tzetkov [BT07], ainsi que ceux de Bouclet [Bou11].

En particulier, Smith et Sogge montrent dans [SS00] en dimension impaire pour les ondes, que sous l'hypothèse de non-captivité, des estimations de Strichartz locales en temps impliquent des estimations globales. Ce résultat est étendu à tout les dimensions d'espace indépendamment par Burq [Bur03] et Metcalfe [Met04]. Un fait remarquable apparaissant notamment dans [Bur03] est que des estimations de décroissance locale de l'énergie pour les ondes

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, \dot{H}^1 \times L^2)} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}, \ \forall \chi \in C_c^\infty,$$

qui sont en particulier vraies dans l'espace libre, permettent de mettre bout à bout des estimations de Strichartz locales en temps pour obtenir une estimation globale. De la même manière, la contrepartie pour Schrödinger de la décroissance locale de l'énergie, soit l'effet régularisant

$$\|\chi u\|_{L^2(\mathbb{R}, H^{1/2})} \lesssim \|u_0\|_{L^2}, \ \forall \chi \in C_c^\infty,$$

qui reflète un gain d'une demi-dérivée, permet de mettre bout à bout des estimations de Strichartz semi-classiques en temps h. Heuristiquement, l'estimation régularisante signifie qu'une solution de l'équation de Schrödinger à fréquence $\sim h^{-1}$ passe un temps $\sim h$ près de l'obstacle, il suffit donc d'obtenir des estimations pour des temps de l'ordre de h. C'est par exemple la stratégie suivie par [Iva10] pour déduire des estimations semi-classiques en temps h sur le billard de Sinaï, des estimations globales et sans perte à l'extérieur d'un obstacle convexe.

Notons qu'un précurseur remarquable à des résultats d'estimations locale de l'énergie à l'extérieur d'obstacles non captifs pour les ondes sont les travaux de Morawetz [Mor61], Morawetz, Raltson et Strauss [MRS77], qui s'appuient sur la construction d'une fonction d'échappement, c'est à dire une fonction de l'espace des phases qui croît selon les rayons et respecte la réflexion au bord. Une telle fonction a été construite par [MRS77] dans le cas d'un obstacle non captant. Sa construction dans le cadre de variétés sans bords est au coeur de travaux plus récents : citons, par exemple le résultat de Datchev et Vasy [DV12].

Burq, Gérard et Tzetkov [BGT04a] montrent le même effet régularisant que dans l'espace libre à l'extérieur d'un obstacle non-captant général. Ils l'obtiennent comme conséquence de l'estimation de résolvante

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon)^2)^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \lesssim |\lambda|^{-1},$$
$$|\lambda| \gg 1,\ 0 < \epsilon \ll 1,$$

pour laquelle l'hypothèse de non-captivité joue un rôle crucial, montrée en grande généralité entre autres par Lax et Philips [LP89], Melrose et Sjöstrand [MS78, MS82], Vainberg [Vn88], Vasy et Zworski [VZ00], et qui permet également, avec une méthode similaire, de déduire la décroissance locale de l'énergie pour les ondes. D'autre part, Burq montre en particulier dans [Bur04], dans un cadre très général, que l'effet régularisant sans perte, y compris local en temps, est *faux* s'il existe une trajectoire captée. Plus précisément, dans le cadre qui nous intéresse, pour tout $t_0 > 0$, l'application

$$u_0 \in C_c^\infty \subset L^2 \longrightarrow \chi e^{it\Delta}u_0 \in L^2([0,t_0], H^{\frac{1}{2}})$$

n'est jamais bornée lorsque χ n'est pas identiquement nulle sur l'ensemble capté.

1.4. Effet régularisant et trajectoires captées

Une question naturelle suite au résultat de [Bur04] évoqué précédemment est de se demander si un effet régularisant *avec perte* serait valable dans certaines géométries captives. On peut, essentiellement, distinguer trois types de trajectoires captives : *elliptiques*, *paraboliques*, et *hyperboliques*. Considérant l'application de Poincaré de premier retour $\mathcal{P}$ dans un voisinage de l'espace des phases d'un point $\rho \in T^\star\Omega$ d'une trajectoire captée, on dit que la trajectoire est elliptique (stable) si toutes les valeurs propres de la différentielle de $\mathcal{P}$ en ρ sont de modules plus petits que un, parabolique (neutre), si ces valeurs propres sont toutes de module égaux à un, ou hyperbolique (instable) si il en existe au moins une de module plus grand que un.

S'il existe une trajectoire elliptique, il est possible de construire des quasimodes à support compact $(e_n)_{n\geq 1}$ associés à $\lambda_n \longrightarrow \infty$

$$-\Delta e_n = \lambda_n e_n + r_n, \ \|r_n\|_{H^s} = O(\lambda_n^{-\infty}),$$

alors, pour tout $t_0 > 0$ et tout $s > 0$

$$\begin{aligned}\|e^{it\Delta}e_n\|_{L^1([0,t_0[,H^s)} &\geq \| \int_0^{t_0} e^{it\Delta}e_n\|_{H^s}dt = t_0\|\lambda_n e_n + r_n\|_{H^s} \\ &\geq t_0\lambda_n - O(\lambda_n^{-\infty}),\end{aligned}$$

et aucun type d'effet régularisant ne peut être vrai.

L'effet régularisant à l'extérieur de plusieurs obstacles strictement convexes

Nous nous intéressons ici à l'exemple canonique du cas hyperbolique, soit l'extérieur de deux obstacles strictement convexes, et sa généralisation à l'extérieur de plusieurs $(N \geq 3)$ obstacles strictement convexes. Un tel cadre a été étudié dans un premier temps par M.Ikawa [Ika82, Ika88] et C. Gérard [G88] dans les années 1980. Ikawa montre en particulier dans [Ika82] à l'extérieur de deux obstacles, résultat qu'il généralise plus tard dans [Ika88] à l'exterieur d'un nombre arbitraire d'obstacles, que la résolvante $\chi(-\Delta_D - \tau^2)^{-1}\chi$, holomorphe sur $\{\mathrm{Im}\tau{>}0\}$, admet un prolongement holomorphe sur sur une bande du demi-espace supérieur $\{|\tau| > 1, \mathrm{Im}\tau \leq \alpha\}$ et y satisfait

$$\|\chi(-\Delta - \tau^2)^{-1}\chi\|_{L^2(\Omega)\to L^2(\Omega)} \leq C|\tau|^N,$$

pour un certain $N \gg 1$, ce qui implique en particulier des estimations de décroissance locales de l'énergie et effets régularisant avec perte d'un nombre fini de dérivées. A l'extérieur de plusieurs $(N \geq 3)$ obstacles, il n'y a non plus une seule, mais une infinité de trajectoires captives. L'hyperbolicité des trajectoires considérées séparemment ne suffit pas et il faut s'intéresser à leur interaction. Ikawa fait dans [Ika88] une hypothèse d'hyperbolicité forte :

Définition 1.2 (Condition d'Ikawa, 1 : hyperbolicité forte)**.** Il existe $\alpha > 0$ tel que

$$\sum_{\gamma\in\mathcal{P}} \lambda_\gamma d_\gamma e^{\alpha d_\gamma} < \infty, \tag{1.4.1}$$

où $\mathcal{P}$ désigne l'ensemble des trajectoires captées, d_γ la longueur de $\gamma \in \mathcal{P}$ et $\lambda_\gamma = \sqrt{\mu_\gamma^1\mu_\gamma^2}$ où $\mu_\gamma^{1,2}$ sont les deux valeurs propres de module plus petit que un de l'application de Poincaré associée à γ.

Cette hypothèse sera plus tard connnue sous le nom de *condition d'Ikawa*. Son travail s'appuie sur une construction de paramétrix pour l'équation des ondes qui sera reprise par Burq [Bur93] et nous même, et sera décrite dans la suite de cette exposition. Cette construction se fait au prix de l'hypothèse technique suivante

Définition 1.3 (Condition d'Ikawa, 2 : pas d'obstacle dans l'ombre)**.** Pour tous i, j, k deux à deux distincts,

$$\mathrm{Conv}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset, \tag{1.4.2}$$

Burq remarque dans [Bur04] que le résultat d'Ikawa, combiné à une estimation standard pour les opérateurs auto-adjoints et à un principe du maximum semiclassique inspiré de Thang et Zworski [TZ00], implique en fait l'estimation plus fine

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \lesssim \frac{\log(2+\sqrt{\lambda})}{1+\sqrt{\lambda}},$$
$$\lambda \in \mathbb{R},\ 0 < \epsilon \ll 1,$$

et donc une estimation régularisante avec une perte arbitrairement petite de $\epsilon > 0$ dérivées. Plus précisément, l'estimation de résolvante précédente implique pour des données localisées en fréquence une perte logarithmique par rapport à la fréquence, soit :

$$\|\chi e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^2(\mathbb{R},H^{1/2})} \lesssim |\log h|^{\frac{1}{2}}\|\psi(-h^2\Delta)u_0\|_{L^2},\ \forall \chi \in C_c^\infty,$$

où $\psi(-h^2\Delta)$ localise à la fréquence h.

Remarque. Le cas intermédiaire entre la perte totale engendrée par une trajectoire elliptique et la perte logarithmique induite par un ensemble capté de type hyperbolique est étudié par Christianson et Wunsch [CW13], qui montrent une estimation régularisante optimale avec perte polynomiale dans une variété ayant une seule trajectoire périodique faiblement hyperbolique.

1.5. Compenser la perte due à une trajectoire captive hyperbolique

Si le lien précis entre effet régularisant sans perte et présence d'une trajectoire captée est maintenant bien compris, suite notamment au résultat de [Bur04], ce n'est pas encore le cas pour les estimations de Strichartz. En particulier, Burq, Guillarmou et Hassel établissent dans [BGH10] des estimations de Strichartz *sans perte* dans une variété (M, g) sans bord, conique à l'infini, de courbure négative et ayant un ensemble capté de type hyperbolique suffisamment petit, établissant ainsi que la présence de trajectoires captées, pourvu qu'elles soient suffisamment instables, n'induisent pas nécessairement une perte au niveau des estimations de Strichartz. Plus précisément, les auteurs considèrent une variété conique à l'infini de courbure strictement négative dans un partie compacte convexe qui inclut l'ensemble capté, ce dernier étant supposé assez petit au sens où sa *pression topologique* en $s = \frac{1}{2}$ est strictement négative :

$$P(\frac{1}{2}) < 0.$$

La pression topologique $P(s)$, dont on pourra trouver une définition précise dans [NZ06] et [BGH10], mesure la compétition entre la densité de l'ensemble capté et l'instabilité du flot. L'hypothèse $P(1/2) < 0$ signifie que c'est l'instabilité qui l'emporte. Il s'agit en fait de l'hypothèse d'Ikawa d'hyperbolicité forte (1.4.1), écrite dans le présent contexte : ce problème est donc un homologue sans bord au problème à l'extérieur de plusieurs obstacles strictement convexes vérifiant la condition d'Ikawa.

Dans une telle géométrie, l'effet régularisant sans perte n'est plus valable d'après [Bur04], et ne peut donc plus être utilisé pour mettre bout à bout des estimations de Strichartz semi-classiques en temps h. Nonnenmacher et Zworski [NZ06], Datchev [Dat09] montrent que de la même manière que pour le problème d'Ikawa, un effet régularisant avec perte logarithmique est valable dans une telle variété :

$$\|\chi e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^2(\mathbb{R},H^{1/2})} \lesssim |\log h|^{\frac{1}{2}}\|\psi(-h^2\Delta)u_0\|_{L^2}, \ \forall\chi\in C_c^\infty.$$

La remarque fondamentale de [BGH10] est que cette perte logarithmique peut être compensée si l'on montre des estimations de Strichartz en temps logarithmique $h|\log h|$, soit

$$\|e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^p([0,h|\log h|])L^q(M)} \lesssim \|\psi(-h^2\Delta)u_0\|_{L^2}.$$

Intuitivement, la perte logarithmique dans l'estimation régularisante signifie que l'onde passe du fait des phénomènes de captivité un temps plus grand près de l'obstacle, de l'ordre maintenant de $h|\log h|$: il s'agit donc de montrer des estimations jusqu'aux temps logarithmiques. En de tels temps, les auteurs adaptent la construction de paramétrix de Anantharaman [Ana08] pour obtenir une estimation de dispersion ponctuelle près de l'ensemble capté. Notons que, dans le cas de l'équation des ondes dans une géométrie similaire, une telle paramétrix a été construite par Bérard [B77].

1.6. Résultat : Estimations de Strichartz en dehors de deux obstacles strictement convexes

Motivés par les travaux de Burq, Guillarmou et Hassel [BGH10] et d'Ikawa [Ika82, Ika88], nous nous intéressons aux estimations de Strichartz pour les équations de Schrödinger et des ondes à l'extérieur de plusieurs obstacles strictement convexes vérifiant la condition d'Ikawa. Nous traitons tout d'abord le cas de de deux obstacles strictement convexes, à l'extérieur desquels nous obtenons des estimations de Strichartz globales et sans perte par rapport à l'espace libre :

Théorème 1.4 ([Laf17b, Laf17a]). *Soient $\Theta_1, \Theta_2 \subset \mathbb{R}^3$ deux obstacles strictement convexes. Alors, des estimations de Strichartz globales et sans pertes sont valables dans $\Omega = \mathbb{R}^3\setminus(\Theta_1\cup\Theta_2)$ pour les équations de Schrödinger et des ondes :*

$$\|u\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|u_0\|_{\dot H^s} + \|u_1\|_{\dot H^{s-1}}, \ (Ondes)$$
$$\|u\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|u_0\|_{L^2}, \ (Schrödinger)$$

le couple (p,q) vérifiant la condition d'admissibilité hors point limite pour l'équation associée.

Nous montrons le résultat pour l'équation de Schrödinger dans [Laf17b]. Ce résultat est ici présenté en section 7. Il est ensuite généralisé à l'équation des ondes dans [Laf17a], généralisation ici présentée en section 8. Nous suivons la méthode suivante, inspirée par les éléments précédents de notre exposition :

1. Grâce à l'estimation régularisante avec perte logarithmique due à [Bur04], aux travaux d'Ivanovici en dehors d'un convexe strict [Iva10], et à la remarque de Burq, Guillarmou et Hassel [BGH10], on se réduit à des estimations de Strichartz en temps logarithmique près de la trajectoire captée,

2. on suit la construction d'Ikawa [Ika82, Ika88], reprise par Burq [Bur93], pour construire une solution approchée près de la trajectoire captée,

3. un argument de phase stationnaire, combiné à un résultat d'Ikawa [Ika82, Ika88] concernant la décroissance d'une telle solution, donne le résultat.

Résumons brièvement notre argument, en premier lieu pour l'équation de Schrödinger :

1.6.1. Réduction du problème

Supposons que des estimations de Strichartz pour Schrödinger soient vérifiées en temps logarithmique près de la trajectoire captée $\mathcal{R}$, soit

$$\|\chi e^{-it\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(0,h|\log h|)L^q(\Omega)} \leq C\|u_0\|_{L^2},$$

où χ est supportée près de la trajectoire captée, et montrons comment obtenir le résultat.

Soient $\chi_{\text{obst}}, \chi_{\text{ray}} \in C_0^\infty$ tel que $\chi_{\text{obst}} = 1$ dans un voisinnage de $\Theta_1 \cup \Theta_2 \cup \mathcal{R}$, et $\chi_{\text{ray}} \in C_0^\infty$ tel que$\chi_{\text{ray}} = 1$ dans un voisinage de $\mathcal{R}$, et soit $\tilde{\psi}$ tel que $\tilde{\psi} = 1$ sur le support de ψ. On décompose :

$$\begin{aligned}\psi(-h^2\Delta)e^{it\Delta_D}u_0 = \tilde{\psi}(-h^2\Delta)(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}\chi_{\text{ray}}\psi(-h^2\Delta)e^{it\Delta_D}u_0,\end{aligned}$$

qui correspondent respectivement aux parties loin des obstacles, près des obstacles mais loin de la trajectoire captée, et près à la fois de l'obstacle et de la trajectoire captée. Ce dernier terme est géré en suivant la remarque de Burq, Guillarmou et Hassel précédemment présentée pour peu que l'on dispose d'une estimation sans perte en dehors de la trajectoire captée, qui, dans cette géométrie particulière, est par exemple conséquence d'estimations de résolvante de [VZ00] en dehors d'un obstacle non captant, puisque $\Theta_1 \cup \Theta_2$ coïncide, en dehors de la trajectoire captée, avec un tel obstacle.

Les deux premiers termes sont controlés en adaptant des arguments de [Iva10]. Le premier terme, $w = (1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0$, vérifie le problème

$$\begin{cases} i\partial_t w + \Delta_D w_h = & -[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ w(t=0) = & (1-\chi_{\text{obst}})\psi(-h^2\Delta)u_0 \end{cases}$$

mais comme $1-\chi_{\text{obst}}=1$ près de $\partial\Omega$, il s'agit d'un problème dans l'espace libre ! La formule de Duhamel combinée avec les estimations de Strichartz dans $\mathbb{R}^n$ et un lemme de Christ-Kiselev fournissent alors

$$\|w\|_{L^pL^q} \lesssim \|(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}^n)} + \|[\chi_{\text{obst}},\Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{-1/2}}.$$

mais $[\chi,\Delta_D]$ est supporté loin de $\mathcal{R}$ et l'estimation régularisante sans perte permet de contrôler le membre de droite. Le second terme, $\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0$ correspond à un problème près de l'obstacle mais loin de la trajectoire captée, il peut donc être controlé en mettant bout à bout les estimations en temps h dans une variété compacte à bord strictement concave obtenues par [Iva10], grâce à l'estimation sans perte.

1.6.2. Construction d'une solution approchée

Un argument de translation inspiré de [Iva10] montre alors que l'on peut de plus se réduire à l'étude de données microlocalement supportées près de la trajectoire captée. Plus précisément, nous définissons l'ensemble capté en temps T d'un voisinage D de $\mathcal{R}$, noté $\hat{\mathcal{T}}_T(D)\subset T^\star\Omega$ comme l'ensemble des points de $D\times\{|\xi|\in[\alpha_0,\beta_0]\}$, qui restent dans D en temps T, et on se réduit à l'étude de données $u_0=\mathrm{Op}(q)u_0$, où q est supporté dans un voisinage de $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\tilde{D})$ et $q=1$ sur $\hat{\mathcal{T}}_{2\epsilon|\log h|}(D)$, où $D\subset\tilde{D}$.

Notons que la nécessité notamment de devoir par la suite contrôler les dérivées de q nous motive à étudier la régularité du flot de billard, et l'on montre que

$$d(\Phi_t(\rho),\Phi_t(\tilde{\rho}))\leq C^t d(\rho,\tilde{\rho})^\mu$$

sauf sur de petits intervalles de temps, la perte de régularité Höldérienne étant due aux rayons qui frôlent des trajectoires tangentes, et une telle régularité suffira pour choisir q de telle sorte que $|\partial^\alpha_{x,\xi}q|\lesssim h^{-c|\alpha|\epsilon}$, ce qui n'induira aucune perte puisque ces dérivées seront par la suite confrontées à des termes d'ordre au moins h.

Un argument de dualité montre qu'il suffit d'obtenir l'estimation de dispersion, pour $t\leq\epsilon|\log h|$

$$|e^{iht\Delta}\delta_y|\lesssim(ht)^{-3/2},\ \text{où } \delta_y=\frac{1}{(2\pi h)^3}\int e^{i(x-y)\cdot\xi/h}q(x,\xi)d\xi,$$

et le flot étant linéaire, il suffit pour obtenir une solution approchée avec donnée δ_y d'en construire une avec donnée $e^{i(x-y)\cdot\xi}q(x,\xi)$, y et ξ étant fixés. Nous suivons pour cela la méthode d'Ikawa [Ika82, Ika88], reprise par Burq [Bur93] pour le contrôle de l'équation des plaques.

On définit les histoires de réflexions $\mathcal{I}$ comme l'ensemble des n-uplet $J=(j_1,\cdots,j_n)$, $n\geq 0$, où $j_{i+1}\neq j_i$, on note $J'=(j_1,\cdots,j_{n-1})$ et une solution est cherchée sous la forme d'une série de Neumann

$$e^{ith\Delta_{D,\Omega}}q(\cdot,\xi)=\sum_{J\in\mathcal{I}}(-1)^J w_J,$$

$$i\partial_t w_J - h\Delta w_J=0,\ \ w_J=w_{J'} \text{ sur } \mathbb{R}^3\backslash\Theta_{j_n},$$

où $w_\emptyset$ correspond à une onde libre et w_J à une onde réfléchie selon l'histoire J puis ignorant les obstacles. Les w_J sont obtenus selon la méthodes BKW : on cherche w_J sous la forme

$$w_J(x,t) = \sum_{k\geq 0} w_k^J(x,t) h^k e^{-i(\varphi_J(x,\xi)|\xi|-t\xi^2)/h}$$

où $\varphi_\emptyset(x,\xi) = (x-y)\cdot\xi/|\xi|$, les phase φ_J, $|J|\geq 1$ vérifient l'équation Eikonale

$$|\nabla\varphi_J| = 1,\ \varphi_J = \varphi_{J'} \ \text{ sur } \mathbb{R}^3\backslash\Theta_{j_n},$$

et w_k^J un transport selon la phase :

$$\begin{gathered}(\partial_t + 2|\xi|\nabla\varphi_J\cdot\nabla + |\xi|\Delta\varphi_J)w_k^J = -i\Delta w_{k-1}^J,\\ w_k^J = w_k^{J'} \text{ sur } \mathbb{R}^3\backslash\Theta_{j_n},\end{gathered}$$

où on a utilisé la convention $w_{-1}^J = 0$. Une solution aux équations Eikonales est construite par la méthode de réflexion des phases d'Ikawa : $\nabla\varphi_J(x,\xi)$ correspondant à la direction d'un rayon arrivé en x après être parti suivant la direction $\frac{\xi}{|\xi|}$ et avoir été réfléchi suivant l'histoire J, en d'autres termes, *le gradient de la phase suit les rayons*.

On peut alors résoudre les équations de transport suivant les rayons. Pour celà, on remarque que $w(\tau) = v(x+2\tau|\xi|\nabla\psi, t+\tau)$ résout l'équation différentielle ordinaire

$$\partial_\tau w = -|\xi|\Delta\psi(x+2\tau\nabla\psi(x))w,$$

mais on peut intégrer ([Lun44]) :

$$\exp(-|\xi|\int_{\tau_0}^{\tau}\Delta\psi(x+2s\nabla\psi(x))ds) = \left(\frac{G\psi(x+2\tau|\xi|\nabla\psi(x))}{G\psi(x+2\tau_0|\xi|\nabla\psi(x))}\right)^{1/2},$$

où $G\psi$ désigne la courbure gaussienne de la surface de niveau de ψ. On a donc, par exemple pour w_0^J

$$w_0^J(x+2\tau|\xi|\nabla\varphi_J(x), t+\tau) = \left(\frac{G\varphi_J(x+2\tau|\xi|\nabla\varphi_J(x))}{G\varphi_J(x)}\right)^{1/2} w_0^J(x,t)$$

pour $x\in\partial\Theta_{j_n}$. Mais $w_0^J = w_0^{J'}$ sur $\partial\Theta_{j_n}$ et on peut réitérer le raisonnement en remontant les rayons jusqu'à $w_0^\emptyset$ pour lequel résoudre les équations de transport est immédiat - la localisation de q nous assurant en particulier que l'on puisse en effet remonter les rayons. On obtient

$$\begin{gathered}w_0^J(x,t) = \Lambda\varphi_J(x,\xi)q(X_{-2t}(x,|\xi|\nabla\varphi_J),\xi),\\ \Lambda\varphi_J(x,\xi) = \left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x,|\xi|\nabla\varphi_J))}\right)^{1/2}\times\cdots\times\left(\frac{G\varphi(X^{-|J|-1}(x,|\xi|\nabla\varphi_J))}{G\varphi(X^{-|J|}(x,|\xi|\nabla\varphi_J))}\right)^{1/2},\end{gathered}$$

où X_t désigne la composante spatiale du flot X^i les points du bord atteints après i réflexions, et des expressions similaires sont valables pour $k\geq 1$.

Notons que la racine carrée du produit des rapports des courbures Gaussiennes $\Lambda\varphi_J$ est l'analogue de la racine carrée du produit des Jacobiennes apparaissant dans [BGH10]. C'est ce terme qui est responsable de la décroissance temporelle du flot : d'après Ikawa [Ika82, Ika88],

$$|\Lambda\varphi_J| \lesssim \lambda^{|J|}$$

où $0 < \lambda < 1$ est la racine carrée du produit des deux valeurs propres de module plus petit que 1 de la différentielle de l'application de Poincaré associée à la trajectoire captive.

1.6.3. Les dérivées directionnelles de la phase

On a donc

$$e^{iht\Delta}\delta_y \simeq \frac{1}{(2\pi h)^3}\sum_{J\in\mathcal{I}}\sum_{k\geq 0}\int w_k^J(x,t,\xi)h^k e^{-i(\varphi_J(x,\xi)|\xi|-t\xi^2)/h}d\xi,$$

où les w_k^J et φ_J ont été construits précédemment. Si un résultat d'Ikawa [Ika82, Ika88] concernant le produit du rapport des courbures Gaussiennes nous assurera la décroissance temporelle du flot, il est nécessaire pour conclure de gagner trois demies puissances de h. Nous utilisons pour cela un argument de phase stationnaire. Il est donc nécessaire, d'une part, de pouvoir contrôler les dérivées *directionnelles* qui surgiront, ainsi que d'étudier les points stationnaires des phases $S_J(x,\xi,t) = \varphi_J(x,\xi)|\xi| - t\xi^2$ et de montrer que sa Hessienne y est non dégénérée.

Pour ce faire, nous différencions l'équation Eikonale par rapport à ξ et nous résolvons suivant les rayons l'équation de transport qui apparaît. Par exemple, dérivant $|\nabla(\varphi_J|\xi|)|^2 = |\xi|^2$ on obtient

$$|\xi|\nabla(\partial_{\xi_i}\varphi_J|\xi|)\cdot\nabla\varphi_J = \frac{\xi_i}{|\xi|},$$

la quantité $\partial_{\xi_i}\varphi_J|\xi|$ qui nous intéresse est donc transportée suivant $\nabla\varphi_J$, c'est à dire suivant les rayons. En intégrant l'équation différentielle ordinaire que vérifie $\tau \longrightarrow \partial_{\xi_i}(\varphi_J|\xi|)(x+\tau\nabla\varphi_J(x,\xi))$, et en utilisant la condition au bord pour les phases $\varphi_J = \varphi_{J'}$ sur $\partial\Theta_{j_n}$, on obtient pour $x\in\partial\Theta_{j_n}$

$$\partial_{\xi_i}(\varphi_J|\xi|)(x+\tau\nabla\varphi_J(x,\xi),\xi) = \partial_{\xi_i}(\varphi_{J'}|\xi|)(x,\xi) + \tau\frac{\xi_i}{|\xi|},$$

et on itère le raisonnement jusqu'à la première phase. On obtient

$$D_\xi\mathcal{S}_J(x,\xi,t) = \hat{X}_{-2t}(x,|\xi|\nabla\varphi_J(x,\xi)) - y$$

et la phase stationne en le vecteur $\xi = s_J(x,t)$ permettant d'atteindre x depuis y en temps $2t$ en commençant par une réflexion sur Θ_{j_1}.

De la même manière, nous résolvons en remontant les rayons l'équation de transport vérifiée par la phase

$$\nabla\left(\partial^2_{\xi_i\xi_j}\psi_J\right)\cdot\nabla\psi_J = \delta_{ij} - \partial_{\xi_i}\nabla\psi_J\cdot\partial_{\xi_j}\nabla\psi_J$$

pour montrer sa non dégénérescence en les points stationnaires pour $t \geq t_0 > 0$.

Finalement, pour contrôler les dérivées directionnelles qui apparaîtront, il s'agit de contrôler les dérivées spatio-directionelles des phases. Pour ce faire, nous suivons la même idée que précédemment : $D_\xi^\alpha \varphi_J$ vérifie

$$\nabla(D_\xi^\alpha \varphi_J) \cdot \nabla \varphi_J = R_J,$$

où R_J ne contient que des dérivées d'ordre inférieur. On résoud cette équation de transport en remontant les rayons, on prend les dérivées spatiales de la solution et on obtient par induction

$$|D_\xi^\alpha D_x^\beta \nabla \varphi_J| \leq D_{\alpha,\beta}^{|J|},$$

un résultat d'Ikawa nous assurant l'initialisation $\alpha = 0$. Si un tel résultat semble brutal, il est en fait suffisant pour obtenir le contrôle qui nous intéresse en des temps logarithmiques : en effet, les dérivées directionnelles qui apparaîtront sont contre des termes d'ordre au moins h, or, pour $w_k^J \neq 0$, on a $|J| \approx t \leq \epsilon |\log h|$, et la perte d'$\epsilon$ puissance de h qui apparaît alors n'induit aucune perte.

1.6.4. Phase stationnaire et conclusion

Ainsi, après un argument de phase stationnaire, on obtient pour $t \geq t_0$

$$e^{iht\Delta}\delta_y \simeq \frac{1}{(2\pi)^3} h^{-3/2} \sum_{J \in \mathcal{I}} \sum_{k \geq 0} \tilde{w}_k^J(x,t,\xi) h^k e^{-is_J(x,t)/h},$$

où $\tilde{w}_0^J = w_0^J$ et, pour $k \geq 1$, $\tilde{w}_k^J$ est une combinaison linéaires de dérivées directionnelles de w_k^J. D'après le résultat d'Ikawa sur les phases précédemment cité et comme pour $w_k^J \neq 0$, on a $|J| \approx t \leq \epsilon |\log h|$,

$$\sum_{J \in \mathcal{I}} |w_0^J(x,t,\xi)| \lesssim \sum_{J \;|\; w_k^J \neq 0} \lambda^{|J|} \lesssim \sum_{r \gtrsim t} \lambda^r \leq e^{-\mu t} \tag{1.6.1}$$

pour un certain $\mu > 0$ ne dépendant que de la géométrie des obstacles. De la même manière, les termes $\tilde{w}_k^J(x,t,\xi)h^k$ admettent une borne similaire puisque comme évoqué précédemment, prendre une dérivée directionnelle de w_k^J induit au pire une perte de $h^{-\epsilon}$ pour des temps logarithmiques. Ainsi donc, pour $t \in (t_0, \epsilon |\log h|)$:

$$|e^{ith\Delta}\delta_y| \lesssim h^{-3/2} e^{-\mu t}$$

et pour des temps petits, $e^{ith\Delta}\delta_y$ est une onde libre (l'argument de translation de [Iva10] assurant que l'on puisse prendre une donnée supportée loin du bord) et on peut utiliser l'estimation de dispersion ponctuelle de $\mathbb{R}^3$, ce qui clôt la preuve.

1.6.5. L'équation des ondes

Dans le cas de l'équation des ondes, la réduction du problème est exactement similaire, en utilisant la décroissance locale de l'énergie plutôt que l'estimation régularisante, et le résultat de Smith et Sogge [SS95] à l'extérieur d'un convexe strict plutôt que le résultat de [Iva10].

La construction de paramétrix, similaire, se fait maintenant pour l'équation des ondes. La différence fondamentale étant que les rayons sont maintenant suivis à vitesse 1, cela se traduisant, au niveau de la phase, par le fait que cette dernière stationne maintenant sur des directions pleines, et on utilise alors un argument de phase stationnaire sur chaque sphère $\mathcal{S}^{d-1}(0, s)$. On ne gagne plus maintenant que $\frac{d-1}{2}$ puissances de h (ce qui traduit le caractère moins dispersif de l'équation des ondes), et on obtient de la même manière que pour Schrödinger l'estimation dispersive près de la trajectoire captée.

1.7. Résultat : L'extérieur de $N \geq 3$ obstacles strictement convexes vérifiant la condition d'Ikawa

Nous généralisons le résultat précédent à l'extérieur de $N \geq 3$ obstacles vérifiant la condition d'Ikawa :

Théorème 1.5 ([Laf18b]). *Soit $(\Theta_i)_{1 \leq i \leq N}$ une famille de convexes stricts de $\mathbb{R}^3$, vérifiant les conditions d'Ikawa (1.4.1) et (1.4.2), et $\Omega = \mathbb{R}^3 \backslash \bigcup_{1 \leq i \leq N} \Theta_i$. Alors, sous la condition d'admissibilité hors point limite, des estimations de Strichartz globales et sans pertes sont valables dans Ω pour les équations de Schrödinger et des ondes :*

$$\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}, \quad \text{(Ondes)}$$
$$\|u\|_{L^q(\mathbb{R}, L^r(\Omega))} \lesssim \|u_0\|_{L^2}. \quad \text{(Schrödinger)}$$

Ce résultat est montré dans [Laf18b], il est ici présenté en section 9. Donnons un aperçu de la preuve dans le cas de l'équation de Schrödinger.

Estimation régularisante sans perte dans la zone non-captée

Le plan de la preuve est le même que pour le cas le plus simple de deux obstacles. Pourvu que l'on dispose d'une estimation régularisante *sans perte* à l'extérieur de l'ensemble capté, l'argument précédent reste valide pour montrer que l'on peut se réduire à montrer des estimations près de l'ensemble capté en temps logarithmique. Cependant, on ne dispose plus d'une géométrie particulière nous permettant d'utiliser une estimation de résolvante en dehors d'un obstacle non-captant, et il nous faut donc donner une preuve générale. Notons, de plus, que nous montrons ici une estimation à l'extérieur de l'ensemble capté *dans l'espace des phases* et non plus simplement de sa projection spatiale, c'est à dire que la fonction de coupure en espace est remplacée par un opérateur dont le micro-support est disjoint de l'ensemble capté vu comme sous-ensemble de l'espace des phases.

Nous utilisons pour cela un argument de commutateur, similaire à celui apparaissant dans [BGH10], dans l'esprit de Morawetz, Raltson et Strauss [MRS77]: pour toute solution u de l'équation de Schrödinger et tout opérateur A on a :

$$\langle Au, u\rangle(T) - \langle Au, u\rangle(0) = \int_0^T \int_\Omega \langle [i\Delta, A]u, u\rangle + \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u\rangle.$$

Si l'on choisit A d'ordre zéro ayant un symbole positif, et elliptique sur l'ensemble qui nous intéresse, et tel que le terme de bord apparaissant a le bon signe à des termes d'ordre inférieur près, on obtient donc l'estimation voulue. La méthode de [MRS77] s'adapte à l'équation de Schrödinger pour calculer le symbole de A *au bord*, et nous reprenons leur construction d'une fonction d'échappement – valable pour un obstacle non-captant générique – à l'extérieur de l'ensemble capté pour construire le symbole de A. Notons que l'on ne dispose d'une expression du symbole au bord que dans une zone $\{\tau - \eta^2 > 0\}$, et pour cette raison, cette méthode ne permet pas de montrer que le terme de bord est positif à des termes d'ordre inférieur près, mais seulement qu'il est*t presque positif* à des termes d'ordre inférieur près, et nous obtenons une estimation de la forme

$$\|\mathrm{Op}_h(\phi)u\|_{L^2H^{1/2}} \leq C(\|u_0\|_{L^2} + \delta\|\mathrm{Op}_h(\tilde{\phi})u\|_{L^2H^{1/2}}) + C_\delta O(h^\infty),$$

où $\tilde{\phi} = 1$ sur le support de ϕ. On itère ce raisonnement après avoir choisit $\delta > 0$ suffisamment petit afin d'obtenir le résultat. Les termes d'ordre inférieur apparaissant sont gérés en utilisant l'estimation *avec perte*.

Notons que, dans l'esprit de [DV12], on peut itérer la preuve et contrôler à chaque étape les termes d'ordre inférieur par l'hypothèse de récurrence, pour montrer que l'effet régularisant sans perte dans la zone non captée est en fait valable dès que l'on dispose d'un effet régularisant avec perte *polynomiale*, assurant l'initialisation du raisonnement.

Réduction au voisinage des trajectoires périodiques

En utilisant l'estimation régularisante dans la zone non captée, on se réduit donc comme précédemment à montrer des estimations de Strichartz sans perte en temps logarithmique, micro-localement près de l'ensemble capté :

$$\|\mathrm{Op}_h(\phi)e^{-it\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(0,h|\log h|)L^q(\Omega)} \leq C\|u_0\|_{L^2},$$

où $\phi \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ est supportée près de l'ensemble capté $\mathcal{K} \subset T^\star\Omega \cup T^\star\partial\Omega$, et, via des arguments de dualité, il suffit donc de montrer l'estimation de dispersion

$$\|\mathrm{Op}_h(\phi)e^{ith\Delta}\psi(-h^2\Delta)\mathrm{Op}_h(\phi)^\star\|_{L^1\to L^\infty} \lesssim (ht)^{-3/2}, \ \forall 0 \leq t \leq \epsilon|\log h|.$$

Pour nous réduire à la construction d'une paramétrix près d'une trajectoire captée, nous écrivons, inspirés par [BGH10]:

$$e^{iTh\Delta} \approx \sum_{\mathbf{k}=(k_1,\cdots,k_L)} e^{its_0\Delta}\Pi_{k_L}e^{i\tau h\Delta}\Pi_{k_{L-1}}\cdots\Pi_{k_1}e^{i\tau h\Delta} + R_1(t) + R_2(t),$$

où les opérateurs Π_k sont des opérateur de coupure microlocaux localisés près des différents segments composant l'ensemble capté. On choisit τ assez petit, et les termes de cette somme correspondent aux parties du flot qui restent près d'une trajectoire périodique. La partie R_1 correspond au flot qui est loin des obstacles et y reste et la partie R_2 correspond à une partie du flot qui est loin des obstacles et y reste, une fois écoulé un temps $T_0 \geq 0$ ne dépendant que des Π_i. Donc, en temps $t \geq T_0$, il suffit de s'intéresser aux parties du flot qui restent près d'une trajectoire périodique.

Pour gérer les temps $0 \leq t \leq T_0$, on choisit ϕ supportée assez près de $\mathcal{K}$ pour qu'en temps $0 \leq t \leq T_0$, les rayons issus des points du support de ϕ ne sont jamais tangents au bord du domaine. Alors la construction de paramétrix d'Ikawa précedemment évoquée permet d'obtenir une estimation dispersive en temps $0 \leq t \leq T_0$: le flot s'écrira en effet comme une somme (dépendant de T_0) d'ondes réfléchies, chacune d'entre elle vérifiant l'estimation dispersive.

Conclusion

Une fois localisés près d'une trajectoire périodique, la construction précédemment évoquée dans le cas de deux obstacles strictement convexes, ainsi que les arguments de phase stationnaire, s'appliquent de la même manière que pour deux obstacles. Cependant, pour terminer la preuve, il s'agit de controler les sommes analogues à (1.6.1), soit

$$\sum_{J \in \mathcal{I}} |w_k^J|,$$

où l'ensemble $\mathcal{I}$ des histoires de réflexion possibles est maintenant beaucoup plus grand, et sa taille entre en compétition avec la décroissance temporelle de chacune des ondes w^J : en d'autres termes, *il y a une compétition entre le nombre de parties de l'onde qui restent piégées entre les obstacles et la décroissance de chacune de ces parties.* L'hypothèse d'Ikawa d'hyperbolicité forte (1.4.1) nous assure que cette compétition a lieu dans un sens favorable, et l'on peut clore la preuve.

Remarque. Si l'hypothèse d'hyperbolicité forte (1.4.1) apparaît de manière naturelle, nous ignorons si elle est optimale : essayer d'obtenir un contre exemple dans un cas où elle n'est pas vérifiée serait digne d'interêt.

Asymptotique en temps long d'équations d'ondes non linéaires 2

2.1. L'équation des ondes non linéaire dans des domaines extérieurs

Considérons l'équation des ondes non linéaire avec conditions au bord de Dirichlet

$$(\partial_t^2 - \Delta)u + \epsilon u|u|^{p-1} = 0, \; u = 0 \text{ sur } \partial\Omega, \tag{NLW}$$
$$(u(0), \partial_t u(0)) = (u_0, u_1) \in \dot{H}^1 \times L^2,$$

où $\epsilon \in \{1, -1\}$, qui admet la quantité conservée :

$$E(u(t)) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |\partial_t u|^2 + \epsilon\frac{1}{p+1}\int |u|^{p+1}. \tag{2.1.1}$$

Nous nous intéresserons en particulier à l'équation tridimensionelle critique : $d = 3$, $p = 6$, pour laquelle l'invariance d'échelle dans le cas de l'espace libre $\Omega = \mathbb{R}^d$

$$u \longrightarrow \lambda^{1/2}u(\lambda\cdot, \lambda\cdot)$$

laisse invariante l'énergie.

Lorsque cette équation admet des solutions pour tout temps, il est naturel de se demander comment les décrire en temps grand ($t \longrightarrow \pm\infty$). Si elles se comportent comme des solutions de l'équation linéaire, on dit qu'il y a diffusion (ou *scattering* dans la littérature anglophone) :

Définition 2.1. Soit u une solution globale de (NLW). On dit que u *diffuse* s'il existe des données $v_\pm$ telles que

$$\|u(t) - S(t)v_\pm\|_{\dot{H}^1} + \|\partial_t(u(t) - S(t)v_\pm)\|_{L^2} \longrightarrow 0$$

lorsque $t \longrightarrow \pm\infty$, où $S(t)v_\pm$ désigne la solution de l'équation des ondes linéaire avec donnée $v_\pm$.

Lorsque $\epsilon = 1$, l'équation est dite défocalisante : la non linéarité va dans le même sens que la décroissance induite par le groupe linéaire, et on s'attend alors à des effets du type diffusion pourvu que le domaine Ω n'induise pas trop d'effets de concentration de l'énergie. C'est en particulier le cas losqu'il s'agit de l'extérieur d'un obstacle non-captant, ou faiblement captant, et l'intuition est qu'il devrait alors y avoir diffusion. L'objet de ce chapitre est l'étude de telles situations. Plus précisément, nous montrons que

1. les solutions diffusent lorsque Ω est l'extérieur d'un obstacle illuminé par un ellipsoïde, notion généralisant la notion d'étoilé,

2. à l'extérieur de deux obstacles strictement convexes, la seule obstruction à la diffusion est la concentration de l'énergie autour de la trajectoire captive, en particulier, les solutions à flot compact ne diffusent pas,

le résultat (2) fournissant un théorème de rigidité dans le cadre d'un argument de concentration-compacité et rigidité qui viendrait compléter la preuve. Ces résultats peuvent être trouvés dans [Laf17a], ils sont ici présentés en section 11.

2.1.1. Existence globale et critère de diffusion

L'existence globale pour des données régulières et radiales dans $\mathbb{R}^d$ est obtenue par Struwe [Str88], puis par Grillakis [Gri90] pour des données régulières non radiales. Shatah et Struwe [SS94] montrent ensuite l'existence globale pour des données H^1 en combinant les estimations de Strichartz avec un argument de non-concentration en tête de cône :

$$\lim_{t\longrightarrow t_0^{-1}} \int_{x\in|x-x_0|<t_0-t} u^6(x,t)dx = 0, \tag{2.1.2}$$

obtenu en intégrant une identité de divergence remarquée par Morawetz [Mor61] dans le cas de l'équation linéaire :

$$\partial_t Q - \mathrm{div} P + \frac{1}{3}|u|^6 = 0, \tag{2.1.3}$$

$$Q = \frac{|\partial_t u|^2 + |\nabla u|^2}{2} + \frac{|u|^6}{6} + \partial_t u(\frac{x}{t}\cdot\nabla)u \tag{2.1.4}$$

$$P = \frac{x}{t}\left(\frac{|\partial_t u|^2 - |\nabla u|^2}{2} + \frac{|u|^6}{6}\right) + \nabla u, \left(\partial_t u(\frac{x}{t}\cdot\nabla)u + \frac{u}{t}\right), \tag{2.1.5}$$

et qui peut être vue comme associée à l'invariance conforme de l'équation des ondes, sur le cône

$$K_S^0 = \{S \leq t \leq 0,\ |x| \leq t\}\,,\ S \longrightarrow 0^-.$$

Bahouri et Shatah [BS98] poursuivent le calcul et intégrent maintenant (2.1.3) sur le grand cône K_0^T, $T \longrightarrow +\infty$ pour montrer la décroissance de la partie non-linéaire de l'énergie, soit

$$\int_{\mathbb{R}^3} |u(t)|^6 \longrightarrow 0 \text{ lorsque } t \longrightarrow \infty,$$

qui, associée aux estimations de Strichartz, donne immédiatement la diffusion.

L'existence globale de solutions dans un domaine est obtenue par Burq, Lebeau et Planchon [BLP08]. Ils remarquent que les estimations de projecteurs spectraux obtenus par [BSS09a] fournissent des estimations de Strichartz suffisantes pour avoir une théorie de Cauchy locale, et généralisent le calcul de Shatah et Struwe à un domaine pour obtenir la non-concentration en tête de cône renversée (2.1.2) et donc obtenir existence globale de solutions. Pour ce faire, ils controlent le terme de bord surgissant maintenant après avoir intégré (2.1.3) : il s'agit pour généraliser le calcul d'obtenir la décroissance

$$\frac{1}{|S|}\int_{((S,0)\times\partial\Omega)\cap K_S^0}(n(x)\cdot x)|\partial_n u|^2 d\sigma(x)dt \longrightarrow 0$$

lorsque $S \longrightarrow 0^-$, or un calcul de commutateur montre que

$$\int_0^{t_0}\int_{\partial\Omega}|\partial_n u|^2 d\sigma dt \lesssim E,$$

et comme de plus $n(x)\cdot x = O(|x|^2)$, les auteurs obtiennent l'existence globale.

Si l'on essaie maintenant d'intégrer l'équation de divergence sur le grand cône K_0^T, $T \longrightarrow \infty$ afin de généraliser l'argument de Bahouri et Shatah et obtenir la décroissance des normes L^6 des solutions et donc la diffusion, le contrôle précédent ne suffit plus car il s'agit maintenant d'obtenir la décroissance en moyenne du terme de bord :

$$\left(\frac{1}{T}\int_0^T\int_{\partial\Omega} tP\cdot n|\partial_n u|^2 d\sigma dt\right)_+ \longrightarrow 0 \text{ lorsque } T \longrightarrow \infty$$

où on a pris la partie positive et $tP = x\cdot n$ si l'on prend le même multiplicateur que dans $\mathbb{R}^d$. Dans le cas d'un obstacle étoilé, ce terme a le bon signe et la preuve se généralise immédiatement. Pour obtenir la diffusion dans des géométries plus générales, deux stratégies se dégagent alors : adapter le multiplicateur afin que ce terme de bord ait le bon signe, ou essayer de le controler et montrer que

$$\frac{1}{T}\int_0^T\int_{\partial\Theta}|\partial_n u|^2 d\sigma dt \longrightarrow 0, \tag{2.1.6}$$

c'est à dire qu'*en moyenne en temps grand, il n'y a plus d'énergie près de l'obstacle*. C'est cette seconde stratégie qui est ici poursuivie.

2.1.2. Résultat : Diffusion à l'extérieur d'un obstacle illuminé par un ellipsoïde

La preuve de Bahouri et Shatah se généralisant immédiatement à l'extérieur d'un obstacle étoilé, il est naturel d'étudier des généralisations de cette notion. Nous nous intéresserons ici à des obstcales *illuminés* :

Définition 2.2. Un sous-ensemble Θ de $\mathbb{R}^n$ est dit illuminé par le convexe $\mathcal{C} \subset \mathbb{R}^n$ si

$$\min_{\partial\Theta} \nabla\rho\cdot\nu > 0$$

où

$$\rho(x) = \inf\{\lambda > 0,\ \text{t.q. } x \in \lambda\mathcal{C}\}$$

est la jauge de $\mathcal{C}$ et ν la dérivée normale sortante à $\partial\Theta$.

Un obstacle étoilé est ainsi un obstacle illuminé par une petite boule. Nous montrons la diffusion à l'extérieur d'obstacles illuminés par des ellipsoïdes :

Théorème 2.3 ([Laf17a]). *Soit $\mathcal{C} \subset \mathbb{R}^3$ l'ellipsoïde d'équation*

$$x^2 + y^2 + \epsilon z^2 = 1,\ 0 < \epsilon \leq 1 \tag{2.1.7}$$

respectivement

$$x^2 + \epsilon y^2 + \epsilon z^2 = 1,\ \frac{1+\sqrt{3}}{4} \leq \epsilon \leq 1 \tag{2.1.8}$$

et Θ un compact de $\mathbb{R}^3$ à frontière régulière illuminé par $\mathcal{C}$. Alors, toute solution de (NLW) dans $\Theta = \mathbb{R}^3 \backslash \Theta$ diffuse dans $\dot{H}^1(\Omega)$.

En particulier, nous obtenons la diffusion pour des os de chien avec un col arbitrairement petit. En adaptant le multiplicateur aboutissant à (2.1.3), Farah Abou Shakra obtient dans [AS14] la diffusion à l'extérieur des obstacles illuminés vérifiant l'hypothèse

$$\min_{\partial\Theta}(s + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0,$$

où les $\rho_1 \leq \rho_2$ sont les rayons de courbures principaux de $\mathcal{C}$, ρ_{2M} le maximum de ρ_2 et s la distance à $\mathcal{C}$. Cependant la quantité non locale ρ_{2M} empêche en particulier de traiter le cas d'os avec un col arbitrairement petit, qui était ici notre motivation afin de nous rapprocher de l'extérieur de deux convexes stricts.

Notre principal outil pour obtenir la décroissance (2.1.6) et donc la diffusion est une identité de moment pouvant être vue comme une autre version de l'identité de divergence (2.1.3) :

$$\begin{aligned}\partial_t \left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2} \Delta\chi u \partial_t u \right) = \int_\Omega (D^2\chi \nabla u, \nabla u) - \frac{1}{4} \int_\Omega u^2 \Delta^2 \chi \\ + \frac{3}{2} \int_\Omega |u|^6 \Delta\chi - \frac{1}{2} \int_{\partial\Omega} |\partial_n u|^2 \partial_n \chi,\end{aligned} \tag{2.1.9}$$

où u est une solution de (NLW) et χ une fonction régulière. L'idée est de trouver un poids χ adapté à la géométrie de l'obstacle permettant d'exploiter cette identité pour contrôler l'énergie au bord. En particulier, si $\nabla\chi$ est borné, le terme de gauche est controlé par l'énergie après intégration en temps, et si de plus χ est convexe, $-\partial_n \chi \geq c > 0$ et $\Delta^2\chi \leq 0$, le terme de droite majore le terme de bord qui nous intéresse et on obtient le contrôle

$$\frac{1}{T} \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt \lesssim \frac{E}{T}, \tag{2.1.10}$$

d'où la diffusion. Le choix naturel pour le poids χ est donc la jauge du convexe par lequel l'obstacle que l'on considère est illuminé. Cependant, si un tel poids vérifie toute les autres hypothèses, l'hypothèse de bilaplacien négatif est évidemment très rigide et restrictive, et nous ne lui trouvons pas de signification géométrique. En particulier, les jauges des ellipsoïdes qui nous intéressent, soit

$$\chi = \epsilon x^2 + y^2 + z^2, \ \epsilon \ll 1,$$

n'ont *pas* un bilaplacien négatif. En revanche, on remarque que c'est le cas si l'on rajoute une dimension, soit pour les ellipsoïdes de dimension quatre $\epsilon x^2 + y^2 + z^2 + w^2$. D'où l'idée d'étendre u comme une solution $\tilde{u}$ d'une équation quadridimensionelle. Le terme de bord associé à $\tilde{u}$ vérifie (2.1.10), et la marge dont on dispose entre cette décroissance très forte et (2.1.6) nous permet de revenir à la solution de l'équation qui nous intéresse. Plus précisément on obtient

$$\int_0^T \int_{\partial\Theta} |\partial_n \tilde{u}|^2 \partial_n \tilde{\chi} d\sigma dt \lesssim E(\tilde{u})$$

où $\tilde{\chi}$ est le multiplicateur tridimensionel. Mais

$$E(\tilde{u}) \lesssim T E(u)$$

alors que

$$\int_0^T \int_{\partial\Theta} |\partial_n \tilde{u}|^2 \partial_n \tilde{\chi} d\sigma dt \gtrsim \ln T \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt$$

D'où

$$\frac{1}{T} \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt \lesssim \frac{1}{\ln T} E(u) \longrightarrow 0,$$

il y a donc diffusion.

2.1.3. Résultat : Théorème de rigidité à l'extérieur de deux convexes stricts

Motivés par le résultat précédent, nous nous intéressons au problème de la diffusion à l'extérieur de deux convexes stricts - problème pour lequel nous disposons des estimations de Strichartz globales et sans pertes. Cependant, il n'y a pas d'espoir pour qu'un calcul de multiplicateur seul permette d'obtenir le résultat : en effet Morawetz, Raltson et Strauss montrent en particulier dans [MRS77] que l'hypothèse de non-captivité équivaut à l'existence d'une *fonction d'échappement* $a \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, c'est à dire une fonction de l'espace des phases vérifiant

$$\xi \cdot \nabla_x a(x, \xi) \geq c > 0 \text{ sur } \Omega,$$
$$\frac{a(x,\xi) - a(x,\xi')}{(\xi - \xi') \cdot n(x)} \geq c > 0 \text{ sur } \partial\Omega,$$

où ξ' est la direction réfléchie associée à ξ par les lois de l'optique géométrique. Or, à un muliplicateur χ convenable au sens de la section précédente, on peut associer une fonction d'échappement

$$a(x, \xi) = \nabla \chi(x) \cdot \xi.$$

Cependant, nous pouvons construire un multiplicateur qui a le bon comportement *partout sauf au voisinnage du rayon capté* pour montrer que :

Théorème 2.4 ([Laf17a]). *Soient Θ_1 et Θ_2 deux convexes stricts de $\mathbb{R}^3$. Alors, il existe une famille $(\mathcal{S}(T))_{T\geq 1}$ de voisinages ouverts du rayon capté $\mathcal{R}$ vérifiant*

$$\mathcal{S}(T) \longrightarrow \mathcal{R} \text{ lorsque } T \longrightarrow +\infty$$

telle que toute solution globale de (NLW) dans $\Omega := \mathbb{R}^3 \setminus (\Theta_1 \cup \Theta_2)$ vérifie, lorsque T va vers l'infini

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))\setminus\mathcal{S}(T)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0.$$

Où le critère

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0$$

équivaut à (2.1.6) – par exemple par application de l'identité de Morawetz (2.1.9). En particulier, les solutions dont le flot $\{u(t),\ t \geq 0\}$ est d'adhérence compacte dans $\dot{H}^1$ diffusent donc, car elles ne peuvent concentrer leur énergie au voisinage du rayon capté. En conséquence de quoi

Corollaire 2.5. *Il n'existe pas de solution à flot compact de (NLW) à l'extérieur de deux convexes stricts qui ne diffuse pas.*

Il s'agit d'un théorème de rigidité pouvant être exploité dans le cadre d'un argument de concentration-compacité/rigidité introduit par Kenig et Merle : supposant qu'il existe une énergie critique finie au dessus de laquelle existent des solutions qui ne diffusent pas, on construit une solution dite critique, qui est de flot compact et non diffusive, et sera par la suite éliminée – c'est là un travail en cours, en collaboration avec C. Laurent [LL].

Dans le cas le plus simple de deux boules centrées en 1 et -1, on choisit le poids (figure 2.1.1)

$$\chi(x) = |x-1| + |x+1|,$$

qui vérifie

$$\nabla\chi \cdot n \geq 0,\ \ \Delta\chi \gtrsim 1 \text{ sur } B(0,A),\ \ \Delta^2\chi = 0.$$

L'identité de moment permet donc d'obtenir

$$\int_0^T \int_\Omega (D^2\chi\nabla u, \nabla u) + \int_0^T \int_{\Omega\cap B(0,A)} |u|^6 \lesssim E,$$

mais la hessienne $D^2\chi$ de χ s'annule sur le rayon capté. L'étude de ses valeurs propres fournit le domaine $S(T)$:

$$D^2\chi \gtrsim \alpha \text{ à l'extérieur d'un voisinnage } S(\alpha) \text{ du rayon capté}$$

et on choisit $\alpha(T)$ tendant vers zéro assez lentement pour obtenir la décroissance qui nous intéresse.

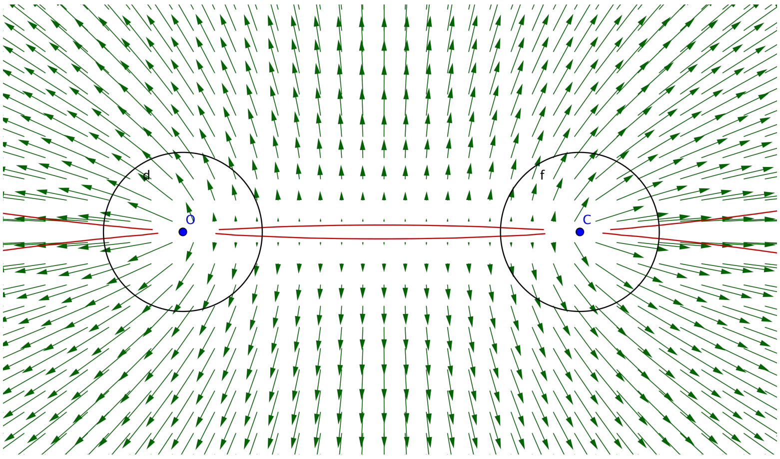

FIG. 2.1.1. : $\nabla\chi$ et $S(\alpha)$ pour deux boules

A l'extérieur de deux convexes stricts quelconques, ce poids n'est évidemment plus adapté et on utilise une famille de poids

$$\chi_c(x) = |x-c| + |x+c|, \ c \longrightarrow \infty$$

qui, pour $c \longrightarrow \infty$ a un gradient de comportement similaire à $\frac{(x_1,x_2)}{|(x_1,x_2)|}$ (figure 2.1.2), dont le produit scalaire avec la dérivée normale sortante aux obstacles est positif (et s'annule sur le rayon capté), et on ajuste les vitesses de convergence $c(T) \longrightarrow +\infty$ et $\alpha(T) \longrightarrow 0$ afin d'obtenir une décroissance suffisante. Notons que, là encore, nous avons exploité la marge dont on dispose entre la décroissance suffisante (2.1.6) et la forte décroissance en $1/T$ donnée par le membre de gauche dans (2.1.9).

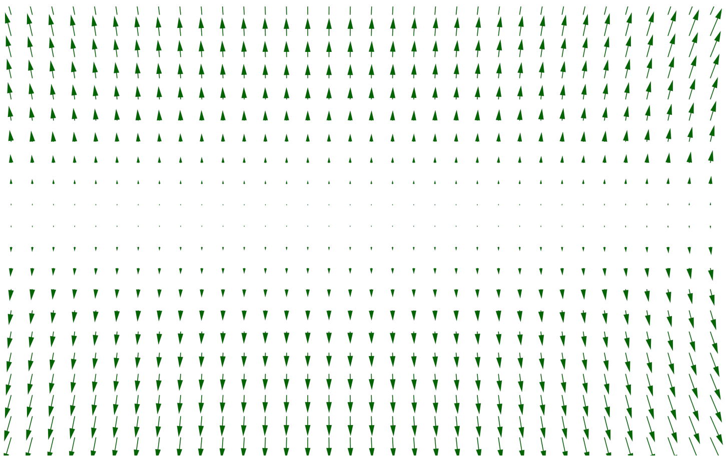

FIG. 2.1.2. : $\nabla\chi_c$ pour $c \gg 1$

2.2. L'équation de Schrödinger non linéaire avec un potentiel

On s'intéresse maintenant à l'équation de Schrödinger non linéaire avec un potentiel, dans l'espace libre $\mathbb{R}^d$

$$i\partial_t u - \Delta u + Vu = \epsilon u|u|^\alpha, \ u(0) = u_0 \in H^1, \tag{NLS$_V$}$$

qui conserve en particulier la norme L^2, ou masse, et l'énergie

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 + \int V|u(t)|^2 + \epsilon\frac{1}{\alpha+2}\int |u(t)|^{\alpha+2} = E(u(0)).$$

Dans le cas libre $V = 0$, les estimations de Strichartz fournissent, à l'aide d'un argument de point fixe, l'existence locale de solutions pour les équations critiques et sous critique en énergie $\alpha \leq \frac{4}{d-1}$. En énergie sous-critique, la durée de vie de la solution dépend de la seule norme de la donnée initiale et la conservation de l'énergie donne l'existence globale pour l'équation défocalisante $\epsilon = 1$ à laquelle nous nous intéressons ici. De la même manière que l'équation des ondes non linéaire, on définit la notion de diffusion (ou *scattering*)

Définition 2.6. Soit u une solution globale de (NLS). On dit que u *diffuse* s'il existe des données $v_\pm$ telles que

$$\|u(t) - e^{-it\Delta}v_\pm\|_{H^1} \longrightarrow 0$$

lorsque $t \longrightarrow \pm\infty$.

Une de nos motivations pour nous intéresser à (NLS$_V$) dans le cadre de l'étude de problèmes extérieurs est la similarité entre le potentiel et un obstacle. L'absence de bord et donc de réflexion rend cependant certains arguments plus faciles à mener à terme et l'étude de (NLS$_V$), en plus de son interêt intrinsèque, peut donc être une source d'idées pour l'étude de problèmes avec obstacles.

Nous montrons la diffusion dans les deux situations suivantes :

1. pour un potentiel répulsif ($x \cdot \nabla V \leq 0$),

2. pour une somme de deux potentiels répulsifs dont les surfaces de niveau sont convexes.

Notons que la situation (1) est analogue à l'extérieur d'un obstacle étoilé, et la situation (2), dans laquelle il existe des trajectoires captées, à l'extérieur de deux convexes stricts. Ce dernier résultat fournit en particulier un résultat de diffusion dans une géométrie captante. Le premier résulat peut être trouvé dans [Laf16], le second dans [Laf18a]. Ils sont ici respectivement présentés en sections 10 et 12.

2.2.1. Concentration-compacité/rigidité

Nakanishi [Nak99] montre le scattering dans l'espace d'énergie H^1 des solutions de l'équation défocalisante homogène ($V = 0$), en utilisant des inégalité du type Morawetz. De telles inégalités, étendues à des versions bilinéaires, continuent par la suite à être exploitées pour obtenir des résultats du type scattering pour l'équation défocalisante : citons notamment les travaux de [PV09], [CGT09], et [Vis09].

Une méthode très robuste pour traiter entre autres les problèmes du type scattering, dite de *concentration-compacité/rigidité* est introduite par Kenig et Merle [KM06]. Elle est motivée notamment par l'étude de l'équation focalisante ($\epsilon = -1$), pour laquelle existe un seuil d'énergie en deçà duquel les solutions diffusent, donné par le soliton $u = e^{it}Q$, où

$$\Delta Q - Q + Q^{\alpha+1} = 0,$$

solution stationnaire de l'équation et l'objet pour lequel la décroissance induite par le groupe linéaire et l'effet focalisant de la non-linéarité se compensent exactement. Cette méthode, étendue pour Schrödinger au cas intercritique dans [HR08], et à toutes les dimensions d'espace dans [FXC11] a engendré un nombre impressionnant de travaux. Nous l'utilisons ici dans le cas d'un potentiel $V \neq 0$. Avant de discuter de la généralisation au cas inhomogène, présentons la brièvement dans le cas défocalisant homogène $V = 0$.

On définit l'énergie critique

$$E_c = \sup \left\{ E > 0 \mid \forall \varphi \in H^1,\ E(\varphi) < E \Rightarrow \mathcal{S}(t)\varphi \in L^p L^r \right\},$$

où $\mathcal{S}(t)\varphi$ désigne la solution de (NLS) avec donnée φ, et on suppose par l'absurde que

$$E_c < \infty.$$

Il existe alors une suite de données $(\varphi_n)_{n\geq 1}$ minimisante pour E_c engendrant des solutions non diffusives :

$$E(\varphi_n) \geq E_c,\ E(\varphi_n) \longrightarrow E_c,\ \mathcal{S}(t)\varphi \notin L^p L^r.$$

L'idée est alors d'extraire de $(\varphi_n)_{n\geq 1}$, qui est bornée dans H^1, une valeur d'adhérence φ_c, qui engendrera une solution à flot compact qui ne diffuse pas. Une première description des suites bornées dans H^1 est donnée par les lemmes de concentration-compacité à la Lions, mais l'avancée majeure qui permet de clore l'argument est la description de telles suites en terme de somme de profils *à un reste diffusif près,* chaque profil étant une solution en un temps donné de l'équation linéaire associée au problème. Une telle décomposition en profils apparaît pour la première fois dans [BG99]. Pour l'équation de Schrödinger elle peut par exemple s'écrire

$$\varphi_n = \sum_{j=1}^{J} e^{-it_j^n \Delta} \psi_j(\cdot - x_j^n) + R_n^J \quad \forall J \in \mathbb{N},$$

les profils $e^{it_j^n A} \tau_{x_n^j} \psi_j$ étant orthogonaux au sens où

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \longrightarrow \infty, \quad \forall j \neq k,$$

et

$$\|u_n\|_{H^1}^2 = \sum_{j=1}^{J} \|\psi_j(\cdot - x_j^n)\|_{H^1}^2 + \|R_n^J\|_{H^1}^2 + o_n(1), \quad \forall J \in \mathbb{N},$$

$$\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \|e^{-it_j^n\Delta}\psi_j(\cdot - x_j^n)\|_{L^p}^p + \|R_n^J\|_{L^p}^p + o_n(1), \quad \forall J \in \mathbb{N},$$

et le reste diffusif au sens où engendrant un flot linéaire petit dans les normes Strichartz

$$\forall \epsilon > 0,\ \exists J \in \mathbb{N}, \quad \limsup_{n\to\infty} \|e^{-it\Delta}R_n^J\|_{L^pL^r} \leq \epsilon.$$

On associe alors à chaque profil un profil *non linéaire* $\tilde{\psi}_j$ tel que

$$\|\mathcal{S}(-t)\tilde{\psi}_j - \psi_j\|_{H^1} \longrightarrow 0$$

lorsque $t \longrightarrow t_j$, et l'orthogonalité des profils permet alors d'approximer

$$\mathcal{S}(t)\varphi_n \approx \sum_{j=1}^{J} \mathcal{S}(t - t_j^n)\tilde{\psi}_j(\cdot - x_j^n) \tag{2.2.1}$$

avec un reste petit au sens des normes Strichartz. Mais, s'il existe plus d'un profil non trivial, chacun d'entre eux emporte une partie non triviale de l'énergie, et pour n assez grand ils sont tous sous l'énergie critique E_c et engendrent donc des solutions qui diffusent ! D'après (2.2.1) φ_n engendre alors une solution qui diffuse, ce qui n'est pas possible. Il n'y a donc qu'un profil non trivial φ_c. On prend alors

$$u_c = \mathcal{S}(t)\varphi_c,$$

c'est notre solution critique. Appliquer une seconde fois le raisonnement au flot $\{u_c(t),\ t \in \mathbb{R}\}$ permet de montrer que ce dernier est d'adhérence compacte dans H^1.

Mais un calcul classique de viriel, qui est un analogue pour Schrödinger de l'identité de Morawetz (2.1.9) :

$$\partial_t^2 \int \chi|u|^2 = 4\int (D^2\chi\nabla u, \nabla u) + \frac{2}{\alpha+2}\int \Delta\chi|u|^{\alpha+2} - \int \Delta^2\chi|u|^2. \tag{2.2.2}$$

permet de montrer qu'une telle solution ne peut exister. On prend classiquement un poids quadratique

$$\chi_R := R^2\phi(\frac{\cdot}{R}) \text{ où } \phi(x) = \begin{cases} x^2 & \text{pour } |x| \leq 1 \\ 0 & \text{pour } |x| \geq 2 \end{cases}$$

alors

$$\left|\partial_t \int \chi_R|u|^2\right| \leq C(E)R,$$

tandis que

$$\partial_t^2 \int \chi_R |u|^2 \geq \delta > 0,$$

pour R assez grand par l'identité (2.2.2), car u étant à flot compact

$$\sup_{t\in\mathbb{R}} \int_{|x|\geq R} \left(|\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^{\alpha+2}\right) dx \longrightarrow 0$$

lorsque $R \longrightarrow \infty$.

Dans le cas de l'équation avec un potentiel, $V \neq 0$, il s'agit de généraliser ces deux points. Or un point clé pour approximer (2.2.1) dans l'argument précédent est l'invariance par translation du flot

$$\mathcal{S}(t - t_j^n)\tilde{\psi}_j(\cdot - x_n^j) = \left(\mathcal{S}(t - t_j^n)\tilde{\psi}_j\right)(\cdot - x_n^j)$$

qui n'est plus vraie en particulier dans le cas $V \neq 0$: il s'agit là de la principale difficulté à surmonter dans l'argument de concentration-compacité. Le calcul de viriel permettant de montrer la rigidité fait quand à lui apparaître un terme supplémentaire du au potentiel

$$-\int \nabla V \cdot \nabla \chi. \tag{2.2.3}$$

Il s'agit là de l'analogue du terme de bord apparaissant pour les ondes dans (2.1.9). Dans le cas d'un potentiel répulsif, ce terme a le bon signe pour le choix de potentiel quadratique et l'argument s'adapte alors immédiatement. Ce n'est plus le cas si le potentiel n'est pas répulsif.

2.2.2. Résultat : Diffusion pour NLS avec un potentiel

Dans [BV16], Banica et Visciglia s'attaquent au problème en s'intéressant à l'équation avec un potentiel de Dirac

$$i\partial_t u - \Delta u + \delta_0 u = u|u|^\alpha, \; u(0) = u_0 \in H^1,$$

qui est singulier mais pour lequel des formules explicites sont disponibles. Motivés par ce résultat, nous montrons la diffusion pour NLS avec un potentiel positif répulsif arbitraire mais moins singulier en dimension un :

Théorème 2.7 ([Laf16]). *Soit $\alpha > 4$ et $V \in L^1_1(\mathbb{R})$ tel que $V' \in L^1_1(\mathbb{R})$. On suppose de plus que V est positif et répulsif : $V \geq 0$ et $xV' \leq 0$. Alors, toute solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ de (NLS_V) avec potentiel V diffuse dans $H^1(\mathbb{R})$.*

Les problèmes du type scattering sont classiquement plus difficiles en petites dimensions, car des dimensions supplémentaires rajoutent des directions de dispersion. C'est pourquoi nous choisissons, pour présenter la preuve de manière plus simple, de nous limiter au cas unidimensionel. Notons que ce résultat est généralisé à toute dimension $d \neq 2$ dans [Laf18a] présenté au paragraphe suivant.

Goldberg et Schlag montrent dans [GS04] l'estimation dispersive

$$\|e^{it(-\Delta+V)}\|_{L^1\longrightarrow L^\infty}\lesssim\frac{1}{t^{1/2}}\tag{2.2.4}$$

pour un potentiel dans $L^1_1(\mathbb{R})$ n'ayant pas de résonance en énergie nulle. Un argument simple montre qu'un potentiel positif vérifie automatiquement cette dernière hypothèse et l'on dispose par un argument $TT^\star$ standard des estimations de Strichartz utiles à l'étude du problème. En particulier, elles permettent d'obtenir l'existence locale de solutions, et donc globales via la conservation de l'énergie.

L'idée clé permettant de surmonter l'absence d'invariance par translation de l'équation et d'obtenir le résultat est qu'*un profil qui part à l'infini ne voit plus le potentiel*, c'est à dire que pour $x_n\longrightarrow\infty$

$$\|e^{-it\Delta}\tau_{x_n}\psi-e^{-it(\Delta-V)}\tau_{x_n}\psi\|_{L^pL^r}\longrightarrow 0,\tag{2.2.5}$$

où τ désigne l'opérateur de translation : $\tau_{x_n}\psi=\psi(\cdot-)$, de la même manière pour le terme de Duhamel

$$\|\int_0^t e^{-i(t-s)\Delta}\left(U_n|U_n|^\alpha\right)(s)ds-\int_0^t e^{-i(t-s)(\Delta-V)}\left(U_n|U_n|^\alpha\right)(s)ds\|_{L^pL^r}\longrightarrow 0,\tag{2.2.6}$$

où $U_n(x,t)=U(x-x_n,t)$, U étant une solution de l'équation non linéaire libre ($V=0$), et enfin, t étant fixé

$$\|e^{it\Delta}\tau_{x_n}\psi-e^{it(\Delta-V)}\tau_{x_n}\psi\|_{H^1}\longrightarrow 0.\tag{2.2.7}$$

Donnons par exemple une idée de la preuve de (2.2.5) - (2.2.6) et (2.2.7) se montrant de manière similaire. L'estimation de dispersion (2.2.4) permet de montrer que

$$\sup_{n\in\mathbb{N}}\|e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(T,\infty)L^r}\longrightarrow 0,$$

lorsque T va vers l'infini, et on est donc réduits à montrer que, à T fixé

$$\|e^{-it\Delta}\tau_{x_n}\psi-e^{-it(\Delta-V)}\tau_{x_n}\psi\|_{L^p(0,T)L^r}\longrightarrow 0,$$

lorsque $n\longrightarrow\infty$. Or la différence

$$v_n:=e^{-it\Delta}\tau_{x_n}\psi-e^{-it(\Delta-V)}\tau_{x_n}\psi$$

vérifie l'équation de Schrödinger linéaire avec donnée nulle

$$i\partial_t u-\Delta u+Vu=Ve^{-it\Delta}\tau_{x_n}\psi,$$

et peut donc être estimée grace aux estimations de Strichartz :

$$\begin{aligned}\|v_n\|_{L^p(0,T)L^r}&\lesssim\|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^{\gamma'}(0,T)L^1}\leq T^{\frac{1}{\gamma'}}\|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,T)L^1}\\&=T^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^1},\end{aligned}\tag{2.2.8}$$

qui tend donc vers 0 lorsque $x_n \longrightarrow \infty$ car V est dans L^1.

L'estimation (2.2.7) et des arguments d'analyse fonctionnelle permettent de voir que l'opérateur $-\Delta + V$ vérifie les hypothèses du théorème de décomposition en profils abstrait obtenu par [BV16], et l'on dispose d'une d'une décomposition en profils associée à l'opérateur qui nous intéresse. Par la suite, les estimations (2.2.5) et (2.2.6) permettent de manière analogue à [BV16], de remplacer $e^{it(-\Delta+V)}$ par $e^{-it\Delta}$ pour des profils qui partent à l'infini. Ce propagateur commute avec les translations, on obtient l'approximation (2.2.1) et une solution critique est construite. Elle est enfin éliminée par le même calcul de viriel que pour le cas homogène $V = 0$, car comme remarqué au paragraphe précédent, le terme supplémentaire (2.2.3) qui surgit dans le calcul a dans le cas d'un potentiel répulsif le bon signe.

2.2.3. Résultat : Diffusion pour NLS avec une somme de deux potentiels répulsifs

Dans le cas d'un potentiel non répulsif, la construction d'un élément critique reste valide, mais ce dernier ne peut être éliminé par les techniques précédentes. Or, de la même manière que pour l'équation des ondes non linéaires à l'extérieur d'obstacles, on a l'intuition qu'il devrait y avoir diffusion pour un potentiel qui ne piège pas l'énergie, ou la piège peu. Une stratégie pour montrer la diffusion dans de telles situations serait alors d'adapter la partie rigidité du raisonnement précédent à de tels potentiels. Or, on a vu que nous sommes capable d'obtenir un théorème de rigidité pour l'équation des ondes non linéaire à l'extérieur de deux obstacles strictement convexes [Laf17a]. Motivés par l'analogie entre le calcul de viriel (2.2.2) et l'identité de Morawetz (2.1.9) pour les ondes grace à laquelle nous avons pu obtenir un tel résultat, nous montrons la rigidité pour une somme de deux potentiels répulsifs dont *les surfaces de niveau sont convexes* et obtenons

Théorème 2.8 ([Laf18a])**.** *Soit $d \geq 3$. Soient V_1 et V_2 deux potentiels répulsifs et positifs dont les surfaces de niveau sont convexes. On suppose de plus que $V = V_1 + V_2$ vérifie*

$$V, \nabla V \in L^{\frac{d}{2}}(\mathbb{R}^d, (1 + |x|^\beta)dx), \;\; \beta > \frac{2}{3} \tag{2.2.9}$$

et l'estimation dispersive

$$\|e^{it(-\Delta+V)}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{|t|^{d/2}}. \tag{2.2.10}$$

Alors, dans le régime intercritique

$$\frac{4}{d} < \alpha < \frac{4}{d-1}, \tag{2.2.11}$$

toute solution de (NLS_V) en dimension d avec potentiel $V = V_1 + V_2$ diffuse dans $H^1(\mathbb{R}^d)$.

L'hypothèse de décroissance (2.2.9) est l'analogue de l'hypothèse $V \in L^1_1(\mathbb{R})$, $V' \in L^1_1(\mathbb{R})$ apparaissant dans [Laf16]. Nous prenons comme boite noire la validité de l'estimation de dispersion ponctuelle (2.2.10), qui permet d'obtenir les estimations de Strichartz adaptées à l'étude du problème et une théorie de Cauchy locale. Notons cependant qu'en dimension $d = 3$, cette estimation est automatiquement vérifiée par de tels potentiels via les travaux de Goldberg et Schlag [GS04] dès lors que $\beta \geq 1$.

La première étape est de généraliser la construction d'une solution critique, et donc les estimations (2.2.5), (2.2.6) et (2.2.7) vues au paragraphe précédent à toutes les dimensions d'espace. Il est nécessaire pour cela disposer des bonnes estimations de Strichartz permettant d'estimer de manière analogue (2.2.8). Il s'agit là des estimations *endpoint*, qui sont fausses en dimension 2, pour laquelle nous n'obtenons donc pas le résultat. Notons que cette première partie, dans laquelle n'est pas utilisée l'hypothèse de convexité, généralise en particulier le theorem 2.7 vu au paragraphe précédent aux dimensions d'espace $d \geq 3$.

L'élimination des solutions à flot compact se fait par un calcul de viriel analogue à ce qui est fait pour montrer le théorème 2.4. Une nouvelle fois, nous utilisons le poids

$$\chi(x) = |x + c| + |x - c|, \ c \gg 1.$$

où la droite $(-c, c)$ relie les centre de V_1 et V_2. Alors,

$$|z'(t)| \lesssim C(E, M),$$

tandis que, pour une solution à flot compact

$$z''(t) \geq \frac{1}{c_1}\mu - \int \nabla\chi \cdot \nabla V |u|^2,$$

où $\mu > 0$. Le terme associé au potentiel qui apparaît n'a plus le bon signe. Cependant, pour $c \gg 1$, $\nabla\chi$ est presque transverse sortant aux surfaces de niveau des potentiels V_1 et V_2 et nous montrons alors que pour c_1 fixé assez grand, et pourvu que le potentiel décroisse suffisamment (d'où l'hypothèse (2.2.9)), on a

$$z''(t) \geq \frac{\mu}{2c_1},$$

d'où la contradiction.

Perspectives 3

Nous avons contribué dans cette thèse à l'étude des équations de Schrödinger et des ondes, à la fois linéaires et non linéaires, dans des géométries captantes instables. C'est un travail de longue haleine qu'il faudra bien entendu poursuivre. Il s'inscrit dans l'objectif à très long terme de comprendre finement le lien entre le comportement d'une onde et la géométrie du domaine ou variété dans lequel elle vit.

Un prolongement naturel à notre travail sur les estimations de Strichartz à l'extérieur de plusieurs convexes stricts serait de généraliser les travaux de Ivanovici et Lebeau [IL17] à l'extérieur de plusieurs obstacles convexes. Plusieurs questions naturelles se posent : l'estimation sans perte est-elle toujours valable en dimension $d = 3$? En dimension $d \geq 4$, l'ajout d'obstacles supplémentaires fait-il apparaitre de nouveaux phénomènes de refocalisation, ou les seuls contre exemples à l'estimation dispersive sont-ils encore les points de Poisson-Arago ? L'estimation dispersive est-elle valable sans perte à l'extérieur de plusieurs convexes génériques ?

Compléter notre résultat de rigidité pour l'équation des ondes non linéaire à l'extérieur de deux obstacles strictement convexes par une description des profils afin de mener à bout la preuve de la diffusion pour les ondes critiques dans une telle géométrie est bien entendu une continuation de notre travail qu'il faudra mener à terme. Par ailleurs, nos travaux sont très dépendants de l'utilisation d'identités du type Morawetz, qui ont une grande rigidité. Nous avons bon espoir que compléter leur utilisation par des raisonnements du type concentration-compacité/rigidité comme nous l'avons ébauché permette de montrer des résultats dans des géométries plus générales. S'intéresser à des questions similaires, et adapter ces raisonnements à des variétés non euclidiennes sans bord nous semble également intéressant. Il sera cependant à terme nécessaire de développer de nouvelles méthodes et outils plus souples : dans la situation où il y a plus de deux obstacles convexes, les calculs de multiplicateur ne semblent par exemple permettre d'obtenir aucune information.

Enfin, très peu est connu sur le problème de la diffusion pour des conditions au bord de Neumann. Il serait digne d'interêt de s'attaquer au problème, en considérant par exemple pour commencer le cas le plus simple de l'extérieur d'une boule. Mentionnons, pour finir, que l'analogie entre obstacle et potentiel nous semble pertinente et que c'est un point de vue qu'il faudra continuer à développer.

Part II.

Introduction (English)

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4 Strichartz estimates outside many convex obstacles

Let $\Omega \subset \mathbb{R}^d$. We are interested in the Schrödinger

$$\begin{cases} i\partial_t u - \Delta u = 0,\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{LS}$$

and wave equations with Dirichlet boundary conditions:

$$\begin{cases} \partial_t^2 u - \Delta u = 0,\ u = 0 \text{ on } \partial\Omega, \\ (u, \partial_t u)(0) = (u_0, u_1). \end{cases} \tag{LW}$$

In order to understand the perturbative theory and the non-linear equations associated with these equations, it is crucial to understand how the *linear* flows decay. In particular, the measure of $L_t^p L_x^q$ Lebesgue space-time norms of solutions, known as Strichartz estimates, turn out to be particularly useful. The purpose of the first part of this thesis is to obtain such estimates, without loss with respect to the free space, in an unstable trapped framework: the exterior of many strictly convex obstacles.

4.1. Strichartz estimates in the free space

In free space $\Omega = \mathbb{R}^d$ for the Schrödinger equation, taking the spatial Fourier transform in (LS) and integrating in time leads to the explicit representation formula

$$u(x,t) = \frac{1}{4i\pi|t|^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy,$$

and thus to the pointwise dispersive estimate:

$$\|e^{it\Delta}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{t^{d/2}},$$

which is a first way of measuring the decay of the linear flow. It permits in particular, together with duality arguments, to obtain the so-called Strichartz estimates:

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2},$$

where (q, r) has to respect an admissibility condition given by the scaling of the equation, namely

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \ (q, r, d) \neq (2, \infty, 2).$$

Locally in time, such an estimate reflects a gain of integrability: if the data is L^2, the solution is L^r, $r > 2$, for almost every time. Globally, it describes a decay effect: the L^r norm of the solution has to go to zero, at least in a L^q way.

For the wave equation in $\mathbb{R}^d$, the Strichartz estimates are written as

$$\|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}$$

where $2 \leq q \leq \infty$ and $2 \leq r < \infty$ follow the scaling invariance

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s,$$

and the wave's admissibility condition

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}.$$

They can be obtained from frequency-localized estimates after Littlewood-Paley summation, which in turn are for example a consequence of the pointwise dispersive estimate for waves, namely

$$\|e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0\|_{L^\infty} \lesssim h^{-\frac{d+1}{2}} \frac{1}{t^{\frac{d-1}{2}}} \|\Psi(h\sqrt{-\Delta})u_0\|_{L^1},$$

where $\Psi(h\sqrt{-\Delta})$ localizes at frequencies $\sim h^{-1}$.

For both Schrödinger and wave equations, such estimates reflect the dispersive nature of the equation: each frequency propagates along different directions, and are worse for the wave equation, which enjoys less dispersion, because each frequency propagates at the same speed. In addition to their intrinsic interest, they turn out to be a very useful and robust tool for the study of the corresponding non-linear equations: they permit, for example, to obtain local existence results via fixed point methods, and provide a framework for the study of the long-time behavior of such equations.

Their long history goes back to the work of Strichartz [Str77] for the $p = q$ case in $\mathbb{R}^d$, extended to all exponents $q \neq 2$ for the Schrödinger equation by Ginibre and Velo [GV85b], and by the same authors [GV85a, GV87, GV95] and Lindblad and Sogge [LS95] to all exponents $r \neq \infty$ for the waves. The end-point estimates, $q = 2$ for Schrödinger, respectively $r = \infty$ for the waves, is more delicate. There were obtained in great generality by Keel and Tao [KT98], and are known to fail for the two-dimensional Schrödinger equation $(p, q, r) = (2, \infty, 2)$ – see for example [MS98], and the three-dimensional wave equation [KM93].

4.2. Strichartz estimates and geometry

We give, in this non exhaustive-section, a brief overview of some of the main results obtained in non euclidian geometries, which seem to us enlightening for the sequel of the exposition.

4.2.1. The wave equation

For the wave equation in a manifold without boundary, singularities propagates along geodesics at constant speed, and the finite speed of propagation shows that it suffises to work in local coordinates to obtain local estimates. It was the path followed for example by [Kap89], [MSS93], [Smi98], and [Tat02]. The case of a domain with boundaries, where propagation of singularities at the boundary have to be dealt with, is more delicate. The propagation of singularities in such a framework was studied by Melrose and Sjöstrand [MS78, MS82], who define the notion of generalized bicharacteristic broken ray, generalizing the notion of geometrical optics ray, along which the propagation occurs.

In the exterior of one strictly convex obstacle, the parametrix construction of Melrose and Taylor [MT87] gives an explicit description of the solution near diffractive points. It is in particular used by Smith and Sogge [SS95] to show that outside one strictly convex obstacle, Strichartz estimates hold with no loss with respect to the flat case.

Such a parametrix is not useful anymore in a general geometry, because of the presence of rays which are reflected multiple times and their limits, the so-called gliding rays. In the interior of a strictly convex domain, we used to believe for a long time that Strichartz estimates should have been true with no loss with respect to the flat case, because gallery modes, which accumulate the energy near the boundary, are responsible for no loss. However, Ivanovici [Iva12] gives a counter example coming from the caustics that may appear near the boundary in small times.

The first local estimates for the wave equations were shown by Burq, Lebeau and Planchon, [BLP08]. They are obtained as a consequence of spectral projector estimates of Smith and Sogge [SS07a], and are then generalized by Blair, Smith and Sogge [BSS09a]. They hold with no derivative loss – the loss coming from tangent directions are compensated by the fact that such directions live in a small frequency cone – but for restricted indices.

Recently, Ivanovici, Lebeau and Planchon [ILP14a] in a model case, and Ivanovici, Lascar, Lebeau and Planchon [ILLP] improve all known results in dimensions $d \geq 3$ inside a strictly convex domain. They show a pointwise dispersive estimate with a loss of $1/4$ derivative and prove that, because of caustic phenomena, this loss is unavoidable. Moreover, showing that such caustics appear sufficiently rarely so that a mean in time is able to reduce the loss they induce, they obtain in [ILP14b] Strichartz estimates with no loss of derivatives, for a reduced range of indices corresponding to a loss of $1/6$ in the pointwise dispersive estimate.

4.2.2. The Schrödinger equation

Finite speed of propagation for the semi-classical equation

In the case of the Schrödinger equation, things turn out to be much more delicate because of the infinite speed of propagation. However, the semi-classical change of variable permits thanks to the method of Lebeau [Leb92] to go back to a finite – but non constant – speed of propagation. We consider a frequency-localized data, of frequencies of order h^{-1}, and perform the time change of variable $t \to ht$. We will consider the so called *semi-classical* equation

$$\begin{cases} i\partial_t u - h\Delta u = 0, \ u = 0 \text{ on } \partial\Omega, \\ u(0) = \Psi(-h^2\Delta)u_0. \end{cases} \tag{LS$_{sc}$}$$

Now, let $(e_n)_{n\geq 0}$ be an $L^2(\Omega)$-orthonormal family of eigenfunctions of $-\Delta_D$ on Ω, with corresponding eigenvalues $(\lambda_n)_{n\geq 0}$. We identify the data

$$u_{k,0} \in \text{Vect}\left\{e_n, \ 2^k\alpha \leq \sqrt{\lambda_n} \leq 2^k\beta\right\}$$

with, for $h_k = 2^{-k}$, the solutions of the system

$$\begin{cases} i\partial_t u_k - h_k\Delta u_k = 0, \ u_k = 0 \text{ on } \partial\Omega, \\ u_k(0) = u_{k,0}. \end{cases} \tag{LS$_{sc}$}$$

We will say that $\rho_0 = (y_0, \eta_0) \in T^\star(\Omega \times \mathbb{R}) \cup T^\star(\partial\Omega \times \mathbb{R})$ is *not* in the wave-front set up to the boundary $WF_b(U)$ of $U = (u_{k,0})_{k\geq 0}$, if there exists an h_k-pseudodifferential operator, tangential at the boundary, with compactly supported symbol p, elliptic in ρ_0, and $\psi \in C_c^\infty$ equal to 1 near y_0 such that

$$\forall\sigma, \forall N, \ \|\text{Op}_{h_k}(p)\psi u_k\|_{H^\sigma(\mathbb{R}\times\Omega)} \leq C_{N,\sigma} h_k^N.$$

Then, according to [Leb92],

$$WF_b(u_k) \subset \Sigma_b \cap \{\tau \in [\alpha, \beta], \ |\xi| \leq D\},$$

where Σ_b denotes the characteristic set $\Sigma_b = \left\{\tau - |\xi|^2 = 0\right\}$. Moreover, if we define $\Theta(U)(x,t,s) = \sum_k e^{-ih_k^{-1}s} u_k(x,t,s)$, $\Theta(U)$ verify

$$\partial^2_{t,s}\Theta(U) - \Delta(U) = 0, \ u = 0 \text{ in } \partial\Omega \times \mathbb{R} \times \mathbb{R},$$

for which Melrose et Sjöstrand's theorem is at hand. From [Leb92], we have

Lemma 4.1. *For all $s_0 \in \mathbb{R}$, for all $\rho_0 \in T^\star(\Omega \times \mathbb{R}) \cup T^\star(\Omega \times \mathbb{R})$,*

$$\rho_0 \in WF_b(U) \iff \theta(\rho_0, s_0) \in WF_b(\Theta(U))$$

where $\theta(\rho, s) := (\rho; s, 1)$.

The wave-front set of the sequence U is therefore invariant by the generalized bicharacteristic flow on $T^\star(\Omega \times \mathbb{R}) \cup T^\star(\partial\Omega \times \mathbb{R})$.

Strichartz estimates

Hence, working on the semi-classical equation with frequency-localized data, we can take advantage of the finite speed of propagation, and, in the case of a manifold without boundary, work in local coordinates. It is for example the strategy followed by Burq, Gérard and Tzetkov [BGT04b]. They work on the semi-classical equation with frequency-localized data of order $\sim h^{-1}$ in a manifold without boundary, for which they construct a parametrix in times h, which gives in turn Strichartz estimate with a loss of $1/p$-derivatives with respect to the flat case:

$$\|e^{it\Delta}u_0\|_{L^p([0,T],L^q(M))} \leq C(T)\|u_0\|_{H^{1/p}(M)}.$$

Notice that such a loss can be seen as natural: indeed, as the energy propagates at speed h^{-1}, we expect to be able to show estimates without loss in times h, which give, after Littlewood-Paley summation, a loss of order $1/p$.

The case of a manifold with boundary, for which reflexions have to be be dealt with, is once again more intricate. A first strategy, introduced by Anton [Ant08] and Blair, Smith and Sogge [BSS08] consists in straightening out the boundary and symmetrizing with respect to it in order to reduce to the case of a manifold without boundary with Lipshitz coefficients. They obtain in this way estimates with a loss of order $4/3p$, that is

$$\|e^{it\Delta}u_0\|_{L^p([0,T],L^q(M))} \leq C(T)\|u_0\|_{H^{4/3p}(M)}.$$

However, such a method does not involve the particular geometry of the domain, and there is no hope to use it to show improved results in particular geometries where better estimates are expected. In a compact manifold with strictly concave boundaries – such as the Sinaï billiard – Ivanovici makes use in [Iva10] of the finite speed of propagation for the semi-classical equation together with the parametrix of Melrose and Taylor, in order to show estimates in times h for spectrally-localized data. Once again, such estimates give local estimates with a loss of $1/p$ derivates after Littlewood-Paley summation. Taking advantage of an argument we will discuss in the next section, they deduce global estimates with no loss with respect to the free space for the Schrödinger equation outside one strictly convex obstacle. Moreover, she shows in [Iva12] that inside a strictly convex domain, a loss of $1/6$ derivatives in unavoidable. Her counter-example is given by gallery modes – let us recall that these modes induce no loss for the wave equation.

Finally, let us notice that generalizing the work of [ILP14a] and [ILLP] to the Schrödinger equation in order to obtain sharp estimates inside a strictly convex domain seems for now out of reach: the non-constant speed of propagation makes multiple reflexions near the boundary extremely difficult to handle.

Remark. Strichartz estimates with no loss does not imply necessarily a pointwise dispersive estimate as good as in the flat case. A remarkable example of this fact is given by Ivanovici and Lebeau, which show in particular in [IL17] that the pointwise dispersive estimate *fails* in the exterior of balls in dimensions $d \geq 4$. As there is no explicit concentration of energy in such a framework, this result is rather surprising. However, by the Huygens principle, every point of the boundary reached by the wave acts as a new source of light:

the wave can therefore refocalise in the experimentally observed, so-called Poisson-Arago spot, which is at the origin of the counter-example given by the authors.

4.3. Smoothing effect, Strichartz estimates and non-captivity

An heuristic obstruction to the establishment of global Strichartz estimates without loss is the presence of *trapped rays:* they should, at least in some sense, induce concentration of energy effects. Therefore, it is natural to work under the *non-trapping assumption*:

Definition. An obstacle $\Theta \subset \mathbb{R}^d$ is said to be non-trapping if all the rays of geometrical optics of $\mathbb{R}^d \backslash \Theta$ escape all compact in finite time. In the same way, a manifold without boundary (M, g) is said to be non-trapping if all the geodesics escape all compact in finite time.

Many authors work under this assumption: let us mention the results of Smith and Sogge [SS00], Staffilani and Tataru [ST02], Burq [Bur03], Metcalfe [Met04], Hassell, Tao and Wunsch [HTW06], Bouclet and Tzetkov [BT07], and Bouclet [Bou11].

In particular, Smith and Sogge show in [SS00] for the wave equation in the odd-dimensional case, that under the non-trapping assumption, local in time Strichartz estimates imply global ones. This result is extended to all space dimensions independently by Burq [Bur03] and Metcalfe [Met04]. A remarkable fact appearing in [Bur03] is that local-energy decay estimates for the wave equation, namely

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R},\dot{H}^1\times L^2)} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2},\ \forall\chi \in C_c^\infty,$$

that hold in particular in the free space, permit to stack local in time Strichartz estimates in order to obtain global ones. In the same way, the Schrödinger counterpart of the local energy decay, the so-called local smoothing estimate

$$\|\chi u\|_{L^2(\mathbb{R},H^{1/2})} \lesssim \|u_0\|_{L^2},\ \forall\chi \in C_c^\infty,$$

that reflects a gain of one-half derivative, permits to stack semi-classical Strichartz estimates in time $\sim h$. Heuristically, the above estimate means that a frequency-localized Schrödinger wave remains for a time of order $\sim h$ near the obstacle: therefore, it suffices to show Strichartz estimates in time $\sim h$. This is for example the strategy followed by [Iva10] in order to deduce global Strichartz estimates without loss outside one strictly convex obstacle from semi-classical ones in time h on the Sinaï billiard.

Notice that a remarkable precursor to local energy estimates in the exterior of non-trapping obstacles for the wave equation are the works of Morawetz [Mor61], Morawetz, Raltson and Strauss [MRS77]. Their results rely on the construction of a so-called escape function, that is, a phase-space function which is non-decreasing along rays and respects the laws of reflection on the boundary. Such a function was constructed by [MRS77] in the exterior of a non-trapping obstacle. Constructing it in manifold without boundaries lies at the heart of more recent works: let us mention, for example, the result of Datchev and Vasy [DV12].

Burq, Gérard and Tzetkov [BGT04a] show that the same smoothing effect as in the free space holds in the exterior of any non-trapping obstacle. Their result is shown as a consequence of the resolvent estimate

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon)^2)^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \lesssim |\lambda|^{-1},$$
$$|\lambda| \gg 1, \ 0 < \epsilon \ll 1,$$

for which the non-trapping assumption plays a crucial role, was shown in great generality by Lax and Philips [LP89], Melrose and Sjöstrand [MS78, MS82], Vainberg [Vn88], Vasy and Zworski [VZ00]. It allows, with a similar method, to deduce local energy decay for waves as well. On the other hand, Burq shows in particular in [Bur04] in a very general setting, that the smoothing effect without loss, even local in time, *fails* as soon as a trapped trajectory exists. More precisely, in the framework we are interested in, for any $t_0 > 0$, the map

$$u_0 \in C_c^\infty \subset L^2 \longrightarrow \chi e^{it\Delta} u_0 \in L^2([0,t_0], H^{\frac{1}{2}})$$

is never bounded if χ is not vanishing on the trapped set.

4.4. Smoothing effect and trapped trajectories

A natural question raised by the previously mentioned result of [Bur04], is if a smoothing effect *with a loss* could be true in some trapped geometries. We can essentially distinguish three kind of trapped trajectories: *elliptic*, *parabolic*, and *hyperbolic* ones. A point $\rho \in T^\star\Omega$ of a captive trajectory been given, let us consider the Poincaré map of first return $\mathcal{P}$ in a phase-space neighborhood of ρ. We say that the trajectory is elliptic (stable) if all the eigenvalues of the differential of $\mathcal{P}$ in ρ have modulus lesser than one, parabolic (neutral) if they are all of modulus one, and hyperbolic (unstable), if a least one of them has a modulus greater than one.

If an elliptic trajectory exists, it is possible to construct compactly-supported quasi-modes $(e_n)_{n\geq 1}$ associated to $\lambda_n \longrightarrow \infty$

$$-\Delta e_n = \lambda_n e_n + r_n, \ \|r_n\|_{H^s} = O(\lambda_n^{-\infty}),$$

and then, for any $t_0 > 0$ and $s > 0$

$$\|e^{it\Delta}e_n\|_{L^1([0,t_0[,H^s)} \geq \|\int_0^{t_0} e^{it\Delta}e_n\|_{H^s}dt = t_0\|\lambda_n e_n + r_n\|_{H^s}$$
$$\geq t_0\lambda_n - O(\lambda_n^{-\infty}),$$

and therefore, no kind of smoothing estimate can be true.

Smoothing effect in the exterior of many strictly convex obstacles

We are now interested in the canonical example of the hyperbolic case, namely, the exterior of two strictly convex obstacles, and its generalization to the exterior of many ($N \geq 3$) strictly convex obstacles. Such a framework was first studied by M. Ikawa [Ika82, Ika88] and C. Gérard [G88] in the eighties. Ikawa shows in particular in [Ika82] in the exterior of two strictly convex obstacles, and in [Ika88] in the exterior of many of them, that the resolvent $\chi(-\Delta_D - \tau^2)^{-1}\chi$, holomorphic on $\{\text{Im}\tau > 0\}$, admits a holomorphic continuation in the strip $\{|\tau| > 1, \text{Im}\tau \leq \alpha\}$ of the upper half-plane an satisfies there

$$\|\chi(-\Delta - \tau^2)^{-1}\chi\|_{L^2(\Omega)\to L^2(\Omega)} \leq C|\tau|^N,$$

for some $N \gg 1$. It implies, in particular, local energy decay and smoothing estimates with a loss of a finite number of derivatives. In the exterior of many ($N \geq 3$) obstacles, there is infinitely many trapped trajectories. The hyperbolicity of trajectories taken separately does not suffice, and we have to consider their interactions. Ikawa makes in [Ika88] a strong-hyperbolicity assumption:

Definition 4.2 (Ikawa condition 1: strongly hyperbolic setting)**.** There exists $\alpha > 0$ such that

$$\sum_{\gamma\in\mathcal{P}} \lambda_\gamma d_\gamma e^{\alpha d_\gamma} < \infty, \tag{4.4.1}$$

where $\mathcal{P}$ is the set of all periodic trajectories, d_γ is the length of $\gamma \in \mathcal{P}$ and $\lambda_\gamma = \sqrt{\mu_\gamma^1 \mu_\gamma^2}$, where $\mu_\gamma^{1,2}$ are the two eigenvalues of modulus lesser than one of the Poincaré map associated with γ.

This assumption is now known as *Ikawa's condition.* His work rely on a parametrix construction for the wave equation, which will be pursued by [Bur93] and ourselves and will be described in the sequel of this exposition. This construction is made at the cost of the technical assumption

Definition 4.3 (Ikawa condition, 2 : no obstacle in shadow)**.** For all i, j, k pairwise distinct,

$$\text{Conv}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset. \tag{4.4.2}$$

Burq notices in [Bur04] that the result of Ikawa, combined with a standard estimate for auto-adjoint operators and a semi-classical maximum principle inspired by Tang and Zworski [TZ00], implies in fact the sharper estimate

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \lesssim \frac{\log(2+\sqrt{\lambda})}{1+\sqrt{\lambda}},$$
$$\lambda \in \mathbb{R},\ 0 < \epsilon \ll 1,$$

and therefore a smoothing estimate with an arbitrary small loss of $\epsilon > 0$ derivatives. More precisely, the above resolvent estimate implies for frequency-localized data an estimate with a logarithmic loss with respect to the frequency, that is

$$\|\chi e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^2(\mathbb{R},H^{1/2})} \lesssim |\log h|^{\frac{1}{2}}\|\psi(-h^2\Delta)u_0\|_{L^2},\ \forall \chi \in C_c^\infty,$$

where $\psi(-h^2\Delta)$ localizes at frequencies $\sim h^{-1}$.

Remark. The intermediate case between the total loss induced by an elliptic trajectory and the trivial one of an hyperbolic trajectory is studied by Christianson and Wunsch [CW13], who show a smoothing estimate with an optimal polynomial loss in a manifold with a single weakly-hyperbolic trapped trajectory.

4.5. Compensate the loss induced by an hyperbolic trapped trajectory

If the precise link between smoothing effect without loss and trapped trajectories is now well understood, thanks in particular to the result of [Bur04], it is not the case for Strichartz estimates yet. In particular, Burq, Guillarmou and Hassell establish in [BGH10] Strichartz estimates *without loss* in a manifold without boundary which is conic at infinity, has negative curvature and a small enough hyperbolic trapped set: therefore, the presence of trapped trajectories, provided they are sufficiently unstable, does not induce necessarily a loss in Strichartz estimates. More precisely, the authors consider a conic at infinity manifold, with strictly negative curvature in a compact convex set containing the trapped set, the later been supposed small enough in the sense that his *topological pressure* in $s = \frac{1}{2}$ is strictly negative:

$$P(\frac{1}{2}) < 0.$$

One can find a precise definition of the topological pressure $P(s)$ in [NZ06] and [BGH10]. It measures the competition between the density of the trapped set and the instability of the flow. The assumption $P(1/2) < 0$ means that the instability wins. It is, in fact, the Ikawa's condition (4.4.1) written in the present setting: this problem is therefore the counterpart without boundary of the problem at exterior of many convex obstacles verifying Ikawa's condition.

According to [Bur04], the smoothing effect without loss fails in such a geometry. It can therefore not be used to stack semi-classical Strichartz in time h anymore. Nonnenmacher and Zworski [NZ06], Datchev [Dat09] show that in the same way as for the Ikawa's framework, a smoothing effect with logarithmic loss holds in such a manifold:

$$\|\chi e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^2(\mathbb{R},H^{1/2})} \lesssim |\log h|^{\frac{1}{2}}\|\psi(-h^2\Delta)u_0\|_{L^2},\ \forall \chi \in C_c^\infty.$$

The fundamental remark of [BGH10] is that this logarithmic loss can be compensated if one shows Strichartz estimates up to times $h|\log h|$, namely

$$\|e^{it\Delta}\psi(-h^2\Delta)u_0\|_{L^p([0,h|\log h|])L^q(M)} \lesssim \|\psi(-h^2\Delta)u_0\|_{L^2}.$$

Heuristically, the logarithmic loss means that the wave, because of trapping phenomena, spents now a time of order $h|\log h|$ near the obstacle: one has therefore to show estimates up to logarithmic times. In such times, the authors adapt the parametrix construction of Anantharaman [Ana08] in order to obtain a pointwise dispersive estimate near the trapped set. Notice that, for the wave equation in a similar geometry, such a parametrix was constructed by Bérard [B77].

4.6. Result: Strichartz estimates in the exterior of two strictly convex obstacles

Motivated by the works of Burq, Guillarmou and Hassel [BGH10] and Ikawa [Ika82, Ika88], we are interested in Strichartz estimates for Schrödinger and wave equations in the exterior of many strictly convex obstacles verifying Ikawa's condition. We deal first with the exterior of two strictly convex obstacles, for which we obtain global Strichartz estimates without loss with respect to the free space:

Theorem 4.4 ([Laf17b, Laf17a]). *Let $\Theta_1, \Theta_2 \subset \mathbb{R}^3$ be two strictly convex obstacles. Then, global Strichartz estimates without loss hold for both Schrödinger and wave equations in $\Omega = \mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2)$:*

$$\|u\|_{L^p(\mathbb{R}, L^q(\Omega))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}, \ (Waves)$$
$$\|u\|_{L^p(\mathbb{R}, L^q(\Omega))} \lesssim \|u_0\|_{L^2}, \ (Schrödinger)$$

where the pair (p, q) verifies the non-endpoint admissibility condition for the associated equation.

We show this result for the Schrödinger equation in [Laf17b]. It is here presented in section 7. This result is then generalized to the wave equation in [Laf17a], and we present this generalization in section 8. Guided by the previous elements of our exposition, we follow the following roadmap:

1. Thanks to the smoothing effect with logarithmic loss due to [Bur04], to the result of Ivanovici outside one strictly convex obstacle[Iva10], and to the remark of Burq, Guillarmou and Hassel [BGH10], we reduce ourselves to Strichartz estimates in logarithmic times near the trapped trajectory,

2. we follow the parametrix construction of [Ika82, Ika88] and Burq [Bur93], in order to obtain an approximate solution near the trapped tarjectory,

3. a stationary phase argument, together with a result of Ikawa [Ika82, Ika88] concerning the time decay of such a solution, gives the result.

Let us briefly summarize our argument, in the case of the Schrödinger equation.

4.6.1. Reduction of the problem

Let us assume that Strichartz estimates in logarithmic times near the trapped ray $\mathcal{R}$ hold true, that is

$$\|\chi e^{-it\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(0,h|\log h|)L^q(\Omega)} \leq C\|u_0\|_{L^2},$$

where χ is supported near $\mathcal{R}$, and let us show the result.

Let $\chi_{\text{obst}}, \chi_{\text{ray}} \in C_0^\infty$ be such that $\chi_{\text{obst}} = 1$ in a neighborhood of $\Theta_1 \cup \Theta_2 \cup \mathcal{R}$, and $\chi_{\text{ray}} \in C_0^\infty$ such that $\chi_{\text{ray}} = 1$ in a neighborhood of $\mathcal{R}$, and let $\tilde{\psi}$ be such that $\tilde{\psi} = 1$ on the support of ψ. We decompose:

$$\begin{aligned}\psi(-h^2\Delta)e^{it\Delta_D}u_0 = \tilde{\psi}(-h^2\Delta)(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}\chi_{\text{ray}}\psi(-h^2\Delta)e^{it\Delta_D}u_0,\end{aligned}$$

which correspond respectively to the parts away from the obstacles, near the obstacles but away the trapped trajectory, and near both the obstacles and the trapped ray. This last term is handled following the remark of Burq, Guillarmou and Hassel previously mentioned, provided a smoothing estimate without loss outside the trapped ray is at hand, but such an estimate, is, in this particular geometry, for example a consequence of resolvent estimates of [VZ00] outside a non-trapping obstacle because $\Theta_1 \cup \Theta_2$ coincides, outside the trapped trajectory, with such an obstacle.

The first two terms are dealt with adapting arguments from [Iva10]. The first one, $w = (1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0$, verifies the non-homogeneous Schrödinger equation

$$\begin{cases} i\partial_t w + \Delta_D w_h = & -[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0, \\ w(t=0) = & (1-\chi_{\text{obst}})\psi(-h^2\Delta)u_0, \end{cases}$$

but, as $1-\chi_{\text{obst}} = 1$ near $\partial\Omega$, it is a problem in the free space. The Duhamel formula together with standard Strichartz estimates on $\mathbb{R}^n$ and a lemma due to Christ and Kiselev gives

$$\|w\|_{L^pL^q} \lesssim \|(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}^n)} + \|[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{-1/2}},$$

but $[\chi, \Delta_D]$ is supported away from $\mathcal{R}$ and the smoothing estimate without loss in the non-trapped region permits to control the right hand side. The second term, namely $\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0$ corresponds to a problem near the obstacle but away from the periodic trajectory, thus, it can be dealt with stacking the estimates in time h in a compact manifold with strictly concave boundaries obtained by [Iva10], thanks to the smoothing estimate without loss in the non-trapped region.

4.6.2. Construction of an approximate solution

A translation argument inspired by [Iva10] then shows that we can moreover reduce ourselves to data micro-locally supported near the trapped trajectory. More precisely, we define the trapped set in time T of a neighborhood D of $\mathcal{R}$, denoted $\hat{\mathcal{T}}_T(D) \subset T^\star\Omega$, as the set of points of $D \times \{|\xi| \in [\alpha_0, \beta_0]\}$ that remain in D in time T, and we reduce ourselves to data $u_0 = \mathrm{Op}(q)u_0$, where q is supported in a neighborhood of $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\tilde{D})$ and $q = 1$ on $\hat{\mathcal{T}}_{2\epsilon|\log h|}(D)$, where $D \subset \tilde{D}$.

Notice that the need to be able to control in the sequel the derivatives of q motivates us to study the regularity of the billiard flow, and we show that

$$d(\Phi_t(\rho), \Phi_t(\tilde{\rho})) \leq C^t d(\rho, \tilde{\rho})^\mu$$

except in small time intervals. The Hölder loss of regularity is coming from the rays that are close to tangent trajectories. Such a regularity will suffice to choose q in a such a way that $|\partial^\alpha_{x,\xi} q| \lesssim h^{-c|\alpha|\epsilon}$, which will induce no loss because these derivatives will be against terms of order at least h.

A duality argument then shows that it suffices to show the dispersive estimate, for $t \leq \epsilon|\log h|$

$$|e^{iht\Delta}\delta_y| \lesssim (ht)^{-3/2}, \text{ où } \delta_y = \frac{1}{(2\pi h)^3}\int e^{i(x-y)\cdot\xi/h} q(x,\xi) d\xi,$$

and, by linearity of the flow, it suffices to construct an approximate solution with data $e^{i(x-y)\cdot\xi} q(x,\xi)$, y and ξ been fixed, in order to obtain an approximate solution with data δ_y. In order to do so, we follow the method of Ikawa [Ika82, Ika88] and [Bur93].

We define the set of all possible stories of reflexions $\mathcal{I}$ as the set of all n-uplets $J = (j_1, \cdots, j_n)$, $n \geq 0$, where $j_{i+1} \neq j_i$, we denote $J' = (j_1, \cdots, j_{n-1})$ and we look for an approximate solution as the Neumann sum

$$e^{ith\Delta_{D,\Omega}} q(\cdot,\xi) = \sum_{J\in\mathcal{I}} (-1)^J w_J,$$

$$i\partial_t w_J - h\Delta w_J = 0, \; w_J = w_{J'} \text{ on } \mathbb{R}^3 \backslash \Theta_{j_n},$$

where $w_\emptyset$ corresponds to a free wave and w_J to a wave reflected following the story J and then ignoring the obstacles. The w_J's are obtained by the BKW method: we look for it as

$$w_J(x,t) = \sum_{k\geq 0} w^J_k(x,t) h^k e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h}$$

where $\varphi_\emptyset(x,\xi) = (x-y)\cdot\xi/|\xi|$, the phases φ_J, $|J| \geq 1$ verifies the Eikonal equation

$$|\nabla\varphi_J| = 1, \; \varphi_J = \varphi_{J'} \text{ on } \mathbb{R}^3 \backslash \Theta_{j_n},$$

and w^J_k a transport equation along the phase

$$(\partial_t + 2|\xi|\nabla\varphi_J \cdot \nabla + |\xi|\Delta\varphi_J) w^J_k = -i\Delta w^J_{k-1},$$

$$w^J_k = w^{J'}_k \text{ on } \mathbb{R}^3 \backslash \Theta_{j_n},$$

where we used the convention $w_{-1}^J = 0$. A solution to the Eikonal equations is constructed using the method of reflected waves of Ikawa: $\nabla\varphi_J(x,\xi)$ corresponds to the direction of a ray that starts in direction $\frac{\xi}{|\xi|}$ and arrives in x after been reflected according to the story J, in other words, *the gradient of the phase follows the rays.*

We can then solve the transport equations along the rays. In order to do so, we remark that $w(\tau) = v(x + 2\tau|\xi|\nabla\psi, t+\tau)$ solves the ordinary differential equation

$$\partial_\tau w = -|\xi|\Delta\psi(x + 2\tau\nabla\psi(x))w,$$

but we can integrate ([Lun44]):

$$\exp(-|\xi|\int_{\tau_0}^{\tau}\Delta\psi(x + 2s\nabla\psi(x))ds) = \left(\frac{G\psi(x + 2\tau|\xi|\nabla\psi(x))}{G\psi(x + 2\tau_0|\xi|\nabla\psi(x))}\right)^{1/2},$$

where $G\psi(x)$ is the Gaussian curvature of the level surface of ψ in x. Therefore we have, for example for w_0^J

$$w_0^J(x + 2\tau|\xi|\nabla\varphi_J(x), t+\tau) = \left(\frac{G\varphi_J(x + 2\tau|\xi|\nabla\varphi_J(x))}{G\varphi_J(x)}\right)^{1/2} w_0^J(x,t)$$

for $x \in \partial\Theta_{j_n}$. But $w_0^J = w_0^{J'}$ on $\partial\Theta_{j_n}$, and we can iterate the argument following the rays back to $w_0^\emptyset$, for which solving the transport equations is immediate – the localization of q ensuring in particular that we can indeed follow the rays back. We obtain

$$w_0^J(x,t) = \Lambda\varphi_J(x,\xi)q(X_{-2t}(x,|\xi|\nabla\varphi_J),\xi),$$

$$\Lambda\varphi_J(x,\xi) = \left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x,|\xi|\nabla\varphi_J))}\right)^{1/2} \times\cdots\times \left(\frac{G\varphi(X^{-|J|-1}(x,|\xi|\nabla\varphi_J))}{G\varphi(X^{-|J|}(x,|\xi|\nabla\varphi_J))}\right)^{1/2},$$

where X_t design the spatial component of the flow, X^i the points of the boundary attained after i reflexions, and similar expression are at hand for $k \geq 1$.

Notice that the square root of the product of Gaussian curvatures quotients $\Lambda\varphi_J$ is the analog of the square root of the product of Jacobians arising in [BGH10]. This term is responsible of the time decay of the flow: according to Ikawa [Ika82, Ika88],

$$|\Lambda\varphi_J| \lesssim \lambda^{|J|}$$

where $0 < \lambda < 1$ is the square root of the product of the two eigenvalues of modulus lesser than one of the differential of the Poincaré map associated with the captive trajectory.

4.6.3. Directional derivatives of the phase

Hence we have

$$e^{iht\Delta}\delta_y \simeq \frac{1}{(2\pi h)^3}\sum_{J\in\mathcal{I}}\sum_{k\geq 0}\int w_k^J(x,t,\xi)h^k e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h}d\xi,$$

where the w_k^J's and φ_J's have been previously constructed. If a result of Ikawa [Ika82, Ika88] concerning the product of the Gaussian curvature quotients will provide the time decay of the flow, it is necessary in order to conclude to gain three-half powers of h. In order to do so, we will perform a stationary phase argument. It will be therefore necessary, on the one hand, to be able to control the arising *directional* derivatives, and, on the other hand, to study the stationary points of the phases $S_J(x,\xi,t) = \varphi_J(x,\xi)|\xi| - t\xi^2$ and show that their Hessian are non-degenerated in such points.

To this purpose, we differentiate the Eikonal equation with respect to ξ and we solve the arising transport equation along the rays. For example, differentiating $|\nabla(\varphi_J|\xi|)|^2 = |\xi|^2$, we obtain

$$|\xi|\nabla(\partial_{\xi_i}\varphi_J|\xi|)\cdot\nabla\varphi_J = \frac{\xi_i}{|\xi|},$$

and the quantity $\partial_{\xi_i}\varphi_J|\xi|$ we are interested in is hence transported along $\nabla\varphi_J$, that is, along the rays. We integrate the ordinary differential equation verified by $\tau \longrightarrow \partial_{\xi_i}(\varphi_J|\xi|)(x+\tau\nabla\varphi_J(x,\xi))$, use the boundary condition for the phases $\varphi_J = \varphi_{J'}$ on $\partial\Theta_{j_n}$, and we obtain, for $x \in \partial\Theta_{j_n}$

$$\partial_{\xi_i}(\varphi_J|\xi|)(x+\tau\nabla\varphi_J(x,\xi),\xi) = \partial_{\xi_i}(\varphi_{J'}|\xi|)(x,\xi) + \tau\frac{\xi_i}{|\xi|},$$

and we iterate the argument up to the first phase. In this way, we get

$$D_\xi\mathcal{S}_J(x,\xi,t) = \hat{X}_{-2t}(x,|\xi|\nabla\varphi_J(x,\xi)) - y$$

and the phase is stationary in the direction $\xi = s_J(x,t)$ which permits to reach x from y in time $2t$, beginning with a reflexion on Θ_{j_1}.

In the same way, we solve following back the rays the transport equation verified by the second directional derivatives of the phase, namely

$$\nabla\left(\partial^2_{\xi_i\xi_j}\psi_J\right)\cdot\nabla\psi_J = \delta_{ij} - \partial_{\xi_i}\nabla\psi_J\cdot\partial_{\xi_j}\nabla\psi_J$$

in order to show its non-degeneracy in stationary points for times $t \geq t_0 > 0$.

Finally, in order to control the directional derivatives who will arise, we need to control the spatio-directional derivatives of the phases. To this purpose, we follow the same idea as previously: $D_\xi^\alpha\varphi_J$ verifies

$$\nabla(D_\xi^\alpha\varphi_J)\cdot\nabla\varphi_J = R_J,$$

where R_J contains only inferior order derivatives. We solve this transport equation following back the rays, take the spatial derivatives of the solution and get by induction

$$|D_\xi^\alpha D_x^\beta\nabla\varphi_J| \leq D_{\alpha,\beta}^{|J|},$$

a result of Ikawa giving the initialization $\alpha = 0$. Such an estimate seems very rough, but it is in fact enough in order to obtain the control we are interested in for logarithmic times: indeed, the arising directional derivatives will be against terms of order at least h, but, for $w_k^J \neq 0$, we have $|J| \approx t \leq \epsilon|\log h|$, and the loss of ϵ powers of h which will appear will induce no loss.

4.6.4. Stationary phase and conclusion

Thus, after a stationary phase argument, we obtain, for $t \geq t_0$

$$e^{iht\Delta}\delta_y \simeq \frac{1}{(2\pi)^3} h^{-3/2} \sum_{J\in\mathcal{I}} \sum_{k\geq 0} \tilde{w}_k^J(x,t,\xi) h^k e^{-is_J(x,t)/h},$$

where $\tilde{w}_0^J = w_0^J$ and, for $k \geq 1$, $\tilde{w}_k^J$ is a linear combination of directional derivatives of w_k^J. According to the previously mentioned result of Ikawa, and since for $w_k^J \neq 0$, we have $|J| \approx t \leq \epsilon|\log h|$,

$$\sum_{J\in\mathcal{I}} |w_0^J(x,t,\xi)| \lesssim \sum_{J \,|\, w_k^J \neq 0} \lambda^{|J|} \lesssim \sum_{r \gtrsim t} \lambda^r \leq e^{-\mu t} \tag{4.6.1}$$

for some $\mu > 0$ depending only of the geometry of the obstacles. In the same way, the terms $\tilde{w}_k^J(x,t,\xi)h^k$, $k > 1$, admit similar bounds because as we discussed above, taking a directional derivative of w_k^J induces a loss of order a most $h^{-\epsilon}$ in logarithmic times. Thus, for $t \in (t_0, \epsilon|\log h|)$:

$$|e^{ith\Delta}\delta_y| \lesssim h^{-3/2} e^{-\mu t},$$

and in small times, $e^{ith\Delta}\delta_y$ is a free wave (the translation argument of [Iva10] ensuring that we can take a data supported away of the boundary) for which we can use the usual pointwise dispersive estimate of $\mathbb{R}^3$: the theorem is therefore demonstrated.

4.6.5. The wave equation

For the wave equation, the reduction of the problem is exactly the same, using the local energy decay rather than the smoothing estimate, and the result of Smith and Sogge [SS95] in the exterior of one strictly convex obstacle instead of the result of [Iva10].

The similar construction of a parametrix is now done for the wave equation, the fundamental difference with the Schrödinger equation been that rays are now followed at speed one. This appears, at the level of the phase, by the fact that it is now stationary in plain lines, and we use a stationary phase argument on each sphere $\mathcal{S}^{d-1}(0,s)$. We gain therefore only $\frac{d-1}{2}$ powers of h – it reflects the lesser dispersive nature of the wave equation – and we obtain in the same way as for Schrödinger a sufficient dispersive estimate near the trapped trajectory.

4.7. Result: The exterior of $N \geq 3$ strictly convex obstacles verifying Ikawa's condition

We generalize the previous result to the exterior of $N \geq 3$ obstacles verifying Ikawa's condition:

Theorem 4.5 ([Laf18b]). *Let $(\Theta_i)_{1\leq i\leq N}$ be a family of strictly convex obstacles of $\mathbb{R}^3$, verifying Ikawa's conditions (4.4.1) and (4.4.2), and $\Omega = \mathbb{R}^3 \backslash \bigcup_{1\leq i\leq N} \Theta_i$. Then, under the* *non-endpoint admissibility condition, global Strichartz estimates with no loss with respect to the free space hold true in* Ω *for both wave and Schrödinger equations:*

$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}, \text{ (Waves)}$$
$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \lesssim \|u_0\|_{L^2}. \text{ (Schrödinger)}$$

This result is shown in [Laf18b], and is here presented in section 9. Let us summarize our argument in the case of the Schrödinger equation.

Smoothing effect without loss in the non-trapped region

The outline of the proof is the same as for the simpler case of two obstacles. Provided a smoothing estimate *without loss* in the exterior of the trapped set is at hand, the previous argument applies in the exact same way to show that we can reduce ourselves to Strichartz estimates near the trapped set in logarithmic times. However, we can not rely on the particular two-obstacles geometry anymore in order to use a resolvent estimate outside a non-trapping obstacle, and we have to give a general proof. Let us notice, moreover, that we show here an estimate outside the trapped set *in the phase-space* and not only his spatial projection: that is, the spatial cut-off function is replaced by an operator with micro-support disjoint from the trapped set as a subset of the phase-space.

In order to do so, we use a commutator argument, similar to the one appearing in [BGH10] and in the spirit of Morawetz, Raltson and Strauss [MRS77]: for any Schrödinger wave u and any operator A we have:

$$\langle Au, u\rangle(T) - \langle Au, u\rangle(0) = \int_0^T \int_\Omega \langle [i\Delta, A]u, u\rangle + \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u\rangle.$$

Therefore, if we choose A of order zero, with a non-negative symbol, elliptic in the subset we are interested in, and such that the arising boundary term has the right sign up to lower order terms, we shall obtain the desired estimate. The method of [MRS77] adapts to the Schrödinger flow in order to derive the symbol of A *at the boundary*, and we use their construction of an escape function – valid in the exterior of a generic non-trapping obstacle – in the exterior of the trapped set to construct the symbol of A. Notice that an explicit expression of the symbol at the border is only available in a zone $\{\tau - \eta^2 > 0\}$, and, for this reason, this method does not permit to show that the border term is non-negative up to lower order terms, but only that it is *almost non-negative* up to lower order terms, and we obtain an estimate of the form

$$\|\mathrm{Op}_h(\phi)u\|_{L^2H^{1/2}} \leq C(\|u_0\|_{L^2} + \delta\|\mathrm{Op}_h(\tilde{\phi})u\|_{L^2H^{1/2}}) + C_\delta O(h^\infty),$$

where $\tilde{\phi} = 1$ on the support of ϕ. We fix $\delta > 0$ small enough and iterate the argument in order to obtain the result. Lower order terms are dealt with using the estimate *with logarithmic loss.*

Notice that we can iterate such a proof in the spirit of [DV12], and control at each step the lower order terms by the induction hypothesis, in order to show that a smoothing estimate without loss in the non-trapped region holds true as soon as a smoothing effect with *polynomial* loss is at hand.

Reduction to the neighborhood of periodic trajectories

Using the smoothing effect in the non-trapped region, we reduce as previously ourselves to show Strichartz estimates in logarithmic times, in a micro-local neighborhood of the trapped set, that is:

$$\|\mathrm{Op}_h(\phi)e^{-it\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(0,h|\log h|)L^q(\Omega)} \leq C\|u_0\|_{L^2},$$

where $\phi \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is supported in a small enough neighborhood of the trapped set $\mathcal{K} \subset T^\star\Omega \cup T^\star\partial\Omega$, and via duality arguments, it suffices to show the following pointwise dispersive estimate:

$$\|\mathrm{Op}_h(\phi)e^{ith\Delta}\psi(-h^2\Delta)\mathrm{Op}_h(\phi)^\star\|_{L^1\to L^\infty} \lesssim (ht)^{-3/2}, \ \forall 0 \leq t \leq \epsilon|\log h|.$$

In order to reduce ourselves to construct a parametrix near a trapped trajectory, we write, inspired by [BGH10]:

$$e^{iTh\Delta} \approx \sum_{\mathbf{k}=(k_1,\cdots,k_L)} e^{its_0\Delta}\Pi_{k_L}e^{i\tau h\Delta}\Pi_{k_{L-1}}\cdots\Pi_{k_1}e^{i\tau h\Delta} + R_1(t) + R_2(t),$$

where Π_k are micro-local cut-off operators localized near the segments that compose the trapped set. We choose τ small enough, and the terms of this sum correspond to the parts of the flow that remain near a periodic trajectory.

The R_1 term corresponds of the part of the flow which is away from the obstacles and remains away, and the R_2 term to a part of the flow which is away from the obstacles and remains away, once spent a time $T_0 > 0$ depending only of the Π_i's. Thus, in times $t \geq T_0$, it suffices to deal with the parts of the flow which remain near a periodic trajectory.

In order to handle $0 \leq t \leq T_0$ times, we choose ϕ supported near enough $\mathcal{K}$ so that in times $0 \leq t \leq T_0$, the rays starting from the support of ϕ are never tangent to the boundary. Then, the previously mentioned parametrix construction of Ikawa permits to obtain the dispersive estimate in times $0 \leq t \leq T_0$: indeed, the flow will write as a finite (depending of T_0) sum of reflected waves, each of them verifying the dispersive estimate.

Conclusion

Once localized near a periodic trajectory, the previously mentioned parametrix construction, as well as the stationary phase arguments, apply in the exact same way as for two obstacles. However, in order to close the proof, we have to control the sums analog to (4.6.1), namely

$$\sum_{J\in\mathcal{I}} |w_k^J|,$$

where the set $\mathcal{I}$ of all possible stories of reflexions is now far bigger. Its size compete with the time decay of each of the w^J waves: in other words, *there is a competition between* *the number of the parts of the flow which remain trapped between the obstacles and the decay of each such part.* Ikawa's assumption (4.4.1) of strong hyperbolicity ensures that this competition occurs in a favorable way, and the can conclude the proof.

Remark. If the strong hyperbolicity assumption (4.4.1) appears in a natural way, we do not know if it is optimal. Hence, it would be worth try to obtain a counter-example in a framework which is not verifying the assumption.

5 Long time asymptotics of non-linear wave equations

5.1. The non-linear wave equation in exterior domains

Let us consider the following non-linear wave equation with Dirichlet boundary conditions

$$(\partial_t^2 - \Delta)u + \epsilon u|u|^{p-1} = 0,\ u = 0 \text{ on } \partial\Omega, \tag{NLW}$$
$$(u(0), \partial_t u(0)) = (u_0, u_1) \in \dot{H}^1 \times L^2,$$

where $\epsilon \in \{1, -1\}$, which admits the conserved quantity

$$E(u(t)) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |\partial_t u|^2 + \epsilon \frac{1}{p+1}\int |u|^{p+1}. \tag{5.1.1}$$

We are in particular interested with the three-dimensional energy-critical equation: $d = 3$, $p = 6$, for which the scaling invariance in the free space $\Omega = \mathbb{R}^d$

$$u \longrightarrow \lambda^{1/2} u(\lambda\cdot, \lambda\cdot)$$

let invariant the energy.

When such an equation admits global-in-time solutions, it is natural to wonder how to describe them in large times ($t \longrightarrow \pm\infty$). If they behave linearly in such times, we say that they *scatter*. More precisely,

Definition 5.1. Let u be a global solution of (NLW). We say that u scatters if there exists data $v_\pm$ such that

$$\|u(t) - S(t)v_\pm\|_{\dot{H}^1} + \|\partial_t(u(t) - S(t)v_\pm)\|_{L^2} \longrightarrow 0$$

as $t \longrightarrow \pm\infty$, where $S(t)v_\pm$ is the solution of the linear wave equation with data $v_\pm$.

When $\epsilon = 1$, the equation is said to be defocusing: the non-linearity goes in the same way as the decay induced by the linear group, and we then expect scattering-type effects, provided the domain Ω does not induce to much concentration effects. It is in particular the case in the exterior of a non-trapping, or a weakly-trapping obstacle, and we expect scattering in such frameworks.

The object of this chapter is to study such situations. More precisely, we show that

1. solutions scatter when Ω is the exterior of an obstacle illuminated by an ellipsoid, notion generalizing star-shaped bodies,

2. in the exterior of two strictly convex obstacles, the only obstruction to scattering is the concentration of energy around the trapped trajectory, in particular, compact-flow solutions does not scatter.

The result (2) can be seen as a rigidity argument in a concentration-compactness/rigidity approach which would complete the proof. These results can be found in [Laf17a], they are here presented in section 11.

5.1.1. Global existence and scattering criterion

The global existence for smooth radial data in $\mathbb{R}^d$ is obtained by Struwe [Str88], then by Grillakis [Gri90] in the smooth, non-radial case. Shatah and Struwe [SS94] then show global existence for H^1 data, combining Strichartz estimates with a non-concentration effect in the edge of the reversed cone:

$$\lim_{t\longrightarrow t_0^{-1}} \int_{x\in|x-x_0|<t_0-t} u^6(x,t)dx = 0. \tag{5.1.2}$$

This non-concentration effect is obtained as a consequence of a divergence identity due to Morawetz [Mor61] for the linear wave equation, namely

$$\partial_t Q - \mathrm{div} P + \frac{1}{3}|u|^6 = 0, \tag{5.1.3}$$

$$Q = \frac{|\partial_t u|^2 + |\nabla u|^2}{2} + \frac{|u|^6}{6} + \partial_t u(\frac{x}{t}\cdot\nabla)u, \tag{5.1.4}$$

$$P = \frac{x}{t}\left(\frac{|\partial_t u|^2 - |\nabla u|^2}{2} + \frac{|u|^6}{6}\right) + \nabla u, \left(\partial_t u(\frac{x}{t}\cdot\nabla)u + \frac{u}{t}\right), \tag{5.1.5}$$

that can be traced back to the conformal invariance of the wave equation, and is integrated over the small cone

$$K_S^0 = \{S \le t \le T,\ |x| \le t\},\ S \longrightarrow 0^-.$$

Bahouri and Shatah [BS98] pursue the computation and integrate now (5.1.3) over the big cone K_0^T, $T \longrightarrow +\infty$, in order to show that the non-linear part of the energy decays to zero, that is

$$\int_{\mathbb{R}^3} |u(t)|^6 \longrightarrow 0 \text{ as } t \longrightarrow \infty,$$

and this decay, together with Strichartz estimates, gives immediately the scattering.

Global existence in a general domain is obtained by Burq, Lebeau and Planchon [BLP08]. They remark that the spectral projector estimates due to [BSS09a] give strong enough Strichartz estimates to have a local Cauchy theory, and generalize the argument of Shatah and Struwe to a domain with boundaries in order to obtain the non-concentration effect

(5.1.2) and therefore global existence of solutions. In order to do so, they have to show that the arising boundary term decays to zero, that is

$$\frac{1}{|S|}\int_{((S,0)\times\partial\Omega)\cap K_S^0}(n(x)\cdot x)|\partial_n u|^2 d\sigma(x)dt \longrightarrow 0$$

as $S \longrightarrow 0^-$. But, a commutator argument shows that

$$\int_0^{t_0}\int_{\partial\Omega}|\partial_n u|^2 d\sigma dt \lesssim E,$$

and, as moreover $n(x)\cdot x = O(|x|^2)$, the above decay holds and the authors obtain global existence.

If one tries now to integrate the divergence identity (5.1.3) over the big cone K_0^T, $T \longrightarrow \infty$ in order to generalize the argument of Bahouri and Shatah and obtain the L^6-decay of solutions and therefore scattering, the above control does not suffices because we have now to obtain the decay in-mean of the boundary term for large times, that is

$$\left(\frac{1}{T}\int_0^T\int_{\partial\Omega}tP\cdot n|\partial_n u|^2 d\sigma dt\right)_+ \longrightarrow 0 \text{ as } T\longrightarrow\infty,$$

where we took the positive part and $tP = x\cdot n$ if we take the same multiplier as in $\mathbb{R}^d$. In the exterior of a star-shaped obstacle, this boundary term has the right sign, and the proof generalizes immediately. To obtain scattering in more general geometries, one can try to follow two kind of strategies: adapt the multiplier in such a way that this boundary term has the right sign, or try to control it and show that

$$\frac{1}{T}\int_0^T\int_{\partial\Theta}|\partial_n u|^2 d\sigma dt \longrightarrow 0, \tag{5.1.6}$$

that is, *that in mean in large times, there is no energy left near the obstacle.* We follow here this second strategy.

5.1.2. Result: Scattering in the exterior of an obstacle illuminated by an ellipsoid

Because the proof of Bahouri and Shatah generalizes immediately to the exterior of a star-shaped obstacle, it seems natural to study generalizations of such obstacles. We are here interested in *illuminated obstacles*:

Definition 5.2. A subset Θ of $\mathbb{R}^n$ is said to be illuminated by a convex $\mathcal{C}\subset\mathbb{R}^n$ if

$$\min_{\partial\Theta}\nabla\rho\cdot\nu > 0$$

where

$$\rho(x) = \inf\{\lambda > 0, \text{ s.t. } x\in\lambda\mathcal{C}\}$$

is the gauge function of $\mathcal{C}$ and ν the outward-pointing normal derivative to $\partial\Theta$.

A star-shaped obstacle corresponds therefore to a body illuminated by a small ball. We show the scattering for critical non-linear waves in the exterior of obstacles illuminated by a class of ellipsoids:

Theorem 5.3 ([Laf17a]). *Let $\mathcal{C} \subset \mathbb{R}^3$ be the ellipsoid of equation*

$$x^2 + y^2 + \epsilon z^2 = 1, \ 0 < \epsilon \leq 1 \tag{5.1.7}$$

resp.

$$x^2 + \epsilon y^2 + \epsilon z^2 = 1, \ \frac{1+\sqrt{3}}{4} \leq \epsilon \leq 1 \tag{5.1.8}$$

and Θ be a compact subset of $\mathbb{R}^3$ illuminated by $\mathcal{C}$. Then, any solution of (NLW) in $\Omega = \mathbb{R}^3 \backslash \Theta$ scatters in $\dot{H}^1(\Omega)$.

In particular, we obtain scattering for dog-bones with arbitrary thin necks. Remark that Farah Abou Shakra adapts in [AS14] the multiplier that permits to obtain (5.1.3) and shows the scattering in the exterior of the obstacles verifying the assumption

$$\min_{\partial\Theta}(s + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0,$$

where $\rho_1 \leq \rho_2$ are the principal rays of curvature of $\mathcal{C}$, ρ_{2M} the maximum of ρ_2 and s the distance to $\mathcal{C}$. However, the non-local quantity ρ_{2M} avoids in particular to handle dog bones with arbitrary thin necks, which was our motivation in order to deal with a situation close to the two-convex framework.

Our principal tool to obtain the decay (5.1.6) and therefore the scattering is a momentum identity that can be seen as another version of the divergence identity (5.1.3):

$$\partial_t \left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2} \Delta\chi u \partial_t u \right) = \int_\Omega (D^2\chi \nabla u, \nabla u) - \frac{1}{4} \int_\Omega u^2 \Delta^2 \chi$$
$$+ \frac{3}{2} \int_\Omega |u|^6 \Delta\chi - \frac{1}{2} \int_{\partial\Omega} |\partial_n u|^2 \partial_n \chi, \tag{5.1.9}$$

where u solves (NLW) and χ is a smooth function. The idea is to find a weight function χ adapted to the geometry of the obstacle, that permits to take advantage of the above identity to show the decay of the energy at the border. In particular, if $\nabla\chi$ is bounded, the left-hand side is bounded by the energy after time-integration, and if moreover χ is convex, $-\partial_n \chi \geq c > 0$ and $\Delta^2 \chi \leq 0$, the right-hand side bounds below the term we are interested in and we obtain the decay

$$\frac{1}{T} \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt \lesssim \frac{E}{T}, \tag{5.1.10}$$

and therefore scattering. The natural choice for the weight function χ is therefore the gauge of the illuminating convex. However, if such a weight verifies all other constraints, the non-negative bilaplacian assumption is very rigid, and seems to have no geometrical interpretation. In particular, the gauges of the ellipsoids we are interested in, namely

$$\chi = \epsilon x^2 + y^2 + z^2,\ \epsilon \ll 1,$$

do *not* have a non-negative bilaplacian. However, it is the case if we add one dimension, that is, for four-dimensional ellipsoids of gauges $\epsilon x^2 + y^2 + z^2 + w^2$. To take advantage of this remark, we extend u as a solution $\tilde{u}$ of a four-dimensional equation. The border-term associated with $\tilde{u}$ verifies (5.1.10), and the room we have between this strong decay and (5.1.6) permits to go back to the original, three-dimensional equation. More precisely, we have

$$\int_0^T \int_{\partial\Theta} |\partial_n \tilde{u}|^2 \partial_n \tilde{\chi} d\sigma dt \lesssim E(\tilde{u})$$

where $\tilde{\chi}$ is the four-dimensional weight. But

$$E(\tilde{u}) \lesssim T E(u),$$

and

$$\int_0^T \int_{\partial\Theta} |\partial_n \tilde{u}|^2 \partial_n \tilde{\chi} d\sigma dt \gtrsim \ln T \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt,$$

therefore

$$\frac{1}{T} \int_0^T \int_{\partial\Theta} |\partial_n u|^2 d\sigma dt \lesssim \frac{1}{\ln T} E(u) \longrightarrow 0,$$

and the solutions scatter.

5.1.3. Result: A rigidity theorem in the exterior of two strictly convex obstacles

Motivated by the previously exposed result, we are interested in the scattering problem in the exterior of two strictly convex obstacles – let us recall that we showed global Strichartz estimates without loss in this framework. However, there is no hope a multiplier argument used alone could give the result: indeed, Morawetz, Raltson and Strauss proved in particular in [MRS77] that the non-captivity assumption is equivalent to the existence of an *escape function* $a \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, that is, a phase-space function verifying

$$\xi \cdot \nabla_x a(x,\xi) \geq c > 0 \text{ in } \Omega,$$
$$\frac{a(x,\xi) - a(x,\xi')}{(\xi - \xi') \cdot n(x)} \geq c > 0 \text{ on } \partial\Omega,$$

where ξ' is the reflected direction associated with ξ by the laws of geometrical optics. But, if a suitable – in the sense of the previous section – weight function χ is at hand, we can construct an escape function

$$a(x,\xi) = \nabla\chi(x) \cdot \xi.$$

However, we can construct a multiplier that behave in the right way *everywhere except in a neighborhood of the trapped ray* in order to show:

Theorem 5.4 ([Laf17a]). *Let Θ_1 and Θ_2 be two strictly convex subsets of $\mathbb{R}^3$. Then, there exists a family $(\mathcal{S}(T))_{T\geq 1}$ of open neighborhoods of the trapped ray $\mathcal{R}$ verifying*

$$\mathcal{S}(T) \longrightarrow \mathcal{R} \text{ as } T \longrightarrow +\infty$$

such that any global solution of (NLW) in $\Omega := \mathbb{R}^3 \setminus (\Theta_1 \cup \Theta_2)$ verifies, as T goes to infinity

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))\setminus \mathcal{S}(T)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0.$$

Where the criterion

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0$$

is equivalent to (5.1.6) – this is for example a consequence of the Morawetz momentum identity (5.1.9). In particular, solutions with pre-compact flow $\{u(t),\ t \geq 0\}$ in $\dot{H}^1$ scatter, because they can't concentrate their energy in the neighborhood of the trapped ray. Therefore,

Corollary 5.5. *There is no compact-flow solution of (NLW) in the exterior of two strictly convex obstacles which does not scatter.*

It is a rigidity theorem that can be taked advantage of in a concentration-compactness/rigidity approach, introduced by Kenig and Merle: we suppose there is a finite energy level above which non-scattering solutions exist, and construct a so-called critical solution, who has a compact flow and does not scatter, and will be eliminated – this is the subject of a work in progress, in collaboration with C. Laurent [LL].

In the simplest case of two balls, with centers 1 and −1, we choose the weight (figure 5.1.1)

$$\chi(x) = |x-1| + |x+1|.$$

It verifies

$$\nabla\chi \cdot n \geq 0,\ \Delta\chi \gtrsim 1 \text{ in } B(0,A),\ \Delta^2\chi = 0.$$

The momentum identity permits to show

$$\int_0^T \int_\Omega (D^2\chi \nabla u, \nabla u) + \int_0^T \int_{\Omega\cap B(0,A)} |u|^6 \lesssim E,$$

but the Hessian matrix $D^2\chi$ of χ vanishes on the trapped ray. The study of its eigenvalues gives $S(T)$:

$$D^2\chi \gtrsim \alpha \text{ in the exterior of a neighborhood } S(\alpha) \text{ of the trapped ray}$$

and we choose $\alpha(T)$ going to zero slowly enough to obtain we decay we need.

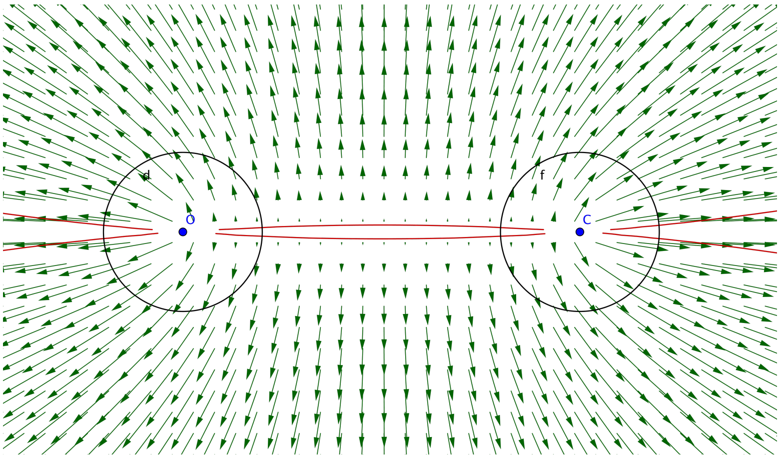

Figure 5.1.1.: $\nabla\chi$ and $S(\alpha)$ for two balls

In the exterior of two strictly convex obstacles, this weight is not adapted anymore and we use a family of weights

$$\chi_c(x) = |x - c| + |x + c|, \ c \longrightarrow \infty.$$

For $c \longrightarrow \infty$, it has a gradient behaving similarly to $\frac{(x_1,x_2)}{|(x_1,x_2)|}$ (figure 5.1.2), for which the scalar product with the outward-pointing derivative to the obstacles is non-negative (and vanishes on the trapped ray), and we choose the speeds of decay $c(T) \longrightarrow +\infty$ and $\alpha(T) \longrightarrow 0$ in such a way we obtain the decay we are interested in. Let us remark that once again, we took advantage of the room at our disposal between the strong $1/T$-decay given by the left-hand side of (5.1.9), and the weaker decay (5.1.6) we need.

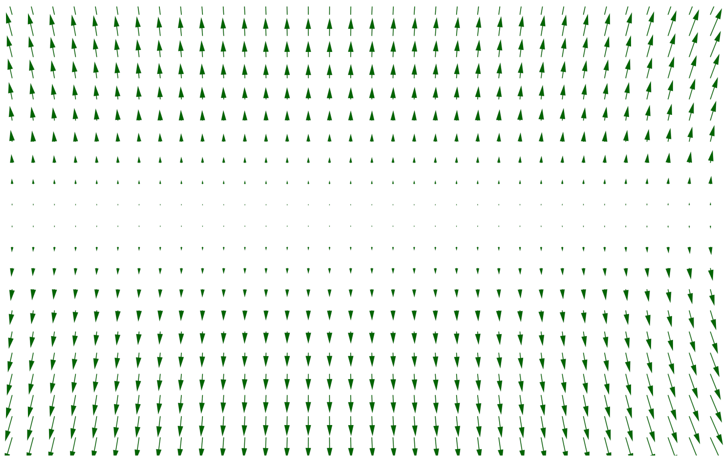

Figure 5.1.2.: $\nabla\chi_c$ for $c \gg 1$

5.2. The non-linear Schrödinger equation with a potential

We are now interested in the following non-linear Schrödinger equation with a potential, in the free space $\mathbb{R}^d$

$$i\partial_t u - \Delta u + Vu = \epsilon u|u|^\alpha, \ u(0) = u_0 \in H^1, \qquad (\text{NLS}_V)$$

that preserves in particular the L^2-norm of solutions – or mass, and the energy

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 + \int V|u(t)|^2 + \epsilon\frac{1}{\alpha+2}\int |u(t)|^{\alpha+2} = E(u(0)).$$

In the homogeneous case $V = 0$, Strichartz estimates together with a fixed point argument gives local existence of solutions in energy-critical and subcritical regimes $\alpha \leq \frac{4}{d-1}$. In the energy sub-critical one, the lifetime of a solution depends only of its norm, and the energy conservation law gives global existence for the defocusing equation $\epsilon = 1$. In the same way as for the non-linear wave equation, we define the scattering as

Definition 5.6. Let u be a global solution of (NLS). We say that u scatters if there exists $v_\pm$ such that

$$\|u(t) - e^{-it\Delta}v_\pm\|_{H^1} \longrightarrow 0$$

as $t \longrightarrow \pm\infty$.

One of our motivations in the study of (NLS_V) is the similarity between a potential and an obstacle. The absence of boundary makes several arguments easier to pursue, and the study of (NLS_V), in addition to its intrinsic interest, can therefore be a source of ideas in order to study problems with obstacles.

We show the scattering in the following two situations:

1. for a repulsive potential ($x \cdot \nabla V \leq 0$),

2. for a sum of two repulsive potentials with convex level surfaces.

Notice that situation (1) is analogous to the exterior of a star-shaped obstacle, and situation (2), who enjoys trapped trajectories, to the exterior of two strictly convex obstacles. This last result gives in particular a scattering result in a trapped setting. The first one can be found in [Laf16], the second one in [Laf18a]. There are here respectively presented in sections 10 and 12.

5.2.1. Concentration-compactness/rigidity

Nakanishi [Nak99] shows the scattering in the energy space H^1 for the defocusing homogeneous equation ($V = 0$), using Morawetz-type inequalities. Such inequalities, extended to bilinear versions, continue then to be used in order to show scattering-type results for the defocusing equation: let us mention the works of [PV09], [CGT09], and [Vis09].

A very robust method that permits to deal in particular with scattering-type problems, known as *concentration-compactness/rigidity*, is introduced by Kenig and Merle [KM06].

It is motivated in particular by the study of the focusing equation ($\epsilon = -1$), for which exists an energy-threshold below which solutions scatter, given by the ground state $u = e^{it}Q$ of the equation, where

$$\Delta Q - Q + Q^{\alpha+1} = 0,$$

for which the decay induced by the linear group and the focusing effect of the non-linearity compensate exactly. This method, extended for Schrödinger to the intercritical regime in [HR08], and to all spatial dimensions by [FXC11], generated an impressive number of works. We use it here in the case of a potential $V \neq 0$. Before we discuss the generalization to the inhomogeneous case, let us briefly summarize the method in the defocusing homogeneous case $V = 0$.

We define the critical energy

$$E_c = \sup \left\{ E > 0 \mid \forall \varphi \in H^1,\ E(\varphi) < E \Rightarrow \mathcal{S}(t)\varphi \in L^pL^r \right\},$$

where $\mathcal{S}(t)\varphi$ is the solution of (NLS) with data φ, and assume by contradiction that

$$E_c < \infty.$$

Then, there exists a minimizing sequence for E_c of data $(\varphi_n)_{n\geq 1}$, that generate non-scattering solutions:

$$E(\varphi_n) \geq E_c,\ E(\varphi_n) \longrightarrow E_c,\ \mathcal{S}(t)\varphi \notin L^pL^r.$$

The idea is to extract from the H^1-bounded sequence $(\varphi_n)_{n\geq 1}$ a converging subsequence. Its limit φ_c will produce a non-scattering compact flow solution who will be eliminated. A first description of H^1-bounded sequences is given by concentration-compactness lemmas in the spirit of Lions, but the major breakthrough which permits to conclude here is the description of such sequences as a sum of profiles *up to a dispersive remainder*, each profile being a solution in a given time of the linear equation associated to the problem. Such a profile decomposition appears for the first time in [BG99], for the wave equation. For the Schrödinger equation it can for example be written as

$$\varphi_n = \sum_{j=1}^{J} e^{-it_j^n\Delta}\psi_j(\cdot - x_j^n) + R_n^J \quad \forall J \in \mathbb{N},$$

the profiles $e^{it_j^n A}\tau_{x_n^j}\psi_j$ verifying the orthogonality condition

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \longrightarrow \infty, \quad \forall j \neq k,$$

and

$$\|u_n\|_{H^1}^2 = \sum_{j=1}^{J} \|\psi_j(\cdot - x_j^n)\|_{H^1}^2 + \|R_n^J\|_{H^1}^2 + o_n(1), \quad \forall J \in \mathbb{N},$$

$$\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \|e^{-it_j^n\Delta}\psi_j(\cdot - x_j^n)\|_{L^p}^p + \|R_n^J\|_{L^p}^p + o_n(1), \quad \forall J \in \mathbb{N},$$

and the remainder being small in Strichartz norms:

$$\forall \epsilon > 0,\ \exists J \in \mathbb{N}, \quad \limsup_{n\to\infty} \|e^{-it\Delta} R_n^J\|_{L^pL^r} \leq \epsilon.$$

We associate to each profile a *nonlinear* profile $\tilde{\psi}_j$ such that

$$\|\mathcal{S}(-t)\tilde{\psi}_j - \psi_j\|_{H^1} \longrightarrow 0$$

as $t \longrightarrow t_j$, and the orthogonality of profiles permits then to approximate

$$\mathcal{S}(t)\varphi_n \approx \sum_{j=1}^{J} \mathcal{S}(t - t_j^n)\tilde{\psi}_j(\cdot - x_j^n) \tag{5.2.1}$$

with a small remainder in Strichartz norms. But, if there is more than one non-trivial profile, each of them takes a non-trivial part of the energy, and for n large enough they are all below the critical energy E_c and therefore produce solutions that scatter! Because of (5.2.1) φ_n hence generates a scattering solution, this is not possible. Therefore, there is only one non-trivial profile, φ_c. We take

$$u_c = \mathcal{S}(t)\varphi_c,$$

it is our critical solution. Apply a second time the argument, now to the flow $\{u_c(t),\ t \in \mathbb{R}\}$, permits to show that it is of compact closure in H^1.

But a classical virial computation, which is a Schrödinger-counterpart of the Morawetz identity (5.1.9):

$$\partial_t^2 \int \chi|u|^2 = 4\int (D^2\chi\nabla u, \nabla u) + \frac{2}{\alpha+2}\int \Delta\chi|u|^{\alpha+2} - \int \Delta^2\chi|u|^2. \tag{5.2.2}$$

permits to show that such a solution cannot exist. To this purpose, one takes a quadratic weight

$$\chi_R := R^2\phi(\frac{\cdot}{R}) \text{ où } \phi(x) = \begin{cases} x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

then

$$\left|\partial_t \int \chi_R|u|^2\right| \leq C(E)R,$$

but by (5.2.2),

$$\partial_t^2 \int \chi_R|u|^2 \geq \delta > 0,$$

for R big enough, because as u has a pre-compact flow it holds that

$$\sup_{t\in\mathbb{R}} \int_{|x|\geq R} \left(|\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^{\alpha+2}\right) dx \longrightarrow 0$$

as $R \longrightarrow \infty$.

In the case of the equation with a potential $V \neq 0$, we have to generalize both parts of the argument. But a crucial point in order to approximate (5.2.1) in the previous argument is the translation invariance of the equation:

$$\mathcal{S}(t-t_j^n)\tilde{\psi}_j(\cdot - x_n^j) = \left(\mathcal{S}(t-t_j^n)\tilde{\psi}_j\right)(\cdot - x_n^j),$$

which *fails* for $V \neq 0$: it is the main difficulty to overcome in the concentration-compactness part. The classical virial computation used in the rigidity part makes appear an additional term

$$-\int \nabla V \cdot \nabla \chi. \tag{5.2.3}$$

It is the counterpart of the boundary term arising in (5.1.9) for the wave equation. For a repulsive potential, this term has the right sign and the rigidity argument adapts immediately. It is not the case anymore if the potential is non repulsive.

5.2.2. Result: Scattering for NLS with a potential

In [BV16], Banica and Visciglia attack the problem, considering the equation with a Dirac potential

$$i\partial_t u - \Delta u + \delta_0 u = u|u|^\alpha, \; u(0) = u_0 \in H^1,$$

which is more singular but for which explicit formulas are at hand. Motivated by their result, we show the scattering for NLS with an arbitrary, but less singular potential, in one space-dimension:

Theorem 5.7 ([Laf16])**.** *Let $\alpha > 4$ and $V \in L_1^1(\mathbb{R})$ be such that $V' \in L_1^1(\mathbb{R})$. We suppose moreover that V is non-negative and repulsive: $V \geq 0$ and $xV' \leq 0$. Then, any solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (NLS$_V$) with potential V scatters in $H^1(\mathbb{R})$.*

Scattering-type problems are classically more difficult in small dimensions, because additional dimensions add dispersion directions. Hence we chose to restrict ourselves to the one-dimensional case in order to present the proof in the simplest possible way. Notice that our result is generalized in any spatial dimension $d \neq 2$ in [Laf18a], presented in the next section.

Goldberg and Schlag show in [GS04] the dispersive estimate

$$\|e^{it(-\Delta+V)}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{t^{1/2}} \tag{5.2.4}$$

for an $L_1^1(\mathbb{R})$ potential with no resonance at zero energy. A simple argument shows that a non-negative potential automatically verifies this last assumption, and via a standard $TT^\star$ argument, we get the Strichartz estimates adapted to the study of the problem. They permit, in particular, to obtain local existence of solutions, and hence global existence via the energy conservation law.

The key idea that permits to overcome the lack of translation-invariance of the equation, is that *a profile escaping to infinity does not see the potential*, that is, for $x_n \longrightarrow \infty$

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{-it(\Delta - V)}\tau_{x_n}\psi\|_{L^pL^r} \longrightarrow 0, \tag{5.2.5}$$

where τ is the translation-operator: $\tau_{x_n}\psi = \psi(\cdot -)$, and in the same way for the Duhamel term

$$\| \int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right)(s)ds - \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^pL^r} \longrightarrow 0, \quad (5.2.6)$$

where $U_n(x,t) = U(x-x_n,t)$ for U a solution of the homogeneous ($V=0$) nonlinear equation, and finally, t been fixed

$$\|e^{it\Delta}\tau_{x_n}\psi - e^{it(\Delta-V)}\tau_{x_n}\psi\|_{H^1} \longrightarrow 0. \quad (5.2.7)$$

Let us give for example an idea of the proof of (5.2.5) – (5.2.6) and (5.2.7) can be obtained in a similar way. The pointwise dispersive estimate (5.2.4) permits to obtain

$$\sup_{n\in\mathbb{N}} \|e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(T,\infty)L^r} \longrightarrow 0,$$

as T goes to infinity, and we are hence reduced to show that for T fixed

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{-it(\Delta-V)}\tau_{x_n}\psi\|_{L^p(0,T)L^r} \longrightarrow 0,$$

as $n \longrightarrow \infty$. But, the difference

$$v_n := e^{-it\Delta}\tau_{x_n}\psi - e^{-it(\Delta-V)}\tau_{x_n}\psi$$

verifies the linear Schrödinger equation with null data

$$i\partial_t u - \Delta u + Vu = Ve^{-it\Delta}\tau_{x_n}\psi,$$

and can therefore be estimated via Strichartz estimates:

$$\begin{aligned} \|v_n\|_{L^p(0,T)L^r} &\lesssim \|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^{\gamma'}(0,T)L^1} \le T^{\frac{1}{\gamma'}}\|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,T)L^1} \\ &= T^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^1}, \end{aligned} \quad (5.2.8)$$

which goes to 0 as $x_n \longrightarrow \infty$ because V is in L^1.

The estimate (5.2.7), together with functional analysis arguments, permits to show that the operator $-\Delta + V$ verifies the assumptions of the abstract profile decomposition of [BV16], and we get a profile decomposition associated with the operator we are interested in. Then, the estimates (5.2.5) and (5.2.6) permit in the same way as in [BV16], to replace $e^{it(-\Delta+V)}$ by $e^{-it\Delta}$ for profiles going to infinity. This propagator commutes with translations, we thus can approximate (5.2.1) and construct a critical solution. It is eliminated thanks to the same virial computation as in the homogeneous case $V=0$, because, as we emphasized in the previous section, the additional term (5.2.3) arising in the computation has the right sign for a repulsive potential.

5.2.3. Result: Scattering for NLS with a sum of two repulsive potentials

If the potential is non repulsive, the construction of a critical element remains valid, but it cannot be eliminated by the above rigidity argument. But, in the same way as for non-linear waves outside obstacles, we have the intuition that for potential which does not trap the energy, or not too much, there should be scattering. A strategy in order to show the scattering in such situations would be to adapt the rigidity part of the previous argument to such potentials. But, we saw that we can obtain such a rigidity theorem for the non-linear wave equation outside two strictly convex obstacles. Motivated by the analogy between the virial computation (5.2.2) and the Morawetz identity (5.1.9) thanks to which we were able to obtain such a result, we show the rigidity part for a sum of two potentials *with convex level surfaces* and obtain

Theorem 5.8 ([Laf18a]). *Let $d \geq 3$ and V_1, V_2 be two repulsive potentials with convex level surfaces. We suppose moreover that $V = V_1 + V_2$ verifies the decay assumption*

$$V, \nabla V \in L^{\frac{d}{2}}(\mathbb{R}^d, (1+|x|^{\beta})dx), \ \beta > \frac{2}{3} \tag{5.2.9}$$

and the pointwise dispersive estimate

$$\|e^{it(-\Delta+V)}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{|t|^{d/2}}. \tag{5.2.10}$$

Then, in the intercritical regime

$$\frac{4}{d} < \alpha < \frac{4}{d-1}, \tag{5.2.11}$$

any solution of (NLS_V) in dimension d with potential $V = V_1 + V_2$ scatters in $H^1(\mathbb{R}^d)$.

The decay assumption (5.2.9) is the analog of the assumption $V \in L^1_1(\mathbb{R})$, $V' \in L^1_1(\mathbb{R})$ arising in [Laf16]. We assume the validity of the pointwise dispersive estimate (5.2.10), which permits to obtain the good set of Strichartz estimates, as a black box. Notice however that in spatial dimension $d = 3$, as soon as $\beta \geq 1$, this estimate is automatically verified by such potentials by the work of Goldberg and Schlag [GS04].

The first step is to generalize the construction of a critical solution, and thus the estimates (5.2.5), (5.2.6) and (5.2.7) of the previous paragraph, to all spatial dimensions. In order to do so in the same way as in [Laf16], we need *endpoint* Strichartz estimates, which fails in dimension two: this is why we don't obtain the result in the two-dimensional case. Notice that this first part, where the convexity assumption is not used, generalizes in particular theorem 5.7 to all spatial dimension $d \geq 3$.

We eliminate compact flow solutions with a virial computation analog to the one we did in order to obtain theorem 5.4. Once again, we use the weight

$$\chi(x) = |x+c| + |x-c|, \ c \gg 1.$$

where $(-c, c)$ joins the centers of repulsivity of V_1 and V_2. Then,

$$|z'(t)| \lesssim C(E, M),$$

and, for a compact flow solution

$$z''(t) \geq \frac{1}{c_1}\mu - \int \nabla\chi \cdot \nabla V |u|^2,$$

with $\mu > 0$. The arising term associated to the potential does not have the right sign. However, for $c \gg 1$, $\nabla\chi$ is almost outward-transverse to the level surfaces of V_1 and V_2, and we show that for c_1 fixed big enough, and as soon as the potential enjoys a sufficient decay (it is the assumption (5.2.9)), we have

$$z''(t) \geq \frac{\mu}{2c_1},$$

and we get a contradiction.

6 Outlooks

We contributed in this dissertation to the study of Schrödinger and wave equations, both linear and non-linear, in unstable trapping geometries. This is a work that we will have to pursue: it is part of the very long-term goal of understanding finely the link between the behavior of a wave and the geometry of the domain or manifold it lives in.

A natural sequel of our work about Strichartz estimates in the exterior of many strictly convex obstacles would be to generalize the results of Ivanovici and Lebeau [IL17] to the exterior of many strictly convex obstacles. Many natural questions arise: is the pointwise dispersive estimate without loss in dimension $d = 3$ still valid? In dimensions $d \geq 4$, is there new phenomena due to the addition of other convex obstacles, or are Poisson-Arago spots still the only counter-examples to the dispersive estimate? Is the estimate valid in the exterior of many generic convex?

Complete our rigidity result for the critical non-linear wave equation in the exterior of two strictly convex obstacles by a profiles description in order to show the scattering for critical waves in such a geometry is of course a continuation of our work we will have to carry through. Moreover, our work relies heavily on very rigid Morawetz-type identities. We have a good hope that using it in concentration-compactness/rigidity framework as we began to do it would lead to prove results in more general geometries. Adapting these arguments to non-euclidian manifolds without boundary would also be interesting. However, it will be sooner or later necessary to develop more robust tools and methods: in the situation with more than two strictly convex obstacles, multiplier arguments seem for example give no information.

Finally, very little is known about the scattering-problem for Neumann boundary conditions. It would be worth trying to attack the problem: one could for example begin with the simplest situation of radial solutions in the exterior of a ball. Let us lastly mention that the analogy between obstacles and potentials seems pertinent and is a point of view we will have to continue to develop.

Part III.

Dispersive estimates

7 Strichartz estimates without loss outside two strictly convex obstacles

7.1. Introduction

The *Strichartz estimates* are a family of dispersive estimates for the Schrödinger flow on a Riemannian Manifold M:

$$\|e^{-it\Delta}u_0\|_{L^p(0,T)L^q(M)} \leq C_T\|u_0\|_{L^2},$$

where the exponents (p,q) have to follow the admissibility condition given by the scaling of the equation:

$$p,q \geq 2,\ \frac{2}{p}+\frac{n}{q}=\frac{n}{2},\ (p,q,n)\neq(2,\infty,2),$$

and n is the dimension of M. If M is a manifold with boundaries, we understand Δ to be the Dirichlet Laplacian.

Locally in time, these estimates describe a regularizing effect, reflected by a gain of integrability. Globally, they describe a decay effect: the L^q_x norm has to decay, at least in a L^p_t sense.

We say that *a loss* occurs in the Strichartz estimates, if the estimates do not hold but we are able to show that

$$\|e^{-it\Delta}u_0\|_{L^p(0,T)L^q(M)} \leq C_T\|u_0\|_{H^s},$$

with $s>0$.

The story of Strichartz estimates in the case of the flat Laplacian on $M=\mathbb{R}^n$ begins with the work of [Str77] for $p=q$, for the wave equation, then generalized for any admissible couple and the Schrödinger equation by [GV85b], and by [KT98] for the endpoint ($p=2$) case. In the variable coeficients case, the situation is more difficult. Strichartz estimates without loss for the Schrödinger equation were obtained in several geometrical situations where all the geodesics go to infinity - we say that the manifold is *nontrapping* - : [BT07], [Bou11], [HTW06], [ST02], in the case of manifold without boundary; and in the case of a manifold with boundary, by [Iva10] who obtained the estimates outside one convex obstacle.

Another tool for analyzing the behavior of the Schrödinger flow is the *local smoothing effect*

$$\|\chi e^{-it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{1/2}(M))} \leq C\|u_0\|_{L^2}$$

where χ is a smooth compactly supported function. The influence of the geometry on the local smoothing effect is now fully understood: [Bur04] showed that a necessary and sufficient condition for this estimate to hold is the non-trapping condition. When trapped geodesics exist, a loss in the smoothing effect has to occur.

But in the case of the Strichartz estimates, the situation is not fully understood yet: [BGH10] showed Strichartz estimates without loss for asymptotically euclidian manifolds without boundary for which the trapped set is small enough and exhibit an hyperbolic dynamics.

We go in the same direction and show global Strichartz estimates without loss outside two convex obstacles, providing a first example of Strichartz estimates without loss in a trapped setting for the problem with boundaries:

Theorem 7.1. *Let Θ_1 and Θ_2 be two smooth, compact, strictly convex subsets of $\mathbb{R}^3$, and $\Omega = \mathbb{R}^3\backslash(\Theta_1 \cup \Theta_2)$. If (p, q) verifies*

$$p > 2, q \geq 2, \; \frac{2}{p} + \frac{3}{q} = \frac{3}{2},$$

then, the following estimate holds:

$$\|e^{-it\Delta_D}u_0\|_{L^p(\mathbb{R},L^q(\Omega))} \leq C\|u_0\|_{L^2}. \tag{7.1.1}$$

Strictly convex is here understood in the sense of [Iva10], that is with a second fondamental form bounded below.

The behavior of the waves equation outside several convex obstacles was first investigated by [Ika88, Ika82], who showed local energy decay. Burq used in [Bur93] some of his results to show a control property on an equation closely related to the Schrödinger equation outside several convex bodies.

For data of frequencies $\sim h^{-1}$, if we are able to prove Strichartz estimates in times h

$$\|e^{-it\Delta_D}u_0\|_{L^p(0,h)L^q(\Omega)} \leq C\|u_0\|_{L^2},$$

then the smoothing estimate provides the Strichartz estimates. But in our framework, a logarithmic loss occur in the estimate, due to the presence of a trapped ray:

$$\|\chi e^{-it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{1/2}(M))} \leq C|\ln h|^{\frac{1}{2}}\|u_0\|_{L^2}.$$

However, [BGH10] remarked that this logarithmic loss can be compensated if we show Strichartz estimates up to times $h|\log h|$. Following their idea and arguments of [Iva10], we first show that we can restrict ourselves to prove the estimate on a neighborhood of the periodic ray and in times $h\log h$, for datas of frequencies $\sim h^{-1}$. Then, we reduce again the problem, to data which micro-locally contain only points of the tangent space which do not escape a given neighborhood of the periodic ray after logarithmic times. Finally, following ideas of [Ika88, Ika82], [Bur93], we construct an approximate solution for such data, and we show that this approximation gives the desired estimate.

Remark 7.2. We restrict here ourselves to the three dimensional case because we use results of [Ika88, Ika82] and [Bur93] who are stated in dimension three. The other parts of the proof hold in any dimension without modification. Because the above mentioned results should be extended without difficulty to the arbitrary dimensional case, our proof should work in any dimension.

Remark 7.3. We do not obtain the endpoint case $(p, q) = (2, 6)$ because we make use of the Christ-Kiselev lemma. However, this restriction should be avoided with a more careful analysis.

7.1.1. Notations

We will use the following notations and conventions:

Θ_1, Θ_2 will be two smooth, compact, strictly convex subsets of $\mathbb{R}^3$, called the obstacles,

$X^i(x, p)$ the i'th point of intersection of the reflected ray starting at x in the direction p with the obstacles,

$\Xi^i(x, p)$ the direction of the reflected ray starting at x in the direction p after i reflections,

$(X_t(x, p), \Xi_t(x, p))$ the point and the direction attained by following the reflected ray starting at x in the direction p during a time t at speed $|p|$,

$\Phi_t(x, p)$ the billiard flow, $\Phi_t(x, p) = (X_t(x, p), \Xi_t(x, p))$.

Moreover, we refer to *the periodic ray,* denoted $\mathcal{R}$, as the segment joining Θ_1 and Θ_2 that is the unique periodic trajectory of $\mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2)$.

Let $h > 0$ be a fixed small parameter. We fix $\alpha_0, \beta_0 > 0$ and a smooth funtion $\psi \in C_0^\infty$ supported in $[\alpha_0, \beta_0]$. For $h > 0$ and $u \in L^2$, $\psi(-h^2\Delta)u$ is the localisation of u in frequencies of sizes $[\alpha_0 h^{-1}, \beta_0 h^{-1}]$. We refer to [Iva10] for the definition of $\psi(-h^2\Delta)$.

Finally, for $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we will design by $\mathrm{Op}(a)$ the microlocal operator with symbol a operating at scale h, defined for $u \in L^2(\mathbb{R}^n)$ by

$$(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi h)^n} \int \int e^{-i(x-y)\cdot\xi/h} a(x, \xi) u(y) d\xi dy.$$

We refer to [Zwo12] for classical properties of these operators.

7.2. Reduction to logarithmic times near the periodic ray

The aim of this section is to show that the following proposition implies theorem 7.1:

Proposition 7.4 (Semiclassical Strichartz estimates on a logarithmic interval near the periodic ray)**.** *There exists $\epsilon > 0$ such that for any $\chi \in C_0^\infty$ vanishing outside a small enough neighborhood of the trapped ray, we have*

$$\|\chi e^{-it\Delta_D} \psi(-h^2\Delta) u_0\|_{L^p(0, \epsilon h |\log h|) L^q(\Omega)} \leq C \|u_0\|_{L^2}. \tag{7.2.1}$$

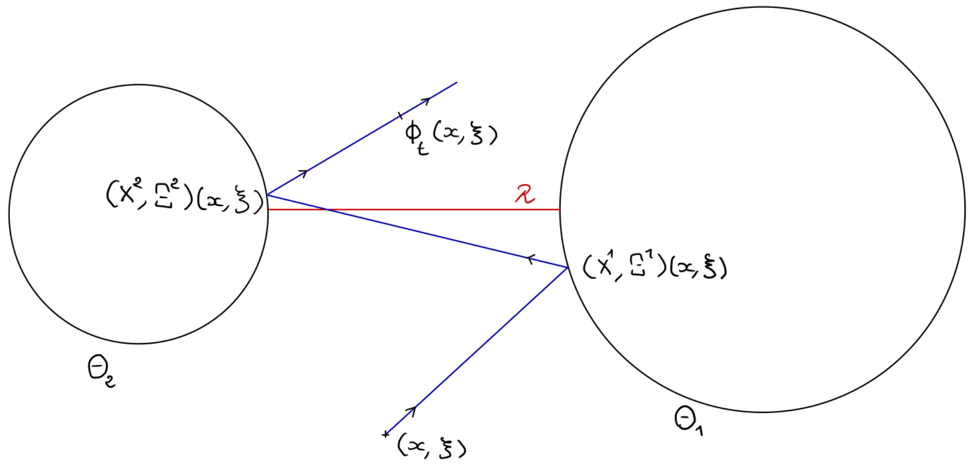

Figure 7.1.1.: Notations

In the rest of this section, we will assume that 7.4 holds with $\epsilon = 1$ and prove theorem 7.1. The proof for other values of $\epsilon > 0$ is the same.

Let $\chi_{\text{obst}}, \chi_{\text{ray}} \in C_0^\infty$ be such that $\chi_{\text{obst}} = 1$ in a neighborhood of $\Theta_1 \cup \Theta_2 \cup \mathcal{R}$, and $\chi_{\text{ray}} \in C_0^\infty$ such that $\chi_{\text{ray}} = 1$ in a neighborhood of $\mathcal{R}$, and let $\tilde{\psi}$ such that $\tilde{\psi} = 1$ on the support of ψ. We decompose $\psi(-h^2\Delta)e^{it\Delta_D}u_0$ in a sum of three terms

$$\begin{aligned}\psi(-h^2\Delta)e^{it\Delta_D}u_0 = \tilde{\psi}(-h^2\Delta)(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ + \tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}\chi_{\text{ray}}\psi(-h^2\Delta)e^{it\Delta_D}u_0. \quad (7.2.2)\end{aligned}$$

We can deal with the first two terms like in [Iva10] if we show a smoothing effect *without loss* for χ equal to 0 near the trapped ray. The third term can be handled with the method of [BGH10]: the smoothing effect *with logarithmic loss* obtained in [Bur04] combined with Proposition 7.4 and the smoothing effect in the non trapping region is sufficient to show Strichartz estimates without loss.

Let us first show

Proposition 7.5 (Local smoothing without loss in the non trapping region)**.** *For any* $\chi \in C_0^\infty$ *vanishing in a neighborhood of the periodic ray, we have*

$$\|\chi e^{-it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{1/2}(\Omega))} \lesssim \|u_0\|_{L^2}. \quad (7.2.3)$$

Proof. Let $\mathcal{K}$ be a smooth, connected, non trapping obstacle, coinciding with $\Theta_1 \cup \Theta_2$ outside a neighborhood of the periodic ray on which χ is vanishing (figure 7.2.1). Note that, in particular, $\mathcal{K}$ coincides with $\Theta_1 \cup \Theta_2$ on the support of χ. We set $\tilde{\Omega} = \mathbb{R}^n \backslash \mathcal{K}$. In particular, $\Delta_{D,\tilde{\Omega}}$ and $\Delta_{D,\Omega}$ coincides on the support of χ. As $\mathcal{K}$ is non trapping, we have

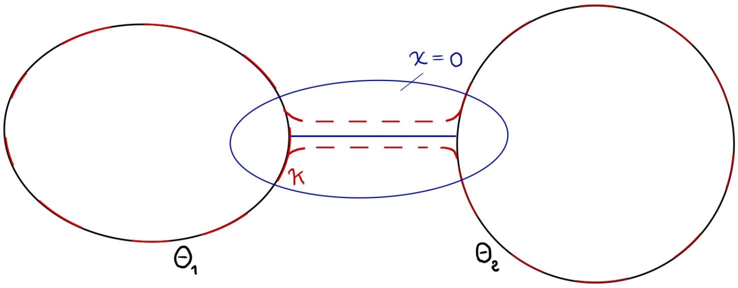

Figure 7.2.1.:

by the work of [VZ00] and [MS78, MS82] for the high frequencies part, [Bur98] for the low frequencies

$$\|\chi(-\Delta_{D,\tilde{\Omega}} - (\lambda \pm \epsilon)^2)^{-1}\chi\|_{L^2(\tilde{\Omega}) \to L^2(\tilde{\Omega})} \leq C|\lambda|^{-1}.$$

Now, if $u \in L^2(\Omega)$, we have

$$\begin{aligned}\|\chi(-\Delta_{D,\Omega} - (\lambda \pm \epsilon)^2)^{-1}\chi u\|_{L^2(\Omega)} &= \|\chi(-\Delta_{D,\tilde{\Omega}} - (\lambda \pm \epsilon)^2)^{-1}\chi u\|_{L^2(\tilde{\Omega})} \\ &\leq C\|u\|_{L^2(\tilde{\Omega})}|\lambda|^{-1} \leq C\|u\|_{L^2(\Omega)}|\lambda|^{-1}.\end{aligned}$$

So, the following resolvent estimate holds

$$\|\chi(-\Delta_{D,\Omega} - (\lambda \pm \epsilon)^2)^{-1}\chi\|_{L^2(\Omega) \to L^2(\Omega)} \leq C|\lambda|^{-1}.$$

And we can deduce the smoothing estimate (7.2.3) exactly like in [BGT04a]. □

7.2.1. The first two terms: away from the periodic ray

In this section, we deal with the first two terms of (7.2.2), following [Iva10].

7.2.1.1. The first term

We set $w_h(x,t) = (1 - \chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0$, who satisfies

$$\begin{cases} i\partial_t w_h + \Delta_D w_h = & -[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ w_h(t=0) = & (1 - \chi_{\text{obst}})\psi(-h^2\Delta)u_0 \end{cases}$$

By the arguments of [Iva10], as $\chi = 1$ near $\partial\Omega$, w_h solve a problem in the full space, and the Duhamel formula combined with the Strichartz estimates for the usual Laplacian on $\mathbb{R}^n$ and the Christ-Kiselev lemma gives the estimate

$$\|w_h\|_{L^pL^q} \lesssim \|(1 - \chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}^n)} + \|[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{-1/2}}.$$

Note that $[\chi, \Delta_D]$ is supported in a compact set and outside some neighborhood of $\Theta_1 \cup \Theta_2 \cup \mathcal{R}$. So, there exists $\tilde{\chi} \in C_0^\infty$ such that $\tilde{\chi} = 1$ on the support of $[\chi_{\text{obst}}, \Delta_D]$ and vanishing in a neighborhood of $\mathcal{R}$. Then, by the smoothing estimate without loss in the non trapping region (7.2.3), we get

$$\begin{aligned} \|[\chi_{\text{obst}}, \Delta_D]\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{-1/2}} &= \|[\chi_{\text{obst}}, \Delta_D]\tilde{\chi}\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{-1/2}} \\ &\leq \|\tilde{\chi}\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2H^{1/2}} \\ &\lesssim \|\psi(-h^2\Delta)u_0\|_{L^2}. \end{aligned}$$

Finally, the estimate

$$\|\tilde{\psi}(-h^2\Delta)w_h\|_{L^r} \leq \|w_h\|_{L^r}$$

of [IP] permits as in [Iva10] to conclude that

$$\|\tilde{\psi}(-h^2\Delta)(1-\chi_{\text{obst}})\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^pL^q} \leq \|\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2}.$$

7.2.1.2. The second term

Following again [Iva10], we localise in space in the following way: let $\varphi \in C_0^\infty(-1,2)$ equal to one on $[0,1]$, and for $l \in \mathbb{Z}$

$$\begin{aligned} v_{h,l} &= \varphi(t/h - l)\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0 \\ V_{h,l} &= \left(\varphi(t/h-l)A + i\frac{\varphi'(t/h-l)}{h}\chi_{\text{obst}}(1-\chi_{\text{ray}})\right)\psi(-h^2\Delta)e^{it\Delta_D}u_0 \end{aligned}$$

where

$$A = \Delta\chi_{\text{obst}}(1-\chi_{\text{ray}}) - \chi_{\text{obst}}\Delta\chi_{\text{ray}} - \nabla\chi_{\text{ray}}\cdot\nabla\chi_{\text{obst}} + (1-\chi_{\text{ray}})\nabla\chi_{\text{obst}}\cdot\nabla - \chi_{\text{obst}}\nabla\chi_{\text{ray}}\cdot\nabla.$$

These quantities verify

$$\begin{cases} i\partial_t v_{h,l} + \Delta_D v_{h,l} = & V_{h,l} \\ v_{h,l|hl+2h>t>hl-h} = & 0 \end{cases}$$

Note that the support of A is included in that of $\chi_{\text{obst}}(1-\chi_{\text{ray}})$. Let $Q \subset \mathbb{R}^3$ be a cube sufficiently large to contain $\Theta_1 \cup \Theta_2$. We denote by S the punctured torus obtained by removing $\Theta_1 \cup \Theta_2$ of Q. Now, let $\tilde{\chi} \in C_0^\infty$ be equal to one on the support of $\chi_{\text{obst}}(1-\chi_{\text{ray}})$, supported outside a neighborhood of $\mathcal{R}$ and in a neighborhood of $\partial\Omega$ such that Δ_D and Δ_S coincides on its support (figure 7.2.2). Writing $v_{h,l} = \tilde{\chi}v_{h,l}$ and $V_{h,l} = \tilde{\chi}V_{h,l}$, the rest of the proof follow as in [Iva10] considering the problem in S, using our smoothing estimate without loss outside the trapping region (7.2.3) instead of the smoothing estimate of [Bur04], and we obtain

$$\|\tilde{\psi}(-h^2\Delta)\chi_{\text{obst}}(1-\chi_{\text{ray}})\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^pL^q} \leq \|\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^2}.$$

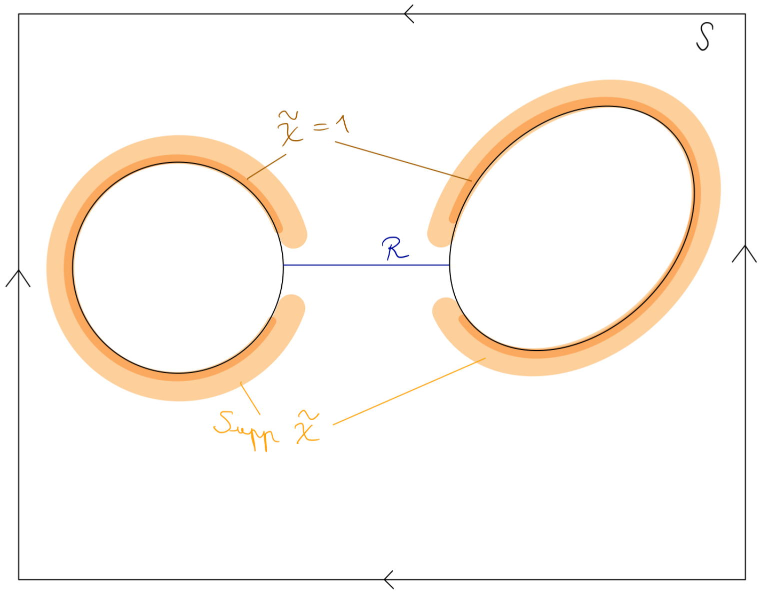

Figure 7.2.2.:

7.2.2. The third term: near the periodic ray

We recall the smoothing effect with logarithmic loss obtained in [Bur04] in the case of the exterior of many convex obstacles verifying the Ikawa's condition - which is always verified in the present framework of the exterior of two balls:

Proposition 7.6. *For any $\chi \in C_0^\infty(\mathbb{R}^d)$ and any $u_0 \in L^2(\Omega)$ such that $u_0 = \psi(-h^2\Delta)u_0$, we have*

$$\|\chi e^{it\Delta_D} u_0\|_{L^2(\mathbb{R},L^2)} \lesssim (h|\log h|)^{\frac{1}{2}} \|u_0\|_{L^2} \tag{7.2.4}$$

Let us denote, in this subsection, $\chi = \chi_{\text{osbt}}\chi_{\text{ray}}$ and $u = \psi(-h^2\Delta)e^{it\Delta_D}u_0$. We will here assume moreover that Proposition 7.4 holds and follow the method of [BGH10] to control the third term of (7.2.2).

In the spirit of [BGH10], we will localize in time intervals of length $h|\log h|$, on which we can apply the semi-classical Strichartz estimates of Proposition 7.4. Consider $\varphi \in C_0^\infty((-1,1))$ satisfying $\varphi \geq 0$, $\varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s-j) = 1$. We decompose

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi(\frac{t}{h|\log h|} - j)\chi u =: \sum_{j\in\mathbb{Z}} u_j.$$

The u_j satisfy the equation

$$(i\partial_t - \Delta_D)u_j = F_j + G_j$$

with

$$F_j = (h|\log h|)^{-1}\varphi'(\frac{t}{h|\log h|} - j)\chi u,$$
$$G_j = 2\varphi(\frac{t}{h|\log h|} - j)\,(\nabla\chi\cdot\nabla u - \Delta_D\chi u)\,.$$

Let us denote

$$v_j(t) = \int_{(j-1)h|\log h|}^{t} e^{i(t-s)\Delta_D}F_j(s)ds,$$
$$w_j(t) = \int_{(j-1)h|\log h|}^{t} e^{i(t-s)\Delta_D}G_j(s)ds.$$

Clearly, $u_j = v_j + w_j$. If we define

$$\tilde v_j(t) = e^{it\Delta_D}\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{-is\Delta_D}F_j(s)ds,$$
$$\tilde w_j(t) = e^{it\Delta_D}\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{-is\Delta_D}G_j(s)ds,$$

the Christ-Kiselev lemma allows us to estimate the L^pL^q norms of $\tilde v_j$ and $\tilde w_j$ instead of v_j and w_j.

We can use the semi-classical Strichartz estimate on logarithmic interval of Proposition 7.4 to estimate $\|\tilde v_j\|_{L^pL^q}$:

$$\|\tilde v_j\|_{L^pL^q} \lesssim \|\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D}F_j(s)ds\|_{L^2}$$

We take $\tilde\chi \in C_0^\infty$ equal to one on the support of χ and use the dual version of (7.2.4) to get

$$\begin{aligned}
\|\tilde v_j\|_{L^pL^q} &\lesssim \|\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D}F_j(s)ds\|_{L^2}\\
&= \frac{1}{h|\log h|}\|\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D}\tilde\chi\varphi'(\frac{t}{h|\log h|} - j)\chi u ds\|_{L^2}\\
&\lesssim \frac{1}{h|\log h|}\times(h|\log h|)^{\frac12}\|\varphi'(\frac{t}{h|\log h|} - j)\chi u\|_{L^2L^2},
\end{aligned}$$

so we get

$$\|\tilde v_j\|_{L^pL^q} \lesssim \frac{1}{(h|\log h|)^{1/2}}\|\varphi'(\frac{t}{h|\log h|} - j)\chi u\|_{L^2L^2}.$$

Let us now estimate $\|\tilde w_j\|_{L^pL^q}$. Again, because of Proposition 7.4, we have

$$\|\tilde w_j\|_{L^pL^q} \lesssim \|\int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D}G_j(s)ds\|_{L^2}.$$

But G_j is localized away from the periodic ray. We take $\tilde{\chi} \in C_0^\infty$ equal to one on the support of $\nabla\chi$ and vanishing near the periodic ray. Then $G_j = \tilde{\chi} G_j$. Hence we can use the dual estimate of the smoothing effect without loss in the non trapping region of Proposition 7.5 to get

$$\begin{aligned}
\|\tilde{w}_j\|_{L^pL^q} &\lesssim \| \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D} G_j(s) ds\|_{L^2} = \| \int_{(j-1)h|\log h|}^{(j+1)h|\log h|} e^{is\Delta_D} \tilde{\chi} G_j(s) ds\|_{L^2} \\
&\lesssim \|G_j\|_{L^2H^{-1/2}} \\
&\lesssim \|\varphi(\frac{t}{h|\log h|} - j)\nabla\chi u\|_{L^2H^{1/2}}.
\end{aligned}$$

We use the Christ-Kiselev lemma twice, take the square and sum to obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^pL^q} &\lesssim \sum_{j\in\mathbb{Z}} \left(\frac{1}{h|\log h|} \|\varphi'(\frac{t}{h|\log h|} - j)\chi u\|^2_{L^2L^2} + \|\varphi(\frac{t}{h|\log h|} - j)\nabla\chi u\|^2_{L^2H^{1/2}} \right) \\
&\lesssim \frac{1}{h|\log h|} \|\chi u\|^2_{L^2L^2} + \|\nabla\chi u\|^2_{L^2H^{1/2}}
\end{aligned}$$

the first term is controlled using the smoothing estimate with logarithmic loss (7.2.4), and the second the smoothing estimate on the non-trapping region (7.2.3). Hence we get

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^pL^q} \lesssim \|\psi(-h^2\Delta)u_0\|^2_{L^2}.$$

But, because of the continuous embedding $l^2(\mathbb{Z}) \hookrightarrow l^p(\mathbb{Z})$ for $p \geq 2$ we know that

$$\|\chi u\|_{L^pL^q} \sim \left(\sum_{j\in\mathbb{Z}} \|u_j\|^p_{L^pL^q} \right)^{1/p} \lesssim \left(\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^pL^q} \right)^{1/2}$$

and we thus can conclude:

$$\|\tilde{\psi}(-h^2\Delta)\chi_{\text{ray}}\chi_{\text{obst}}\psi(-h^2\Delta)e^{it\Delta_D}u_0\|_{L^pL^q} \lesssim \|\psi(-h^2\Delta)u_0\|^2_{L^2}.$$

7.2.3. Conclusion

We conclude from the two previous subsections that Proposition 7.4 implies the estimate

$$\|e^{-it\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(\mathbb{R},L^q(\Omega))} \leq \|\psi(-h^2\Delta)u_0\|_{L^2}.$$

Like in [Iva10], we can remove the frequency cut-off to get

$$\|e^{-it\Delta_D}u_0\|_{L^p(\mathbb{R},L^q(\Omega))} \leq \|u_0\|_{L^2}.$$

Hence Proposition 7.4 implies theorem 7.1. Thus, the rest of the paper will be devoted to prove Proposition 7.4.

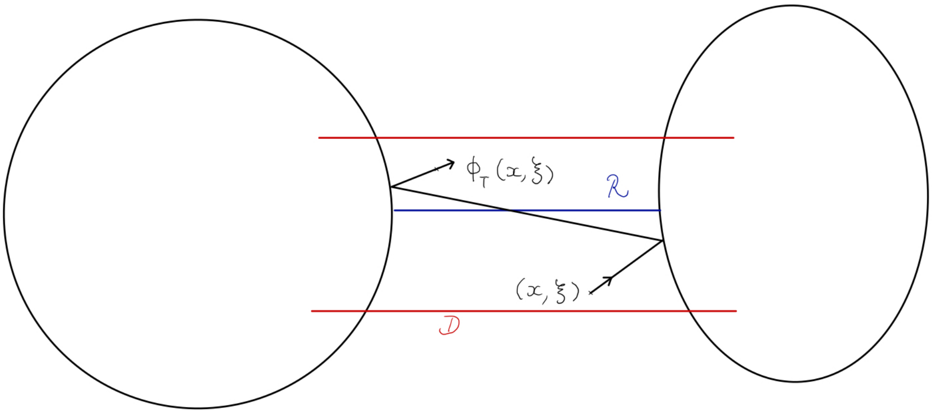

Figure 7.3.1.: The trapped set

7.3. Reduction to the trapped rays

Let D be a neighborhood of the trapped ray. For technical reasons, we suppose that D is an open cylinder with the trapped ray for axis. For $T > 0$, we define the trapped set of D in time T:

Definition 7.7. We say that $(x, \xi) \in T^\star\Omega \cap (\Omega \times \{|\xi| \in [\alpha_0, \beta_0]\})$ belongs to the trapped set of D in time T, denoted $\mathcal{T}_T(D)$ if and only if there exists a broken bicharacteristic γ starting from D and $t < -T$ such that $\gamma(t) = (x, \xi)$. Moreover, we define $\hat{\mathcal{T}}_T(D) := \mathcal{T}_T(D) \cap \{D \times \mathbb{R}^n\}$.

In other words, $\mathcal{T}_T(D)$ is composed of the points of $\Omega \times \{|\xi| \in [\alpha_0, \beta_0]\}$ that lie in D after some time bigger than T, and $\hat{\mathcal{T}}_T(D)$ is composed of the points of $D \times \{|\xi| \in [\alpha_0, \beta_0]\}$ that still lie in D after a time T (figure 7.3.1).

We will say that $u \in L^2$ is micro-locally supported in $U \subset T^\star\Omega$ if $\mathrm{Op}(a)u = u$ for all $a \in C_0^\infty(T^\star\Omega)$ such that $a = 1$ in U. The aim of this section is to show that it is sufficent to prove (7.2.1) for data micro-locally supported in $\hat{\mathcal{T}}_{2\epsilon|\log h|}(D)$.

7.3.1. Some properties of the billiard flow

We first need some properties of the billiard flow associated with $\mathbb{R}^n \backslash (\Theta_1 \cup \Theta_2)$. More precisely, we are interested in the regularity in (x, ξ) of the flow $\Phi_t(x, \xi)$. We first show:

Lemma 7.8. *Let Θ be a smooth, strictly convex compact subset of $\mathbb{R}^n$, with no infinite order contact point, and*

$$W(\Theta) := \{(x, \xi), x \in \mathbb{R}^n \backslash \Theta, \xi \in \mathbb{R}^n, \ s.t. \ \exists t \geq 0, x + t\xi \in \Theta\}.$$

We denote by t the application

$$t : (x, \xi) \in W(\Theta) \longrightarrow \textit{the smallest } t' \textit{ such that } x + t'\xi \in \partial\Omega,$$

and by

$$W_{tan}(\Theta) = \{(x,\xi) \in W \ s.t.\ \xi \cdot n(x + t(x,\xi)\xi) = 0\},$$

the tangent rays. Then:

1. *on* $W(\Theta)\backslash W_{tan}(\Theta)$*,* t *is* C^∞*,*

2. t *is locally Hölder on* $W(\Theta)$*.*

Proof. Let $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that $\partial\Theta$ is given by the implicit relation $g(x) = 0$. We denote, for $x, \xi \in \mathbb{R}^n$, $t \in \mathbb{R}$

$$h(x,\xi,t) = g(x + t\xi).$$

Away from the tangent rays: we pick $(x_1, \xi_1) \in W\backslash W_{\tan}$. Let $t_1 = t(x_1,\xi_1)$. Then, $\partial_t h(x_1,\xi_1,t_1) = \xi_1 \cdot \nabla g(x_1 + t_1\xi_1) \neq 0$ by definition of $W\backslash W_{\tan}$. Therefore, by the implicit functions theorem, t is a C^∞ function of (x,ξ) for (x,ξ) in a neighborhood of (x_1,ξ_1).

General case: Let $(x_1,\xi_1) \in W$ and $t_1 = (x_1,\xi_1)$. Then $h(x_1,\xi_1,t_1) = 0$. Let $k \geq 1$ be the smallest integer such that

$$h((x_1,\xi_1),t_1) = 0, \partial_t h((x_1,\xi_1),t_1) = 0, \ldots, \partial_t^k h((x_1,\xi_1),t_1) \neq 0.$$

Note that k exists because Θ has no infinite order contact point. By the C^∞ preparation theorem due to Malgrange [Mal64], there exists $a_1, \ldots, a_{k-1} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $c \in C^\infty(\mathbb{R}\times\mathbb{R}^n \times \mathbb{R}^n)$ not vanishing on $((x_1,\xi_1),t_1)$, such that, on a neighborhood of $((x_1,\xi_1),t_1)$, h writes

$$h(x,\xi,t) = c(x,\xi,t)(t^k + a_{k-1}(x,\xi)t^{k-1} + \cdots + a_0(x,\xi)).$$

Then, for (x,ξ,t) in a neighborhood of (x_1,ξ_1,t_1),

$$t = t(x,\xi) \iff t^k + a_{k-1}(x,\xi)t^{k-1} + \cdots + a_0(x,\xi) = 0.$$

But, by [Bri10], the roots of a monic polynomial are Hölder with respect to the coefficients - of power one over the muliplicity of the root. Because the a_i, $0 \leq i \leq k-1$ are C^∞ with respect to (x,ξ), we conclude that t is Hölder in a neighborhood of (x_1,ξ_1). □

Remark 7.9. The worst power in the Hölder inequality in a neighborhood of a tangent ray is $\frac{1}{k}$, where k is the order of contact of the tangent ray. Note that in our framework of strictly geodesically convex obstacles, $k = 2$.

Let $\eta > 0$. We adopt the following notations for the tangents sets to $\Theta_1 \cup \Theta_2$ and their η-neighborhood:

$$W_{\tan} = (W_{\tan}(\Theta_1) \cup W_{\tan}(\Theta_2)) \cap \{|\xi| \in [\alpha_0, \beta_0]\}$$

$$W_{\tan,i,\eta} = \{(x,\xi) \in W(\Theta_i),\ |\xi| \in [\alpha_0,\beta_0] \text{ s.t. } |\xi \cdot n(x + t(x,\xi)\xi)| \leq \eta\},$$

$$W_{\tan,\eta} = W_{\tan,1,\eta} \cup W_{\tan,2,\eta}.$$

We show that a ray cannot pass a small enough-neighborhood of the tangent set more than twice, that is

Lemma 7.10. *There exists $\eta > 0$ such that any ray cannot cross $W_{tan,\eta}$ more than twice.*

Proof. If it is not the case, for all $n \geq 0$, there exists $(x_n, \xi_n) \in K \times \mathcal{S}^2$, where K is a compact set strictly containing the obstacles, such that $\Phi_t(x_n, \xi_n)$ cross $W_{\tan,\frac{1}{n}}$ at least three times. Extracting from (x_n, ξ_n) a converging subsequence, by continuity of the flow, letting n going to infinity we obtain a ray that is tangent to $\Theta_1 \cup \Theta_2$ in at least three points. Therefore, it suffises to show that such a ray cannot exists.

Remark that, because there is only two obstacles, if $(x, \xi) \in W_{\tan}$, if we consider the ray starting from (x, ξ) and the ray starting from $(x, -\xi)$, one of the two do not cross any obstacle in positive times. But, if there is a ray tangent to the obstacles in at least three points, if we consider the second tangent point (x_0, ξ_0), both rays starting from (x_0, ξ_0) and $(x_0, -\xi_0)$ have to cross an obstacle, therefore, this is not possible. □

Now, we can control how much two rays starting from different points and directions can diverge:

Lemma 7.11. *Let V be a bounded open set containing the convex hull of $\Theta_1 \cup \Theta_2$. Then, there exists $\alpha > 0$, $C > 0$ and $\tau > 0$ such that, for all $x, \tilde{x} \in V$, all $\xi, \tilde{\xi}$ such that $|\xi|, |\xi'| \in [\alpha_0, \beta_0]$, for all $t > 0$ there exists t' verifying $|t' - t| \leq \tau$ such that*

$$d(\Phi_{t'}(\tilde{x}, \tilde{\xi}), \Phi_{t'}(x, \xi))) \leq C^{t'} d((\tilde{x}, \tilde{\xi}), (x, \xi))^{\alpha}. \tag{7.3.1}$$

Proof. **Preliminary notations and remarks.** For $i = 1, 2$, let t_i be the application associated to $W(\Theta_i)$ by lemma 7.8. According to lemma 7.10, we choose $\eta > 0$ small enough so that any ray cannot cross $W_{\tan,\eta}$ more than twice. Note that, by lemma 7.8, because W_i is compact, t_i is globally Hölder on W_i. We denote by $\mu > 0$ the smallest of the two Hölder powers. Moreover, t_i is C^∞ on $W \backslash W_{\tan,\mathrm{i},2\eta}$ thus in particular globally Lipschitz on $W \backslash W_{\tan,i,\eta}$.

Case A: $(\tilde{x}, \tilde{\xi}), (x, \xi)$ are far. We pick $\epsilon_0 > 0$ to be choosen later. Note that we always have

$$d(\Phi_t(\tilde{x}, \tilde{\xi}), \Phi_t(x, \xi)) \leq d(x, \tilde{x}) + 2\beta_0 t + 2\beta_0,$$

therefore, if $d((\tilde{x}, \tilde{\xi}), (x, \xi)) \geq \epsilon_0$,

$$d(\Phi_t(\tilde{x}, \tilde{\xi}), \Phi_t(x, \xi)) \leq d(x, \tilde{x}) + d((\tilde{x}, \tilde{\xi}), (x, \xi)) \frac{2\beta_0}{\epsilon_0}(t+1), \tag{7.3.2}$$

and the estimate holds.

Case B: $(\tilde{x}, \tilde{\xi}), (x, \xi)$ are close. Now, we suppose that $d((\tilde{x}, \tilde{\xi}), (x, \xi)) < \epsilon_0$. Let t_0, resp. t_1, be the first time where $\{\Phi_t(x, \xi), t \geq 0\}$, resp. $\left\{\Phi_t(\tilde{x}, \tilde{\xi}), t \geq 0\right\}$, cross an obstacle, with the convention that $t_0 = +\infty$, resp. $t_1 = +\infty$ if this does not happen. We denote by X_0 and X_1 the eventual points of intersection with the obstacles. We suppose that $t_0 \leq t_1$. Note that, for $0 \leq t \leq t_0$, $\Phi_t(\tilde{x}, \tilde{\xi}) = (\tilde{x} + t\tilde{\xi}, \tilde{\xi})$, $\Phi_t(x, \xi) = (x + t\xi, \xi)$ and thus

$$d(\Phi_t(\tilde{x}, \tilde{\xi}), \Phi_t(x, \xi)) \leq (1 + t) d((\tilde{x}, \tilde{\xi}), (x, \xi)), \text{ for } 0 \leq t \leq t_0. \tag{7.3.3}$$

Now, we would like to understand what happens for $t \geq t_0$.

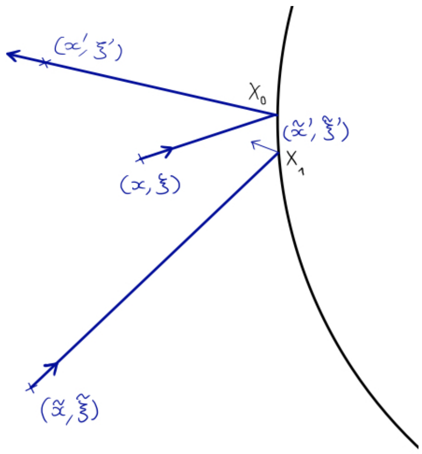

Figure 7.3.2.: Case B.1

Case B.1: $t_0, t_1 < \infty$ and X_0 and X_1 belongs to the same obstacle Θ_i.

We have, because t is μ-Hölder on W_i

$$|t_0 - t_1| \leq Cd((\tilde{x}, \tilde{\xi}), (x, \xi))^\mu \leq C\epsilon_0^\mu \tag{7.3.4}$$

Where C depends only of the geometry of the obstacles. We choose ϵ_0 small enough so that

$$C\epsilon_0^\mu \leq \frac{1}{4}\frac{d}{\beta_0} \tag{7.3.5}$$

where d is the distance between the obstacles. Then, at the time t_1, $\{\Phi_t(x,\xi), t \geq 0\}$ has not crossen another obstacle yet. Thus, $\Phi_{t_1}(x,\xi) = (x', \xi')$ and $\Phi_{t_{1+}}(\tilde{x}, \tilde{\xi}) = (\tilde{x}', \tilde{\xi}')$ are given by, after reflection

$$\begin{cases} (x', \xi') & = (x + t_0\xi + (t_1 - t_0)\xi',\ \xi - 2(n \cdot \xi)\xi) \\ (\tilde{x}', \tilde{\xi}') & = \left(\tilde{x} + t_1\tilde{\xi},\ \tilde{\xi} - 2(\tilde{n} \cdot \tilde{\xi})\tilde{\xi}\right) \end{cases}$$

with

$$\begin{cases} n = & n(x + t_0\xi) \\ \tilde{n} = & n(\tilde{x} + t_1\tilde{\xi}) \end{cases}.$$

Note that, because $X \in \partial\Theta_i \to n(X)$ is C^∞,

$$\begin{aligned} |n - \tilde{n}| &\leq |n(x + t_0\xi) - n(\tilde{x} + t_0\tilde{\xi})| + |n(\tilde{x} + t_0\tilde{\xi}) - n(\tilde{x} + t_1\tilde{\xi})| \\ &\leq C|x + t_0\xi - \tilde{x} - t_0\tilde{\xi}| + C|t_1 - t_0||\tilde{\xi}| \\ &\leq C|x - \tilde{x}| + |t_0||\xi - \tilde{\xi}| + C|t_1 - t_0||\tilde{\xi}|, \end{aligned}$$

moreover, note that

$$|t_i| \leq \frac{\text{diam}(V)}{\alpha_0}, \tag{7.3.6}$$

thus, because of lemma 7.8, we get

$$\begin{aligned}|n-\tilde{n}| &\leq |x-\tilde{x}| + |t_0||\xi-\tilde{\xi}| + C|\tilde{\xi}|d((\tilde{x},\tilde{\xi}),(x,\xi))^{\mu_0}\\ &\leq Cd((\tilde{x},\tilde{\xi}),(x,\xi))^{\mu_0}.\end{aligned} \tag{7.3.7}$$

where

$$\begin{cases}\mu_0=\mu & \text{if } (\tilde{x},\tilde{\xi})\in W_{\tan,\eta} \text{ or } (x,\xi)\in W_{\tan,\eta},\\ \mu_0=1 & \text{else.}\end{cases}$$

Now, to control $d((x',\xi'),(\tilde{x}',\tilde{\xi}'))$ we only have to write

$$\begin{cases}\tilde{\xi}'-\xi' &= \tilde{\xi}-\xi-2(\tilde{n}\cdot\tilde{\xi})(\tilde{\xi}-\xi)+2(\tilde{n}-n)\cdot\tilde{\xi}\xi+n\cdot(\tilde{\xi}-\xi)\xi\\ \tilde{x}'-x' &= \tilde{x}-x+(t_1-t_0)\tilde{\xi}-t_0(\xi-\tilde{\xi})-(t_1-t_0)\xi'\end{cases}$$

and because of (7.3.7), (7.3.6), and lemma 7.8 we obtain

$$d(\Phi_{t_1}(x,\xi),\Phi_{t_{1+}}(\tilde{x},\tilde{\xi})) \leq Cd((x,\xi),(\tilde{x},\tilde{\xi}))^{\mu_0} \tag{7.3.8}$$

with, because of (7.3.4) and (7.3.5)

$$|t_0-t_1| \leq \frac{1}{4}\frac{d}{\beta_0}.$$

Case B.2: $t_0<\infty$ and $t_1=+\infty$; or $t_0,t_1<\infty$ and X_0 and X_1 belongs to different obstacles. Suppose for example that $X_0\in\Theta_1$. For the sake of simplicity, we do the proof in dimension 2 and then explain how to adapt it in dimension 3.

Suppose that the ray starting from $(x,\tilde{\xi})$ do not cross Θ_1. We can always take ξ_1 to be such that $(x,\xi_1)\in W_{\tan,1}$ with $|\xi_1|=|\xi|$, and

$$|\xi-\xi_1| \leq |\xi-\tilde{\xi}|,\ |\tilde{\xi}-\xi_1| \leq |\xi-\tilde{\xi}|, \tag{7.3.9}$$

that is, we choose ξ_1 such that the ray starting from (x,ξ_1) is tangent to Θ_1 and between the ray starting from (x,ξ) and the ray starting from $(x,\tilde{\xi})$ for $t\leq t_0$ (figure 7.3.3).

Remark that, as a consequence of (7.3.9) because $(x,\xi_1)\in W_{\tan}$ and $|\xi-\tilde{\xi}|\leq\epsilon_0$, taking $\epsilon_0\leq\frac{1}{2}\eta$ assure that, necessarily, $(x,\xi)\in W_{\tan,\eta}$.

Let t_0' be the time for which the ray starting from (x,ξ_1) is tangent to Θ_1. Note that, because the application t is Hölder, in the same way that (7.3.4), and because of (7.3.9), taking again ϵ_0 small enough so that (7.3.5) is verified, we have

$$|t_0-t_0'| \leq \frac{1}{4}\frac{d}{\beta_0}.$$

Obviously

$$d(\Phi_{t_{0'}}(x,\xi),\Phi_{t_{0'}}(x,\tilde{\xi})) \leq d(\Phi_{t_0'}(x,\xi),\Phi_{t_0'}(x,\xi_1)) + d(\Phi_{t_0'}(x,\xi_1),\Phi_{t_0'}(x,\tilde{\xi})). \tag{7.3.10}$$

But, using the case B.1

$$d(\Phi_{t_0'}(x,\xi),\Phi_{t_0'}(x,\xi_1)) \leq Cd((x,\xi),(x,\xi_1))^{\mu} \tag{7.3.11}$$

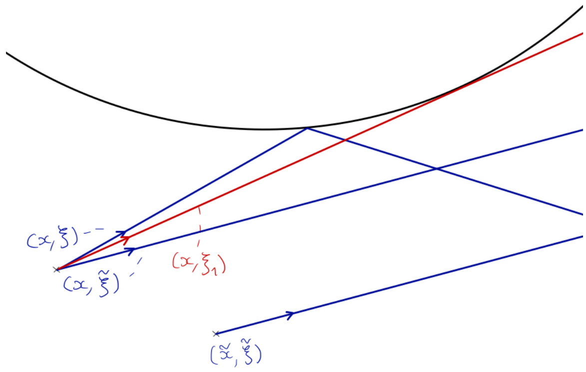

Figure 7.3.3.: Case B.2.a - the ray from $(x, \tilde{\xi})$ do not cross Θ_1

and moreover, in the same way than (7.3.3), because of (7.3.6),

$$d(\Phi_{t_0'}(x, \tilde{\xi}), \Phi_{t_0'}(x, \xi_1)) \leq Cd((x, \tilde{\xi}), (x, \xi_1)). \tag{7.3.12}$$

Combining (7.3.9), (7.3.10), (7.3.11) and (7.3.12) we obtain

$$d(\Phi_{t_0'}(x, \tilde{\xi}), \Phi_{t_0'}(x, \xi)) \leq C|\xi - \tilde{\xi}|^\mu.$$

But, in the same way than (7.3.3) again, because none of the rays starting from $(\tilde{x}, \tilde{\xi})$ or $(x, \tilde{\xi})$ cross obstacles,

$$d(\Phi_{t_0'}(\tilde{x}, \tilde{\xi}), \Phi_{t_0'}(x, \tilde{\xi})) \leq Cd((x, \tilde{\xi}), (x, \xi_1))$$

and we conclude that

$$d(\Phi_{t_0'}(\tilde{x}, \tilde{\xi}), \Phi_{t_0'}(x, \xi)) \leq Cd((\tilde{x}, \tilde{\xi}), (x, \xi))^\mu.$$

Now, suppose that the ray starting from $(x, \tilde{\xi})$ do cross Θ_1. Then, there exists $x_1 \in [x, \tilde{x}]$ such that the ray starting from $(x_1, \tilde{\xi})$ is tangent to Θ_1 (figure 7.3.4). We do the exact same study as in the previous case, now taking the ray starting from $(x_1, \tilde{\xi})$ as intermediary to obtain again

$$d(\Phi_{t_0'}(\tilde{x}, \tilde{\xi}), \Phi_{t_0'}(x, \xi)) \leq Cd((\tilde{x}, \tilde{\xi}), (x, \xi))^\mu, \ |t_0 - t_0'| \leq \frac{1}{4}\frac{d}{\beta_0}. \tag{7.3.13}$$

Remark that, taking again $\epsilon_0 \leq \frac{1}{2}\eta$, we know in the same way that necessarily,$(x, \xi) \in W_{\tan,\eta}$.

In dimension 3, we do the same proof with another intermediate point, in order to treat only coplanar rays three by three.

Conclusion: To conclude, we iterate this argument up to time t: As soon as $d(\Phi_{T_0}(x, \xi), \Phi_{T_0}(\tilde{x}, \tilde{\xi})) \geq \epsilon_0$, we use (7.3.2) for times larger than T_0. Between reflections, we use (7.3.3) together

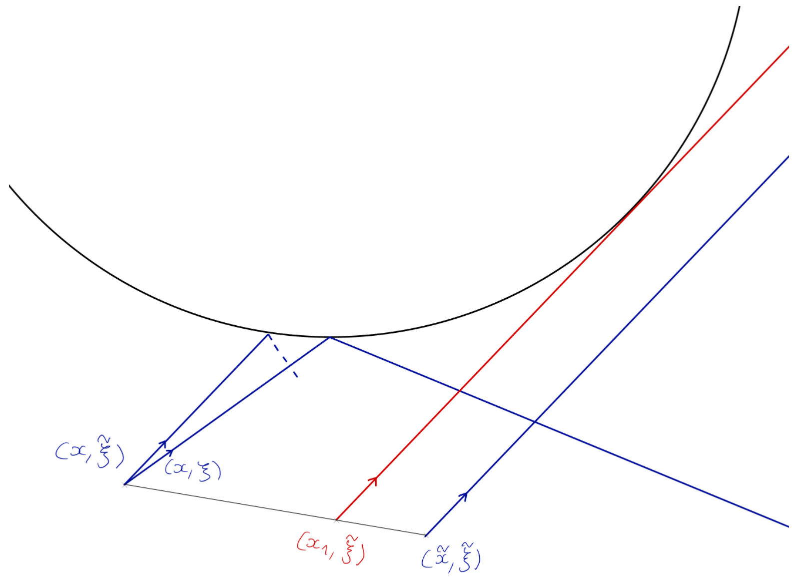

Figure 7.3.4.: Case B.2.b - the ray from $(x, \tilde{\xi})$ cross Θ_1

with (7.3.6). At reflections, we use (7.3.8) (case B.1) or (7.3.13) (case B.2). Note that because the rays cannot cross $W_{\tan,\eta}$ more than twice, we are in the case where $\mu_0 = \mu$ at most four times. So we get, with $N(t)$ the number of reflections we have encountered in time t:

$$d(\Phi_t(\tilde{x}, \tilde{\xi}), \Phi_t(x, \xi)) \leq C^{N(t)}(1+t)d((x,\xi),(\tilde{x},\tilde{\xi}))^{\mu^4}, \tag{7.3.14}$$

for all $t \geq 0$ except thoses in the intervals $[t_0, t_1]$ when the case B.1. is encountered and thoses in the intervals $[t_0, t_0']$ when the case B.2 is encountered. Note that we always have

$$|t_0 - t_1|, |t_0 - t_0'| \leq \frac{1}{4}\frac{d}{\beta_0},$$

and the time separing the t_1's (resp t_0's) from another such forbiden interval - that is, from another reflection of one of the two rays - is at least $\frac{d}{\beta_0} - |t_0 - t_1| \geq \frac{d}{\beta_0} - \frac{1}{4}\frac{d}{\beta_0} > 0$ (resp $\frac{d}{\beta_0} - |t_0 - t_0'|$); therefore (7.3.14) holds for all t except in disjoints intervals of lenght at most $\frac{\tau}{2} = \frac{d}{4\beta_0}$. To conclude, it suffise to note that $N(t) \leq 2\frac{\beta_0}{d}t$. □

Remark 7.12. Because the directional component of the flow is not continuous with respect to time, we can not have (7.3.1) for all time. However, we can take $\tau > 0$ as small as we want at the cost to take a bigger constant C in (7.3.1) - that corresponds to take ϵ_0 smaller.

Remark 7.13. In our framework of strictly geodesically convex obstacles, according to Remark 7.9 and of (7.3.14), we have $\alpha = 2^4 = 16$.

7.3.2. Properties of the trapped set

We now investigate some properties of the trapped set we have defined in the begining of the section.

Lemma 7.14. *For all bicharacteristic γ starting from D with speed in $[\alpha_0, \beta_0]$, we have*

$$d(\gamma(t), \mathcal{T}_T(D)^c) > 0 \ \forall t \in [-T-1, -T]$$

Proof. Notice that, because of the continuity of the flow, $\mathcal{T}_T(D)$ is open. Now, let γ be a bicharacteristic starting from D with speed in $[\alpha_0, \beta_0]$. The set $\{\gamma(t),\ t \in [-T-1, -T]\}$ is compact and $\mathcal{T}_T(D)^c$ is closed, so the distance between them is attained. But by definition of $\mathcal{T}_T(D)$, these two sets never cross. Therefore, the proposition holds. ☐

The following crucial lemma is a consequence of lemma 7.11

Lemma 7.15. *For all $D, \tilde{D}$, there exists $T^\star > 0$, $c > 0$ such that for all $T \geq 0$:*

$$d(\mathcal{T}_{T-T^\star}(D)^c, \mathcal{T}_T(D)) \geq e^{-cT}, \tag{7.3.15}$$

and, if $D \subset \tilde{D}$

$$d(\mathcal{T}_T(\tilde{D})^c, \mathcal{T}_T(D)) \geq \frac{1}{4} e^{-cT} d(\tilde{D}^c, D). \tag{7.3.16}$$

Proof. Let us show (7.3.15). Let $T^\star > 0$. We argue by contradiction. Suppose that the property is false. Then, for all $n \geq 1$ there exists $T_n \geq 0$, $(x_n, \xi_n) \in \mathcal{T}_{T_n}(D)$ and $(\tilde{x}_n, \tilde{\xi}_n) \in \mathcal{T}_{T_n - T^\star}(D)^c$ such that

$$d((\tilde{x}_n, \tilde{\xi}_n), (x_n, \xi_n)) \leq e^{-nT_n}. \tag{7.3.17}$$

By lemma 7.11, there exists $T_n' \in [T_n - \tau, T_n + \tau]$ such that

$$d(\Phi_{T_n'}(\tilde{x}_n, \tilde{\xi}_n), \Phi_{T_n'}(x_n, \xi_n))) \leq d((\tilde{x}_n, \tilde{\xi}_n), (x_n, \xi_n))^\alpha C^{T_n'},$$

then, because of (7.3.17),

$$d(\Phi_{T_n'}(\tilde{x}_n, \tilde{\xi}_n), \Phi_{T_n'}(x_n, \xi_n))) \leq e^{-\alpha n T_n} C^{T_n + \tau} \longrightarrow 0 \tag{7.3.18}$$

as n goes to infinity.

We would like to know how far $\Phi_{T_n'}(\tilde{x}_n, \tilde{\xi}_n)$ can be from D. Let

$$(y_n, \eta_n) = \Phi_{T_n' - T^\star + \tau}(\tilde{x}_n, \tilde{\xi}_n).$$

By definition of the trapped set, $\Phi_t(\tilde{x}_n, \tilde{\xi}_n) \notin D$ for all $t \geq T_n - T^\star$. In particular, this is true for all $t \geq T_n' - T^\star + \tau$. Thus, the ray starting from (y_n, η_n) never cross D. Therefore, because we have chosen D to be a cylinder with the periodic trajectory for axis, (y_n, η_n) is in the case where $\Phi_{T_n'}(\tilde{x}_n, \tilde{\xi}_n) = \Phi_{T^\star - \tau}(y_n, \eta_n)$ is the closest to D, such that (figure 7.3.5):

- η_n is parallel to the periodic trajectory,

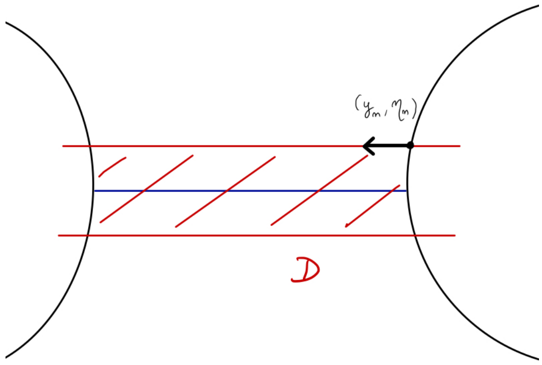

Figure 7.3.5.: (y_n, η_n) in the worst case scenario

- $|\eta_n| = |\xi_n| = \alpha_0$,
- $y_n \in \partial\Theta_i$

in this worst case scenario, after time $T^\star - \tau$, $\Phi_{T_n}(\tilde{x}_n, \tilde{\xi}_n) = \Phi_{T^\star - \tau}(y_n, \eta_n)$ is at a distance at least $c(T^\star) > 0$ of D, with $c(T^\star)$ depending only of D, the minimal curvature of the obstacles, α_0, β_0, and such that $c(T^\star) \to \infty$ as $T^\star \to \infty$. On the other hand, $\Phi_{T_n}(x_n, \xi_n) \in D$, so, because $|T_n' - T_n| \leq \tau$, $\Phi_{T_n'}(x_n, \xi_n)$ is at distance at most $\beta_0\tau$ of D. We can thus choose $T^\star > 0$ large enough so that for all $n \geq 0$

$$d(\Phi_{T_n'}(\tilde{x}_n, \tilde{\xi}_n), \Phi_{T_n'}(x_n, \xi_n)) \geq 1. \tag{7.3.19}$$

which contradicts (7.3.18) and (7.3.15) holds.

Let us now show (7.3.16). If $(x, \xi) \in \mathcal{T}_T(\tilde{D})^c$ and $(\tilde{x}, \tilde{\xi}) \in \mathcal{T}_T(D)$ we have $\Phi_T(x, \xi) \notin \tilde{D}$ and $\Phi_T(\tilde{x}, \tilde{\xi}) \in D$, therefore

$$d(\Phi_T(x, \xi), \Phi_T(\tilde{x}, \tilde{\xi})) > d(\tilde{D}^c, D). \tag{7.3.20}$$

On the other hand, by lemma 7.14, there exist T' such that $|T - T'| \leq \tau$ verifying

$$d(\Phi_{T'}(x, \xi), \Phi_{T'}(\tilde{x}, \tilde{\xi})) \leq C^{T'} d((x, \xi), (\tilde{x}, \tilde{\xi})). \tag{7.3.21}$$

According to Remark 7.12, we can suppose, up to enlarge the constant C, that τ is small enough so that $\beta_0\tau \leq \frac{1}{4} d(\tilde{D}^c, D)$. Then (7.3.20) implies

$$d(\Phi_{T'}(x, \xi), \Phi_{T'}(\tilde{x}, \tilde{\xi})) \geq \frac{1}{2} d(\tilde{D}^c, D). \tag{7.3.22}$$

Therefore, by (7.3.21) and (7.3.22), if $d((x, \xi), (\tilde{x}, \tilde{\xi})) \leq \frac{1}{4} C^{-T} d(\tilde{D}^c, D)$, we cannot have $(x, \xi) \in \mathcal{T}_T(\tilde{D})^c$ and $(\tilde{x}, \tilde{\xi}) \in \mathcal{T}_T(D)$. Thus (7.3.16) is verified. $\square$

7.3.3. Reduction of the problem

We are now in position to show that we can reduce ourselves to prove the following proposition:

Proposition 7.16. *There exists $\epsilon > 0$, a small open neighborhood D of the trapped ray, such that, $\forall u_0 \in L^2$ microlocally supported in $\hat{\mathcal{T}}_{2\epsilon|\log h|}(D)$, and away from a small enough neighborhood of $\partial(\Theta_1 \cup \Theta_2)$, we have for all $\chi \in C_0^\infty$ supported in D*

$$\|\chi e^{-it\Delta}\psi(-h^2\Delta)u_0\|_{L^p(0,\epsilon h|\log h|)L^q} \leq C\|\psi(-h^2\Delta)u_0\|_{L^2}. \tag{7.3.23}$$

Proof of Proposition 7.16 implies theorem 7.1. We suppose that Proposition 7.16 holds and show our main result. By the work of the previous section and the semi-classical change of variable, it suffices to show that there exists $\epsilon' > 0$ such that

$$\|\chi e^{-ith\Delta_D}\psi(-h^2\Delta)u_0\|_{L^p(0,\epsilon'|\log h|)L^q(\Omega)} \leq Ch^{-1/p}\|u_0\|_{L^2}, \tag{7.3.24}$$

for χ supported in a neighborhood of the trapped ray.

We are here inspired by [Iva10], Section 2, p.265-266. As [Iva10] recalls, by the semiclassical finite speed of propagation due to [Leb92], we can restrict ourselves to bicharacteristic travelled at speed in $[\alpha_0, \beta_0]$. We refer to [Leb92], [?] Appendix B, and [Iva10] Appendix A for the propagation properties of the Schrödinger flow in the semi-classical regime for the problem with boundaries.

Let D_1, D_2, D_3 be open neighborhoods of the trapped ray such that

$$D_3 \subsetneq D_2 \subsetneq D_1 \subsetneq D.$$

Let $T = 2\epsilon|\log h|$. We will show that, for χ supported in D_3 and for some $\epsilon_3 < \epsilon_2 < \epsilon_1 < \epsilon$:

1. if (7.3.24) holds for all data microlocally supported in $D_1 \cap \mathcal{T}_{T+1}(D_1)$ with $\epsilon' = \epsilon_1$ then it holds for all data supported in D_2 with $\epsilon' = \epsilon_2$,

2. there exists a small neighborhood of the boundary $\mathcal{V}$ such that, if (7.3.24) holds for all data microlocally supported in $D \cap \mathcal{T}_T(D) \cap \mathcal{V}^c$ with $\epsilon' = \epsilon$ then it holds for all data supported in $D_1 \cap \mathcal{T}_{T+1}(D_1)$ with $\epsilon' = \epsilon_1$,

3. if (7.3.24) holds for all data supported in D_2 with $\epsilon' = \epsilon_2$ then it holds for all data with $\epsilon' = \epsilon_3$,

and the proposition will follow.

(1) Reduction to the trapped sets. Let $\psi(-h^2\Delta)u_0$ be supported in D_2. By the semiclassical finite speed of propagation, there exists $\tau > 0$ small enough such that, modulo $O(h^\infty)$ non contributing terms, $e^{-ith\Delta}\psi(-h^2\Delta)u_0$ is supported in D_1 for all $t \in [-\tau, 0]$.

Let $T^\star$ be given by lemma 7.15. Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $a = 1$ in $\mathcal{T}_{T+1+T^\star}(D_1)$ and $a = 0$ outside $\mathcal{T}_{T+1}(D_1)$. For the convenience of the reader, let us denote

$$\hat{T} := T + 1 + T^\star.$$

$1-a$ is supported in $\mathcal{T}_{\hat{T}}(D_1)^c$, so, every bicharacteristic γ starting from D_1 at time $t=0$ with speed in $[\alpha_0, \beta_0]$ verifies by lemma 7.14

$$d(\gamma(t), \mathrm{Supp}(1-a)) > 0 \ \forall t \in [-\hat{T}-\tau, -\hat{T}]. \tag{7.3.25}$$

Let $\Psi \in C^\infty(\mathbb{R})$ such that $\Psi(t)=0$ for $t \leq -\tau$, $\Psi(t)=1$ for $t \geq 0$, and $\Psi' \geq 0$, and set $\Psi_T(t) = \Psi(t+\hat{T})$. We define

$$w(t,x) = \Psi_T(t) e^{-i(t+\hat{T})h\Delta} \psi(-h^2\Delta) u_0.$$

Then w satisfies

$$\begin{aligned} i\partial_t w - h\Delta w &= i\Psi_T'(t) e^{-i(t+\hat{T})h\Delta} \psi(-h^2\Delta) u_0 \\ w_{|\partial\Omega} &= 0,\ w|_{t \leq -\hat{T}-\tau} = 0 \end{aligned}$$

by the Duhamel formula, as $w(t) = e^{-i\hat{T}h\Delta} u(t) := e^{-i(t+\hat{T})h\Delta}\psi(-h^2\Delta)u_0$ for $t \geq -\hat{T}$ and as Ψ_T' is supported in $[-\hat{T}-\tau, -\hat{T}]$, we have, for $t \geq -\hat{T}$

$$e^{-i\hat{T}h\Delta} u(t,x) = \int_{-\hat{T}-\tau}^{-\hat{T}} e^{-i(t-s)h\Delta} i\Psi_T'(s) e^{-i(s+\hat{T})h\Delta} \psi(-h^2\Delta) u_0 ds.$$

Denote, for $Q \in \{\mathrm{Op}(a), \mathrm{Op}(1-a)\}$

$$u_Q(t,x) = \int_{-\hat{T}-\tau}^{-\hat{T}} e^{-i(t-s)h\Delta} i\Psi_T'(s) Q\left(e^{-i(s+\hat{T})h\Delta} \psi(-h^2\Delta) u_0\right) ds.$$

which is solution of

$$i\partial_t u_Q - h\Delta u_Q = i\Psi_T'(t) Q\left(e^{-i(t+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right).$$

We will see that the term χu_{1-A} does not contribute. To this purpose, we use the b-wave front set of u_{1-A}, $WF_b(u_{1-A})$. We refer to [Iva10] for the definition of this notion. By proposition A.8 of [Iva10], if $\rho_0 \in WF_b(u_{1-A})$, then the broken characteristic starting from $\rho_0 \in WF_b(u_{1-A})$ must intersect the wave front set

$$WF_b(\mathrm{Op}(1-a) e^{-i\hat{T}h\Delta} u) \cap \{t \in [-\hat{T}-\tau, -\hat{T}]\}. \tag{7.3.26}$$

We are interested in estimating u only on D_3 so it is enough to consider $\rho_0 \in WF_b(\chi u_{1-A})$. In particular, if γ is a broken characteristic starting from $\rho_0 \in WF_b(\chi u_{1-A})$, there exists $t \in [-\hat{T}-\tau, -\hat{T}]$ such that $\gamma(t)$ intersect (7.3.26). Because of (7.3.25), this is not possible, and

$$\|\chi u_{1-A}\|_{H^\sigma(\Omega \times R)} = O(h^\infty) \|\psi(-h^2\Delta)u_0\|_{L^2} \tag{7.3.27}$$

for all $\sigma \geq 0$, and this term does not contribute.

On the other hand, remark that

$$\begin{aligned}
&\|\chi u_A\|_{L^p(-\hat{T},-\hat{T}-\tau+T_1)L^q} \\
&\quad\leq \int_{-\hat{T}-\tau}^{-\hat{T}} \|\chi e^{-i(t-s)h\Delta}\Psi'_T(s)A\left(e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right)\|_{L^p(-\hat{T},-\hat{T}-\tau+T_1)L^q}ds \\
&\quad= \int_{-\hat{T}-\tau}^{-\hat{T}} \|\chi e^{-ith\Delta}\Psi'_T(s)A\left(e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right)\|_{L^p(-\hat{T}-s,-\hat{T}-\tau+T_1-s)L^q}ds \\
&\qquad\qquad\leq \int_{-\hat{T}-\tau}^{-\hat{T}} \|\chi e^{-ith\Delta}\Psi'_T(s)A\left(e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right)\|_{L^p(0,T_1)L^q}ds.
\end{aligned}$$

Thus, if Strichartz estimates (7.3.23) in time $T_1 = \epsilon_1|\log h|$ hold true for any data microlocally supported where $a \neq 0$ and spatially supported in D_1, that is in $\mathcal{T}_{T+1}(D_1)\cap D_1$, we have, because $\Psi'_T(s)A\left(e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right)$ is, for $s\in[-\hat{T}-\tau,-\hat{T}]$ and modulo non contributing terms, such a data

$$\begin{aligned}
\|\chi u_A\|_{L^p(-\hat{T},-\hat{T}-\tau+T_1)L^q} &\lesssim h^{-1/p}\int_{-\hat{T}-\tau}^{-\hat{T}} \|\Psi'_T(s)A\left(e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\right)\|_{L^2}ds \\
&\leq h^{-1/p}\int_{-\hat{T}-\tau}^{-\hat{T}} \Psi'_T(s)\|\mathrm{Op}(a)\|_{L^2\to L^2}\|e^{-i(s+\hat{T})h\Delta}\psi(-h^2\Delta)u_0\|_{L^2}ds \\
&= h^{-1/p}\int_{-\hat{T}-\tau}^{-\hat{T}} \Psi'_T(s)\|\mathrm{Op}(a)\|_{L^2\to L^2}\|\psi(-h^2\Delta)u_0\|_{L^2}ds \\
&= h^{-1/p}\|\mathrm{Op}(a)\|_{L^2\to L^2}\|\psi(-h^2\Delta)u_0\|_{L^2}.
\end{aligned}$$

Where we used that $\int_{-\hat{T}-\tau}^{-\hat{T}}\Psi'_T(s)ds = 1$. And therefore, because of (7.3.27)

$$\|\chi e^{-i\hat{T}h\Delta}u\|_{L^p(-\hat{T},-\hat{T}-\tau+T_1)L^q} \lesssim h^{-1/p}\|\mathrm{Op}(a)\|_{L^2\to L^2}\|\psi(-h^2\Delta)u_0\|_{L^2}.$$

It remains to estimate $\|\mathrm{Op}(a)\|_{L^2\to L^2}$. We have, according to [Zwo12]:

$$\|\mathrm{Op}(a)\|_{L^2\to L^2} \;\lesssim\; \sup|a| \,+\, h^{1/2}\sum_{|\alpha|\leq 2n+1}\sup|\partial^{|\alpha|}a| \;\leq\; 1 \,+\, h^{1/2}\sum_{|\alpha|\leq 2n+1}\sup|\partial^{|\alpha|}a|.$$

By lemma 7.15, a can be chosen in such a way that

$$|\partial^{|\alpha|}a| \leq 2e^{c|\alpha|T}$$

so, for $T = 2\epsilon|\log h|$:

$$|\partial^{|\alpha|}a| \lesssim h^{-2c|\alpha|\epsilon}$$

we take $\epsilon > 0$ small enough so that $2c(2n+1)\epsilon \leq \frac{1}{2}$ and we get

$$\|\chi e^{-i\hat{T}h\Delta}u\|_{L^p(-\hat{T},-\hat{T}-\tau+T_1)L^q} \lesssim h^{-1/p}\|\psi(-h^2\Delta)u_0\|_{L^2},$$

that is

$$\|\chi e^{-ith\Delta}\psi(-h^2\Delta)u_0\|_{L^p(0,T_1-\tau)L^q} \lesssim h^{-1/p}\|\psi(-h^2\Delta)u_0\|_{L^2},$$

and thus

$$\|\chi e^{-ith\Delta}\psi(-h^2\Delta)u_0\|_{L^p(0,\epsilon_2|\log h|)L^q} \lesssim h^{-1/p}\|\psi(-h^2\Delta)u_0\|_{L^2}$$

for $\epsilon_2 < \epsilon_1$ small enough.

(2) Restriction to data supported away from the boundary. Let $\eta > 0$. We remark that there exists $t_1(\eta) > 0$, $t_2(\eta) > 0$ such that $t_{1,2}(\eta) \to 0$ as $\eta \to 0$ and, if we denote

$$E_1(\eta) = \{d(x, \partial(\Theta_1 \cup \Theta_2) < \eta\}\,,\ E_2(\eta) = \{d(x, \partial(\Theta_1 \cup \Theta_2) \geq \eta\}\,,$$

then all bicharacteristic starting from $E_1 \cap D$ with speed in $[\alpha_0, \beta_0]$ verifies

$$d(\gamma(t), \mathcal{V} \cap D) > 0\ \forall t \in [-2t_1, -t_1],$$

and all bicharacteristic starting from $E_2 \cap D$ with speed in $[\alpha_0, \beta_0]$ verifies

$$d(\gamma(t), \mathcal{V} \cap D) > 0\ \forall t \in [-t_2, 0],$$

with $\mathcal{V}(\eta)$ a small neighborhood of the boundary.

Let $\psi(-h^2\Delta)u_0$ microlocally supported in $\mathcal{T}_{T+1}(D_1) \cap \{D_1 \times \mathbb{R}^n\}$. Let $\tau_0(\eta) = \max(2t_1, t_2)$. We take $\eta > 0$ sufficently small depending only of D and D_1 so that $\tau_0 \leq 1/2$ and modulo non contributing terms, $e^{-ith\Delta}\psi(-h^2\Delta)u_0$ is microlocally supported in $\mathcal{T}_{T+1-2\tau_0}(D_1) \cap \{D \times \mathbb{R}^n\}$ for all $t \in [-\tau_0, 0]$, and thus in $\mathcal{T}_T(D) \cap \{D \times \mathbb{R}^n\}$ for all $t \in [-\tau_0, 0]$.

We define $\chi_b \in C_0^\infty$ such that $\chi_b(x) = 1$ if $x \in E_1(\eta/2)$ and $\chi_b(x) = 0$ if $x \notin E_1(\eta)$. We estimate $\chi\chi_b u$ on the one hand with the previous strategy and the translation on $[-2t_1, -t_1]$, and $\chi(1-\chi_b)u$ on the other hand with the translation $[-t_2, 0]$.

(3) Reduction to data supported in D. We remark that for all bicharacteristic starting from $D_3 \subsetneq D_2$ with speed in $[\alpha_0, \beta_0]$, we have for $t_0 \geq 0$ small enough

$$d(\gamma(t), D_2^c) > 0,\ \ \forall t \in [0, t_0],$$

and we follow the previous strategies. □

7.4. Approximation of the solution near the trapped set

7.4.1. Phase functions

Following the works of Iwaka [Ika88, Ika82] and Burq [Bur93], we define the phase functions and their reflected phases in the following way.

We call $\varphi : \mathcal{U} \to \mathbb{R}$ a phase function on the open set $\mathcal{U} \subset \mathbb{R}^3$ if φ is C^∞ on $\mathcal{U}$ and verifies $|\nabla\varphi| = 1$. We say that φ verifies (P) on $\partial\Theta_p$ if

1. The principal curvatures of the level surfaces of φ with respect to $-\nabla\varphi$ are non-negative in every point of $\mathcal{U}$,

2. We have, for $j \neq p$

$$\Theta_j \subset \{y + \tau\nabla\varphi(x) \text{ s.t. } \tau \geq 0, y \in \mathcal{U} \cap \partial\Theta_p, \nabla\varphi(y) \cdot n(y) \geq 0\},$$

3. For all $A \in \mathbb{R}$, the set $\{\varphi \leq A\}$ is empty or convex.

Let $\delta_1 \geq 0$ and φ be a phase function. We set

$$\begin{aligned}\Gamma_p(\varphi) &= \{x \in \partial\Theta_p \text{ s.t. } -n(x) \cdot \nabla\varphi(x) \geq \delta_1\},\\ \mathcal{U}_p(\varphi) &= \bigcup_{X^1(x,\nabla\varphi(x))\in\Gamma_p(\varphi)} \{X^1(x, \nabla\varphi(x)) + \tau\Xi(x, \nabla\varphi(x)),\ \tau \geq 0\}.\end{aligned}$$

Then, there exists $\delta_1 \geq 0$ such that, if φ is a phase function verifying (P) on $\partial\Theta_p$, we can define the reflected phase φ_j on the obstacle Θ_j on the open set $\mathcal{U}_j(\varphi)$, verifying (P) on $\partial\Theta_j$, by the following relation, for $X^1(x, \nabla\varphi(x)) \in \Gamma_p(\varphi)$:

$$\varphi_j(X^1(x, \nabla\varphi) + \tau\Xi^1(x, \nabla\varphi)) = \varphi(X^1(x, \nabla\varphi)) + \tau.$$

Note in particular the following simple but fondamental fact, simply differentiating this relation with respect to τ, for all x such that $X^1(x, \nabla\varphi(x)) \in \Gamma_p(\varphi)$:

$$\nabla\varphi_j(X^1(x, \nabla\varphi) + \tau\Xi^1(x, \nabla\varphi)) = \Xi^1(x, \nabla\varphi).$$

We call a finite sequence $J = (j_1, \cdots, j_n)$, $j_i \in \{1, 2\}$ with $j_i \neq j_{i+1}$ a story of reflections, and will denote $\mathcal{I}$ the set of all the stories of reflection. By induction, we can define the phases φ_J for any $J \in \mathcal{I}$, on the sets $\mathcal{U}_J(\varphi)$.

For $f \in C^\infty(\mathcal{U})$ and $m \in \mathbb{N}$, let

$$|f|_m(\mathcal{U}) = \max_{(a_i)\in(\mathcal{S}^2)^m} \sup_{\mathcal{U}} |(a_1 \cdot \nabla) \cdots (a_m \cdot \nabla) f|.$$

We recall the following estimate due to [Ika88, Ika82, Bur93]:

Proposition 7.17. *For every $m \geq 0$ we have*

$$|\nabla\varphi_J|_m \leq C_m |\nabla\varphi|_m.$$

Moreover, according to [Bur93]:

Proposition 7.18. *There exists $M > 0$ such that, for each $(i, j) \in \{1, 2\}^2$, there exists open sets containing the trapped ray $\mathcal{U}_{i,j}$ such that, if $J = \{i, \cdots, j\}$ verifies $|J| \geq M$, and φ verifies (P), φ_J can be defined in $\mathcal{U}_{i,j}$.*

We set

$$\hat{\mathcal{U}}_\infty = \mathcal{U}_{11} \cap \mathcal{U}_{12} \cap \mathcal{U}_{21} \cap \mathcal{U}_{22},$$

and $\mathcal{U}_\infty \subset \hat{\mathcal{U}}_\infty$ to be an open cylinder having for axis the periodic trajectory and contained in $\hat{\mathcal{U}}_\infty$. It will be shrinked in the sequel if necessary.

7.4.2. The microlocal cut-off

According to Section 3, we are reduced to show Proposition 7.16.

By lemma 7.15, we can construct a small shrinking of $\mathcal{U}_\infty$, $\tilde{\mathcal{U}}_\infty \subset \mathcal{U}_\infty$, and $\tilde{q}_{\epsilon,h} \in C^\infty(T^\star\Omega)$ such that $\tilde{q}_{\epsilon,h} = 1$ in an open neighborhood of $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$, $\tilde{q}_{\epsilon,h} = 0$ outside $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\mathcal{U}_\infty)$ in such a way that, for all multi-index α,

$$|\partial_\alpha \tilde{q}_{\epsilon,h}| \lesssim h^{-2|\alpha|c\epsilon}. \tag{7.4.1}$$

It suffices to show Strichartz estimates (7.3.23) in time $\epsilon h|\log h|$ for L^2 functions microlocally supported in $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$ and spatially supported away from a small neighborhood $\mathcal{V}$ of $\partial(\Theta_1 \cup \Theta_2)$. Let $\chi_0 \in C^\infty$ such that $\chi_0 = 0$ near $\partial(\Theta_1 \cup \Theta_2)$ and $\chi_0 = 1$ outside $\mathcal{V}$. For functions microlocally supported in $\hat{\mathcal{T}}_{2\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$ and spatially supported away from $\mathcal{V}$, $\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h})u = u$, thus it suffices to show

$$\|\chi e^{-it\Delta}\psi(-h^2\Delta)\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h})u\|_{L^q(0,\epsilon h|\log h|)L^r} \lesssim \|\psi(-h^2\Delta)u\|_{L^2}. \tag{7.4.2}$$

for all $\chi \in C^\infty$ supported in $\tilde{\mathcal{U}}_\infty$. We will show the stronger estimate:

$$\|e^{-it\Delta}\psi(-h^2\Delta)\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h})u\|_{L^q(0,\epsilon h|\log h|)L^r} \lesssim \|\psi(-h^2\Delta)u\|_{L^2}. \tag{7.4.3}$$

By the $TT^\star$ method - see for example [KT98] - it suffise to show the dispersive estimate, for $0 \leq t \leq \epsilon|\log h|$:

$$\|Q^\star_{\epsilon,h} e^{-ith\Delta} Q_{\epsilon,h}\|_{L^1 \to L^\infty} \lesssim \frac{1}{(ht)^{3/2}} \tag{7.4.4}$$

where

$$Q_{\epsilon,h} := \psi(-h^2\Delta)\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h}).$$

The symbol associated to $\mathrm{Op}(\tilde{q}_{\epsilon,h})\chi_0\psi(-h^2\Delta)$ admits by [Zwo12] the development

$$\sum_{k=0}^{N} \frac{(ih)^k}{k!} \langle D_\xi, D_y\rangle^k \left(\tilde{q}_{\epsilon,h}(x,\xi)\chi_0(y)\psi(\eta)\right)_{|\eta=\xi, y=x} + O(h^{N+1}). \tag{7.4.5}$$

Let us define $q_{\epsilon,h,N} \in C^\infty(T^\star\Omega)$ by

$$q_{\epsilon,h,N}(x,\xi) = \sum_{k=0}^{N} \frac{(ih)^k}{k!} \langle D_\xi, D_y\rangle^k \left(\tilde{q}_{\epsilon,h}(x,\xi)\chi_0(y)\psi(\eta)\right)_{|\eta=\xi, y=x}. \tag{7.4.6}$$

Then, to show (7.4.4), it suffises to show

$$\|\mathrm{Op}(q_{\epsilon,h,N})^\star e^{-ith\Delta} \mathrm{Op}(q_{\epsilon,h,N})\|_{L^1 \to L^\infty} \lesssim \frac{1}{(ht)^{3/2}} \tag{7.4.7}$$

for N large enough.

Note that, in particular,

$$\mathrm{Supp} q_{\epsilon,h,N} \subset \hat{\mathcal{T}}_{2\epsilon|\log h|}(\mathcal{U}_\infty) \cap \{|\xi| \in [\alpha_0, \beta_0]\} \tag{7.4.8}$$

and $q_{\epsilon,h,N}$ is spatially supported outside a small neighborhood of $\partial\left(\Theta_1 \cup \Theta_2\right)$ not depending of ϵ, h, N.

Finally, set

$$\delta^y_{\epsilon,h,N}(x) = \frac{1}{(2\pi h)^3}\int e^{-i(x-y)\cdot\xi/h} q_{\epsilon,T,N}(x,\xi)d\xi,$$

in order to have, for $u \in L^2$

$$\left(\mathrm{Op}(q_{\epsilon,h,N})u\right)(x) = \int \delta^y_{\epsilon,h,N}(x)u(y)dy.$$

Notice that

$$\mathrm{Op}(q_{\epsilon,h,N})^\star e^{-ith\Delta}\mathrm{Op}(q_{\epsilon,h,N})u(x) = \int \mathrm{Op}(q_{\epsilon,h,N})^\star e^{-ith\Delta}\delta^y_{\epsilon,T,N}(x)u(y)dy,$$

thus, to show (7.4.7), it suffises to study $\delta^y_{\epsilon,h,N}$ and to show that, for N large enough

$$|\mathrm{Op}(q_{\epsilon,h,N})^\star e^{-ith\Delta}\delta^y_{\epsilon,h,N}| \lesssim \frac{1}{(ht)^{\frac{3}{2}}}, \text{ for } 0 \leq t \leq \epsilon|\log h|.$$

Let $\mathcal{V}_1$ be a small neighborhood of $\partial(\Theta_1\cup\Theta_2)$ on wich $q_{\epsilon,h,N}$ is vanishing and $\chi_+ \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi_+ = 1$ on $\mathcal{U}_\infty \cap \mathcal{V}_1^c$. We choose χ_+ to be supported on $\mathrm{Conv}(\Theta_1\cup\Theta_2)\backslash(\Theta_1\cup\Theta_2)$ and away from a small enough neighborhood of $\partial(\Theta_1 \cup \Theta_2)$, Conv denoting the convex hull. Note that in particular, $\mathrm{Op}(q_{\epsilon,h,N})^\star = \mathrm{Op}(q_{\epsilon,h,N})^\star\chi_+$. The symbol of $\mathrm{Op}(q_{\epsilon,T,N})^\star$ enjoys the development

$${q_{\epsilon,h,N}}^\star(x,\xi) = e^{ih\langle D_x, D_\xi\rangle}q_{\epsilon,h,N}.$$

Thus, by (7.4.1), taking $\epsilon > 0$ small enough, we have $|q^{\star(\alpha)}_{\epsilon,T,N}| \lesssim 1$ for all $|\alpha| \leq n+1 = 4$. Moreover, $q^{\star(\alpha)}_{\epsilon,T,N}$ is compactly supported in frequencies. Therefore, by [ANV04], Section 4, $\mathrm{Op}(q_{\epsilon,T,N})$ is bounded on $L^\infty \to L^\infty$ independently of h. Therefore, we only have to show, for all $0 \leq T \leq \epsilon|\log h|$

$$|\chi_+ e^{-ith\Delta}\delta^y_{\epsilon,h,N}| \lesssim \frac{1}{(ht)^{\frac{3}{2}}}, \text{ for } 0 \leq t \leq \epsilon|\log h| \tag{7.4.9}$$

for N large enough.

In order to do so, we will construct a parametrix - that is, an approximate solution - in time $0 \leq t \leq \epsilon|\log h|$ for the semi-classical Schrödinger equation with data $\delta^y_{\epsilon,h,N}$. The first step will be to construct an approximate solution of the semi-classical Schrödinger equation with data

$$e^{-i(x-y)\cdot\xi/h}q_{\epsilon,h,N}(x,\xi)$$

where $\xi \in \mathbb{R}^n, \xi \in \mathrm{Supp}q_{\epsilon,h,N}$ is fixed and considered as a parameter. This is the aim of this section.

From now on, we will denote q for $q_{\epsilon,h,N}$.

7.4.3. Approximate solution

We look for the solution in positives times of the equation

$$\begin{cases}(i\partial_t w - h\Delta w) & = 0 \text{ in } \Omega \\ w(t=0)(x) & = e^{-i(x\cdot\xi - t\xi^2)/h} q(x,\xi) \\ w_{|\partial\Omega} & = 0\end{cases}$$

as the Neumann series

$$w = \sum_{J\in\mathcal{I}} (-1)^{|J|} w^J$$

where

$$\begin{cases}(i\partial_t w^{\emptyset} - h\Delta w^{\emptyset}) & = 0 \text{ in } \mathbb{R}^n \\ w(t=0)(x) & = e^{-i((x-y)\cdot\xi - t\xi^2)/h} q(x,\xi)\end{cases}$$

and, for $J \neq \emptyset$, $J = (j_1, \cdots, j_n)$, $J' = (j_1, \cdots, j_{n-1})$

$$\begin{cases}(i\partial_t w^J - h\Delta w^J) & = 0 \text{ in } \mathbb{R}^n \backslash \Theta_{j_n} \\ w(t=0) & = 0 \\ w^J_{|\partial\Theta_{j_n}} & = w^{J'}_{|\partial\Theta_{j_n}}.\end{cases} \tag{7.4.10}$$

We will look for the w^J's as power series in h. In the sake of conciseness, these series will be considered at a formal level in this section, and we will introduce their expression as a finite sum plus a remainder later, in the last section.

7.4.3.1. Free space solution

We look for $u^{\emptyset}$ as

$$\begin{aligned} w^{\emptyset} &= \sum_{k\geq 0} h^k w_k^{\emptyset} e^{-i((x-y)\cdot\xi - t\xi^2)/h} \\ w_0^{\emptyset}(t=0) &= q(x,\xi) \\ w_k^{\emptyset}(t=0) &= 0 \end{aligned}$$

solving the transport equations gives immediately

$$\begin{aligned} w_0^{\emptyset} &= q(x - 2t\xi, \xi) \\ w_k^{\emptyset} &= -i \int_0^t \Delta w_{k-1}(x - 2(s-t)\xi, s) ds \quad k \geq 1 \end{aligned}$$

7.4.3.2. Reflected solutions

We would like to reflect $w^{\emptyset}$ on the obstacle. To this purpose, starting from the phase $\varphi(x) = \frac{(x-y)\cdot\xi}{|\xi|}$, we would like to define the reflected phases as explained in subsection 4.1.

We decompose the set of the stories of reflections as

$$\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$$

where $\mathcal{I}_1$ are all stories begining with a reflection on Θ_1, that is of the form $(1, \cdots)$, and $\mathcal{I}_2$ begining with a reflection on Θ_2, that is of the form $(2, \cdots)$.

Let e be a unit vector with the same direction as $\mathcal{R}$. We take e oriented from Θ_1 to Θ_2. Notice that, for $\frac{\xi}{|\xi|}$ in a small enough neighborhood V of $\{e, -e\}$

- if $\xi \cdot e > 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_1,
- if $\xi \cdot e < 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_2.

We remark that the support of $w^{\emptyset}$ never cross Θ_1 in any time in the first case, and never cross Θ_2 in any time in the second case. Let $(i_0, i_1) = (1, 2)$ if we are in the first situation, and $(2, 1)$ if we are in the second one. We set

$$w^J = 0, \ \forall J \in \mathcal{I}_{i_0}.$$

Then, (7.4.10) is satisfied for all $J \in \mathcal{I}_{i_0}$: indeed, because the support of $w^{\emptyset}$ never cross Θ_{i_0}, we have for all time $0 = w^{\emptyset}_{|\partial\Theta_{i_0}} = w^{\{i_0\}}_{|\partial\Theta_{i_0}}$, and so on. Thus, we are reduced construct the w^J's for $J \in \mathcal{I}_{i_1}$.

Denoting by π the projection on the directions space, we take a spatial neighborhood $\mathcal{U}$ of $\mathcal{R}$ sufficently small so that if $\xi \in \pi(\hat{\mathcal{T}}_{2\epsilon|\ln h|}(\mathcal{U}))$, then $\frac{\xi}{|\xi|} \in V$, and if necessary, we reduce $\mathcal{U}_\infty$ so that it is contained in $\mathcal{U}$. For $\xi \in \pi\mathrm{Supp}q \subset \pi(\hat{\mathcal{T}}_{2\epsilon|\ln h|}(\mathcal{U}_\infty))$, starting with $\varphi(x) = \frac{(x-y)\cdot\xi}{|\xi|}$, who verifies (P) on Θ_{i_0}, we construct the reflected phases φ_J, $J \in \mathcal{I}_{i_1}$ defined in $\mathcal{U}_J(\varphi)$ as we explained in the begining of the section. Reducing again $\mathcal{U}_\infty$ if necessary, for all $\xi \in \pi\mathrm{Supp}q \subset \pi(\hat{\mathcal{T}}_{2\epsilon|\ln h|}(\mathcal{U}_\infty))$, a ray starting in $\mathcal{U}_\infty$ with direction ξ verifies $-n(X^i(x,\xi)) \cdot \Xi^i(x,\xi) \geq 2\delta_1$ for all $i \leq M$. Thus, for $|J| \leq M$, $\mathcal{U}_J \supset \mathcal{U}_\infty$, and therefore $\mathcal{U}_J \supset \mathcal{U}_\infty$ for all J.

We look for w^J as

$$\begin{aligned} w^J &= \sum_{k\geq 0} h^k w^J_k e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h}, \\ w^J_k|_{t\leq 0} &= 0, \\ w^J_{k|\partial\Theta_{j_n}} &= w^{J'}_{k|\partial\Theta_{j_n}}. \end{aligned}$$

For $x \in \mathcal{U}_J(\varphi)$, we have

$$\begin{cases} (\partial_t + 2|\xi|\nabla\varphi_J \cdot \nabla + |\xi|\Delta\varphi_J)w^J_0 & = 0 \\ w^J_{0|\Theta_{j_n}} & = w^{J'}_{0|\Theta_{j_n}} \\ w^J_0|_{t\leq 0} & = 0 \end{cases}$$

and

$$\begin{cases} (\partial_t + 2|\xi|\nabla\varphi_J \cdot \nabla + |\xi|\Delta\varphi_J)w_k^J & = -i\Delta w_{k-1}^J \\ w_{k|\Theta_{jn}}^J & = w_{k|\Theta_{jn}}^{J'} \\ w_k^J|_{t\leq 0} & = 0. \end{cases}$$

We can solve the transport equation along the rays:

Lemma 7.19. *Let $\psi \in C^\infty(D)$ be such that $|\nabla\psi| = 1$. Then, when $\{x + \tau|\xi|\nabla\psi(x);\ \tau \in [0, \tau_0]\} \subset D$, the solution of the equation*

$$(\partial_t + 2|\xi|\nabla\psi \cdot \nabla + |\xi|\Delta\psi)v = h$$

is represented as

$$v(x + 2\tau|\xi|\nabla\psi(x), t + \tau) = \left(\frac{G\psi(x + 2\tau|\xi|\nabla\psi(x))}{G\psi(x)}\right)^{1/2} v(x,t) + \int_0^\tau \left(\frac{G\psi(x + 2\tau|\xi|\nabla\psi(x))}{G\psi(x + 2s|\xi|\nabla\psi(x))}\right)^{1/2} h(x + 2s|\xi|\nabla\psi(x), t + s)ds.$$

where $G\psi$ denote the gaussian curvature of the level surfaces of ψ.

Proof. If we take $w(\tau) = v(x + 2\tau|\xi|\nabla\psi, t + \tau)$, w solves the following ordinary differential equation

$$\partial_\tau w = -|\xi|\Delta\psi(x + 2\tau\nabla\psi(x))w.$$

But - see for example [Lun44]:

$$\exp(-|\xi|\int_{\tau_0}^\tau \Delta\psi(x + 2s\nabla\psi(x))ds) = \left(\frac{G\psi(x + 2\tau|\xi|\nabla\psi(x))}{G\psi(x + 2\tau_0|\xi|\nabla\psi(x))}\right)^{1/2},$$

so the formula holds. □

Now, we can find w_0^J. Indeed, as $w_{0|\partial\Theta_{jn}}^J = w_{0|\partial\Theta_{jn}}^{J'}$, because of the previous formula we have

$$w_0^J(x + 2\tau|\xi|\nabla\varphi_J(x), t + \tau) = \left(\frac{G\varphi_J(x + 2\tau|\xi|\nabla\varphi_J(x))}{G\varphi_J(x)}\right)^{1/2} w_0^{J'}(x,t)$$

for all $x \in \Gamma_{jn}$. So, for all $x \in \mathcal{U}_J(\varphi)$ we get

$$w_0^J(x,t) = \left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x, |\xi|\nabla\varphi_J))}\right)^{1/2} \times w_0^{J'}\left(X^{-1}(x, |\xi|\nabla\varphi_J), t - \frac{\varphi_J(x) - \varphi_J(X^{-1}(x, |\xi|\nabla\varphi_J))}{2|\xi|}\right).$$

Iterating the process and doing the same for $k \geq 1$ we get the following expressions of w_k^J for $x \in \mathcal{U}_J(\varphi)$

Proposition 7.20. *We denote by $\hat{X}_{-2t}(x,|\xi|\nabla\varphi_J)$ the backward spatial component of the flow starting from $(x,|\xi|\nabla\varphi_J)$, defined in the same way as $X_{-2t}(x,|\xi|\nabla\varphi_J)$, at the difference that we ignore the first obstacle encountered if it's not Θ_{j_n}, and we ignore the obstacles after $|J|$ reflections. Moreover, for $J=(j_1=i_1,\dots,j_n)\in\mathcal{I}_{i_1}$, denote by*

$$J(x,t,\xi)=\begin{cases}(j_1,\cdots,j_k) & \text{if } \hat{X}_{-2t}(x,|\xi|\nabla\varphi_J) \text{ has been reflected } n-k \text{ times,}\\ \emptyset & \text{if } \hat{X}_{-2t}(x,|\xi|\nabla\varphi_J) \text{ has been reflected } n \text{ times.}\end{cases}$$

Then, the w_k^J's are given by, for $t\geq 0$ and $x\in\mathcal{U}_J(\varphi)$

$$w_0^J(x,t)=\Lambda\varphi_J(x,\xi)q(\hat{X}_{-2t}(x,|\xi|\nabla\varphi_J),\xi)$$

where

$$\Lambda\varphi_J(x,\xi)=\left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x,|\xi|\nabla\varphi_J))}\right)^{1/2}\times\cdots\times\left(\frac{G\varphi(X^{-|J|-1}(x,|\xi|\nabla\varphi_J))}{G\varphi(X^{-|J|}(x,|\xi|\nabla\varphi_J))}\right)^{1/2},$$

and, for $k\geq 1$, and $x\in\mathcal{U}_J(\varphi)$

$$w_k^J(x,t)=-i\int_0^t g_{\varphi_J}(x,t-s,\xi)\Delta w_{k-1}^{J(x,\xi,t-s)}(\hat{X}_{-2(t-s)}(x,|\xi|\nabla\varphi_J),s)ds$$

where

$$g_{\varphi_J}(x,\xi,t)=\left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x,|\xi|\nabla\varphi_J))}\right)^{1/2}\times\cdots\times\left(\frac{G\varphi_{J(x,t,\xi)}(X^{-|J(x,t,\xi)|-1}(x,|\xi|\nabla\varphi_J))}{G\varphi_{J(x,t,\xi)}(\hat{X}_{-2t}(x,|\xi|\nabla\varphi_J))}\right)^{1/2}.$$

The following lemma permits to study the support of the w_k^J's:

Lemma 7.21. *For $x\in\mathcal{U}_J(\varphi)$*

$$w_k^J(x,t)\neq 0\implies(\hat{X}_{-2t}(x,|\xi|\nabla\varphi_J),\xi)\in \mathit{Supp}q. \tag{7.4.11}$$

And moreover

$$\mathit{Supp}w_k^J\subset\{J(x,\xi,t)=\emptyset\}. \tag{7.4.12}$$

Proof. We prove both properties by induction. The first, (7.4.11), is obviously true for $k=0$. Now, suppose that it is true at the rank $k-1$. If $w_k^J(x,t)\neq 0$, there exists $s\in[0,t]$ such that $w_{k-1}^{J(x,\xi,t-s)}(\hat{X}_{-2(t-s)}(x,|\xi|\nabla\varphi_J),s)\neq 0$. By the induction hypothesis, we deduce that

$$\left(\hat{X}_{-2s}(\hat{X}_{-2(t-s)}(x,|\xi|\nabla\varphi_J),|\xi|\nabla\varphi_{J(x,\xi,t-s)}),\xi\right)\in\mathrm{Supp}q.$$

But $\hat{X}_{-2s}(\hat{X}_{-2(t-s)}(x,|\xi|\nabla\varphi_J),|\xi|\nabla\varphi_{J(x,\xi,t-s)})=\hat{X}_{-2t}(x,|\xi|\nabla\varphi_J)$, thus (7.4.11) is shown for all $k\geq 0$.

Let us now prove (7.4.12) . For $J \in \mathcal{I}_{i_0}$, $w_k^J = 0$ thus it suffises to consider $J \in \mathcal{I}_{i_1}$. We prove again the formula by induction on k.

We have , with $X^{-|J|} := X^{-|J|}(x, \nabla\varphi_J(x)|\xi|)$:

$$w_0^J = \Lambda\varphi_J(x,\xi) q \left(X^{-|J|} - \left(t - (\varphi_J(x) - \varphi_J(X^{-|J|}) \right) \xi, \xi \right).$$

If $J(x,\xi,t) \neq \emptyset$, then $t - (\varphi_J(x) - \varphi_J(X^{-|J|}) \leq 0$. Because $X^{-|J|} \in \partial\Theta_{i_1}$, ξ points toward Θ_{i_1}, and q is spatially supported in a neighborhood of $\mathcal{R}$ and away of the boundary, we deduce that $X^{-|J|} - \left(t - (\varphi_J(x) - \varphi_J(X^{-|J|})\right)\xi$ does not belong to the spatial support of q, thus $w_0^J(x,t) = 0$ and the formula holds for $k = 0$.

Now, let $k \geq 1$ and suppose that (7.4.12) is true for all $J \in \mathcal{I}$ for w_{k-1}. Suppose that $w_k^J(x,\xi,t) \neq 0$. Then, there exists $s \in [0,t]$ such that

$$w_{k-1}^{J(x,\xi,t-s)}(\hat{X}_{-2(t-s)}(x, |\xi|\nabla\varphi_J), s) \neq 0.$$

We denote $J = (j_1, \dots, j_n)$ and $J(x,\xi,t-s) = (j_1, \dots, j_k)$. By the induction hypothesis, $J(\hat{X}_{-2(t-s)}(x,|\xi|\nabla\varphi_J), \xi, t-s) = \emptyset$. That means that for $t - s \leq t' \leq t$, $\hat{X}_{-2t'}(x,|\xi|\nabla\varphi_J)$ is reflected k times. But note that by definition of $J(x,\xi,t-s)$, for $0 \leq t' \leq t-s$,$\hat{X}_{-2t'}(x,|\xi|\nabla\varphi_J)$ is reflected $n-k$ times; so, for $0 \leq t' \leq t$, $\hat{X}_{-2t'}(x,|\xi|\nabla\varphi_J)$ is reflected n times, therefore $J(x,\xi,t) = \emptyset$ and (7.4.12) holds. □

We deduce from the previous lemma that we can extend the explicit expressions of Proposition 7.20 by zero outside $\mathcal{U}_J(\varphi)$ in logarithmic times:

Proposition 7.22. *For $x \notin \mathcal{U}_J(\varphi)$ and $0 \leq t \leq \epsilon|\log h|$ we have $w_k^J(x,t) = 0$.*

Proof. It suffices to show that $w_k^J(x,t) = 0$ for $x \in \mathcal{U}_J(\varphi)$ near the border of $\mathcal{U}_J(\varphi)$. By (7.4.11), it thus suffices to prove that for x close to the border of $\mathcal{U}_J(\varphi)$, $(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \notin$ Supp$q$, or equivalently, that $(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \in$ Supp$q$ implies that x is away from the border of $\mathcal{U}_J(\varphi)$. To this purpose, let us pick x such that $(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \in$ Supp$q$.

In time $s \leq t \leq \epsilon|\log h|$, we follow $\hat{X}_{2s}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi)$. Let s_0 be such that $\hat{X}_{2s_0}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi)$ belongs to $\partial(\Theta_1 \cup \Theta_2)$ and has been reflected exactly $|J|$ times. For $s \geq s_0$, $\hat{X}_s$ will ignore the obstacles. Because $(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \in \hat{\mathcal{T}}_{2\epsilon|\log h|}(\mathcal{U}_\infty)$ and $s_0 \leq \epsilon|\log h|$, we know that

$$\hat{X}_{2s_0}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \in \mathcal{U}_\infty.$$

But $\mathcal{U}_\infty \cap \partial\Theta_i$ is strictly included in Γ_i by our construction. Moreover, by the convexity of the obstacles $|\Xi_{2s_0}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi) \cdot e| \leq |\nabla\varphi_J(y) \cdot e|$ for all $y \in \Gamma_i \backslash (\mathcal{U}_\infty \cap \partial\Theta_i)$. Therefore, for all $s \geq s_0$, $\hat{X}_{2s}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi)$ is away from the border of $\mathcal{U}_J(\varphi)$, and in particular $x = \hat{X}_{2t}(\hat{X}_{-2t}(x,\nabla\varphi_J(x)|\xi|),\xi)$, so the proposition hold. □

Thus, Propositions 7.20 and 7.22 together gives the explicit expression for w_k^J in the full space in logarithmic times. Now, the following two lemmas investigate respectively the temporal and spatial support of $\chi_+ w_k^J$:

Lemma 7.23. *There exists $c_1, c_2 > 0$ such that for every $J \in \mathcal{I}$, the support of w_k^J is included in $\{c_1|J| \leq t\}$ and which of $\chi_+ w_k^J$ is included in $\{c_1|J| \leq t \leq c_2(|J|+1)\}$.*

Proof. Recall that q is supported away from $\partial(\Theta_1 \cup \Theta_2)$. Let

$$\delta_0 = d(\mathrm{Supp} q, \partial(\Theta_1 \cup \Theta_2)) > 0.$$

In order to be reflected $|J|$ times and then be in the support of q, a ray with speed in $[\alpha_0, \beta_0]$ needs a time a least $\frac{d}{2\beta_0}(|J|-1) + \frac{\delta_0}{2\beta_0}$. Moreover, a ray starting in the support of χ_+ needs a time a most $2\frac{D}{\alpha_0}(|J|+1)$ to be reflected $|J|$ times, where D is the maximal distance between the obstacles, and then it leaves the support of q in a time at most $2\frac{d}{\alpha_0}$. Therefore, by lemma 7.21, the temporal support of $\chi_+ w_k^J$ is included in

$$\frac{d}{2\beta_0}(|J|-1) + \frac{\delta_0}{2\beta_0} \leq t \leq 2\frac{D}{\alpha_0}(|J|+1) + 2\frac{d}{\alpha_0},$$

and which of w_k^J in

$$\frac{d}{2\beta_0}(|J|-1) + \frac{\delta_0}{2\beta_0} \leq t$$

and the lemma holds with (say) $c_1 = \frac{\delta_0}{2\beta_0}$ and $c_2 = 4\frac{D}{\alpha_0}$. □

Lemma 7.24. *In times $0 \leq t \leq \epsilon|\log h|$, $\chi_+ w_k^J$ is supported in $\mathcal{U}_\infty$.*

Proof. Let $x \in \mathrm{Supp}\chi_+$ and suppose that $x \notin \mathcal{U}_\infty$.

By (7.4.12), we only have to consider $\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|)$ after $|J|$ reflections. Recall that $x \in \mathrm{Conv}(\Theta_1 \cup \Theta_2) \backslash (\Theta_1 \cup \Theta_2)$. Therefore, before X_{-2t} has done a $|J|+1$'th reflection, $\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|)$ coincide with $X_{-2t}(x, \nabla\varphi_J(x)|\xi|)$. Moreover, after $|J|$ reflections, $\Xi_{-2t}(x, \nabla\varphi_J(x)|\xi|) = \xi$. But, by definition of the trapped set, as $x \notin \mathcal{U}_\infty$, $(X_{-2t}(x, \nabla\varphi_J(x)|\xi|), \xi) \notin \hat{\mathcal{T}}_{2t}(\mathcal{U}_\infty)$. For $t \leq \epsilon|\log h|$, $\hat{\mathcal{T}}_{2t}(\mathcal{U}_\infty) \supset \hat{\mathcal{T}}_{2\epsilon|\log h|}(\mathcal{U}_\infty) \supset \mathrm{Supp} q$ and therefore

$$(\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|), \xi) = (X_{-2t}(x, \nabla\varphi_J(x)|\xi|), \xi) \notin \mathrm{Supp} q.$$

Moreover, after $X_{-2t}(x, \nabla\varphi_J(x)|\xi|)$ has done a $|J|+1$'th reflection and stop to coincide with $\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|)$, $\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|)$ ignore the obstacles and leaves the support of q: in both cases we have $(\hat{X}_{-2t}(x, \nabla\varphi_J(x)|\xi|), \xi) \notin \mathrm{Supp} q$. By (7.4.11), this implies that $w_k^J(x,t) = 0$. $\chi_+ w_k^J$ is thus supported in $\mathcal{U}_\infty$ in times $0 \leq t \leq \epsilon|\log h|$. □

7.4.4. The ξ derivatives

Let us now investigate the derivatives of the phases with respect to ξ. We show the following non-degeneracy property in order to be able to perform a stationary phase on the solutions we are building:

Lemma 7.25. *Let $J \in \mathcal{I}$ and $\mathcal{S}_J(x,t,\xi) := \varphi_J(x,\xi)|\xi| - t\xi^2$. For all $t > 0$ and there exists at most one $s_J(x,t)$ such that $D_\xi \mathcal{S}_J(x,t,s_J(x,t)) = 0$. Moreover, for all $t_0 > 0$, there exists $c(t_0) > 0$ such that, for all $t \geq t_0$ and all $J \in \mathcal{I}$*

$$w^J(x,t,\xi) \neq 0 \implies |\det D^2_\xi \mathcal{S}_J(x,t,\xi)| \geq c(t_0) > 0. \tag{7.4.13}$$

Proof. For $J = \emptyset$, an explicit computation gives, for $t > 0$, $s_\emptyset(x,t) = \frac{(x-y)}{2t}$ and (7.4.13) with $c(t_0) = 2t_0$. Thus we are reduced to the case $|J| \geq 1$. Let $J = (j_1, \cdots, j_n)$.

Non degeneracy of $D^2\mathcal{S}_J$. We first show (7.4.13). Let $\psi_J(x,\xi) = \varphi_J(x,\xi)|\xi|$. Because $|\nabla\psi_J|^2 = |\xi|^2$, $\partial^2_{\xi_i\xi_j}\psi_J$ verifies the transport equation with source term

$$\nabla\left(\partial^2_{\xi_i\xi_j}\psi_J\right) \cdot \nabla\psi_J = \delta_{ij} - \partial_{\xi_i}\nabla\psi_J \cdot \partial_{\xi_j}\nabla\psi_J.$$

For $x \in \Gamma_{j,n}$, we have $\nabla\psi_J(x+\tau\nabla\psi_J(x,\xi)) = \nabla\psi_J(x,\xi)$ and therefore, $g(s) = \partial^2_{\xi_i\xi_j}\psi_J(x+s\nabla\psi_J(x,\xi),\xi)$ verifies

$$g'(s) = \delta_{ij} - \partial_{\xi_i}\nabla\psi_J(x+s\nabla\psi_J(x,\xi),\xi) \cdot \partial_{\xi_j}\nabla\psi_J(x+s\nabla\psi_J(x,\xi),\xi)$$

and thus

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\psi_J(x+\tau\nabla\psi_J) &= \partial^2_{\xi_i\xi_j}\psi_J(x) + \tau\delta_{ij} \\ &\quad - \int_0^\tau \partial_{\xi_i}\nabla\psi_J(x+s\nabla\psi_J(x,\xi),\xi) \cdot \partial_{\xi_j}\nabla\psi_J(x+s\nabla\psi_J(x,\xi),\xi)ds.\end{aligned}$$

Noting that $\partial^2_{\xi_i\xi_j}\psi_J = \partial^2_{\xi_i\xi_j}\psi_{J'}$ on Γ_{j_n} and iterating this argument up to $\partial^2_{\xi_i\xi_j}\psi_\emptyset = 0$ on Γ_{j_1} we get, for all $x \in \mathcal{U}_J(\varphi)$

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\psi_J(x,\xi) &= \frac{l}{|\xi|}\delta_{ij} \\ &\quad - \int_0^{\frac{l}{|\xi|}} \partial_{\xi_i}\nabla\psi_{J^{(s)}}(X_{-s}(x,|\xi|\nabla\varphi_J(x,\xi)),\xi) \cdot \partial_{\xi_j}\nabla\psi_{J^{(s)}}(X_{-s}(x,|\xi|\nabla\varphi_J(x,\xi)),\xi)ds\end{aligned}$$

where we denoted $J^{(s)} = J(x,\xi,s/2)$ with the notations of Proposition 7.20 and $l = l_J(x,\xi)$ given by

$$\begin{aligned}l_J(x,\xi) &= d(x, X^{-1}(x,\nabla\varphi_J(x,\xi))) + d(X^{-1}(x,\nabla\varphi_J(x,\xi)), X^{-2}(x,\nabla\varphi_J(x,\xi))) \\ &\quad + \cdots + d(X^{-|J|-1}(x,\nabla\varphi_J(x,\xi)), X^{-|J|}(x,\nabla\varphi_J(x,\xi))).\end{aligned}$$

Thus we have

$$\begin{aligned}&D^2_\xi\mathcal{S}_J(x,\xi,t) = (\frac{l}{|\xi|} - 2t)Id \\ &- \sum_{k=1}^3 \int_0^{\frac{l}{|\xi|}} D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,|\xi|\nabla\varphi_J(x,\xi)),\xi)\left(D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,|\xi|\nabla\varphi_J(x,\xi)),\xi)\right)^t.\end{aligned}$$

where by y^t xe denoted the transposed of y. Now, remartk that $w^J(x,t) \neq 0$ implies by lemma 7.21

$$\left(\hat{X}_{-2t}(x, |\xi|\nabla\varphi_J(x,\xi)), \xi\right) \in \mathrm{Supp} q$$

from the other hand

$$\hat{X}_{-\frac{l}{|\xi|}}(x, |\xi|\nabla\varphi_J(x,\xi)) \in \partial(\Theta_1 \cup \Theta_2)$$

therefore, because q is supported at distance at least $\delta_0 > 0$ of $\partial(\Theta_1 \cup \Theta_2)$,

$$2t - \frac{l}{|\xi|} \geq \frac{\delta_0}{|\xi|}. \tag{7.4.14}$$

Finally, remark that the matrices

$$D_\xi \partial_{x_k} \psi_J(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi) \left(D_\xi \partial_{x_k} \psi_J(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\right)^t$$

are positives, and we conclude that, for $|J| \geq 1$

$$w^J(x,t) \neq 0 \implies \det D_\xi^2 \mathcal{S}_J(x,\xi,t) \leq -\frac{\delta_0}{\beta_0} < 0.$$

Critical points. We now show the first part of the statement. Differentiating $|\nabla\psi_J|^2 = |\xi|^2$ we get in the same way than before, for $x \in \Gamma_{j_n}$

$$\partial_{\xi_i}(\varphi_J|\xi|)(x + \tau\nabla\varphi_J(x,\xi), \xi) = \partial_{\xi_i}(\varphi_{J'}|\xi|)(x,\xi) + \tau\frac{\xi_i}{|\xi|},$$

and, iterating this argument up to $D_\xi(\varphi_\emptyset|\xi|)(x,\xi) = x - y$:

$$D_\xi(\varphi_J|\xi|)(x,\xi) = X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - y + l_J(x,\xi)\frac{\xi}{|\xi|}.$$

Therefore

$$D_\xi \mathcal{S}_J(x,\xi,t) = X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - (2t - \frac{l_J(x,\xi)}{|\xi|})\xi - y,$$

that is, if $w_J(x,t) \neq 0$, by lemma 7.21

$$D_\xi \mathcal{S}_J(x,\xi,t) = \hat{X}_{-2t}(x, |\xi|\nabla\varphi_J(x,\xi)) - y.$$

Therefore, if $D_\xi \mathcal{S}_J(x,\xi,t) = 0$ then

$$\hat{X}_{2t}(y,\xi) = x.$$

and thus $\xi = s_J(x,t)$ is the vector allowing reaching x from the vector y in time $2t$ beginning with a reflection on Θ_{j_1}. □

Moreover, we will need to control the directional derivatives of w_k^J. To this purpose we show

Proposition 7.26. *For all multi-indices α, β there exists a constant $D_{\alpha,\beta} > 0$ such that the following estimate holds on $\mathcal{U}_\infty$:*

$$|D_\xi^\alpha D_x^\beta \nabla\varphi_J| \leq D_{\alpha,\beta}^{|J|}.$$

Proof. Remark that $|D_x^\beta \varphi_J| \leq C$ for all multi-indices β. Thus, we will show the estimate by induction on the size of α. Let α be a multi-indice. We assume that $|D_\xi^\gamma D_x^\eta \varphi_J| \leq C^{|J|}$ for $|\gamma| \leq |\alpha| - 1$ and all multi-indices η. Differentiating $|\nabla\varphi_J|^2 = 1$ with respect to D_ξ^α , we get an equation of the form

$$\nabla(D_\xi^\alpha \varphi_J) \cdot \nabla\varphi_J = R_J$$

where

$$R_J = \sum_{|\gamma|,|\gamma'| \leq |\alpha|-1} a_{\gamma,\gamma'} D_\xi^\gamma \nabla\varphi_J \cdot D_\xi^{\gamma'} \nabla\varphi_J$$

with $a_{\gamma,\gamma'} \in \mathbb{Z}$. Let $x \in \Gamma_{j_n}$. We denote $g(s) = D^\alpha \varphi_J(x + s\nabla\varphi_J(x,\xi), \xi)$. Because $\nabla\varphi_J(x + s\nabla\varphi_J(x,\xi),\xi) = \nabla\varphi_J(x,\xi)$, g verifies $g'(s) = R(x + s\nabla\varphi_J)$. Therefore we have

$$D_\xi^\alpha \varphi_J(x + \tau\nabla\varphi_J(x,\xi),\xi) = D_\xi^\alpha \varphi_J(x,\xi) + \int_0^\tau R_J(x + s\nabla\varphi_J(x,\xi),\xi) ds.$$

But $D_\xi^\alpha \varphi_J = D_\xi^\alpha \varphi_{J'}$ on Γ_{j_n}, so iterating this process we get, for all $x \in \mathcal{U}_\infty$

$$\begin{aligned} D_\xi^\alpha \varphi_J(x,\xi) &= D_\xi^\alpha \varphi(X^{-|J|}(x, \nabla\varphi_J(x,\xi)),\xi,\xi) \\ - \int_0^{d_1} R_J(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi) ds &- \int_{d_1}^{d_2} R_{J'}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi) ds \\ - \dots &- \int_{d_{|J|-1}}^{d_{|J|}} R_{(j_1)}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi) ds. \end{aligned}$$

with

$$d_1 = d(x, X^{-1}(x,\nabla\varphi_J(x,\xi)), d_i = d(X^{-i}(x,\nabla\varphi_J(x,\xi), X^{-i-1}(x,\nabla\varphi_J(x,\xi)).$$

Now, we differentiate this identity with respect to D_x^β, β been arbitrary. Remark that

- The derivatives of $\nabla\varphi_J$ with respect to x are bounded uniformly with respect to J by Proposition 7.17,
- for $x \in \mathcal{U}_\infty$ and $\xi \in \pi\mathrm{Supp}q$, $(x, \nabla\varphi_J)$ is in a compact set away from $W_{\tan}$ introduced in lemma 7.8. Therefore, by lemma 7.8, X^{-1} is C^∞ on this compact set so his derivatives will be bounded by a constant C, and theses of $X^{-i} = X^{-1} \circ \cdots \circ X^{-1}$ by C^i, and all of them by $C^{|J|}$,
- $d(x, X^{-1}(x,\nabla\varphi_J(x,\xi))$ is nothing but $t(x,\nabla\varphi_J(x,\xi))$ where t is the function introduced lemma 7.8, which is C^∞ away from $W_{\tan}$,

- X_{-s} consists to follow the straight line between X^{-i} and X^{-i-1} for $d_i \leq s \leq d_{i-1}$ (with the convention $d_0 = 0$)
- $(x, y) \in \Theta_1 \cup \Theta_2 \to d(x, y)$ is C^∞,
- By the induction hypothesis the derivatives with respect to x of $D_\xi^\gamma \nabla \varphi_{(j_1, \cdots, j_k)}$ for $|\gamma| \leq |\alpha| - 1$ are bounded by $C^{|J|}$.

So, the left hand side will be bounded by

$$|D_\xi^\alpha D_x^\beta \varphi_J| \leq C + |J| C^{6|J|} \lesssim D^{|J|}$$

with (say) $D = C^7$, and the lemma holds. □

Thus we have

Corollary 7.27. *We following bounds hold on $\mathcal{U}_\infty$*

$$|D_\xi^\alpha w_k^J| \lesssim C_\alpha^{|J|} h^{-(2k+|\alpha|)c\epsilon}.$$

Proof. We have

$$\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x, \nabla\varphi_J(x,\xi))} = \frac{1}{1 + lH(x,\xi) + l^2 G(x,\xi)}$$

where $l = d(x, X^{-1}(x, \nabla\varphi_J(x,\xi))$, H is the trace and G the determinant of the second fundamental form of the level surface of φ_J, that is

$$D_x^2 \varphi_J(X^{-1}(x, \nabla\varphi_J(x,\xi)).$$

Therefore, differentiating the explicit expressions of Proposition 7.20, together with the estimates of Proposition 7.26, Proposition 7.17, and (7.4.1), gives the result. □

7.4.5. Decay of the reflected solutions

The principal result which permits to estimate the decay of the reflected solutions is the convergence of the product of the Gaussian curvatures $\Lambda\varphi_J$ obtained by [Ika88, Ika82] and [Bur93]. In the present framework of two obstacles, it can be expressed as:

Proposition 7.28. *Let $0 < \lambda < 1$ be the product of the two eigenvalues lesser than one of the Poincaré map associated with the periodic trajectory. Then, there exists $0 < \alpha < 1$, and for $I = (1,2)$ and $I = (2,1)$, for every $l \in \{\{1\},\{2\},\emptyset\}$, there exists a C^∞ function $a_{I,l}$ defined in $\mathcal{U}_\infty$, such that, for all $J = (\underbrace{I, \dots, I}_{r \text{ times}}, l)$, we have*

$$\sup_{\mathcal{U}_\infty} |\Lambda\varphi_J - \lambda^r a_{I,l}|_m \leq C_m \lambda^r \alpha^{|J|}.$$

We deduce the following bounds:

Proposition 7.29. *We following bounds hold on $\mathcal{U}_\infty$:*

$$|w_k^J|_m \leq C_k \lambda^{|J|} h^{-(2k+m)c\epsilon}.$$

Moreover, on the whole space, $|w_k^J|_m \leq C_k h^{-(2k+m)c\epsilon}$.

Proof. Making use of the explicit expressions for the w_k^J given by Proposition 7.20, we get by Proposition 7.28, using the remarks made in the proof of Corollary 7.27

$$|w_k^J| \leq C_{2k+m} |q_{\epsilon,h}|_{2k+m} \lambda^{|J|},$$

and (7.4.1) permits to control $|q_{\epsilon,h}|_{2k+m}$ by $h^{-(2k+m)c\epsilon}$. The estimate in the whole space is obtained in the same way without using Proposition 7.28. □

7.5. Proof of theorem 1

Let $K \geq 0$. By the previous section, the function

$$(x,t) \to \frac{1}{(2\pi h)^3} \sum_{J \in \mathcal{I}} \int \sum_{k=0}^{K} h^k w_k^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

satisfies the approximate equation

$$\partial_t u - ih\Delta u = -ih^K \frac{1}{(2\pi h)^3} \sum_{J \in \mathcal{I}} \int \Delta w_{K-1}^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

with data $\delta^y_{\epsilon,h,N}$. Using the fact that $e^{-i(t-s)h\Delta}$ is an H^m-isometry and the Duhamel formula, the difference from the actual solution $e^{-ith\Delta}\delta^y$ is bounded in H^m norm by

$$C \times |t| \times h^{K-3} \times \sup_{t,\xi} \sum_{J \in \mathcal{I}} \|\Delta w_{K-1}^J(\cdot,t,\xi) e^{-i(\varphi_J(\cdot,\xi)|\xi| - t\xi^2)/h}\|_{H^m}.$$

So,

$$e^{-ith\Delta}\delta^y(x) = S_K(x,t) + R_K(x,t) \tag{7.5.1}$$

with

$$S_K(x,t) = \frac{1}{(2\pi h)^3} \sum_{J \in \mathcal{I}} \int \sum_{k=0}^{K} h^k w_k^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

and, for $0 \leq t \leq \epsilon |\log h|$

$$\|R_K(\cdot,t)\|_{H^m} \lesssim |\log h| h^{K-3} \sup_{t,\xi} \sum_{J \in \mathcal{I}} \|\Delta w_{K-1}^J(\cdot,t,\xi) e^{-i(\varphi_J(\cdot,\xi)|\xi| - t\xi^2)/h}\|_{H^m}. \tag{7.5.2}$$

The remainder

We first deal with the remainder term R_K. Let us denote

$$W^J_{K-1}(x,t) = \Delta w^J_{K-1}(\cdot,t,\xi)e^{-i(\varphi_J(\cdot,\xi)|\xi|-t\xi^2)/h}$$

Notice that, by construction of the w_k's, w^J_k is supported in a set of diameter $(C+\beta_0 t)$. Therefore, using Proposition 7.29 to control the derivatives coming from w_{K-1} and Proposition 7.17 to control the derivatives coming from the phase we get:

$$\|\partial^m W^J_{K-1}\|_{L^2} \lesssim C_K(1+\beta_0 t)^{\frac{1}{2}}\|\partial^m W^J_{K-1}\|_{L^\infty} \lesssim C_K(1+t)^{\frac{1}{2}}h^{-m}\times h^{-(2K+m+2)c\epsilon}$$

and thus, by (7.5.2) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, for $0\leq t\leq \epsilon|\log h|$

$$\|R_K\|_{L^\infty} \lesssim |\log h|^{\frac{3}{2}}h^{K(1-2c\epsilon)-5-4c\epsilon}|\left\{J\in\mathcal{I},\ \text{s.t } w^J_{K-1}\neq 0\right\}|. \tag{7.5.3}$$

Note that $w^J_{K-1}(t)\neq 0$ implies by lemma 7.23 that $|J|\leq c_1 t$, and $|\left\{J\in\mathcal{I},\ \text{s.t } w^J_{K-1}\neq 0\right\}|$ is bounded by the number of elements in

$$\left\{\emptyset,(1),(2),(1,2),(2,1),(1,2,1),(2,1,2),\dots,(1,2,\dots),(\underbrace{2,1,\dots}_{\lceil c_1 t\rceil})\right\},$$

that is $1+2\lceil c_1 t\rceil$, so

$$|\left\{J\in\mathcal{I},\ \text{s.t } w^J_{K-1}\neq 0\right\}| \lesssim (1+t) \tag{7.5.4}$$

and therefore, according to (7.5.3), for $0\leq t\leq\epsilon|\log h|$

$$\begin{aligned}\|R_K\|_{L^\infty} &\lesssim C_K|\log h|^{\frac{5}{2}}h^{K(1-2c\epsilon)-5-4c\epsilon}\\ &\lesssim C_K h^{K(1-2c\epsilon)-6-4c\epsilon}.\end{aligned}$$

We take $\epsilon>0$ small enough so that $2c\epsilon\leq\frac{1}{2}$ and we get

$$\|R_K\|_{L^\infty}\leq C_K h^{\frac{K}{2}-7}.$$

Let us fix $K=15$. Then, $\|R_K\|_{L^\infty}\leq C_K h^{-\frac{1}{2}}$. Therefore, as $t\leq\epsilon|\log h|$ implies $h\leq e^{-\frac{t}{\epsilon}}$, we get

$$\|R_K\|_{L^\infty}\leq C_K h^{-\frac{3}{2}}e^{-\frac{t}{\epsilon}} \tag{7.5.5}$$

for $0\leq t\leq\epsilon|\log h|$.

Times $t\geq t_0>0$

Let us now deal with the approximate solution S_K, K been fixed and x in Suppχ_+. Let $t_0>0$ to be chosen later. For $t\geq t_0$, by lemma 7.25 we can perform a stationary phase on each term of the J sum, up to order h. We obtain, for $t \geq t_0$

$$S_K(x,t) = \frac{1}{(2\pi h)^{3/2}} \sum_{J\in\mathcal{I}} e^{-i(\varphi_J(x,s_J(t,x))|s_J(t,x)|-ts_J(t,x)^2)/h} \left(w_0^J(t,x,s_J(t,x)) + h\tilde{w}_1^J(t,x)\right)$$

$$+ \frac{1}{h^{3/2}} \sum_{J\in\mathcal{I}} R^J_{\text{st.ph.}}(x,t) + \frac{1}{(2\pi h)^3} \sum_{J\in\mathcal{I}} \int \sum_{k=2}^{K} h^k w_k^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi|-t\xi^2)/h} d\xi \qquad (7.5.6)$$

where $s_J(t,x)$ is a possible unique critical point of the phase (if it does not exist, the corresponding term is $O(h^\infty)$ and by (7.5.4) it does not contribute). The term $\tilde{w}_1^J$ is a linear combination of

$$D_\xi^2 w_0^J(t,x,s_J(t,x)), w_1^J(t,x,s_J(t,x)),$$

and $R^J_{\text{st.ph.}}$ is the remainder involved in the stationary phase, who verifies (see for example to [Zwo12], Theorem 3.15)

$$|R^J_{\text{st.ph.}}(x,t)| \leq h^2 \sum_{|\alpha|\leq 7} \sup |D_\xi^\alpha w_k^J(x,\cdot,t)|. \qquad (7.5.7)$$

We recall that by lemma 7.24, for $0 \leq t \leq \epsilon|\log h|$, $\chi_+ w_k^J$ is supported in $\mathcal{U}_\infty$. Therefore, for $0 \leq t \leq \epsilon|\log h|$ and all $0 \leq k \leq K-1$, we have, if $x \in \operatorname{Supp}\chi_+$, using the estimate of Proposition 7.29, because $w_k^J(x,\xi,\cdot)$ is supported in $\{c_1|J| \leq t \leq c_2(|J|+1)\}$ by lemma 7.23,

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} \sum_{J \mid w_k^J\neq 0} \lambda^{|J|} \leq C_k h^{-2kc\epsilon} \sum_{r\geq \frac{t}{c_2}} \lambda^{r-1} \lesssim C_k h^{-2kc\epsilon} \lambda^{\frac{t}{c_2}}$$

so, for $0 \leq t \leq \epsilon|\log h|$,

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} e^{-\mu t}.$$

for a certain $\mu > 0$. We take $\epsilon > 0$ small enough so that $2Kc\epsilon \leq \frac{1}{2}$. We get

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-\frac{1}{2}} e^{-\mu t}, \ 1 \leq k \leq K-1, \qquad (7.5.8)$$

$$\sum_{J\in\mathcal{I}} |w_0^J| \lesssim e^{-\mu t}. \qquad (7.5.9)$$

Moreover, using (7.5.7) together with (7.5.4), lemma 7.23 and Corollary 7.27 we obtain, fot $t \leq \epsilon|\log h|$

$$\sum_{J\in\mathcal{I}} |R^J_{\text{st.ph.}}(x,t)| \leq h^2 \sum_{J\in\mathcal{I}} \sum_{|\alpha|\leq 7} \sup |D_\xi^\alpha w_k^J(x,\cdot,t)|$$

$$\leq h^{2-(2K+7)c\epsilon} |\left\{J \in \mathcal{I}, \text{ s.t } w_{K-1}^J \neq 0\right\}| C^{\frac{t}{c_1}} \lesssim h^{2-(2K+7)c\epsilon}(1+t)C^{\frac{t}{c_1}}$$

$$\leq h^{2-(2K+7)c\epsilon}|\log h| h^{-\eta\epsilon}$$

where $\eta > 0$ depends only of α_0, β_0, and the geometry of the obstacles. Therefore, choosing $\epsilon > 0$ small enough

$$\sum_{J \in \mathcal{I}} |R^J_{\text{st.ph.}}(x,t)| \lesssim h \leq e^{-t/\epsilon}. \tag{7.5.10}$$

for $t \leq \epsilon |\log h|$. In the same way we get, taking $\epsilon > 0$ small enough

$$\sum_{J \in \mathcal{I}} |D^2_\xi w^J_0| \lesssim (1+t) C^{\frac{t}{c_1}} \lesssim |\log h| h^{-\nu\epsilon} \leq h^{-1/4}$$

and therefore

$$\sum_{J \in \mathcal{I}} |D^2_\xi w^J_0| \leq h^{-\frac{1}{2}} e^{-t/4\epsilon}. \tag{7.5.11}$$

So, combining (7.5.8), (7.5.9), (7.5.10) and (7.5.11) with (7.5.6), we obtain, for some $\nu > 0$

$$|\chi_+ S_K(x,t)| \lesssim \frac{e^{-\nu t}}{h^{3/2}} \text{ for } t_0 \leq t \leq \epsilon |\log h|. \tag{7.5.12}$$

Small times

It remains now to deal with the case $0 \leq t \leq t_0$. We take $t_0 = \frac{1}{2} c_1$, where c_1 is given by lemma 7.23. Then, for $0 \leq t \leq t_0$, $w^J_k = 0$ for all J such that $|J| \geq 1$ and all $k \in \mathbb{N}$, that is

$$S_K(x,t) = \frac{1}{(2\pi h)^3} \int \sum_{k=0}^{K} h^k w^{\emptyset}_k(x,t,\xi) e^{-i((x-y)\cdot\xi - t\xi^2)/h} d\xi, \text{ for } 0 \leq t \leq t_0$$

which is simply the approximate expression of the solution of the Schrödinger equation with data δ^y, in the free space:

$$S_K(x,t) = \left(e^{-ith\Delta_0} \delta^y \right)(x) + R^{\emptyset}_K(x,t) \text{ for } 0 \leq t \leq t_0$$

where Δ_0 denote the Laplacian in the free space and $\|R^{\emptyset}_K\|_{H^m} \lesssim h^{K-3} \|\Delta w^{\emptyset}\|_{H^m}$. The usual $L^1 \to L^\infty$ estimate for the Schrödinger equation in the free space gives

$$|e^{-ith\Delta_0} \delta^y| \lesssim \frac{1}{(ht)^{3/2}} \|\delta^y\|_{L^1} \lesssim \frac{1}{(ht)^{3/2}},$$

and, dealing with $R^{\emptyset}_K$ as we did for R_K we get

$$|S_K| \lesssim \frac{1}{(ht)^{3/2}}, \text{ for } 0 \leq t \leq t_0. \tag{7.5.13}$$

Conclusion

Collecting (7.5.5), (7.5.12) and (7.5.13) we obtain

$$|\chi_+ e^{-ith\Delta} \delta^y_{\epsilon,h,N}| \lesssim \frac{1}{(ht)^{3/2}}, \text{ for } 0 \leq t \leq \epsilon |\log h|$$

that is (7.4.9). This estimate suffises to obtain theorem 7.1 by the work of reduction done in Sections 2 and 3, as explained in subsection 4.2. The theorem 7.1 is thus demonstrated.

8 The case of the wave equation

8.1. Introduction

Let (M, g) be a Riemaniann manifold of dimension d. We are interested in the linear wave equation on M

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 \\ (u(0), \partial_t u(0)) = (f, g). \end{cases} \tag{8.1.1}$$

where Δ_g design the Laplace-Beltrami operator. In order to study the perturbative theory and the nonlinear problems associated with this equation, it is crucial to estimate the size and the decay of the solutions. Such estimates are the so called *Strichartz estimates*

$$\|u\|_{L^p(0,T)L^q(\Omega)} \le C_T \left(\|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}} \right), \tag{8.1.2}$$

where (p, q) has to follow the admissibility condition given by the scaling of the equation

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \tag{8.1.3}$$

and

$$\frac{1}{p} \le \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{p} \right). \tag{8.1.4}$$

We say that the estimates hold with a loss of order $\lambda > 0$ if they hold for (p, q) satisfying the scaling condition (8.1.3), and

$$\frac{1}{p} \le \left(\frac{d-1}{2} - \lambda \right) \left(\frac{1}{2} - \frac{1}{p} \right).$$

Strichartz estimates were first introduced and established in [Str77] for the $p = q$ case in $\mathbb{R}^n$, then extended to all exponents in [GV85b], [LS95], and [KT98]. As usual, the variable coefficient case is more difficult. In the case of a manifold without boundary, the finite speed of propagation shows that it suffices to obtain the estimates in local coordinates to obtain local Strichartz estimates. Such estimates were obtained by [Kap89], [MSS93], [Smi98], and [Tat02]. The estimates outside one convex obstacle were obtained by [SS95], following the parametrix construction of Melrose and Taylor. Local estimates on a general domain were first proved by [BLP08] for certain ranges of (p, q), using spectral estimates of [SS07b]. The range of indices was then extended by [BSS09a]. This range cannot recover all indices satisfying (8.1.4) : [Iva12] showed indeed that a loss have to occur if some concavity is met. Recently, [ILP14a] proved in a model case local Strichartz estimates inside a convex domain with a loss close to the sharpest one. Their result is extended in [ILLP] to the wave equation.

Phenomenons such as closed geodesics can be obstacles to the establishment of global estimates. Under a non trapping assumption, [SS00] proved in the odd dimensional case that local estimates can be extended to global ones. This result was extended to the even dimensions independently by [Bur03] and [Met04].

However, [BGH10] showed that Strichartz estimates without loss for the Schrödinger equation hold for an asymptotically euclidian manifold without boundary for which the trapped set is sufficiently small and exhibit an hyperbolic dynamic.

Going in the same direction for the problem with boundaries, we recently showed in [Laf17b] global Strichartz estimates without loss for the Schrödinger equation outside two convex obstacles. The aim of this paper is to extend this result to the wave equation. More precisely, we prove

Theorem 8.1. *Let Θ_1 and Θ_2 be two compact, strictly convex subsets of $\mathbb{R}^n$, u be a solution of (8.1.1) in $\Omega = \mathbb{R}^n \backslash (\Theta_1 \cup \Theta_2)$ and (p, q, γ) verifying (8.1.3) and (8.1.4). Then*

$$\|u\|_{L^p(\mathbb{R}, L^q)} \leq C \left(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} \right). \tag{8.1.5}$$

The crucial remark of [SS00] is that local Strichartz estimates combined with the exponential decay of the energy permits to obtain global Strichartz estimates. For the exterior of one convex obstacle in odd dimension, this decay holds and global Strichartz estimates without loss are obtained. In even dimension, such an exponential decay do not hold anymore. However, [Bur03] remarked that it can be replaced by weaker estimates of L^2-integrability of the local energy

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, L^2 \times H^{-1})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}}, \tag{8.1.6}$$

where χ is any compactly supported function, and such an estimate for the complementary of a convex obstacle is a direct consequence of well-known resolvent estimates.

But in the case of the exterior of two convex obstacles, (8.1.6) do not hold anymore: a logarithmic loss occurs due to the trapped geodesic and we only have

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, L^2 \times H^{-1})} \lesssim |\log h| \left(\|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}} \right). \tag{8.1.7}$$

for data supported in frequencies $\sim h^{-1}$. The L^2-integrability of the local energy is the waves-analog of the *smoothing effect* for the Schrödinger equation, for which a loss occurs in the same way. [BGH10] remarked that such a loss can be compensated if we show Strichartz estimates in logarithmic times and we followed this idea in [Laf17b]. We follow here the same roadmap and show that this logarithmic loss can be compensated if we show Strichartz estimates in logarithmic times in the neighborhood of the trapped ray

$$\|\chi u\|_{L^p(0,|\log h|)L^q} \lesssim \|u_0\|_{L^2} + \|u_1\|_{H^{-1}}.$$

Then, we reduce again the problem, to data which micro-locally contains only points of the tangent space which do not escape a given neighborhood of the periodic ray after logarithmic times. Finally, we construct an approximate solution for such data, inspired by [Ika88, Ika82], [Bur93], and we show that this approximation gives the desired estimate.

Note that a large part of the construction we are doing here is similar to which we did in [Laf17b], and we will extensively use results of this previous paper. On the one hand, the wave equation enjoys an exact speed of propagation, and all the results who related on the semiclassical finite speed of propagation of the Schrödinger flow hold with simplificated proofs. On the other hand, the phases of the approximate solution we are building stationate now in whole lines, instead of points, and it is a little more subtle to close the final argument.

8.2. Reduction of the problem

8.2.1. Estimates of L^2-integrability of the local energy

We first show the following two estimates of the L^2-integrability of the local energy, that we will need in the sequel. Their are the analogs of the smoothing estimates for the Schrödinger flow, and were introduced by [Bur03] in the non-trapping case. The first one is an estimate without loss away of the trapped ray. The second one holds in the whole exterior domain, but with a logarithmic loss.

Proposition 8.2 (Global L^2-integrability with no loss away from the trapped ray)**.** *Let $\chi \in C_0^\infty$ be supported outside a small enough neighborhood of the trapped ray. Then, if u is the solution of (8.1.1) with data (f, g):*

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R},\dot{H}^\gamma\times\dot{H}^{\gamma-1})} \lesssim \|f\|_{\dot{H}^\gamma} + \|f\|_{\dot{H}^{\gamma-1}} \tag{8.2.1}$$

Proof. As [Bur03] show it in section 2, it suffices to obtain such an estimate to show the resolvent estimate

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2\to L^2} \lesssim \frac{1}{1+\sqrt{|\lambda|}}.$$

In the spirit of [Laf17b], let K be a non-trapping obstacle such that K and $\Theta_1 \cup \Theta_2$ coincide in the support of χ. In particular, $\Delta_\Omega = \Delta_{\mathbb{R}^n\setminus K}$ on the support of χ. As, moreover, the resolvent estimate is well-known in the non-trapping case (see [VZ00] and [MS78, MS82] for the high frequencies part, [Bur98] for the low frequencies), we have

$$\|\chi(-\Delta_\Omega - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2\to L^2} = \|\chi(-\Delta_{\mathbb{R}^n\setminus K} - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2\to L^2} \lesssim \frac{1}{1+\sqrt{|\lambda|}},$$

and the Proposition is shown. □

Proposition 8.3 (Global L^2-integrability with logarithmic loss)**.** *Let $\chi \in C_0^\infty$. Then, if f, g verifies $\psi(-h^2\Delta)f = f$, $\psi(-h^2\Delta)g = g$ and u is the solution of (8.1.1) with data (f, g):*

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R}, \dot{H}^\gamma \times \dot{H}^{\gamma-1})} \lesssim |\log h|^{1/2} \left(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}\right) \tag{8.2.2}$$

Proof. Denote

$$\begin{aligned} \dot{H}^{\gamma,-} &= D((-\Delta_D)^{s/2} \log(2I - \Delta)^{-1/2}), \\ H^{\gamma,-} &= D((I - \Delta_D)^{s/2} \log(2I - \Delta)^{-1/2}), \end{aligned}$$

by $\dot{H}^{-\gamma,+}$ and $H^{-\gamma,+}$ their dual, and

$$\mathcal{H}^{\gamma,-} = \dot{H}^{\gamma,-} \times \dot{H}^{\gamma-1,-},\ \mathcal{H}^{-\gamma,+} = \dot{H}^{-\gamma,+} \times \dot{H}^{-(\gamma-1),+}.$$

Finally, let us denote

$$A = i \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}.$$

We will show the estimate

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R}, \dot{H}^{\gamma,-} \times \dot{H}^{\gamma-1,-})} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}. \tag{8.2.3}$$

By a classical $TT^\star$ argument (see for example [Bur03] and [Bur04]), (8.2.3) is a direct consequence of the following proposition

Proposition 8.4. *The resolvent $\chi(A - (x - i\epsilon)^{-1})\chi$ is uniformly bounded in*

$$\mathcal{H}^{-s,+} \to \mathcal{H}^{s,-}$$

for $x \in \mathbb{R}$ and $0 < \epsilon < 1$.

Which, in turn, we will obtain as a consequence of the following resolvent estimate obtained by [Bur04]:

$$\|\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi\|_{L^2 \to L^2} \lesssim \frac{2\log(2 + |\lambda|)}{1 + \sqrt{|\lambda|}}. \tag{8.2.4}$$

As

$$(A - z)^{-1} = \begin{pmatrix} -z(\Delta + z^2)^{-1} & i(\Delta + z^2)^{-1} \\ i\Delta(\Delta + z^2)^{-1} & -z(\Delta + z^2)^{-1} \end{pmatrix},$$

we have to show that the following norms are uniformly bounded, for all $s \in \mathbb{R}$

$$\begin{gathered} \|\chi(1 + |z|)(\Delta + z^2)^{-1}\chi\|_{H^{-s,+} \to H^{s,-}}, \\ \|\chi(\Delta + z^2)^{-1}\chi\|_{H^{-s,+} \to H^{s+1,-}}, \\ \|\chi\Delta(\Delta + z^2)^{-1}\chi\|_{H^{-s,+} \to H^{s-1,-}}. \end{gathered}$$

With the same arguments as [Bur03], theses bounds are all consequences of the first one for $s = 0$, that is of

$$\|\chi(1+|z|)(\Delta+z^2)^{-1}\chi\|_{\mathcal{H}^{0,+}\to\mathcal{H}^{0,-}}. \tag{8.2.5}$$

To show (8.2.5), we follow [Bur04], Section 4. Let

$$u = (1+|z|)(\Delta+z^2)^{-1}\chi f.$$

For $\Psi \in C_0^\infty(-1/2, 2)$ equal to one close to 1, we decompose

$$u = \Psi(-\frac{\Delta}{z^2})u + \left(1 - \Psi(-\frac{\Delta}{z^2})\right)u.$$

On the one hand,

$$\|\left(1 - \Psi(-\frac{\Delta}{z^2})\right)u\|_{L^2} \lesssim \|\left(1 - \Psi(-\frac{\Delta}{z^2})\right)\chi f\|_{L^2}.$$

On the other hand, as

$$\Psi(-\frac{\Delta}{z^2})u = (1+|z|)(\Delta+z^2)^{-1}\Psi(-\frac{\Delta}{z^2})\chi f,$$

we have from (8.2.4)

$$\|\chi\Psi(-\frac{\Delta}{z^2})u\|_{L^2} \lesssim \log(2+z^2)\|\Psi(-\frac{\Delta}{z^2})\chi f\|_{L^2},$$

and thus

$$\log(2+z^2)^{-1/2}\|\chi\Psi(-\frac{\Delta}{z^2})u\|_{L^2} \lesssim \log(2+z^2)^{1/2}\|\Psi(-\frac{\Delta}{z^2})\chi f\|_{L^2}.$$

Finally, like in [Bur04], the localization in frequencies allows us to replace the weights in z by the $H^{0,\pm}$ norms, and we get (8.2.5). □

8.2.2. Reduction to logarithmic times near the trapped ray

The aim of this section is to show that the following proposition implies theorem 8.1

Proposition 8.5. *There exists $\epsilon > 0$ and a small neighborhood D of the trapped ray, such that, for all $\chi \in C_0^\infty$ supported in D, if f, g are such that $\psi(-h^2\Delta)f = f$, $\psi(-h^2\Delta)g = g$ and u is the solution of (8.1.1) with data (f, g):*

$$\|\chi u\|_{L^p(0,\epsilon|\log h|)L^q} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}.$$

Thus, we will assume the previous Proposition and show theorem 8.1. As the value of $\epsilon > 0$ does not play any role, we assume here that $\epsilon = 1$.

In the spirit of [Laf17b], let $\chi_{\text{obst}}, \chi_{\text{ray}} \in C_0^\infty$ be such that $\chi_{\text{obst}} = 1$ in a neighborhood of $\Theta_1 \cup \Theta_2 \cup \mathcal{R}$, and $\chi_{\text{ray}} \in C_0^\infty$ such that $\chi_{\text{ray}} = 1$ in a neighborhood of $\mathcal{R}$. We decompose u as the sum

$$u = (1-\chi_{\text{obst}})u + \chi_{\text{obst}}(1-\chi_{\text{ray}})u + \chi_{\text{obst}}\chi_{\text{ray}}u \tag{8.2.6}$$

8.2.2.1. The first term: away from the trapped ray and the obstacles

Let $v = (1-\chi_{\text{obst}})u$. Then v verifies

$$\begin{aligned}\left(\partial_t^2 - \Delta_D\right) v &= -[\Delta_D, \chi_{\text{obst}}]u,\\ (v(0), \partial_t v(0)) &= ((1-\chi_{\text{obst}})f, (1-\chi_{\text{obst}})g).\end{aligned}$$

As v is supported away from the obstacle, it solves a problem in the full space and we can replace the Laplacian in $\left(\partial_t^2 - \Delta_D\right)$ by $\Delta_{\mathbb{R}^n}$. Therefore, by the Duhamel formula

$$v(t) = \cos(t\sqrt{-\Delta_{\mathbb{R}^n}})(1-\chi_{\text{obst}})f + \frac{\sin t\sqrt{-\Delta_{\mathbb{R}^n}}}{\sqrt{-\Delta_{\mathbb{R}^n}}}(1-\chi_{\text{obst}})g \\ - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta_{\mathbb{R}^n}})}{\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi]u(s)ds. \quad (8.2.7)$$

The first two terms are handled thanks to the Strichartz estimates for the waves in $\mathbb{R}^n$:

$$\|\cos(t\sqrt{-\Delta_{\mathbb{R}^n}})(1-\chi_{\text{obst}})f + \frac{\sin t\sqrt{-\Delta_{\mathbb{R}^n}}}{\sqrt{-\Delta_{\mathbb{R}^n}}}(1-\chi_{\text{obst}})g\|_{L^p(\mathbb{R},L^q)} \\ \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}. \quad (8.2.8)$$

And by Christ-Kiselev lemma, cuting the sine in half wave operators and the Strichartz estimates in the full space again

$$\begin{aligned}\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta_{\mathbb{R}^n}})}{\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{L^pL^q} \\ \lesssim \|\int_{\mathbb{R}} \frac{\sin((t-s)\sqrt{-\Delta_{\mathbb{R}^n}})}{\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{L^pL^q} \\ \lesssim \|\frac{e^{-it\sqrt{-\Delta_{\mathbb{R}^n}}}}{\sqrt{-\Delta_{\mathbb{R}^n}}}\int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{L^pL^q} \\ \lesssim \|\int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{\dot{H}^{\gamma-1}}. \quad (8.2.9)\end{aligned}$$

Now, thanks to the dual version of the L^2 estimate (8.2.1) in $\mathbb{R}^n$ (Proposition 8.3 replacing Ω by $\mathbb{R}^n$, which enjoys no trapped geodesic) we get

$$\begin{aligned}\|\int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^n}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{\dot{H}^{\gamma-1}} \\ = \|\int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^n}}}\tilde{\chi}[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{\dot{H}^{\gamma-1}} \lesssim \|[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{L^2\dot{H}^{\gamma-1}}, \quad (8.2.10)\end{aligned}$$

where $\tilde{\chi} = 1$ on the support of $\nabla\chi_{\text{obst}}$. But, using the L^2 estimate away from the trapped ray (8.2.1), because $\nabla\chi_{\text{obst}}$ is supported away from the trapped ray:

$$\|[\Delta_D, \chi_{\text{obst}}]u(s)ds\|_{L^2\dot{H}^{\gamma-1}} \lesssim \|f\|_{\dot{H}^{\gamma-1}} + \|g\|_{\dot{H}^{\gamma-2}} + \|\nabla f\|_{\dot{H}^{\gamma-1}} + \|\nabla g\|_{\dot{H}^{\gamma-2}} \\ \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}. \quad (8.2.11)$$

Collecting (8.2.7), (8.2.8), (8.2.9), (8.2.10) and (8.2.11) we conclude that

$$\|(1-\chi_{\text{obst}})u\|_{L^pL^q} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}. \tag{8.2.12}$$

8.2.2.2. The second term: away from the trapped ray and near the obstacles

Let us now deal with

$$w := \chi_{\text{obst}}(1-\chi_{\text{ray}})u.$$

We denote here $\chi := \chi_{\text{obst}}(1-\chi_{\text{ray}})$ and consider $\varphi \in C_0^\infty((-1,1))$ satisfying $\varphi \geq 0$, $\varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s-j) = 1$. We decompose

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi(t-j)\chi u =: \sum_{j\in\mathbb{Z}} u_j.$$

Because χ is supported away from the trapped ray, using the L^2 estimate away from the trapped ray (8.2.1) combined with the local Strichartz estimates in time 1 for each u_j allows us to recover the estimate in the full space, with the exact same proof as [Bur03], the only difference been using (8.2.1) instead of his the L^2 estimate for non trapping geometries and we get:

$$\|\chi_{\text{obst}}(1-\chi_{\text{ray}})u\|_{L^p(\mathbb{R},L^q)} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}. \tag{8.2.13}$$

8.2.2.3. The third term: near the trapped ray

We will denote here $\chi = \chi_{\text{obst}}\chi_{\text{ray}}$. We will cut u in time intervals of length $|\log h|$. Consider $\varphi \in C_0^\infty((-1,1))$ satisfying $\varphi \geq 0$, $\varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s-j) = 1$. We decompose

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi(\frac{t}{|\log h|} - j)\chi u =: \sum_{j\in\mathbb{Z}} u_j.$$

The u_j satisfy the equation

$$(\partial_t^2 - \Delta)u_j = F_j + G_j$$

where

$$F_j = |\log h|^{-2}\varphi''(\frac{t}{|\log h|} - j)\chi u + 2|\log h|^{-1}\varphi'(\frac{t}{|\log h|} - j)\chi\partial_t u, \tag{8.2.14}$$

$$G_j = -\varphi(\frac{t}{|\log h|} - j)[\Delta, \chi]u. \tag{8.2.15}$$

We denote

$$v_j(t) = \int_{(j-1)|\log h|}^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} F_j(s)ds,$$

$$w_j(t) = \int_{(j-1)|\log h|}^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} G_j(s)ds,$$

in such a way that $u_j = v_j + w_j$. By the L^2-global integrability estimate near the trapped ray (8.2.2) and (8.2.14) we get

$$\sum_{j\in\mathbb{Z}} \||\log h|F_j\|^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim |\log h| \left(\|u_0\|^2_{\dot{H}^\gamma} + \|u_1\|^2_{\dot{H}^{\gamma-1}}\right),$$

and therefore

$$\sum_{j\in\mathbb{Z}} \|F_j\|^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim |\log h|^{-1} \left(\|u_0\|^2_{\dot{H}^\gamma} + \|u_1\|^2_{\dot{H}^{\gamma-1}}\right). \tag{8.2.16}$$

On the other hand, by the Strichartz estimate on logarithmic interval

$$\|v_j\|_{L^pL^q} \lesssim \|F_j\|_{L^1\dot{H}^{\gamma-1}}. \tag{8.2.17}$$

But, as F_j is support on a time interval of size proportional to $|\log h|$, by the Cauchy-Schwarz inequality we get

$$\|F_j\|_{L^1\dot{H}^{\gamma-1}} \lesssim |\log h|^{1/2}\|F_j\|_{L^2\dot{H}^{\gamma-1}}. \tag{8.2.18}$$

Therefore, by (8.2.16), (8.2.17) and (8.2.18)

$$\sum_{j\in\mathbb{Z}} \|v_j\|^2_{L^pL^q} \lesssim \left(\|u_0\|^2_{\dot{H}^\gamma} + \|u_1\|^2_{\dot{H}^{\gamma-1}}\right). \tag{8.2.19}$$

Now, let us deal with w_j. Let us define

$$\tilde{w}_j^\pm = e^{-it\sqrt{-\Delta}} \int_{(j-1)|\log h|}^{(j+1)|\log h|} \frac{e^{is\sqrt{-\Delta}}}{\sqrt{-\Delta}} G_j(s)ds.$$

Decomposing the sine operator in half wave operators and make use of the Christ-Kiselev lemma allows us to estimate the norm of $\tilde{w}_j^\pm$ instead of these of w_j. By the Strichartz estimates on logarithmic interval we get

$$\|\tilde{w}_j^\pm\|_{L^pL^q} \leq \| \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} G_j(s)ds\|_{\dot{H}^{\gamma-1}}.$$

Now, remark that $[\Delta, \chi]$ is supported away from the periodic way. Let $\tilde{\chi}$ be equal to 1 in the support of $\nabla\chi$ and vanishing on the trapped ray. By the dual version of the L^2-global integrability estimate outside the trapped ray (8.2.1) we get

$$\begin{aligned}\|\tilde{w}_j^\pm\|_{L^pL^q} &\leq \| \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} G_j(s)ds\|_{\dot{H}^{\gamma-1}} \\ &= \| \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} \tilde{\chi} G_j(s)ds\|_{\dot{H}^{\gamma-1}} \lesssim \|G_j\|_{L^2\dot{H}^{\gamma-1}}.\end{aligned}$$

And now, by the L^2-global integrability estimate outside the trapped ray (8.2.1) itself:

$$\sum_{j\in\mathbb{Z}} \|G_j\|^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim \left(\|u_0\|^2_{\dot{H}^\gamma} + \|u_1\|^2_{\dot{H}^{\gamma-1}}\right).$$

Therefore we get:

$$\sum_{j\in\mathbb{Z}} \|w_j\|^2_{L^pL^q} \lesssim \left(\|u_0\|^2_{\dot{H}^\gamma} + \|u_1\|^2_{\dot{H}^{\gamma-1}}\right). \tag{8.2.20}$$

Thus, combining (8.2.19) and (8.2.20) we conclude thanks to the embedding $l^2(\mathbb{Z}) \hookrightarrow l^p(\mathbb{Z})$ (we recall that $p \geq 2$) :

$$\|\chi u\|_{L^pL^q} \sim \left(\sum_{j\in\mathbb{Z}} \|u_j\|^p_{L^pL^q}\right)^{\frac{1}{p}} \lesssim \left(\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^pL^q}\right)^{\frac{1}{2}} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}}.$$

Combining this last estimate with equation (8.2.12) and equation (8.2.13) we conclude that

$$\|u\|_{L^pL^q} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}}.$$

Finally, standard methods permit to remove the frequencies cut-off. Therefore Proposition 8.5 implies our main theorem.

8.2.3. Reduction to the trapped set

Let D be an open-neighborhood of the trapped ray, chosen to be a cylinder with the trapped ray for axis. We define the trapped set of D in time T

Definition 8.6. $(x,\xi) \in T^\star\Omega$ belongs to the trapped set of D in time T, denoted $\mathcal{T}_T(D)$, if and only if one of the the ray starting from $(x, \frac{\xi}{|\xi|})$ and $(x, -\frac{\xi}{|\xi|})$ belongs to D after time T.

Note that the only differences with the definition of [Laf17b] are that the rays are all followed at speed one instead of $|\xi| \in [\alpha_0, \beta_0]$. Therefore, with the same proofs, we get

Lemma 8.7. *For all bicharacteristic γ starting from D with speed one, we have*

$$d(\gamma(t), \mathcal{T}_T(D)^c) > 0 \ \forall t \in [-T-1, -T]$$

and

Lemma 8.8. *For all $D, \tilde{D}$, there exists $T^\star > 0$, $c > 0$ such that for all $T \geq 0$:*

$$d(\mathcal{T}_{T-T^\star}(D)^c, \mathcal{T}_T(D)) \geq e^{-cT}, \tag{8.2.21}$$

and, if $D \subset \tilde{D}$

$$d(\mathcal{T}_T(\tilde{D})^c, \mathcal{T}_T(D)) \geq \frac{1}{4}e^{-cT} d(\tilde{D}^c, D). \tag{8.2.22}$$

We say that $f \in L^2$ is microlocally supported in $U \subset T^\star\Omega$, if for all $a \in C^\infty(T^\star\Omega)$ such that $a = 1$ in U we have $\mathrm{Op}(a)f = f$. Using the same time translations as in [Laf17b] combined with the finite speed of propagation, the following Proposition implies our main theorem:

Proposition 8.9. *There exists $\epsilon > 0$ and a small neighborhood D of the trapped ray, such that, for all $\chi \in C_0^\infty$ supported in D, if f, g are such that $\psi(-h^2\Delta)f = f$, $\psi(-h^2\Delta)g = g$, are microlocally supported in $T_{\epsilon|\log h|}(D)$ and spatially in D and away from $\partial(\Theta_1 \cup \Theta_2)$, and u is the solution of (8.1.1) with data (f, g), we have:*

$$\|\chi u\|_{L^p(0,\epsilon|\log h|)L^q} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}.$$

The rest of the paper is thus devoted to prove Proposition 8.9.

8.3. Construction of an approximate solution

8.3.1. The phase functions

We recall here the definition of the phase functions we used in [Laf17b] following the works of Iwaka [Ika88, Ika82] and Burq [Bur93]. We call $\varphi : \mathcal{U} \to \mathbb{R}$ a phase function on the open set $\mathcal{U} \subset \mathbb{R}^3$ if φ is C^∞ on $\mathcal{U}$ and verifies $|\nabla\varphi| = 1$. We say that φ verifies (P) on $\partial\Theta_p$ if

1. The principal curvatures of the level surfaces of φ with respect to $-\nabla\varphi$ are non-negative in every point of $\mathcal{U}$,
2. We have, for $j \neq p$
$$\Theta_j \subset \{y + \tau\nabla\varphi(x) \text{ s.t. } \tau \geq 0, y \in \mathcal{U} \cap \partial\Theta_p, \nabla\varphi(y) \cdot n(y) \geq 0\},$$
3. For all $A \in \mathbb{R}$, the set $\{\varphi \leq A\}$ is empty or convex.

Let $\delta_1 \geq 0$ and φ be a phase function. We set

$$\begin{aligned}
\Gamma_p(\varphi) &= \{x \in \partial\Theta_p \text{ s.t. } -n(x) \cdot \nabla\varphi(x) \geq \delta_1\}, \\
\mathcal{U}_p(\varphi) &= \bigcup_{X^1(x,\nabla\varphi(x)) \in \Gamma_p(\varphi)} \{X^1(x, \nabla\varphi(x)) + \tau\Xi(x, \nabla\varphi(x)),\ \tau \geq 0\}.
\end{aligned}$$

Then, there exists $\delta_1 \geq 0$ such that, if φ is a phase function verifying (P) on $\partial\Theta_p$, we can define the phase φ_j reflected on the obstacle Θ_j on the open set $\mathcal{U}_j(\varphi)$, verifying (P) on $\partial\Theta_j$, by the following relation, for $X^1(x, \nabla\varphi(x)) \in \Gamma_p(\varphi)$:

$$\varphi_j(X^1(x, \nabla\varphi) + \tau\Xi^1(x, \nabla\varphi)) = \varphi(X^1(x, \nabla\varphi)) + \tau.$$

We call a finite sequence $J = (j_1, \cdots, j_n)$, $j_i \in \{1,2\}$ with $j_i \neq j_{i+1}$ a story of reflections, and will denote $\mathcal{I}$ the set of all the stories of reflection. By induction, we can define the phases φ_J for any $J \in \mathcal{I}$, on the sets $\mathcal{U}_J(\varphi)$.

For $f \in C^\infty(\mathcal{U})$ and $m \in \mathbb{N}$, let

$$|f|_m(\mathcal{U}) = \max_{(a_i)\in(\mathcal{S}^2)^m} \sup_{\mathcal{U}} |(a_1 \cdot \nabla) \cdots (a_m \cdot \nabla) f|.$$

The following estimate is due to [Ika88, Ika82, Bur93]:

Proposition 8.10. *For every $m \geq 0$ we have*

$$|\nabla \varphi_J|_m \leq C_m |\nabla \varphi|_m.$$

Moreover, according to [Bur93]:

Proposition 8.11. *There exists $M > 0$ such that, for each $(i,j) \in \{1,2\}^2$, there exists open sets containing the trapped ray $\mathcal{U}_{i,j}$ such that, if $J = \{i, \cdots, j\}$ verifies $|J| \geq M$, and φ verifies (P), φ_J can be defined in $\mathcal{U}_{i,j}$.*

We set

$$\hat{\mathcal{U}}_\infty = \mathcal{U}_{11} \cap \mathcal{U}_{12} \cap \mathcal{U}_{21} \cap \mathcal{U}_{22},$$

and $\mathcal{U}_\infty \subset \hat{\mathcal{U}}_\infty$ to be an open cylinder having for axis the periodic trajectory and contained in $\hat{\mathcal{U}}_\infty$. It will be shrinked in the sequel if necessary. Finally, we recall the following estimate concerning the derivatives with respect to ξ of the phases builded beginning with $\varphi = (x-y) \cdot \frac{\xi}{|\xi|}$ we obtained in [Laf17b]:

Proposition 8.12. *Let $\varphi(x) = (x-y) \cdot \frac{\xi}{|\xi|}$. We denote $\varphi_J(x,\xi)$ the reflected phase we build beginning with φ. Then, for all multi-indices α, β there exists a constant $D_{\alpha,\beta} > 0$ such that the following estimate holds on $\mathcal{U}_\infty$:*

$$|D_\xi^\alpha D_x^\beta \nabla \varphi_J| \leq D_{\alpha,\beta}^{|J|}.$$

8.3.2. The microlocal cut-off

According to Section 2, we are reduced to show Proposition 8.9. By lemma 8.8, we can construct a small shrinking of $\mathcal{U}_\infty$, $\tilde{\mathcal{U}}_\infty \subset \mathcal{U}_\infty$, and $\tilde{q}_{\epsilon,h} \in C^\infty(T^\star\Omega)$ such that $\tilde{q}_{\epsilon,h} = 1$ in an open neighborhood of $\mathcal{T}_{2\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$, $\tilde{q}_{\epsilon,h} = 0$ outside $\mathcal{T}_{2\epsilon|\log h|}(\mathcal{U}_\infty)$ in such a way that, for all multi-indice α,

$$|\partial_\alpha \tilde{q}_{\epsilon,h}| \lesssim h^{-2|\alpha|c\epsilon}. \tag{8.3.1}$$

It suffices to show Strichartz estimates in time $\epsilon|\log h|$ for data microlocally supported in $\mathcal{T}_{\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$ and spatially supported in $\tilde{\mathcal{U}}_\infty$ and away from a small neighborhood $\mathcal{V}$ of $\partial(\Theta_1 \cup \Theta_2)$. Let $\chi_0 \in C^\infty$ such that $\chi_0 = 0$ near $\partial(\Theta_1 \cup \Theta_2)$ and $\chi_0 = 1$ outside $\mathcal{V}$. For such functions, $\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h}) f = f$, thus it suffices to show

$$\|\chi e^{-it\sqrt{-\Delta}} \chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h}) f\|_{L^p(0,\epsilon|\log h|)L^q} \lesssim \|f\|_{\dot{H}^\gamma}, \tag{8.3.2}$$

for all $\chi \in C^\infty$ supported in $\tilde{\mathcal{U}}_\infty$. We will show the strongest estimate:

$$\|e^{-it\sqrt{-\Delta}}\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h})f\|_{L^p(0,\epsilon|\log h|)L^q} \lesssim \|f\|_{\dot{H}^\gamma}. \tag{8.3.3}$$

by the $TT^\star$ method - see for example [KT98] - it suffices to show the dispersive estimate, for $0 \leq t \leq \epsilon|\log h|$:

$$\|Q^\star_{\epsilon,h} e^{-it\sqrt{-\Delta}} Q_{\epsilon,h}\|_{L^1\to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}. \tag{8.3.4}$$

where

$$Q_{\epsilon,h} := \psi(-h^2\Delta)\chi_0 \mathrm{Op}(\tilde{q}_{\epsilon,h}).$$

Then, to show (8.3.4), it suffices to show

$$\|\mathrm{Op}(q_{\epsilon,h,N})^\star e^{-it\sqrt{-\Delta}} \mathrm{Op}(q_{\epsilon,h,N})\|_{L^1\to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}. \tag{8.3.5}$$

for N large enough. Note that, in particular,

$$\mathrm{Supp} q_{\epsilon,h,N} \subset \mathcal{T}_{2\epsilon|\log h|}(\mathcal{U}_\infty) \cap \mathcal{U}_\infty \times \{|\xi| \in [\alpha_0, \beta_0]\} \tag{8.3.6}$$

and $q_{\epsilon,h,N}$ is spatially supported outside a small neighborhood of $\partial(\Theta_1 \cup \Theta_2)$ not depending of ϵ, h, N.

It suffices to obtain

$$\|\mathrm{Op}(q_{\epsilon,h,N})^\star \cos(t\sqrt{-\Delta}) \mathrm{Op}(q_{\epsilon,h,N})\|_{L^1\to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}},$$
$$\|\mathrm{Op}(q_{\epsilon,h,N})^\star \sin(t\sqrt{-\Delta}) \mathrm{Op}(q_{\epsilon,h,N})\|_{L^1\to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}.$$

We will deal for example with the cosine part, the sine is handled in the same way. We set

$$\delta^y_{\epsilon,h,N}(x) = \frac{1}{(2\pi h)^d}\int e^{-i(x-y)\cdot\xi/h} q_{\epsilon,T,N}(x,\xi)d\xi,$$

in order to have, for $u \in L^2$

$$(\mathrm{Op}(q_{\epsilon,h,N})u)(x) = \int \delta^y_{\epsilon,h,N}(x)u(y)dy.$$

Notice that

$$\mathrm{Op}(q_{\epsilon,h,N})^\star \cos(t\sqrt{-\Delta})\mathrm{Op}(q_{\epsilon,h,N})u(x) = \int \mathrm{Op}(q_{\epsilon,h,N})^\star \cos(t\sqrt{-\Delta})\delta^y_{\epsilon,T,N}(x)u(y)dy,$$

thus, to show (8.3.5), it suffices to study $\delta^y_{\epsilon,h,N}$ and to show that, for N large enough

$$|\mathrm{Op}(q_{\epsilon,h,N})^\star \cos(t\sqrt{-\Delta})\delta^y_{\epsilon,h,N}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \text{ for } 0 \leq t \leq \epsilon|\log h|.$$

Let $\mathcal{V}_1$ be a small neighborhood of $\partial(\Theta_1 \cup \Theta_2)$ on which $q_{\epsilon,h,N}$ is vanishing and $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi_0 = 1$ on $\mathcal{U}_\infty \cap \mathcal{V}_1^c$. We choose χ_+ to be supported on $\mathrm{Conv}(\Theta_1 \cup \Theta_2) \backslash (\Theta_1 \cup \Theta_2)$ and away from a small enough neighborhood of $\partial(\Theta_1 \cup \Theta_2)$, Conv denoting the convex hull. Note that in particular, $\mathrm{Op}(q_{\epsilon,h,N})^\star = \mathrm{Op}(q_{\epsilon,h,N})^\star \chi_+$. The symbol of $\mathrm{Op}(q_{\epsilon,T,N})^\star$ enjoys the development

$$q_{\epsilon,h,N}{}^\star(x,\xi) = e^{ih\langle D_x, D_\xi \rangle} q_{\epsilon,h,N}.$$

Thus, by (8.3.1), taking $\epsilon > 0$ small enough, we have $|q_{\epsilon,T,N}^{\star(\alpha)}| \lesssim 1$ for all $|\alpha| \leq n+1 = 4$. Moreover, $q_{\epsilon,T,N}^{\star(\alpha)}$ is compactly supported in frequencies. Therefore, by [ANV04], Section 4, $\mathrm{Op}(q_{\epsilon,T,N})$ is bounded on $L^\infty \to L^\infty$ independently of h. Therefore, we only have to show, for all $0 \leq T \leq \epsilon |\log h|$

$$|\chi_0 \cos(t\sqrt{-\Delta}) \delta^y_{\epsilon,h,N}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \text{ for } 0 \leq t \leq \epsilon |\log h| \tag{8.3.7}$$

for N large enough.

In order to do so, we will construct a parametrix in time $0 \leq t \leq \epsilon |\log h|$ for the wave equation with data $(\delta^y_{\epsilon,h,N}, 0)$. The first step will be to construct an approximate solution of wave equation with data

$$(e^{-i(x-y)\cdot\xi/h} q_{\epsilon,h,N}(x,\xi), 0)$$

where $\xi \in \mathbb{R}^n, \xi \in \mathrm{Supp} q_{\epsilon,h,N}$ is fixed and considered as a parameter.

8.3.3. Approximation of the solution

8.3.3.1. The Neumann sum

We look for the solution w of

$$\begin{cases} \partial_t^2 w - \Delta w & = 0 \text{ in } \Omega \\ w(t=0)(x) & = e^{-i(x-y)\cdot\xi/h} q(x,\xi) \\ \partial_t w(t=0)(x) & = 0 \\ w_{|\partial\Omega} & = 0 \end{cases}$$

as the Neumann serie

$$w = \sum_{J \in \mathcal{I}} (-1)^{|J|} w^J$$

where

$$\begin{cases} \partial_t^2 w^\emptyset - \Delta w^\emptyset & = 0 \text{ in } \mathbb{R}^n \\ w(t=0)(x) & = e^{-i(x-y)\cdot\xi/h} q(x,\xi) \\ \partial_t w(t=0)(x) & = 0 \end{cases} \tag{8.3.8}$$

and, for $J \neq \emptyset$, $J = (j_1, \cdots, j_n)$, $J' = (j_1, \cdots, j_{n-1})$

$$\begin{cases} \partial_t^2 w^J - \Delta w^J & = 0 \text{ in } \mathbb{R}^n \backslash \Theta_{j_n} \\ w(t=0), \partial_t w(t=0) & = 0 \\ w^J_{|\partial\Theta_{j_n}} & = w^{J'}_{|\partial\Theta_{j_n}}. \end{cases} \tag{8.3.9}$$

Let us denote

$$\varphi_J^+(x,\xi) = \varphi_J(x,\xi) \text{ and } \varphi_J^-(x,\xi) = \varphi_J(x,-\xi),$$

that is, φ_J^+ is the reflected phase constructed with $\varphi(x,\xi) = (x-y)\cdot\frac{\xi}{|\xi|}$ and φ_J^+ is the reflected phase constructed with $\varphi(x,-\xi) = -(x-y)\cdot\frac{\xi}{|\xi|}$. We look for w^J as the sum of the two series

$$w^J = w^{J,+} + w^{J,-}$$
$$= \sum_{k\geq 0} w_k^{J,+}(x,t) e^{-i(\varphi_J^+|\xi| - t|\xi|)/h} (-i\frac{h}{|\xi|})^k + \sum_{k\geq 0} w_k^{J,-}(x,t) e^{i(\varphi_J^-|\xi| - t|\xi|)/h} (i\frac{h}{|\xi|})^k.$$

If $w_0^{J,\pm}$ solves the transport equations

$$(2\partial_t + \nabla\varphi_J^\pm \cdot \nabla + \Delta\varphi_J) w_0^{J,\pm} = 0,$$

and for $k \geq 1$

$$(2\partial_t + \nabla\varphi_J^\pm \cdot \nabla + \Delta\varphi_J) w_k^{J,\pm} = -\Box w_{k-1}^{J,\pm},$$

with, for $J = \emptyset$

$$w_0^{\emptyset,+}(x,0) = w_0^{\emptyset,-}(x,0) = \frac{1}{2} q(x,\xi),$$
$$w_k^{\emptyset,\pm}(x,0) = 0, \ \forall k \geq 1,$$
$$\partial_t w_k^{\emptyset,+}(x,0) + \partial_t w_k^{\emptyset,-}(x,0) = 0, \ \forall k \geq 0,$$

and, for $|J| \geq 1$

$$w^{J,\pm}_{k|\partial\Theta_{j_n}} = w^{J',\pm}_{k|\partial\Theta_{j_n}},$$
$$\partial_t w_k^{J,\pm}(x,0) = w_k^{J,\pm}(x,0) = 0,$$

then w^J solves (8.3.8), (8.3.9).

Solving the transport equations for $J = \emptyset$ gives immediately

$$w_0^{\emptyset,+} = \frac{1}{2} q(x - t\frac{\xi}{|\xi|}, \xi),$$
$$w_0^{\emptyset,-} = \frac{1}{2} q(x + t\frac{\xi}{|\xi|}, \xi),$$
$$w_k^{\emptyset,\pm} = -\int_0^t \Box w_{k-1}^{\pm}(x \mp (s-t)\frac{\xi}{|\xi|}, s) ds \quad k \geq 1.$$

8.3.3.2. Reflections on the obstacles

Now, we would like to reflect $w^{\emptyset,\pm}$ on the obstacle. To this purpose, starting from the phases $\varphi(x,\xi) = \frac{(x-y)\cdot\xi}{|\xi|}$ and $\varphi(x,-\xi) = -\frac{(x-y)\cdot\xi}{|\xi|}$ we would like to define the reflected phases as explained in the first subsection.

We decompose the set of the stories of reflections as

$$\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$$

where $\mathcal{I}_1$ are all stories beginning with a reflection on Θ_1, that is of the form $(1,\cdots)$, and $\mathcal{I}_2$ beginning with a reflection on Θ_2, that is of the form $(2,\cdots)$. Let e be a unit vector with the same direction as $\mathcal{R}$. We take e oriented from Θ_1 to Θ_2. For $\frac{\xi}{|\xi|}$ in a small enough neighborhood V of $\{e,-e\}$ we have

1. if $\xi\cdot e > 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_1 and $-\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_2
2. if $\xi\cdot e < 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_2 and $-\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_1

Remark that

- In situation (1), the support of $w^{\emptyset,+}$ never cross Θ_1 and the support of $w^{\emptyset,-}$ never cross Θ_2 in any time,
- in situation (2), the support of $w^{\emptyset,+}$ never cross Θ_2 and the support of $w^{\emptyset,-}$ never cross Θ_1 in any time.

We set

- In situation (1): $(\mathcal{I}_+,\mathcal{I}_-) := (\mathcal{I}_2,\mathcal{I}_1)$,
- in situation (2): $(\mathcal{I}_+,\mathcal{I}_-) := (\mathcal{I}_1,\mathcal{I}_2)$.

Then, (8.3.9) is satisfied for $w^{\pm}$ for all $J\in\mathcal{I}_{\mp}$: indeed, because the support of $w^{\emptyset,\pm}$ never cross $\Theta_{i_\mp}$, we have for all time $0 = w^{\emptyset,\pm}_{|\partial\Theta_{i_\mp}} = w^{\{i_\mp\}}_{|\partial\Theta_{i_\mp}}$, and so on. Thus, we are reduced to construct the $w^{J,\pm}$'s for $J\in\mathcal{I}_\pm$. In the same way as in [Laf17b], shrinking $\mathcal{U}_\infty$ if necessary, all the phases we we will be dealing with are well defined according to the previous remarks.

Then, in the exact same way as [Laf17b], we solve the transport equations along the rays:

Proposition 8.13. *We denote by $\hat{X}_{-t}(x,\nabla\varphi_J^{\pm})$ the backward spatial component of the flow starting from $(x,\nabla\varphi_J^{\pm})$, defined in the same way as $X_{-t}(x,\nabla\varphi_J^{\pm})$, at the difference that we ignore the first obstacle encountered if it's not Θ_{j_n}, and we ignore the obstacles after $|J|$ reflections. Moreover, for $J=(j_1=i_1,\dots,j_n)\in\mathcal{I}_\pm$, denote by*

$$J(x,t,\xi) = \begin{cases} (j_1,\cdots,j_k) & \text{if } \hat{X}_{-t}(x,\nabla\varphi_J^{\pm}) \text{ has been reflected } n-k \text{ times,} \\ \emptyset & \text{if } \hat{X}_{-t}(x,\nabla\varphi_J^{\pm}) \text{ has been reflected } n \text{ times.} \end{cases}$$

Then, the w_k^J's are given by, for $t \geq 0$ and $x \in \mathcal{U}_J(\varphi)$

$$w_0^{J,\pm}(x,t) = \Lambda\varphi_J^{\pm}(x,\xi)q(\hat{X}_{-t}(x,\nabla\varphi_J^{\pm}),\xi)$$

where

$$\Lambda\varphi_J^{\pm}(x,\xi) = \frac{G\varphi_J^{\pm}(x)}{G\varphi_J^{\pm}(X^{-1}(x,\nabla\varphi_J^{\pm}))} \times \cdots \times \frac{G\varphi^{\pm}(X^{-|J|-1}(x,\nabla\varphi_J^{\pm}))}{G\varphi^{\pm}(X^{-|J|}(x,\nabla\varphi_J^{\pm}))},$$

and, for $k \geq 1$, and $x \in \mathcal{U}_J(\varphi)$

$$w_k^{J,\pm}(x,t) = \int_0^t g_{\varphi_J}(x,(t-s),\xi)\Box w_{k-1}^{J(x,\xi,\pm(t-s))}(\hat{X}_{-(t-s)}(x,\nabla\varphi_J^{\pm}),s)ds$$

where

$$g_{\varphi_J}^{\pm}(x,\xi,t) = \frac{G\varphi_J^{\pm}(x)}{G\varphi_J^{\pm}(X^{-1}(x,\nabla\varphi_J^{\pm}))} \times \cdots \times \frac{G\varphi_{J(x,t,\xi)}^{\pm}(X^{-|J(x,t,\xi)|-1)}(x,\nabla\varphi_J^{\pm}))}{G\varphi_{J(x,t,\xi)}^{\pm}(\hat{X}_{-t}(x,\nabla\varphi_J^{\pm}))}.$$

With the exact same proofs as in [Laf17b], subsection 4.3.2, following the rays at speed one instead of speed $|\xi| \in [\alpha_0,\beta_0]$, $w^{J,\pm}$ verify the following properties:

Proposition 8.14. *We have*

1. $w_k^J(x,t) \neq 0 \implies (\hat{X}_{-t}(x,\nabla\varphi_J),\xi) \in Supp q$.
2. *For $x \notin \mathcal{U}_J(\varphi)$ and $0 \leq t \leq \epsilon|\log h|$, $w_k^{J,\pm}(x,t) = 0$,*
3. *there exists $c_1, c_2 > 0$ such that for every $J \in \mathcal{I}$, the support of $w_k^{J,\pm}$ is included in $\{c_1|J| \leq t\}$ and which of $\chi_0 w_k^{J,\pm}$ is included in $\{c_1|J| \leq t \leq c_2(|J|+1)\}$,*
4. *in times $0 \leq t \leq \epsilon|\log h|$, $\chi_0 w_k^{J,\pm}$ is supported in $\mathcal{U}_\infty$.*

Moreover, Proposition 8.10 combined with Proposition 8.13 and (8.3.1) gives immediately:

Proposition 8.15. *The following bound hold on $\mathcal{U}_\infty$*

$$|D_\xi^\alpha w_k^{J,\pm}| \lesssim C_\alpha^{|J|} h^{-(2k+|\alpha|)c\epsilon}.$$

8.3.4. Decay estimates

We recall the principal result who permits to estimate the decay of the reflected solutions, namely the convergence of the product of the Gaussian curvatures $\Lambda\varphi_J$ obtained by [Ika88, Ika82] and [Bur93]. In the present framework of two obstacles, it writes:

Proposition 8.16. *Let $0 < \lambda < 1$ be the product of the two eigenvalues lesser than one of the Poincaré map associated with the periodic trajectory. Then, there exists $0 < \alpha < 1$, and for $I = (1, 2)$ and $I = (2, 1)$, for every $l \in \{\{1\}, \{2\}, \emptyset\}$, there exists a C^∞ function $a_{I,l}$ defined in $\mathcal{U}_\infty$, such that, for all $J = (\underbrace{I, \dots, I}_{r \text{ times}}, l)$, we have*

$$\sup_{\mathcal{U}_\infty} |\Lambda \varphi_J - \lambda^r a_{I,l}|_m \le C_m \lambda^r \alpha^{|J|}.$$

Combined with the explicit expressions of Proposition 8.13 and (8.3.1), this result gives as in [Laf17b] the following decay:

Proposition 8.17. *We following bounds hold on $\mathcal{U}_\infty$:*

$$|w_k^{J,\pm}|_m \le C_k \lambda^{|J|} h^{-(2k+m)c\epsilon}.$$

Moreover, on the whole space, $|w_k^J|_m \le C_k h^{-(2k+m)c\epsilon}$.

8.3.5. Critical points of the phase

We need to study the critical points of the phase in order to be able to perform a stationary phase argument on the solution we are building. At the difference of [Laf17b], the phases here stationate in whole directions. Therefore, we will perform a stationary phase on each sphere $\mathcal{S}^{n-1}(0, s)$. To this purpose, we need

Proposition 8.18. *Let us denote*

$$\mathcal{S}_J^\pm(x, \xi, t) = \varphi_J^\pm(x, \xi)|\xi| - t|\xi|.$$

Then, there exists $\eta > 0$ such that for all $|J| \ge 1$,

$$\begin{aligned} d(X^{-|J|}(x, \nabla \varphi_J^\pm(x, \xi)), y) \le \eta \text{ and } w^{J,\pm}(x, t, \xi) \neq 0 \\ \implies D_\xi \mathcal{S}_J^\pm(x, \xi, t) \neq 0. \end{aligned} \tag{8.3.10}$$

Moreover, as soon as $d(X^{-|J|}(x, \nabla \varphi_J(x, \xi)), y) \ge \eta$, for all $s > 0$ and x there exists a unique $s_J(x, s) \in \mathcal{S}^{n-1}(0, s)$ such that, for all $t \ge 0$

$$w^{J,\pm}(x, \cdot, s_J^\pm(x, s)) \neq 0 \text{ and } D_{\mathcal{S}^{n-1}(0,s)} \mathcal{S}_J^\pm(x, s_J^\pm(x, s), t) = 0. \tag{8.3.11}$$

Furthermore, if $d(X^{-|J|}(x, \nabla \varphi_J(x, \xi)), y) \ge \eta$,

$$\det D^2_{\mathcal{S}^{n-1}(0,s)} \mathcal{S}^\pm{}_J(x, s_J^\pm(x, s), t) > c. \tag{8.3.12}$$

Proof. For the sake of readability, we denote $\mathcal{S}_J = \mathcal{S}_J^+$, $\varphi_J := \varphi_J^+$, $w^J = w^{J,+}$ and we make the proof for the positive part of the wave, $w^{J,+}$: for $w^{J,-}$, the proof is the same.

In the same way as in [Laf17b], we obtain, differentiating $|\nabla\varphi_J(x,\xi)|\xi||^2 = |\xi|^2$ with respect to ξ and integrating the transport equation obtained along the rays up to the first phase:

$$D_\xi \mathcal{S}_J^{\pm}(x,\xi,t) = X^{-|J|}(x,\nabla\varphi_J(x,\xi)) - y - (t - l_J(x,\xi))\frac{\xi}{|\xi|}. \tag{8.3.13}$$

Note that, by Proposition (8.14), (1), $w^{J,\pm}(x,\xi,t) \neq 0$ implies that, because q is supported away of the boundary, for $|J| \geq 1$

$$t - l_J(x,\xi) \geq \delta_0 > 0$$

and thus, we get (8.3.10). Moreover, we deduce that

$$D_{\mathcal{S}^{n-1}(0,s)}\mathcal{S}_J(x,\xi)$$
$$= X^{-|J|}(x,\nabla\varphi_J^{\pm}(x,\xi)) - y - \left(\left(X^{-|J|}(x,\nabla\varphi_J(x,\xi)) - y\right)\cdot\frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|}.$$

Thus, if ξ is such that $D_{\mathcal{S}^{n-1}(0,s)}\mathcal{S}_J(x,\xi) = 0$,

$$X^{-|J|}(x,\nabla\varphi_J(x,\xi)) - y \,/\!/\, \frac{\xi}{|\xi|},$$

hence $\frac{\xi}{|\xi|}$ is a direction allowing reaching the point x from the point y following the story of reflection J. Note that there is a priori two such vectors on $\mathcal{S}^{n-1}(0,s)$: one and its opposite, but because $w^{J,+} = 0$ for $J \in \mathcal{I}_-$, we have $w^J(x,t,\xi) = 0$ for one of them. We thus get (8.3.11). Note that the critical point ξ such that $w^J(x,t,\xi) \neq 0$ is the one verifying

$$\left(X^{-|J|}(x,\nabla\varphi_J(x,\xi)) - y\right)\cdot\frac{\xi}{|\xi|} > 0. \tag{8.3.14}$$

The Hessian in $\mathbb{R}^n$ of $\mathcal{S}_J$ is derivated like in [Laf17b], differentiating $|\nabla\varphi_J(x,\xi)|\xi||^2 = |\xi|^2$ with respect to ξ and integrating the transport equation obtained along the rays once again:

$$D_\xi^2 S_J^{\pm}(x,\xi,t) = \frac{l}{|\xi|}Id - \left(\frac{Id}{|\xi|} - \frac{\xi\xi^t}{|\xi|^3}\right)t$$
$$- \sum_{k=1}^{3}\int_0^{\frac{l}{|\xi|}} D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi)\,(D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi))^t\,.$$

We would like to deduce an expression of $D^2_{\mathcal{S}^{n-1}(0,s)}S_J(x,\xi,t)$ for $\xi = s_J(x,t)$. To this purpose, we recall that

Lemma 8.19. *Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a submersion in 0 and $M := g^{-1}(0)$. Moreover, let $f : \mathbb{R}^n \to \mathbb{R}$ and F be its restriction to M. We suppose that F has a critical point in $a \in M$. Then, the Hessian of F in a is the restriction of*

$$d^2 f_a - \lambda \circ d^2 g_a$$

to T_aM, where λ is the Lagrange multiplier of f with respect to g in a, that is the unique linear form $\lambda \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ such that $df_a = \lambda \circ dg_a$.

Here, we can take $g(\xi) := |\xi|^2 - s^2$. Then $Dg(\xi) = 2\xi$. The Lagrange multiplier of $\mathcal{S}_J(x, \cdot)$ with respect to g in $\xi := s_J(x, s)$ is the unique $\lambda \in \mathbb{R}$ such that

$$D_\xi \mathcal{S}_J(x, t, \xi) = 2\lambda\xi.$$

Therefore, according to (8.3.13)

$$2\lambda = \Big(X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - y\Big) \cdot \frac{\xi}{|\xi|} - (t - l_J(x,\xi))\frac{1}{|\xi|}.$$

On the other hand, by lemma 8.19

$$D^2_{\mathcal{S}^{n-1}(0,s)} S_J(x, \xi, t) = \big(D^2_\xi S_J(x,\xi,t) - 2\lambda \mathrm{Id}\big)_{|T_\xi S^{n-1}(0,s)}$$

Thus, $D^2_{\mathcal{S}^{n-1}(0,s)} S_J(x,\xi,t)$ is the restriction to $T_\xi \mathcal{S}^{n-1}(0,s)$ of

$$-(X^{-|J|}(x, \nabla\varphi^\pm_J(x,\xi)) - y) \cdot \frac{\xi}{|\xi|} Id + \frac{\xi\xi^t}{|\xi|^3}$$
$$-\sum_{k=1}^{3} \int_0^{\frac{l}{|\xi|}} D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\left(D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\right)^t.$$

But, the quadratic form $\xi\xi^t$ vanishes in $T_\xi\mathcal{S}^{n-1}$. Therefore, $D^2_{\mathcal{S}^{n-1}(0,s)} S_J(x,\xi,t)$ is the restriction to $T_\xi\mathcal{S}^{n-1}(0,s)$ of

$$-(X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - y) \cdot \frac{\xi}{|\xi|}\mathrm{Id}$$
$$-\sum_{k=1}^{3} \int_0^{\frac{l}{|\xi|}} D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\left(D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\right)^t.$$

.

And as the matrices

$$D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\left(D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi)\right)^t$$

are positives, and according to (8.3.14), we get the last part of the statement. □

8.4. Proof of the main result

Let $K \geq 0$. By the previous section, the function

$$(x,t) \to \frac{1}{(2\pi h)^d} \sum_\pm \sum_{J\in\mathcal{I}} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi^\pm_J(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi$$

satisfies the approximate equation

$$\partial_t^2 u - \Delta u = (\mp h)^K \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J\in\mathcal{I}} \int \Box w_{K-1}^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} |\xi|^{-K} d\xi$$

with data

$$(u(0), \partial_t u(0)) = (\delta_{\epsilon,h,N}^y, 0).$$

By the Duhamel formula, the difference from the actual solution, that is from $\cos(t\sqrt{-\Delta})\delta^y$, is bounded in H^m norm by

$$C \times h^{K-d} \times |t| \sup_{s\in[0,t],\xi\in[\alpha_0,\beta_0]} \sum_{\pm} \sum_{J\in\mathcal{I}} \|\Box w_{K-1}^{J,\pm}(\cdot,s,\xi) e^{\mp i(\varphi_J^{\pm}(\cdot,\xi)|\xi| - t|\xi|)/h}\|_{H^m}.$$

So, for $0 \le t \le \epsilon |\log h|$

$$\left(\cos(t\sqrt{-\Delta})\delta^y\right)(x) = S_K(x,t) + R_K(x,t) \tag{8.4.1}$$

with

$$S_K(x,t) = \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J\in\mathcal{I}} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi \tag{8.4.2}$$

and

$$\|R_K\|_{H^m} \lesssim |\log h| h^{K-d} \sup_{s,\xi} \sum_{\pm} \sum_{J\in\mathcal{I}} \|\Box w_{K-1}^{J,\pm}(\cdot,s,\xi) e^{\mp i(\varphi_J^{\pm}(\cdot,\xi)|\xi| - t|\xi|)/h}\|_{H^m}. \tag{8.4.3}$$

The remainder

We first deal with the remainder term R_K. Let us denote

$$W_{K-1}^{J,\pm}(x,s,\xi) = \Box w_{K-1}^{J,\pm}(\cdot,s,\xi) e^{\mp i(\varphi_J^{\pm}(\cdot,\xi)|\xi| - t|\xi|)/h}$$

Notice that, by construction of the w_k's, w_k^J is supported in a set of diameter $(C+t)$. Therefore, using Proposition 8.17, Proposition 8.10 and the derivative of a product we get:

$$\sum_{J\in\mathcal{I}} \|\partial^m W_{K-1}^{J,\pm}\|_{L^2} \lesssim C_K (1+t)^{\frac{d}{2}} \sum_{J\in\mathcal{I}} \|\partial^m W_{K-1}^{J,\pm}\|_{L^\infty} \lesssim C_K (1+t)^{\frac{d}{2}} h^{-m} \sum_{J\in\mathcal{I}} h^{-(2K+m)c\epsilon}$$

and thus, by the Sobolev embedding $H^d \hookrightarrow L^\infty$ and (8.4.3)

$$\|R_K\|_{L^\infty} \lesssim C_K |\log h| h^{K-2d} (1+t)^{\frac{d}{2}} h^{-(2K+d)c\epsilon} |\left\{J\in\mathcal{I},\ \text{s.t } w_{K-1}^J \neq 0\right\}|.$$

By Proposition 8.14 we get in the same way as in [Laf17b]

$$|\left\{J\in\mathcal{I},\ \text{s.t } w_{K-1}^J \neq 0\right\}| \lesssim (1+t), \tag{8.4.4}$$

and therefore

$$\|R_K\|_{L^\infty} \lesssim C_K |\log h| h^{K(1-2c\epsilon)-d(2+c\epsilon)}(1+t)^{\frac{d}{2}+1} \tag{8.4.5}$$

Thus, for $0 \leq t \leq \epsilon|\log h|$

$$\|R_K\|_{L^\infty} \lesssim C_K |\log h|^{\frac{d}{2}+1} h^{K(1-2c\epsilon)-d(2+c\epsilon)} \leq C_K h^{K(1-2c\epsilon)-d(1+c\epsilon)-1}.$$

We take $\epsilon > 0$ small enough so that $2c\epsilon \leq \frac{1}{2}$ and we get

$$\|R_K\|_{L^\infty} \leq C_K h^{\frac{K}{2}-3d-1}.$$

Let us now fix K large enough so that

$$\frac{K}{2} - 3d - 1 \geq -\frac{d+1}{2} + 1.$$

Then, as $0 \leq t \leq \epsilon|\log h|$ is equivalent to $h \leq e^{-\frac{t}{\epsilon}}$, we obtain

$$\|R_K\|_{L^\infty} \leq C_K h^{-\frac{d+1}{2}} e^{-\frac{t}{\epsilon}}. \tag{8.4.6}$$

for $0 \leq t \leq \epsilon|\log h|$.

The free wave $J = \emptyset$

Let us denote

$$S_K^{\emptyset}(x,t) = \frac{1}{(2\pi h)^d} \sum_{\pm} \int \sum_{k=0}^{K} (\mp ih/|\xi|)^k w_k^{\emptyset,\pm}(x,t,\xi) e^{-i((x-y)\cdot\xi \mp t|\xi|)/h} d\xi$$

the free part of the wave, and

$$S_K^r = \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{|J|\geq 1} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi,$$

the reflected waves, in such a way that

$$S_K = S_K^{\emptyset} + S_K^r.$$

Note that $S_K^{\emptyset}$ is simply the approximate expression of the solution of the wave equation with data $(\delta^y, 0)$, in the free space:

$$S_K^{\emptyset}(x,t) = \left(\cos(t\sqrt{-\Delta_0})\delta^y\right)(x) + R_K^{\emptyset}(x,t)$$

where Δ_0 denote the Laplacian in the free space and by the Duhamel formula, for $0 \leq t \leq \epsilon|\log h|$

$$\|R_K^{\emptyset}\|_{H^m} \lesssim h^{K-d} |\log h| \sup_{s,\xi} \|\Box w^{\emptyset}(\cdot, s, \xi)\|_{H^m}$$

The usual dispersive estimate for the waves in the free space gives, by the frequencies localization of δ^y

$$|\cos(t\sqrt{-\Delta_0})\delta^y| \lesssim \frac{1}{h^{\frac{d+1}{2}}t^{\frac{d-1}{2}}}$$

and thus dealing with $R_K^\emptyset$ as we did for R_K we get

$$|S_K^\emptyset| \lesssim \frac{1}{h^{\frac{d+1}{2}}t^{\frac{d-1}{2}}}, \text{ for } 0 \leq t \leq \epsilon|\log h|. \tag{8.4.7}$$

The reflected waves $|J| \geq 1$

According to Proposition 8.18, the parts

$$d(X^{-|J|}(x, \nabla\varphi_J^\pm), y) \leq \eta$$

enjoys a rapid decay and we thus have

$$S_K^r = \frac{1}{(2\pi h)^d}\sum_\pm \sum_{|J|\geq 1}\int \sum_{k=0}^K w_k^{J,\pm}(x,t,\xi)e^{\mp i(\varphi_J^\pm(x,\xi)|\xi|-t|\xi|)/h} 1_{d(X^{-|J|}(x,\nabla\varphi_J^\pm),y)\geq\eta}(\mp ih/|\xi|)^k d\xi + \sum_{|J|\geq 1} O(h^\infty).$$

Note that, by (8.4.4), the $O(h^\infty)$ part does not contribute. We write the remaining part of S_K^r as, illegitimately omiting $1_{d(X^{-|J|}(x,\nabla\varphi_J^\pm),y)\geq\eta}$ for the seek of lisibility:

$$\begin{aligned} S_K^r(x,t) &= \frac{1}{(2\pi h)^d}\sum_\pm \sum_{J\in\mathcal{I}}\int \sum_{k=0}^K w_k^{J,\pm}(x,t,\xi)e^{\mp i(\varphi_J^\pm(x,\xi)|\xi|-t|\xi|)/h}(\mp ih/|\xi|)^k d\xi \\ &= \frac{1}{(2\pi h)^d}\sum_\pm \sum_{J\in\mathcal{I}}\int_{s=\alpha_0}^{\beta_0}\int_{|\xi|=s}\sum_{k=0}^K w_k^{J,\pm}(x,t,\xi)e^{\mp i(\varphi_J^\pm(x,\xi)|\xi|-t|\xi|)/h}(\mp ih/s)^k d\xi ds. \end{aligned}$$

According to Proposition 8.18, we can perform a stationary phase on each sphere $\{|\xi| = s\}$, for each term of the sum $\sum_\pm \sum_{|J|\geq 1}$, up to order h^{k_0}. We obtain, as the sphere is of dimension $d-1$, for $t \geq 0$

$$\begin{aligned} S_K^r(x,t) = \frac{1}{(2\pi h)^d}h^{\frac{d-1}{2}} &\sum_{\pm,J\in\mathcal{I}}\sum_{k=0}^{k_0}\int_{\alpha_0}^{\beta_0} e^{\mp i(\varphi_J^\pm(x,s_J(t,x))s-ts)/h}(\mp ih/s)^k\tilde{w}_k^{J,\pm}(t,x)ds \\ &+ \frac{1}{(2\pi h)^d}h^{\frac{d-1}{2}}\sum_{\pm,J\in\mathcal{I}}\int_{\alpha_0}^{\beta_0} R_{\text{st.ph.}}^{J,\pm}(x,t,s)ds \\ &+ \frac{1}{(2\pi h)^d}\sum_{\pm,J\in\mathcal{I}}\int \sum_{k=k_0+1}^K w_k^J(x,t,\xi)e^{\mp i(\varphi_J^\pm(x,\xi)|\xi|-t|\xi|)/h}(\mp ih/|\xi|)^k d\xi. \end{aligned} \tag{8.4.8}$$

where for $0 \leq k \leq k_0$, the term $\tilde{w}_k^{J,\pm}$ is a linear combination of

$$D_\xi^{2k} w_0^{J,\pm}(t,x,s_J(t,x,s)), D_\xi^{2(k-1)} w_1^{J,\pm}(t,x,s_J(t,x,s)), \dots, w_k^{J,\pm}(t,x,s_J(t,x,s)),$$

where $s_J(t,x,s)$ is the stationary point of the phase on the hypersphere $\{|\xi| = s\}$ and $R^{J,\pm}_{\text{st.ph.}}$ is the remainder involved in the stationary phase. Then the proof proceed as in [Laf17b], Section 5, and we obtain in the exact same way, combining the decays estimates of Proposition 8.17 with the informations on the temporal support of w^J given by Proposition 8.14, for $\epsilon > 0$ small enough depending only of α_0, β_0 and of the geometry of the obstacles, for $0 \leq t \leq \epsilon |\log h|$ and $x \in \text{Supp}\chi_0$:

$$\sum_{J \in \mathcal{I}} |w_k^{J,\pm}| \leq C_k h^{-\frac{1}{2}} e^{-\mu t}, \ 1 \leq k \leq K-1,$$

$$\sum_{J \in \mathcal{I}} |w_0^{J,\pm}| \lesssim e^{-\mu t},$$

$$\sum_{J \in \mathcal{I}} |D_\xi^{2k} w_0^{J,\pm}| \lesssim C_k h^{-\frac{1}{2}} e^{-t/4\epsilon}, \ 1 \leq k \leq k_0,$$

$$\sum_{J \in \mathcal{I}} |R^{J,\pm}_{\text{st.ph.}}| \lesssim e^{-t/\epsilon}.$$

for some $\mu = \mu(\epsilon)$. Therefore we obtain, taking $k_0 = \lceil \frac{d-1}{2} \rceil$, for some $\nu > 0$ depending only of α_0, β_0 and of the geometry of the obstacles

$$|\chi_0 S_K^r(x,t)| \lesssim h^{-\frac{d+1}{2}} e^{-\nu t} \quad \text{for } 0 \leq t \leq \epsilon |\log h|. \tag{8.4.9}$$

Conclusion

Thus, collecting (8.4.1), (8.4.6), (8.4.7) and (8.4.9) we get

$$|\chi_0 \cos(t\sqrt{-\Delta}) \delta^y_{\epsilon,h,N}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \ \text{for } 0 \leq t \leq \epsilon |\log h|.$$

That is (8.3.7). Thus, theorem 8.1 is proved by the work of reduction of the previous sections.

9 The exterior of many convex obstacles

9.1. Introduction

Let (M,g) be a Riemaniann manifold of dimension d. We are interested in the Schrödinger

$$\begin{cases} i\partial_t u - \Delta_g u = 0 \\ u(0) = u_0 \end{cases} \tag{9.1.1}$$

and wave equations on M

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 \\ (u(0), \partial_t u(0)) = (f,g), \end{cases} \tag{9.1.2}$$

where Δ_g denotes the Laplace-Beltrami operator. A key to study the perturbative theory and the nonlinear problems associated with these equations is to understand the size and the decay of the linear flows. One tool to quantify these decays is the so-called *Strichartz estimates*

$$\|u\|_{L^q(0,T)L^r(M)} \leq C_T \left(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} \right), \text{ (Waves)}$$
$$\|u\|_{L^q(0,T)L^r(M)} \leq C_T \|u_0\|_{L^2}, \text{ (Schrödinger)}$$

where (p,q) have to follow an admissibility condition given by the scaling of the equation, respectively

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \; \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4},$$
$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \; (q,r,d) \neq (2,\infty,2),$$

for the Schrödinger and wave equations.

These estimates have a long story, beginning with the work of [Str77] for the $p=q$ case in $\mathbb{R}^n$, extended to all exponents by [GV85b], [LS95], and [KT98]. For the wave equation in a manifold without boundary, the finite speed of propagation shows that it suffices to work in local coordinates to obtain local Strichartz estimates. This path was followed by

[Kap89], [MSS93], [Smi98], and [Tat02]. The case of a manifold with boundary, where reflexions have to be dealt with, is more difficult. Estimates outside one convex obstacle for the wave equation were obtained by [SS95], following the parametrix construction of Melrose and Taylor, which give an explicit representation of the solution near diffractive points, and for the Schrödinger equation later by [Iva10].

The first local estimates for the wave equation on a general domain where shown by [BLP08] for certain ranges of indices, then extended by [BSS09a]. These estimates cannot be as good as in the flat case : [Iva12] showed indeed that a loss have to occur if some concavity is met, because of the formation of caustics. Recently, [ILP14a] and [ILLP] obtained almost sharp local Strichartz estimates inside a convex domain.

One obstruction to the establishment of global estimates without loss are trapped geodesics. Under a non trapping assumption, such estimates were established for the wave equation by the works of [SS00], [Bur03] and [Met04]. For the Schrödinger flow in the boundary less case, [BT07], [Bou11], [HTW06], [ST02] obtained the estimates in several non-trapping geometries.

When trapped geodesics are met, [Bur04] showed that a loss with respect to the flat case have to occur for the wave equation in the global L^2 integrability of the flow, and his counterpart, the smoothing estimate, for the Schrödinger equation:

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, \dot{H}^s \times \dot{H}^{s-1})} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} \text{ (Waves)},$$
$$\|\chi u\|_{L^2(\mathbb{R}, H^{1/2})} \lesssim \|u_0\|_{L^2} \text{ (Schrödinger)}.$$

Despite this loss in the smoothing estimate, [BGH10] showed Strichartz estimates without loss for the Schrödinger equation in an asymptotically euclidian manifold without boundary for which the trapped set is sufficiently small and exhibit an hyperbolic dynamic.

Following this breakthrough, we recently proved in [Laf17b, Laf17a] global Strichartz estimates without loss for Schrödinger and wave equations outside two strictly convex obstacles, exhibiting in the boundary case the first trapped situation where no loss occurs. The goal of this paper is to extend this result to the case of the exterior of $N \geq 3$ convex obstacles, which is in many aspects a counterpart with boundaries of the framework studied in [BGH10] .

In this N-convex obstacles setting, there is infinitely many trapped rays. Therefore, there is a competition between the large number of parts of the flow that remain trapped between the obstacles and the decay of each such part. For a sufficient decay to hold, this competition has to occur in a favorable way. This is the so called Ikawa condition:

Definition 9.1 (Ikawa condition, 1: strongly hyperbolic setting)**.** There exists $\alpha > 0$ such that the following condition hold

$$\sum_{\gamma \in \mathcal{P}} \lambda_\gamma d_\gamma e^{\alpha d_\gamma} < \infty. \tag{9.1.3}$$

Here $\mathcal{P}$ design the set of all primitive closed trajectories, d_γ the length of the trajectory γ and $\lambda_\gamma = \sqrt{\mu_\gamma \mu'_\gamma}$, where μ_γ and μ'_γ are the two eigenvalues modulus smaller than one of the Poincaré map associated with γ. This condition was first introduced by [Ika82] when investigating the decay of the local energy of the wave equation. Notice that it is the analog of the topologic pressure condition arising in [BGH10]. It is, for example, always verified for two obstacles, or for N obstacles if their are suffisently far away from each other.

We will moreover suppose the second part of the Ikawa condition to be verified, namely, denoting by Θ_i the obstacles:

Definition 9.2 (Ikawa condition, 2: no obstacle in shadow)**.** For all i, j, k pairwise distinct,

$$\mathrm{Conv}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset. \tag{9.1.4}$$

At the difference of the first one, and excepting the degenerated situation where a periodic trajectory is tangent to an obstacle, this condition may be purely technical (it permits to construct solutions without been preoccupied by the shadows induced by the obstacle) and should be avoided with a more careful analysis.

We are now in position to state our result.

Theorem 9.3. *Let $(\Theta_i)_{1\leq i\leq N}$ be a finite family of strictly convex subsets of $\mathbb{R}^3$, such that Ikawa's conditions (9.1.3) and (9.1.4) hold, and $\Omega = \mathbb{R}^3 \backslash \underset{1\leq i\leq N}{\cup} \Theta_i$. Then, under the non-endpoint admissibility conditions:*

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s, \;\; \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \;\; q \neq \infty, \;\; \textit{(Waves)}$$
$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \;\; (q,r) \neq (2,6), \;\; \textit{(Schrödinger)}$$

global Strichartz estimates without loss hold for both Schrödinger and wave equations in Ω :

$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}, \;\; \textit{(Waves)}$$
$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \lesssim \|u_0\|_{L^2}. \;\; \textit{(Schrödinger)}$$

Overview of the proof

We generalize the approach introduced in [Laf17a, Laf17b].

As we dealt with the Schrödinger equation outside two convex obstacles in [Laf17b] and showed in [Laf17a] how to adapt the work to the wave equation, the main novelty of this note is how to handle the N-convex framework, and therefore we present a detailed proof of our main result in the more intricate case of the Schrödinger equation, and briefly explain how to adapt it to the wave equation with the material of [Laf17a] in the last section.

In the flat case, the smoothing estimate permits to stack Strichartz estimates in time $\sim h$ for data of frequency $\sim h^{-1}$ to show global estimates. As remarked in [BGH10], the logarithmic loss can be compensated if we show Strichartz estimates in time $h|\log h|$

instead of h near the trapped set, provided a smoothing estimate without loss in the non trapping region is at hand. Therefore, our first section is devoted to prove such an estimate, using a commutator argument together with the escape function construction of Morawetz, Raltson and Strauss [MRS77]. We then show that we can reduce ourselves to data micro-locally supported near trapped trajectories, and that remain in a neighborhood of it in logarithmic times. We extend to the N-convex framework the construction of an approximate solution for such data done in [Laf17a, Laf17b] following ideas of [Ika88, Ika82] and [?], and finally, we show that under the strong hyperbolic setting assumption (9.1.3), this construction gives a sufficient decay.

Notations

- We denote by $\mathcal{K} \subset T^\star\Omega \cup T^\star\partial\Omega$ the trapped set, which is is composed of infinitely many periodic trajectories,
- and by $\mathcal{P}$ the set of all primitive periodic trajectories, that is, followed only once,
- the operator $\psi(-h^2\Delta)$ localizes at frequencies $|\xi| \in [\alpha_0 h^{-1}, \beta_0 h^{-1}]$, we refer to [Iva10] for the definition of this operator,
- the set $\mathcal{I}$ is the set of all stories of reflexions, that is all finites sequences $(j_1, \cdots, j_k)$ with values in $[\![1, \cdots, N]\!]$ such that $j_i \neq j_{i+1}$,
- moreover, we will adopt all the notations introduced in [Laf17b]. Let us in particular recall that
$$\Phi_t : T^\star\Omega \cup T^\star\partial\Omega \longrightarrow T^\star\Omega \cup T^\star\partial\Omega$$
denotes the billiard flow on Ω: $\Phi_t(x, \xi)$ is the point attained after a time t from the point x in the direction $\frac{\xi}{|\xi|}$ at the speed $|\xi|$, following the laws of geometrical optics,
- finally, let us recall that the spatial and directional components of Φ_t are respectively denoted X_t and Ξ_t.

9.2. Smoothing effect without loss outside the trapped set

Let us recall the smoothing effect with logarithmic loss obtained in [Bur04] in our framework of a family of strictly convex obstacle verifying Ikawa's condition:

Proposition 9.4. *For any $\chi \in C_c^\infty(\mathbb{R}^3)$ and any $u_0 \in L^2(\Omega)$ such that $u_0 = \psi(-h^2\Delta)u_0$, we have*
$$\|\chi e^{it\Delta_D} u_0\|_{L^2(\mathbb{R},L^2)} \lesssim (h|\log h|)^{\frac{1}{2}} \|u_0\|_{L^2}. \tag{9.2.1}$$

The aim of this first section is to prove a smoothing effect without loss outside the trapped set:

Proposition 9.5 (Local smoothing without loss in the non trapping region)**.** *Let $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be supported in the complementary of the trapped set, $\mathcal{K}^c$. Then we have, for $u_0 = \psi(-h^2\Delta)u_0$*

$$\|Op_h(\phi)e^{-it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{1/2}(\Omega))} \lesssim \|u_0\|_{L^2}. \tag{9.2.2}$$

Proof. We will use the same strategy as in [BGH10] lemma 2.2, adapting the proof in the case of a domain with boundary. Notice that, for any operator A,

$$\langle Au,u\rangle(T) - \langle Au,u\rangle(0) = \int_0^T \int_\Omega \langle [i\Delta, A]u, u\rangle + \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u\rangle. \tag{9.2.3}$$

Thus, if we find an operator A of order 0 such that $[i\Delta, A]$ is elliptic and positive on the support of ϕ and such that the border term

$$B = \int_0^T \int_{\partial\Omega} \langle Au, \partial_n u\rangle d\sigma dt$$

is essentially positive, we shall obtain the desired estimate.

Notations

If $b \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is a real symbol such that $b \geq 0$, and $b \geq \alpha$ on U, we use Garding inequality on symbols of the form

$$b - \alpha\frac{a\bar{a}}{(\sup|a|)^2} \geq 0,$$

where $a \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is supported in U. Notice that we have $\mathrm{Op}_h(a\bar{a}) = \mathrm{Op}_h(a)\mathrm{Op}_h(a)^\star + O(h)$.

Moreover, we will denote, in this section and this section only, Φ for the operator associated to ϕ.

The symbol of A at the border as an operator acting on Schrödinger waves

We perform the semi-classical time change of variable to write:

$$B = h^{-1}\int_0^{hT} \int_{\partial\Omega} \langle A(e^{iht}u_0), \partial_n(e^{iht}u_0)\rangle d\sigma dt$$

We use the strategy of [MRS77] to derive the symbol of A at the border as an operator acting on Schrödinger waves. Let us consider A as an operator acting on $\partial\Omega \times \mathbb{R}$. Notice that, because $\partial\Omega \times \mathbb{R}$ is nowhere characteristic for the semi-classical Schrödinger flow, there exists an operator Q of order zero such that for any semi-classical Schrödinger wave v

$$Av_{|\partial\Omega\times\mathbb{R}} = Q(\partial_n v). \tag{9.2.4}$$

Let q be the symbol of this operator. Let $(x_0, t_0) \in \partial\Omega \times \mathbb{R}$ and $(\eta, \tau) \in T_{(x_0,t_0)}(\partial\Omega \times \mathbb{R})$. We denote by $\psi_\pm$ the two distinct solutions of the Eikonal equations

$$\begin{aligned} |\nabla\psi(x)|^2 &= -\tau, \\ \psi_\pm(x) &= x \cdot \eta \text{ on } \partial\mathcal{O}, \end{aligned}$$

that are well defined in a neighborhood of x_0 as soon as $\tau - \eta^2 > 0$: indeed, extending n in a small neighborhood of the border, one can always take

$$\psi_\pm = x \cdot \eta \pm \sqrt{\tau - \eta^2} n.$$

For $\lambda > 0$, consider, extending $\partial_n \psi_\pm$ in a neighborhood of x_0 in Ω

$$v_\lambda = \frac{e^{i\lambda(\psi_+ + t\tau)} - e^{i\lambda(\psi_- + t\tau)}}{(i\lambda)(\partial_n\psi_+ - \partial_n\psi_-)},$$

which is solution of an approximate semi-classical Schrödinger equation

$$\begin{aligned} i\partial_t v_\lambda - \lambda^{-1}\Delta v_\lambda &= O(\lambda^{-1}), \\ v_\lambda &= 0 \text{ on } \partial\mathcal{O}. \end{aligned}$$

verifying, in a neighborhood of x_0 in $\partial\mathcal{O}$

$$\partial_n v_\lambda = e^{i\lambda(x\cdot\eta + t\tau)}.$$

But, the principal symbol of Q can be computed as

$$\begin{aligned} q(x_0, t_0, \eta, \tau) &= \lim_{\lambda\to\infty} e^{-i\lambda(x_0\cdot\eta + t_0\cdot\tau)} Q(e^{i\lambda(x\cdot\eta + t\tau)})(x_0, t_0) \\ &= \lim_{\lambda\to\infty} e^{-i\lambda(x_0\cdot\eta + t_0\cdot\tau)} Q(\partial_n v_\lambda)(x_0, t_0). \end{aligned}$$

By the Duhamel formula, the difference between v_λ and the solution of the actual equation w_λ is bounded in a neighborhood of (x_0, t_0) by

$$|w_\lambda - v_\lambda| \lesssim \lambda^{-1},$$

therefore, we can replace v_λ by w_λ, which is an exact Schrödinger wave, in the limit and make use of (9.2.4) to get:

$$\begin{aligned} q(x_0, t_0, \eta, \tau) &= \lim_{\lambda\to\infty} e^{-i\lambda(x_0\cdot\eta + t_0\cdot\tau)} Q(\partial_n w_\lambda)(x_0, t_0) \\ = \lim_{\lambda\to\infty} e^{-i\lambda(x_0\cdot\eta + t_0\cdot\tau)} A(w_\lambda)(x_0, t_0) &= \lim_{\lambda\to\infty} e^{-i\lambda(x_0\cdot\eta + t_0\cdot\tau)} A(v_\lambda)(x_0, t_0) \\ &= \left(\frac{a(x, d\psi_+) - a(x, d\psi_-)}{2(\partial_n\psi_+ - \partial_n\psi_-)} \right)(x_0, t_0). \end{aligned}$$

And we conclude that, this computation being valid for $\tau - \eta^2 > 0$

$$q(x_0, t_0, \eta, \tau) = \left(\frac{a(x_0, \xi_+) - a(x_0, \xi_-)}{(\xi_+ - \xi_-) \cdot n(x_0)} \right), \tag{9.2.5}$$

$$\xi_\pm = \eta \pm \sqrt{\tau - \eta^2} n(x_0). \tag{9.2.6}$$

Notice that $\xi_\pm$ is a pair of reflected rays.

The escape function

Let $(y,\eta)\notin\mathcal{K}$. The generalized broken ray starting from (y,η) is composed of a finite number of segments, thus, the construction of [MRS77], Section 5, holds to construct a ray function starting from (y,η), that is, a function p_0 satisfying

$$\xi\cdot\nabla p_0(x,\xi)\geq 0,\ \frac{p_0(x,\xi)-p_0(x,\xi')}{(\xi-\xi')\cdot n(x)}\geq 0,$$

and

$$\eta\cdot\nabla p_0(y,\eta)>0,\ \frac{p_0(y,\eta)-p_0(y,\eta')}{(\eta-\eta')\cdot n(y)}>0.$$

Therefore, by compactness, we can construct a function a such that

$$\xi\cdot\nabla a(x,\xi)\geq 0, \frac{a(x,\xi)-a(x,\xi^{'})}{(\xi-\xi^{'})\cdot n(x)}\geq 0 \tag{9.2.7}$$

$$\xi\cdot\nabla a(x,\xi)>0,\ \frac{a(x,\xi)-a(x,\xi^{'})}{(\xi-\xi^{'})\cdot n(x)}>0,\ \text{on } V\supset\supset \text{supp }\phi. \tag{9.2.8}$$

Finally, notice that, because the construction of [MRS77] follows the rays and because the trapped set is invariant by the flow, we can construct a in such a way that

$$a=0 \text{ near } \mathcal{K}. \tag{9.2.9}$$

Remark that, as in [MRS77], such an a can be approximated by a polynomial in order to justify the above integration by parts.

A first estimate

Let $\delta>0$. Because of (9.2.7), (9.2.5), q is real-valued and positive on $\{\tau-\eta^2\geq 0\}$, therefore, there exists $\epsilon>0$ small enough so that, on $\{\tau-\eta^2\geq -\epsilon\}$ we have, with the notations of (9.2.5)

$$\Re e\frac{a(x_0,\xi_+)-a(x_0,\xi_-)}{(\xi_+-\xi_-)\cdot n(x_0)}\geq -\delta/2. \tag{9.2.10}$$

and, for $|\alpha|\leq 2(d+1)=8$

$$\left|\Im m\ \partial^\alpha_{x,t,\xi,\tau}\frac{a(x_0,\xi_+)-a(x_0,\xi_-)}{(\xi_+-\xi_-)\cdot n(x_0)}\right|\leq\delta/2 \tag{9.2.11}$$

Now, let χ be positive and supported in $\{\tau-\eta^2\geq -2\epsilon\}$ and such that $\chi=1$ in $\{\tau-\eta^2\geq -\epsilon\}$. We decompose a as the sum

$$a=\chi a+(1-\chi)a.$$

Note that $(1-\chi)a$ is supported away from the characteristic set $\{\tau=\eta^2\}$ of the semi-classical Schrödinger flow. Therefore,

$$\|\mathrm{Op}_h((1-\chi)a)u\|_{H^\sigma(\mathbb{R}\times\Omega)}=O(h^\infty)\|u_0\|_{L^2},$$

and using a trace theorem

$$B = \int_0^T \int_{\partial\Omega} \langle R(\partial_n(e^{it\Delta}u_0)), \partial_n(e^{it\Delta}u_0)\rangle d\sigma dt + O(h^\infty)\|u_0\|_{L^2},$$

where $R = \mathrm{Op}(\chi a)$. Notice that a pair of reflected rays share the same norm, therefore, by (9.2.5), the symbol of R is

$$r(x_0, t_0, \eta, \tau) = \chi(\eta, \tau)\left(\frac{a(x_0, \xi_+(\eta,\tau)) - a(x_0, \xi_-(\eta,\tau))}{(\xi_+(\eta,\tau) - \xi_-(\eta,\tau))\cdot n(x_0)}\right),$$
$$\xi_\pm = \eta \pm \sqrt{\tau - \eta^2}n(x_0).$$

Therefore, by (9.2.10), (9.2.8) and (9.2.9), we can use the Garding inequality for the real part, the Calderon-Vaillancourt theorem for the imaginary part in order to write

$$B \geq -\delta \int_0^T \int_{\partial\Omega} |\tilde{\Phi}u|^2 d\sigma dt - c_{\mathrm{Gard}}\|\chi_b u\|_{L^2([0,T],H^{-1/2}(\partial\Omega))} + O(h^\infty)\|u_0\|_{L^2}. \qquad (9.2.12)$$

where $\tilde{\phi} \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is supported in $\mathcal{K}^c$ and $\tilde{\phi} = 1$ on the support of ϕ, and $\chi_b \in C_c^\infty(\mathbb{R}^3)$ is such that $\chi_b = 1$ on $\partial\Omega$.

Moreover, by the same procedure as in [MRS77], we may suppose that for $|x| \geq R \gg 1$, a is given by $a(x,\xi) = hx\cdot\xi$. Let $\chi_R \in C_c^\infty$ be such that $\chi_R = 1$ on $\{|x| \leq 2R\}$ and $\chi_R = 0$ on $\{|x| \geq 3R\}$. We decompose

$$\int_\Omega \langle [i\Delta, A]u, u\rangle = \int_\Omega \langle [i\Delta, A]\chi_R u, \chi_R u\rangle$$
$$+ \int_\Omega \langle [i\Delta, A]\chi_R u, (1-\chi_R)u\rangle + \int_\Omega \langle [i\Delta, A](1-\chi_R)u, \chi_R u\rangle$$
$$+ \int_\Omega \langle [i\Delta, A](1-\chi_R)u, (1-\chi_R)u\rangle.$$

Because the commutator is truly non-negative for functions supported in $\{|x| \geq 2R\}$, the last term is non-negative. Moreover, the integrand of both intermediate terms are supported in $\{2R \leq |x| \leq 3R\}$. Therefore, taking R large enough, the long-range smoothing estimate, which is for example a consequence of the long-range resolvent estimate of Cardoso and Vodev [CV02] by the procedure of [BGT04a], allows us to control them:

$$\left|\int_{\mathbb{R}} \int_\Omega \langle [i\Delta, A]\chi_R u, (1-\chi_R)u\rangle + \langle [i\Delta, A](1-\chi_R)u, \chi_R u\rangle\right|$$
$$\lesssim \|\tilde{\chi}u\|_{L^2H^{1/2}} \lesssim \|u_0\|_{L^2},$$

where $\tilde{\chi} \in C_c^\infty$ is equal to one in $\{2R \leq |x| \leq 3R\}$ and supported in $\{|x| \geq R\}$. Finally, by the Garding inequality again, using (9.2.8):

$$\int_0^T \int_\Omega \langle [i\Delta, A]\chi_R u, \chi_R u\rangle \geq C\|\Phi u\|_{L^2H^{1/2}} - c_{\mathrm{Gard}}\|\chi_R u\|_{L^2L^2}, \qquad (9.2.13)$$

Thus, combining (9.2.3), (9.2.12), and (9.2.13), using the trace theorem and controlling the lower order terms with the estimate *with logarithmic loss* we get

$$\|\Phi u\|_{L^2H^{1/2}} \leq C(\|u_0\|_{L^2} + \delta\|\tilde{\Phi}u\|_{L^2H^{1/2}}) + C_\delta O(h^\infty). \qquad (9.2.14)$$

Iteration and conclusion

To conclude, we would like to take $\delta > 0$ small enough and iterate (9.2.14). In order to do so, we have to take care of the potential dependency in $\phi, \tilde{\phi}, \tilde{\tilde{\phi}}, \dots, \overset{\sim(k)}{\phi}, \cdots$ and δ of the constants appearing in this estimate. Let us first remark that we take all the $\overset{\sim(k)}{\phi}$ in a given small neighborhood of the support of ϕ - this neighborhood is a subset of V of (9.2.8). Thus, there exists $A \geq 1$ such that, for $|\alpha + \beta| \leq N$

$$\|\partial_{x,\xi}^{\alpha,\beta} \overset{\sim(k)}{\phi} \|_{L^\infty} \leq A^k.$$

Therefore, the Garding constants c_{Gard} in (9.2.12), (9.2.13) at the k-th iteration can be taken as A^k. Moreover, by (9.2.8), $\xi \cdot \nabla a$ is bounded below by a constant C uniformly on the support of all the $\overset{\sim(k)}{\phi}$, so we can choose the same constant C in (9.2.13) at all iteration. Finally, the $O(h^\infty)$ term depends only of δ.

Therefore, we can precise the constants in (9.2.14) at the k-th iteration:

$$\| \overset{\sim(k)}{\Phi} u\|_{L^2H^{1/2}} \leq (C + A^k)\|u_0\|_{L^2} + C\delta \| \overset{\sim(k+1)}{\Phi} u\|_{L^2H^{1/2}} + C_\delta O(h^\infty),$$

where C and A have no dependencies in k and δ and C_δ depends only of δ. Thus we get

$$\begin{aligned}
\|\Phi u\|_{L^2H^{1/2}} \leq & \left[C\frac{1-(C\delta)^{k+1}}{1-C\delta} + \frac{(C\delta A) - (C\delta A)^{k+1}}{1 - C\delta A} \right] \|u_0\|_{L^2} \\
& + (C\delta)^k \| \overset{\sim(k+1)}{\Phi} u\|_{L^2H^{1/2}} + C_\delta \frac{1-(C\delta)^{k+1}}{1-C\delta} O(h^\infty) \\
\leq & \left[C\frac{1-(C\delta)^{k+1}}{1-C\delta} + \frac{(C\delta A) - (C\delta A)^{k+1}}{1 - C\delta A} \right] \|u_0\|_{L^2} \\
& + (C\delta)^k \|\chi_0 u\|_{L^2H^{1/2}} + C_\delta \frac{1-(C\delta)^{k+1}}{1-C\delta} O(h^\infty)
\end{aligned}$$

where χ_0 is compactly supported. We fix δ small enough so that $C\delta A < 1$ and let k go to infinity to obtain the result. □

Remark 9.6. Notice that the exact same proof holds for any arbitrary domain for which a smoothing estimate with logarithmic loss holds. Moreover, as remarked by [DV13], we can iterate such a proof and therefore it suffices to assume a smoothing estimate with polynomial loss. More precisely, we initiate the argument controlling the lower order terms by the smoothing estimate with polynomial loss, and then iterate the proof and control the lower order terms by the previous estimate at each step, until we reach h^0. Thus we obtain the more general:

Proposition 9.7. *Let Ω be such that the following smoothing estimate with polynomial loss holds: there exists $k > 0$ such that for all $\chi \in C_c^\infty(\mathbb{R}^d)$ and all $u_0 \in L^2$ such that $u_0 = \psi(-h^2\Delta)u_0$, we have:*

$$\|\chi e^{-it\Delta_D} u_0\|_{L^2(\mathbb{R}, H^{1/2}(\Omega))} \lesssim h^{-k}\|u_0\|_{L^2}.$$

Then, a smoothing estimate without loss holds outside the trapped set $\mathcal{K}$: that is, for all $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ supported in $\mathcal{K}^c$, we have

$$\| Op_h(\phi) e^{-it\Delta_D} u_0 \|_{L^2(\mathbb{R}, H^{1/2}(\Omega))} \lesssim \|u_0\|_{L^2}.$$

9.3. Reduction to the logarithmic trapped set

Because of Proposition 9.4 and Proposition 9.5, the exact same proof as in [Laf17b], section 2, show that the following proposition implies our main result for the Schrödinger equation:

Proposition 9.8 (Strichartz estimates on a logarithmic interval near the trapped set)**.** *There exists $\epsilon > 0$ such that for all $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ supported in a small enough neighborhood of $\mathcal{K}$, we have*

$$\| Op_h(\phi) e^{-it\Delta_D} \psi(-h^2\Delta) u_0 \|_{L^p(0, \epsilon h |\log h|) L^q(\Omega)} \leq C \|u_0\|_{L^2}. \tag{9.3.1}$$

Notice that, by a classical $TT^\star$ argument, Proposition 9.8 is a consequence of the following pointwise dispersive estimate:

$$\| A e^{ith\Delta} \psi(-h^2\Delta) A^\star \|_{L^1 \to L^\infty} \lesssim (ht)^{-3/2}, \ \forall 0 \leq t \leq \epsilon |\log h|, \tag{9.3.2}$$

where we denoted, here and in the sequel of this section

$$A := \mathrm{Op}_h(\phi)$$

in the sake of readability.

Thus, the rest of the paper will be devoted to prove such an estimate. The aim of this section is to show that we can reduce ourselves to data micro-locally supported to the points that remain near the trapped trajectories in logarithmic times. In order to do so, we first need to generalizes some properties of the billiard flow shown in [Laf17b]:

9.3.1. Regularity of the billiard flow

We first need the following lemma, where we denoted by $W_{\tan,\eta}$ an η-neighborhood of the tangent rays:

Lemma 9.9. *There exists $\eta > 0$ such that any ray cannot cross $W_{tan,\eta}$ more than twice.*

Proof. If it is not the case, for all $n \geq 0$, there exists $(x_n, \xi_n) \in K \times \mathcal{S}^2$, where K is a compact set strictly containing the obstacles, such that $\Phi_t(x_n, \xi_n)$ cross $W_{\tan,\frac{1}{n}}$ at least three times. Extracting from (x_n, ξ_n) a converging subsequence, by continuity of the flow, letting n going to infinity we obtain a ray that is tangent to $\cup\Theta_i$ in at least points. Therefore, it suffices to show that such a ray cannot exist.

Remark that, because of the non-shadows condition (9.1.4), if $(x, \xi) \in W_{\tan}$, if we consider the ray starting from (x, ξ) and the ray starting from $(x, -\xi)$, one of the two do not cross any obstacle in positive times. But, if there is a ray tangent to the obstacles in at least three points, if we consider the second tangent point (x_0, ξ_0), both rays starting from (x_0, ξ_0) and $(x_0, -\xi_0)$ have to cross an obstacle, therefore, this is not possible. □

Together with lemma 3.2 of [Laf17b], which gives the (Hölder) regularity of the billiard flow near tangent points for a domain with no infinite order of contact points, we obtain, with the exact same proof as in this previous paper - the only assumption made been which given by lemma 9.9:

Lemma 9.10. *Let V be a bounded open set containing the convex hull of $\cup\Theta_i$. Then, there exists $\mu > 0$, $C > 0$ and $\tau > 0$ such that, for all $x, \tilde{x} \in V$, all $\xi, \tilde{\xi}$ such that $|\xi|, |\xi'| \in [\alpha_0, \beta_0]$, for all $t > 0$ there exists t' verifying $|t' - t| \leq \tau$ such that*

$$d(\Phi_{t'}(\tilde{x}, \tilde{\xi}), \Phi_{t'}(x, \xi)) \leq C^{t'} d((\tilde{x}, \tilde{\xi}), (x, \xi))^{\mu}. \tag{9.3.3}$$

Remark 9.11. It is crucial, in the proof of this previous lemma, that a ray cannot cross $W_{\tan,\eta}$ infinitely many times: indeed, regularity is lost at each tangent point. Therefore, in the case which does not enters the framework of Ikawa condition, 2: no obstacle in shadow of a trapped ray which is tangent to an obstacle, this proof does not hold, and we do not know if such a regularity of the flow is true. As this regularity is crucial in the sequel, we think that this “non shadow” condition may not be only technical, at least in the degenerated situation previously mentioned.

Finally, let us remark that

Lemma 9.12. *Let $\delta > 0$ and D_δ be a δ-neighborhood of $\mathcal{P}$. Then, for all compact K, $\Phi_t(\rho) \longrightarrow \infty$ as $t \longrightarrow \pm\infty$ uniformly with respect to $\rho \in K \cap D_\delta^c$.*

Proof. It suffices to prove that the length of all trajectories in $K \cap D_\delta^c$ are uniformly bounded. If it is not the case, there exists $\rho_n \in D_\delta^c \cap K$ such that

$$\text{lenght}\, \{\Phi_t(\rho_n)\}_{t\geq 0} \cap K \longrightarrow +\infty$$

as n goes to infinity. Up to extract a subsequence, $\rho_n \longrightarrow \rho^\star \in D_\delta^c$. Necessarily, lenght $\{\Phi_t(\rho^\star)\}_{t\geq 0} \cap K = \infty$, thus $\rho^\star \in \mathcal{P}$, this is not possible. □

Lemma 9.13. *$\mathcal{K}$ is closed.*

Proof. Let $\rho_n \in \mathcal{K}$, $\rho_n \longrightarrow \rho$. There exists $A > 0$ such that for any t, $d(\pi_x \Phi_t(\rho_n), 0_{\mathbb{R}^3}) \leq A$. $\pi_x \Phi_t(\cdot)$ been continuous for any fixed t, it suffices to pass to the limit $n \longrightarrow \infty$ in the previous inequality to obtain $\rho \in \mathcal{K}$. □

9.3.2. Reduction of the problem

We now show that we can reduce ourselves to points that remain near trapped trajectories in logarithmic times $T_0 \leq t \leq \epsilon|\log h|$ in order to prove the pointwise dispersive estimate (9.3.2) in times $[T_0, \epsilon|\log h|]$. In contrast to [Laf17b], where we used a translation argument in the spirit of [Iva10], we are here inspired by [BGH10].

Let $\delta > 0$. By lemma 9.13, the projection on $\mathbb{R}^3 \times \mathcal{S}^2$ of the trapped set is compact, thus there exists a finite number of phase-space segments $(S_k)_{1\leq k\leq N_\delta}$, $S_k = s_k \times \mathbb{R}\xi_k \subset T^\star\Omega$, s_i been a segment of $\mathbb{R}^3$, such that $\mathcal{K}$ is contained in a δ-neighborhood of $\cup S_k$. The small quantity $\delta > 0$ may be reduced a finite number of time in the sequel.

We will now define a microlocal partition of unity (Π_k). Let $p_k \in C_0^\infty(T^\star\Omega)$, $0 \le p_k \le 1$ be a family of functions such that p_k is supported in a neighborhood W_k of S_k and

$$\sum_{1\le k\le N_\delta} p_k = 1 \text{ in a neighborhood of } \mathcal{K}.$$

Let us define

$$\Pi_k = \mathrm{Op}_h(p_k),\ \forall 1 \le k \le N_\delta.$$

Now, let $\chi_0 \in C^\infty(\mathbb{R}^3)$, $0 \le \chi_0 \le 1$ such that χ_0 is supported sufficiently far from $\mathrm{Con}\cup\Theta_i$, and equal one far from the origin. Notice that any broken bicharacteristic entering the support of χ_0 from its complement remains in it for all times. We take

$$\Pi_0 = \chi_0$$

and let

$$\Pi_{-1} = \mathrm{Op}_h\left(1 - \chi_0 - \sum_{1\le k\le N_\delta} p_k\right).$$

Π_{-1} is defined in such a way that his symbol verifies

$$d(\mathrm{Supp}p_{-1}, \mathcal{K}) \ge d_1 > 0,$$

therefore, by lemma 9.12, there exists $T_0 > 0$ such that

$$\pi_x\Phi_t(\mathrm{Supp}p_{-1}) \subset \mathrm{Supp}\chi_0,\ \forall |t| \ge T_0.$$

Now, let $\tau > 0$. It will be fixed in the sequel. In the spirit of [BGH10], we decompose $T = (L-1)\tau + s_0$, where $L \in \mathbb{N}$ and $s_0 \in [0,\tau)$. We have

$$e^{iTh\Delta} = e^{its_0\Delta}\left(e^{i\tau h\Delta}\right)^{L-1},\ e^{i\tau h\Delta} = e^{i\tau h\Delta}\sum_{-1\le k\le N_\delta}\Pi_k.$$

and thus

$$e^{iTh\Delta} = \sum_{\mathbf{k}=(k_1,\cdots,k_L)} e^{its_0\Delta}\Pi_{k_L}e^{i\tau h\Delta}\Pi_{k_{L-1}}\cdots\Pi_{k_1}e^{i\tau h\Delta},$$

where the sum is taken over all multi-indice $\mathbf{k} \in [\![-1, N_\delta]\!]^L$. Let us remark that, because the wavefront set of the semi-classical Schrödinger flow is invariant by the generalized bicharacteristic flow, denoting

$$\sigma_{\mathbf{k}} = Ae^{its_0\Delta}\Pi_{k_L}e^{i\tau h\Delta}\Pi_{k_{L-1}}\cdots\Pi_{k_1}e^{i\tau h\Delta}\psi(-h^2\Delta)A^\star,$$

it holds that

$$\rho \in WF_h(\sigma_{\mathbf{k}}) \implies \begin{cases} \pi_x\rho \in \mathrm{Supp}\phi, \\ \Phi_{j\tau}(\rho) \in \mathrm{Supp}q_{k_j} & \forall 1 \le j \le L, \\ \pi_x\Phi_T(\rho) \in \mathrm{Supp}\phi. \end{cases} \tag{9.3.4}$$

Thus we have

Lemma 9.14. *Let* $\mathbf{k} \in [\![-1, N_\delta]\!]^L$*. If there exists* $1 \leq j \leq L$ *such that* $k_j = 0$ *or* $k_j = -1$*, then* $\sigma_{\mathbf{k}} = O(h^\infty)$ *as an* $L^1 \to L^\infty$ *operator.*

Proof. As remarked in [BGH10], by virtue of Sobolev embeddings it suffices to show that $\sigma_{\mathbf{k}} = O(h^\infty)$ as an $L^2 \to L^2$ operator, thus has null operator wavefront set. Let us suppose first that there exists j such that $k_j = 0$. We choose j to be the the first such indice. Suppose that $\rho \in WF_h(\sigma_{\mathbf{k}})$. There exists $t_0 \in [(j-1)\tau, j\tau]$ such that the spatial projection of $\Phi_{j\tau}(\rho)$ enters the support of χ_0 from its complementary, thus it does not leave it. Therefore $\pi_x \Phi_T(\rho) \in \mathrm{Supp}\chi_0$, this is not possible. Thus $WF_h(\sigma_{\mathbf{k}}) = \emptyset$.

Now, suppose that there exists $j \in [1, L - \frac{T_0}{\tau}]$ such that $k_j = -1$. Let $\rho \in WF_h(\sigma_{\mathbf{k}})$. $\Phi_{j\tau}(\rho) \in \mathrm{Supp}\Pi_{-1}$, hence

$$\pi_x \Phi_{j\tau+t}(\rho) \in \mathrm{Supp}\chi_0, \ \forall t \geq T_0,$$

and we are reduced thus to the previous case. In the same way, we exclude $j \in [\frac{T_0}{\tau}, L]$ using the property for all $t \leq -T_0$. □

But, as the $\mathbf{k}$-sum contains at most $(N_\delta + 2)^{\frac{\epsilon}{\tau}|\log h|}$ – that is, a negative power of h – terms, we have

$$\sum_{\mathbf{k}} O(h^\infty) = O(h^\infty),$$

and therefore we deduce from the previous lemma that, as an $L^1 \to L^\infty$ operator

$$Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star = \sum_{\mathbf{k}, k_j \geq 1} \sigma_{\mathbf{k}} + O(h^\infty).$$

Now, we will choose $\tau > 0$ small enough given by the following lemma:

Lemma 9.15. *For all* $\delta > 0$*, there exists* $\tau > 0$ *small enough so that, for every trajectory* $\gamma \in \mathcal{P}$*, we have*

$$d(\rho, \gamma) < \delta, \ d(\Phi_\tau(\rho), \gamma) < \delta \implies \forall t \in [0, \tau], d(\Phi_t(\rho), \gamma) < 3\delta.$$

Proof. Let $\tilde{\rho}$ realizing the distance from ρ to γ. We denote

$$t_0 = \inf\{t \geq 0, \ \text{s.t. } \pi_x \Phi_t(\rho) \in \Theta\}, \ \tilde{t}_0 = \inf\{t \geq 0, \ \text{s.t. } \pi_x \Phi_t(\tilde{\rho}) \in \Theta\}.$$

We assume that, for example, $\tilde{t}_0 > t_0$. Notice that, by the proof of lemma 9.10 from [Laf17b], we have

$$\forall t \in [0, \tau] \backslash (t_0, \tilde{t}_0), \ d(\Phi_t(\rho), \Phi_t(\tilde{\rho})) \leq C^\tau \delta.$$

Moreover, for $t \in [t_0, \tilde{t}_0]$,

$$d(\Phi_t(\rho), \Phi_t(\tilde{\rho})) \leq d(\Phi_t(\rho), \Phi_{t_0}(\rho)) + d(\Phi_{t_0}(\rho), \Phi_{t_0}(\tilde{\rho})) + d(\Phi_{t_0}(\tilde{\rho}), \Phi_t(\tilde{\rho})),$$

but, as $\{(\Phi_t(\rho)\}_{t \in [t_0, \tilde{t}_0]}$ and $\{(\Phi_t(\tilde{\rho})\}_{t \in [t_0, \tilde{t}_0]}$ are straight lines

$$d(\Phi_t(\rho), \Phi_{t_0}(\rho)) \leq |t - t_0||\pi_\xi \rho| \leq \tau\beta_0,$$

and similarly for $\tilde{\rho}$. Therefore

$$d(\Phi_t(\rho), \Phi_t(\tilde{\rho})) \leq 2\tau\beta_0 + C^\tau \delta.$$

We take $\tau > 0$ small enough so that $2\tau\beta_0 \leq \delta$ and $C^\tau \leq 2$ and we get the result. □

The segment S_{k_j} joins the obstacles Θ_{a_j} and Θ_{b_j}. Choosing $\delta > 0$ small enough, by (9.3.4), $\sigma_{\mathbf{k}}$ is not $O(h^\infty)$ only if, for all j

$$(a_j = a_{j+1} \text{ and } b_j = b_{j+1}) \text{ or } (a_{i+1} = b_j).$$

that is, only if $\gamma_{\mathbf{k}} = S_{k_1} \circ S_{k_2} \circ \cdots \circ S_{k_L}$ is a trajectory. Let, if it is the case, $J_{\mathbf{k}}$ be the corresponding story of reflexions. We extract from $J_{\mathbf{k}}$ the primitive story $I_{\mathbf{k}}$, that is, $J_{\mathbf{k}} = lI_{\mathbf{k}} + r$, $I_{\mathbf{k}}$ been primitive.

We now introduce the trapped set of an open subset in time T:

Definition 9.16. Let D be an open subset of $(T^\star\Omega \cup T^\star\partial\Omega) \cap \{|\xi| \in [\alpha_0, \beta_0]\}$ and $T > 0$. We define the trapped set of D in time T, denoted $\mathcal{T}_T(D)$, in the following way

$$\rho \in \mathcal{T}_T(D) \iff \forall t \in [0,T],\ \Phi_T(\rho) \in D.$$

Let us denote by $D_{I_{\mathbf{k}},\delta}$ a δ-neighborhood of $\gamma_{\mathbf{k}} \cap \{|\xi| \in [\alpha_0, \beta_0]\}$. For I a primitive story of reflexions, let $q_{I,T} \in C_0^\infty$ be such that

$$q_{I,T} = 0 \text{ outside } \mathcal{T}_T(D_{I,4\delta}),\ q_{I,T} = 1 \text{ in } \mathcal{T}_T(D_{I,3\delta}), \tag{9.3.5}$$

and denote

$$Q_I^T := \mathrm{Op}_h(q_{I,T}).$$

We have, by (9.3.4) and the choice of $\tau > 0$ permitted by lemma 9.15

$$\sigma_{\mathbf{k}} = \sigma_{\mathbf{k}} Q_{I_{\mathbf{k}}}^T + O(h^\infty).$$

Now, remark that for I a primitive story of reflexions

$$Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star Q_I^T = \sum_{\mathbf{k}, I_{\mathbf{k}}=I} \sigma_{\mathbf{k}} Q_I^T + O(h^\infty),$$

and therefore we recover

$$\sum_{I \text{ primitive}} Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star Q_I^T = Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star + O(h^\infty).$$

Let us finally remark that for $T \leq \epsilon|\log h|$, we have $h \leq e^{-\frac{T}{\epsilon}}$, thus the $O(h^\infty)$ term verifies the dispersive estimate. Therefore, we have proven that:

Lemma 9.17. *If the following dispersive estimate holds true*

$$\| \sum_{I \text{ primitive}} Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star Q_I^T \|_{L^1 \longrightarrow L^\infty} \lesssim (hT)^{-\frac{3}{2}},\ \forall T_0 \leq T \leq \epsilon|\log h|,$$

then the dispersive estimate (9.3.2) is true in times $[T_0, \epsilon|\log h|]$.

9.3.3. Times $0 \leq t \leq T_0$ and conclusion of the section

Finally, notice that the construction of Q_I^T does not depend of ϕ. We choose ϕ supported in a small enough neighborhood of $\mathcal{K}$ so that, in times $0 \leq t \leq T_0$ and for $|\xi| \in [\alpha_0, \beta_0]$, the bicharacteristic flow $\Phi_t(\rho)$ starting from ρ has only hyperbolic points of intersection with the boundary. But, for such points, we can use the parametrix construction of Ikawa [Ika82, Ika88], adapted to this problem in [Laf17b] and explained in the next section in the N-convex framework to show that the dispersive estimate holds true in times $0 \leq t \leq T_0$, with a constant depending on T_0: indeed, the flow can be written as a finite (depending on T_0) sum of reflected waves, each of them verifying the dispersive estimate.

Thus, by lemma 9.17, we are reduced to show the following dispersive estimate in order to obtain our main result, namely, we have

Lemma 9.18. *If the following dispersive estimate holds true*

$$\| \sum_{I \text{ primitive}} A e^{iTh\Delta} \psi(-h^2\Delta) A^\star Q_I^T \|_{L^1 \longrightarrow L^\infty} \lesssim (hT)^{-\frac{3}{2}}, \ \forall T_0 \leq T \leq \epsilon |\log h|, \tag{9.3.6}$$

then Strichartz estimates of Theorem 9.3 hold true for the Schrödinger equation.

where the symbols of Q_I^T were defined by (9.3.5). The sequel of the paper is devoted to doing so.

Let us remark that, with the same proof as in [Laf17b], we have, as a consequence of lemma 9.10,

$$d(\mathcal{T}_T(\tilde{D})^c, \mathcal{T}_T(D)) \geq \frac{1}{4} e^{-cT} d(\tilde{D}^c, D), \ \forall D \subset \tilde{D}$$

and therefore q_I^T can, and will be constructed in such a way that, for $0 \leq T \leq \epsilon |\log h|$

$$|\partial_\alpha q_I^T| \lesssim h^{-2|\alpha| c\epsilon}. \tag{9.3.7}$$

9.4. Construction of an approximate solution

9.4.1. The microlocal cut off

We will use the reflected-phase construction of [Ika88, Ika82] and [Bur93]. It is summed up in [Laf17b], let us recall that φ_J is the reflected phase obtained from φ after the story of reflexions J.

According to [Bur93] (remark 3.17') there exists $M > 0$ such that if $J \in \mathcal{I}$, $J = rI + l$ verifies $|J| \geq M$, and φ verifies (P), φ_J can be defined in $\mathcal{U}_{I,l}^\infty$. We choose $\delta > 0$ small enough so that, according to the construction of the previous section

$$D_{I,4\delta} \subset \bigcup_{|l| \leq |I|-1} \mathcal{U}_{I,l}^\infty,$$

moreover, we will take $T_0 \geq 2\beta_0 M$.

Let us recall that we are reduced to show the following dispersive estimate:

$$\| \sum_{I \text{ primitive}} Ae^{iTh\Delta}\psi(-h^2\Delta)A^\star Q_I^T \|_{L^1 \longrightarrow L^\infty} \lesssim (hT)^{-\frac{3}{2}}, \ \forall T_0 \le T \le \epsilon|\log h|.$$

For all primitive story I, let us define

$$\delta_I^y(x) = \frac{1}{(2\pi h)^3}\int e^{-i(x-y)\cdot\xi/h} p_{I,T}(x,\xi)d\xi,$$

where $p_{I,T}$ is the symbol associated with $P_I^T := \psi(-h^2\Delta)A^\star Q_I^T$. Then we have, for $u_0 \in L^2$

$$\psi(-h^2\Delta)A^\star Q_I^T u_0(x) = \int \delta_I^y(x)u_0(y)dy.$$

Then, by linearity of the flow

$$Ae^{ith\Delta}\psi(-h^2\Delta)A^\star Q_I^T u_0 = \int Ae^{ith\Delta}\delta_I^y u_0(y)dy,$$

and it therefore suffices to show that

$$\sum_{I \text{ primitive}} |Ae^{iTh\Delta}\delta_I^y(x)| \lesssim (hT)^{-3/2}, \ \forall T_0 \le T \le \epsilon|\log h|.$$

Finally, notice that as the operator A is bounded in $L^\infty \to L^\infty$ in the same way as in [Laf17b], it suffices only to show that

$$\sum_{I \text{ primitive}} |\chi e^{iTh\Delta}\delta_I^y(x)| \lesssim (hT)^{-3/2}, \ \forall T_0 \le T \le \epsilon|\log h|, \tag{9.4.1}$$

where $\chi \in C_c^\infty(\mathbb{R}^3)$ is supported in a neighborhood of the spatial projection of the support of ϕ and equal to one on it.

In order to do so, we will construct a parametrix, that is, an approximate solution, in time $0 \le t \le \epsilon|\log h|$ for the semi-classical Schrödinger equation with data δ_I^y. The first step will be to construct an approximate solution of the semi-classical Schrödinger equation with data

$$e^{-i(x-y)\cdot\xi/h}p_{I,T}(x,\xi)$$

where $\xi \in \mathbb{R}^n$ is fixed and considered as a parameter. Now that we are localized around a trajectory, the construction is exactly the same as in [Laf17b]. Let us sum it up briefly. In the sequel of this section, $p_{I,T}$ will be denoted p in the seek of conciseness.

9.4.2. Approximate solution

We look for the solution in positives times of the equation

$$\begin{cases} (i\partial_t w - h\Delta w) & = 0 \text{ in } \Omega \\ w(t=0)(x) & = e^{-i(x-y)\cdot\xi/h}p(x,\xi) \\ w_{|\partial\Omega} & = 0 \end{cases}$$

as the Neumann serie

$$w = \sum_{J\in\mathcal{I}} (-1)^{|J|} w^J$$

where

$$\begin{cases} (i\partial_t w^{\emptyset} - h\Delta w^{\emptyset}) & = 0 \text{ in } \mathbb{R}^n \\ w^{\emptyset}(t=0)(x) & = e^{-i(x-y)\cdot\xi/h} p(x,\xi) \end{cases}$$

and, for $J \neq \emptyset$, $J = (j_1, \cdots, j_n)$, $J' = (j_1, \cdots, j_{n-1})$

$$\begin{cases} (i\partial_t w^J - h\Delta w^J) & = 0 \text{ in } \mathbb{R}^n \backslash \Theta_{j_n} \\ w^J(t=0) & = 0 \\ w^J_{|\partial\Theta_{j_n}} & = w^{J'}_{|\partial\Theta_{j_n}}. \end{cases} \tag{9.4.2}$$

We will look for the w^J's as power series in h. In the sake of conciseness, these series will be considered at a formal level in this section, and we will introduce their expression as a finite sum plus a remainder later, in the last section.

We look for $w^{\emptyset}$ as

$$w^{\emptyset} = \sum_{k\geq 0} h^k w_k^{\emptyset} e^{-i((x-y)\cdot\xi - t\xi^2)/h},$$

$$w_0^{\emptyset}(t=0) = q(x,\xi), \;\; w_k^{\emptyset}(t=0) = 0.$$

Solving the transport equations gives immediately

$$\begin{aligned} w_0^{\emptyset} &= p(x - 2t\xi, \xi), \\ w_k^{\emptyset} &= -i \int_0^t \Delta w_{k-1}^{\emptyset}(x - 2(s-t)\xi, s) ds \quad k \geq 1. \end{aligned}$$

Now, starting from the phase $\varphi(x) = \frac{(x-y)\cdot\xi}{|\xi|}$, we define the reflected phases as before and we look for w^J as:

$$w^J = \sum_{k\geq 0} h^k w_k^J e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h},$$

$$w_k^J|_{t\leq 0} = 0, \;\; w^J_{k|\partial\Theta_{j_n}} = w^{J'}_{k|\partial\Theta_{j_n}}.$$

For $x \in \mathcal{U}_J(\varphi)$, we have

$$\begin{cases} (\partial_t + 2|\xi|\nabla\varphi_J\cdot\nabla + |\xi|\Delta\varphi_J) w_0^J & = 0 \\ w^J_{0|\Theta_{j_n}} & = w^{J'}_{0|\Theta_{j_n}} \\ w_0^J|_{t\leq 0} & = 0 \end{cases}$$

and

$$\begin{cases} (\partial_t + 2|\xi|\nabla\varphi_J\cdot\nabla + |\xi|\Delta\varphi_J) w_k^J & = -i\Delta w_{k-1}^J \\ w^J_{k|\Theta_{j_n}} & = w^{J'}_{k|\Theta_{j_n}} \\ w_k^J|_{t\leq 0} & = 0. \end{cases}$$

Solving the transport equations along the rays by the procedure explained in [Laf17b], we get the exact same following expressions of w_k^J for $x \in \mathcal{U}_J(\varphi)$:

Proposition 9.19. *We denote by $\hat{X}_{-2t}(x, |\xi|\nabla\varphi_J)$ the backward spatial component of the flow starting from $(x, |\xi|\nabla\varphi_J)$, defined in the same way as $X_{-2t}(x, |\xi|\nabla\varphi_J)$, at the difference that we ignore the first obstacle encountered if it's not Θ_{j_n}, and we ignore the obstacles after $|J|$ reflections. Moreover, for $J = (j_1, \dots, j_n) \in \mathcal{I}$, we denote by*

$$J(x,t,\xi) = \begin{cases} (j_1, \cdots, j_k) & \text{if } \hat{X}_{-2t}(x, |\xi|\nabla\varphi_J) \text{ has been reflected } n-k \text{ times,} \\ \emptyset & \text{if } \hat{X}_{-2t}(x, |\xi|\nabla\varphi_J) \text{ has been reflected } n \text{ times.} \end{cases}$$

Then, the w_k^J's are given by, for $t \geq 0$ and $x \in \mathcal{U}_J(\varphi)$

$$w_0^J(x,t) = \Lambda\varphi_J(x,\xi)p(\hat{X}_{-2t}(x, |\xi|\nabla\varphi_J), \xi)$$

where

$$\Lambda\varphi_J(x,\xi) = \left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x, |\xi|\nabla\varphi_J))}\right)^{1/2} \times \cdots \times \left(\frac{G\varphi(X^{-|J|-1}(x, |\xi|\nabla\varphi_J))}{G\varphi(X^{-|J|}(x, |\xi|\nabla\varphi_J))}\right)^{1/2},$$

and, for $k \geq 1$, and $x \in \mathcal{U}_J(\varphi)$

$$w_k^J(x,t) = -i\int_0^t g_{\varphi_J}(x, t-s, \xi)\Delta w_{k-1}^{J(x,\xi,t-s)}(\hat{X}_{-2(t-s)}(x, |\xi|\nabla\varphi_J), s)ds$$

where

$$g_{\varphi_J}(x,\xi,t) = \left(\frac{G\varphi_J(x)}{G\varphi_J(X^{-1}(x, |\xi|\nabla\varphi_J))}\right)^{1/2} \times \cdots \times \left(\frac{G\varphi_{J(x,t,\xi)}(X^{-|J(x,t,\xi)|-1}(x, |\xi|\nabla\varphi_J))}{G\varphi_{J(x,t,\xi)}(\hat{X}_{-2t}(x, |\xi|\nabla\varphi_J))}\right)^{1/2}.$$

And, by the same proof again as in [Laf17b] it implies in particular the following three results. The first of them is about the support of the solutions:

Lemma 9.20. *For $x \in \mathcal{U}_J(\varphi)$*

$$w_k^J(x,t) \neq 0 \implies (\hat{X}_{-2t}(x, |\xi|\nabla\varphi_J), \xi) \in Supp p. \tag{9.4.3}$$

And moreover

$$Supp w_k^J \subset \{J(x,\xi,t) = \emptyset\}. \tag{9.4.4}$$

It implies that we can extend it by zero outside the domains of definition of the phases:

Proposition 9.21. *For $x \notin \mathcal{U}_J(\varphi)$ and $0 \leq t \leq T$ we have $w_k^J(x,t) = 0$.*

And that the have $|J| \approx t$:

Lemma 9.22. *There exists $c_1, c_2 > 0$ such that for every $J \in \mathcal{I}$, the support of w_k^J is included in $\{c_1|J| \leq t\}$ and which of χw_k^J is included in $\{c_1|J| \leq t \leq c_2(|J|+1)\}$.*

Now, let us recall that $q = q_{I,T}$ where I is a given primitive trajectory. We have:

Lemma 9.23. *If J is not of the form $rI + l$, then $w_k^J = 0$ for $0 \leq t \leq \epsilon|\log h|$.*

Proof. If $w_k^J(x,\xi) \neq 0$, it follows from lemma 9.20 that there exists a broken ray joining $(x, |\xi|\nabla\varphi_J)$ and a point of the support of $p_{I,T}$ in time t following the complete story of reflexions J. By definition of the trapped set and because Supp$p \subset \mathcal{T}_T(D_{I,4\delta})$, this broken ray remains in a neighborhood of the trajectory γ corresponding to I, thus J can only be of the form $rI + l$. ☐

Finally, let us notice that

Lemma 9.24. *In times $0 \leq t \leq T$, for $J = rI + l$, χw_k^J is supported in $\mathcal{U}_{I,l}^\infty$.*

Proof. From (9.4.3), the support of w_k^J consists of the support of $q(.,\xi)$, transported along the billiard flow with initial direction ξ along the story of reflexion J and then ignoring the obstacles. Because of the non-shadow condition (9.1.4), the part ignoring the obstacles is cut off by χ, thus we obtain the result. ☐

9.4.3. The ξ derivatives

The following results about the directional derivatives of the phase and the solution has been proven in [Laf17b], where the proof does not involve the particular two obstacles geometry. The first one involves the critical points of the phase and its non-degeneracy:

Lemma 9.25. *Let $J \in \mathcal{I}$ and $\mathcal{S}_J(x,t,\xi) := \varphi_J(x,\xi)|\xi| - t\xi^2$. For all $t > 0$ and there exists at most one $s_J(x,t)$ such that $D_\xi \mathcal{S}_J(x,t,s_J(x,t)) = 0$. Moreover, for all $t_0 > 0$, there exists $c(t_0) > 0$ such that, for all $t \geq t_0$ and all $J \in \mathcal{I}$*

$$w^J(x,t,\xi) \neq 0 \implies |\det D_\xi^2 \mathcal{S}_J(x,t,\xi)| \geq c(t_0) > 0. \tag{9.4.5}$$

The last two permits to control the directional derivatives of the solutions:

Proposition 9.26. *For all multi-indices α, β there exists a constant $D_{\alpha,\beta} > 0$ such that the following estimate holds on $\mathcal{U}_{I,l}^\infty$:*

$$|D_\xi^\alpha D_x^\beta \nabla\varphi_J| \leq D_{\alpha,\beta}^{|J|}.$$

Corollary 9.27. *We following bounds hold on $\mathcal{U}_{I,l}^\infty$*

$$|D_\xi^\alpha w_k^J| \lesssim C_\alpha^{|J|} h^{-(2k+|\alpha|)c\epsilon}.$$

9.4.4. Decay of the reflected solutions

The principal result which permits us to estimate the decay of the reflected solutions is the convergence of the product of the Gaussian curvatures $\Lambda\varphi_J$ obtained by [Ika88, Ika82] and [Bur93]. It writes, in this setting

Proposition 9.28. *Let $0 < \lambda_I < 1$ be the square-root of the product of the two eigenvalues lesser than one of the Poincaré map associated with the periodic trajectory I. Then, there exists $0 < \alpha < 1$ and a C^∞ function $a_{I,l}$ defined in $\mathcal{U}_{I,l}^\infty$, such that, for all $J = rI + l$, we have*

$$\sup_{\mathcal{U}_{I,l}^\infty} |\Lambda\varphi_J - \lambda_I^r a_{I,l}|_m \leq C_m \lambda_I^r \alpha^{|J|}.$$

In the same way as in [Laf17b], it implies in particular:

Proposition 9.29. *If $J = rI + l$, where I is a primitive trajectory and $l \leq |I|$, then the following bounds hold on $\mathcal{U}_{I,l}^\infty$:*

$$|w_k^J|_m \leq C_k \lambda_I^{|J|} h^{-(2k+m)c\epsilon}.$$

Moreover, on the whole space, $|w_k^J|_m \leq C_k h^{-(2k+m)c\epsilon}$.

9.5. Proof of the main result

Let $K \geq 0$. By the previous section, the function

$$(x,t) \to \frac{1}{(2\pi h)^3} \sum_{J=rI+l} \int \sum_{k=0}^{K} h^k w_k^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

satisfies the approximate equation

$$\partial_t u - ih\Delta u = -ih^K \frac{1}{(2\pi h)^3} \sum_{J=rI+l} \int \Delta w_{K-1}^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

with data $\delta_{I,T}^y$. Because $e^{-i(t-s)h\Delta}$ is an H^m-isometry and by the Duhamel formula, the difference from the actual solution $e^{-ith\Delta}\delta^y$ is bounded in H^m norm by

$$C \times |t| \times h^{K-3} \times \sup_{t,\xi} \sum_{J=rI+l} \|\Delta w_{K-1}^J(\cdot,t,\xi) e^{-i(\varphi_J(\cdot,\xi)|\xi| - t\xi^2)/h}\|_{H^m}.$$

Therefore,

$$\sum_{I \text{ primitive}} e^{-ith\Delta}\delta_I^y(x) = S_K(x,t) + R_K(x,t) \tag{9.5.1}$$

with

$$S_K(x,t) = \frac{1}{(2\pi h)^3} \sum_{J\in\mathcal{I}} \int \sum_{k=0}^{K} h^k w_k^J(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi| - t\xi^2)/h} d\xi$$

and, for $0 \leq t \leq \epsilon |\log h|$

$$\|R_K(\cdot, t)\|_{H^m} \lesssim |\log h| h^{K-3} \sup_{t,\xi} \sum_{J \in \mathcal{I}} \|\Delta w^J_{K-1}(\cdot, t, \xi) e^{-i(\varphi_J(\cdot,\xi)|\xi| - t\xi^2)/h}\|_{H^m}, \tag{9.5.2}$$

where w^J_k is understood to be constructed from $p_{I,T}$ when $J = rI + l$.

The remainder

We first deal with the remainder term R_K. Let us denote

$$W^J_{K-1}(x,t) = \Delta w^J_{K-1}(\cdot, t, \xi) e^{-i(\varphi_J(\cdot,\xi)|\xi| - t\xi^2)/h}$$

Notice that, by construction of the w_k's, w^J_k is supported in a set of diameter $(C + \beta_0 t)$. Therefore, using Proposition 9.29 to control the derivatives coming from w_{K-1} and the estimate

$$|\nabla \varphi_J|_m \leq C_m |\nabla \varphi|_m$$

from [Ika88] to control the derivatives coming from the phase we get:

$$\|\partial^m W^J_{K-1}\|_{L^2} \lesssim C_K (1 + \beta_0 t)^{\frac{1}{2}} \|\partial^m W^J_{K-1}\|_{L^\infty} \lesssim C_K (1+t)^{\frac{1}{2}} h^{-m} \times h^{-(2K+m+2)c\epsilon}$$

and thus, by (9.5.2) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, for $0 \leq t \leq \epsilon |\log h|$

$$\|R_K\|_{L^\infty} \lesssim |\log h|^{\frac{3}{2}} h^{K(1-2c\epsilon)-5-4c\epsilon} |\left\{ J \in \mathcal{I},\ \text{s.t } w^J_{K-1} \neq 0 \right\}|. \tag{9.5.3}$$

Note that $w^J_{K-1}(t) \neq 0$ implies by lemma 9.22 that $|J| \leq c_1 t$, and $|\left\{ J \in \mathcal{I},\ \text{s.t } w^J_{K-1} \neq 0 \right\}|$ is bounded by the number of elements in

$$\alpha_{\lceil c_1 t \rceil}$$

where

$$\alpha_k = \{\text{sequences } s \text{ in } [\![1, N]\!] \text{ of length } \leq k \text{ s.t } s_{i+1} \neq s_i\}$$

But

Lemma 9.30. *The number of elements in α_k admits the bound*

$$|\alpha_k| \leq C_N N^k.$$

Proof. Let us denote

$$\beta_k = \{\text{sequences } s \text{ in } [\![1, N]\!] \text{ of lenght } k \text{ s.t } s_{i+1} \neq s_i\}.$$

We have

$$|\beta_1| = N$$

and

$$|\beta_{k+1}| = (N-1)|\beta_k|.$$

Therefore

$$|\beta_k| = N(N-1)^{k-1},\ |\alpha_k| = \sum_{i=1}^{k} \beta_i + 1 = N\frac{(N-1)^k - 1}{N-2} + 1,$$

and the bound holds. □

Thus

$$|\left\{J \in \mathcal{I},\ \text{s.t } w^J_{K-1} \neq 0\right\}| \lesssim N^t \tag{9.5.4}$$

and therefore, according to (9.5.3), for $0 \leq t \leq \epsilon|\log h|$

$$\begin{aligned}\|R_K\|_{L^\infty} &\lesssim C_K |\log h|^{\frac{3}{2}} h^{K(1-2c\epsilon)-5-4c\epsilon} h^{-\epsilon \log N} \\ &\lesssim C_K h^{K(1-2c\epsilon)-6-4c\epsilon-\epsilon\log N}.\end{aligned}$$

We take $\epsilon > 0$ small enough so that $2c\epsilon \leq \frac{1}{2}$ and $\epsilon \log N \leq 1$ in order to get

$$\|R_K\|_{L^\infty} \leq C_K h^{\frac{K}{2}-8}.$$

Let us fix $K = 15$. Then, $\|R_K\|_{L^\infty} \leq C_K h^{-\frac{1}{2}}$. Therefore, as $t \leq \epsilon|\log h|$ implies $h \leq e^{-\frac{t}{\epsilon}}$, we get

$$\|R_K\|_{L^\infty} \leq C_K h^{-\frac{3}{2}} e^{-\frac{t}{\epsilon}} \tag{9.5.5}$$

for $0 \leq t \leq \epsilon|\log h|$.

Times $t \geq t_0 > 0$

Let us now deal with the approximate solution S_K, K been fixed and x in Suppχ. Let $t_0 > 0$ to be chosen later. For $t \geq t_0$, by lemma 9.25 we can perform a stationary phase on each term of the J sum, up to order h. We obtain, for $t \geq t_0$

$$\begin{aligned}S_K(x,t) = \frac{1}{(2\pi h)^{3/2}} \sum_{J\in\mathcal{I}} e^{-i(\varphi_J(x,s_J(t,x))|s_J(t,x)|-ts_J(t,x)^2)/h} \left(w^J_0(t,x,s_J(t,x)) + h\tilde{w}^J_1(t,x)\right) \\ + \frac{1}{h^{3/2}} \sum_{J\in\mathcal{I}} R^J_{\text{st.ph.}}(x,t) + \frac{1}{(2\pi h)^3} \sum_{J\in\mathcal{I}} \int \sum_{k=2}^{K} h^k w^J_k(x,t,\xi) e^{-i(\varphi_J(x,\xi)|\xi|-t\xi^2)/h} d\xi\end{aligned} \tag{9.5.6}$$

where $s_J(t,x)$ is an eventual unique critical point of the phase (if it does not exist, the corresponding term is $O(h^\infty)$ and by (9.5.4) it does not contribute). The term $\tilde{w}^J_1$ is a linear combination of

$$D^2_\xi w^J_0(t,x,s_J(t,x)), w^J_1(t,x,s_J(t,x)),$$

and $R^J_{\text{st.ph.}}$ is the remainder involved in the stationary phase, who verifies (see for example to [Zwo12], Theorem 3.15)

$$|R^J_{\text{st.ph.}}(x,t)| \leq h^2 \sum_{|\alpha|\leq 7} \sup |D^\alpha_\xi w^J_k(x,\cdot,t)|. \tag{9.5.7}$$

We recall that by lemma 9.24, for $0 \leq t \leq \epsilon|\log h|$, χw_k^J is supported in $\mathcal{U}_{I,l}^{\infty}$. Therefore, for $0 \leq t \leq \epsilon|\log h|$ and all $0 \leq k \leq K-1$, we have, if $x \in \mathrm{Supp}\chi$, using the estimate of Proposition 9.29, because $w_k^J(x,\xi,\cdot)$ is supported in $\{c_1|J| \leq t \leq c_2(|J|+1)\}$ by lemma 9.22,

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} \sum_{\substack{J=rI+s \ | \ w_k^J\neq 0 \\ I \text{ primitive}, \ |s|\leq |I|-1}} \lambda_I^{|J|}.$$

Thus

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} \sum_{I \text{ primitive}} \sum_{\substack{r\geq 0 \\ 0\leq s\leq |I|-1}} \lambda_I^{\rho_k(I)+r}\lambda_I^s,$$

where we denoted

$$\rho_k(I) = \inf\left\{r \geq 1 \text{ s.t. } \exists s, \ w_k^{rI+s} \neq 0\right\},$$

and we get

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} \sum_{\substack{I \text{ primitive} \\ \rho_k(I)\neq\infty}} \frac{1}{1-\lambda_I}\lambda_I^{\rho_k(I)}|I|. \tag{9.5.8}$$

Moreover, as

$$\rho_k(I) \lesssim \frac{t}{|I|} \tag{9.5.9}$$

and, because as remarked in [Bur93], if γ is the trajectory associated to I

$$\frac{d_\gamma}{\mathrm{diam}\mathcal{C}} \leq \mathrm{card}\gamma = |I| \leq \frac{d_\gamma}{d_{\min}} \tag{9.5.10}$$

where $\mathcal{C}$ is the convex hull of $\cup\Theta_i$. Therefore, combining (9.5.8) with (9.5.9) and (9.5.10)

$$\sum_{J\in\mathcal{I}} |w_k^J| \lesssim C_k h^{-2kc\epsilon} \sum_{\gamma \text{ primitive}} d_\gamma \lambda_\gamma^{D_k \frac{t}{d_\gamma}}. \tag{9.5.11}$$

But, by Ikawa condition (9.1.3), there exists $\alpha > 0$ such that

$$\sum_{\gamma \text{ primitive}} d_\gamma \lambda_\gamma e^{\alpha d_\gamma} < \infty.$$

Let us denote

$$C_\gamma = \lambda_\gamma e^{\alpha d_\gamma}.$$

Notice that, because d_γ is bounded from below by $d_{\min}$ uniformly with respect to γ, we have a fortiori

$$\sum C_\gamma < \infty.$$

Therefore, all C_γ but a finite number are lesser than one. Reducing α if necessary and taking it small enough, we can thus assume that

$$0 \leq C_\gamma \leq 1, \ \forall\gamma.$$

Hence, for $t \geq \frac{d_{\min}}{D_k}$ we have

$$C_\gamma^{D\frac{t}{d_\gamma}} \leq C_\gamma,$$

thus, by (9.5.12), for $t \geq \frac{d_{\min}}{D_k}$

$$\begin{aligned}\sum_{J\in\mathcal{I}} |w_k^J| &\lesssim C_k h^{-2kc\epsilon} \sum_{\gamma \text{ primitive}} d_\gamma \left(C_\gamma e^{-\alpha d_\gamma}\right)^{D_k \frac{t}{d_\gamma}} \\ &\lesssim C_k h^{-2kc\epsilon} \sum_{\gamma \text{ primitive}} d_\gamma C_\gamma^{D\frac{t}{d_\gamma}} e^{-\alpha D_k t} \leq C_k h^{-2kc\epsilon} e^{-\alpha D_k t} \sum_{\gamma \text{ primitive}} d_\gamma C_\gamma,\end{aligned}$$

and hence, because of (9.1.3),

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-2kc\epsilon} e^{-\mu_k t} \text{ for } \frac{d_{\min}}{D_k} \leq t \leq \epsilon |\log h|. \tag{9.5.12}$$

for some $\mu_k > 0$. Now, remark that for $t \leq \frac{d_{\min}}{D}$, by (9.5.11) we have

$$\sum_{J\in\mathcal{I}} |w_k^J| \lesssim C_k h^{-2kc\epsilon} \sum_{\gamma \text{ primitive}} d_\gamma \lambda_\gamma$$

but because d_γ are bounded below, (9.1.3) implies a fortiori

$$\sum_{\gamma \text{ primitive}} d_\gamma \lambda_\gamma < \infty$$

and thus

$$\sum_{J\in\mathcal{I}} |w_k^J| \lesssim C_k h^{-2kc\epsilon} \text{ for } t_0 \leq t \leq \frac{d_{\min}}{D_k}. \tag{9.5.13}$$

Combining (9.5.12) and (9.5.13) we get

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C'_k h^{-2kc\epsilon} e^{-\mu_k t} \text{ for } t_0 \leq t \leq \epsilon |\log h|$$

Let us take $\epsilon > 0$ small enough so that $2Kc\epsilon \leq \frac{1}{2}$. We get, for $t_0 \leq t \leq \epsilon|\log h|$

$$\sum_{J\in\mathcal{I}} |w_k^J| \leq C_k h^{-\frac{1}{2}} e^{-\mu t},\ 1 \leq k \leq K-1, \tag{9.5.14}$$

$$\sum_{J\in\mathcal{I}} |w_0^J| \lesssim e^{-\mu t}. \tag{9.5.15}$$

with

$$\mu = \min_{0\leq k\leq K-1} \mu_k > 0.$$

Moreover, using (9.5.7) together with (9.5.4), lemma 9.22 and Corollary 9.27 we obtain, for $t \leq \epsilon|\log h|$

$$\sum_{J\in\mathcal{I}} |R^J_{\text{st.ph.}}(x,t)| \leq h^2 \sum_{J\in\mathcal{I}} \sum_{|\alpha|\leq 7} \sup |D^\alpha_\xi w^J_k(x,\cdot,t)|$$
$$\leq h^{2-(2K+7)c\epsilon} |\left\{J \in \mathcal{I},\ \text{s.t } w^J_{K-1} \neq 0\right\}| C^{\frac{t}{c_1}} \lesssim h^{2-(2K+7)c\epsilon} N^t C^{\frac{t}{c_1}}$$
$$\leq h^{2-(2K+7)c\epsilon} h^{-\eta\epsilon}$$

where $\eta > 0$ depends only of α_0, β_0, and the geometry of the obstacles. Therefore, choosing $\epsilon > 0$ small enough

$$\sum_{J\in\mathcal{I}} |R^J_{\text{st.ph.}}(x,t)| \lesssim h \leq e^{-t/\epsilon}. \tag{9.5.16}$$

for $t \leq \epsilon|\log h|$. In the same way we get, taking $\epsilon > 0$ small enough and $t \leq \epsilon|\log h|$

$$\sum_{J\in\mathcal{I}} |D^2_\xi w^J_0| \lesssim N^t C^{\frac{t}{c_1}} \lesssim h^{-1/4}$$

and therefore

$$\sum_{J\in\mathcal{I}} |D^2_\xi w^J_0| \leq h^{-\frac{1}{2}} e^{-t/4\epsilon}. \tag{9.5.17}$$

So, combining (9.5.14), (9.5.15), (9.5.16) and (9.5.17) with (9.5.6), we obtain, for some $\nu > 0$

$$|\chi S_K(x,t)| \lesssim \frac{e^{-\nu t}}{h^{3/2}} \quad \text{for } t_0 \leq t \leq T. \tag{9.5.18}$$

Conclusion

Combining the above estimate (9.5.18) with the control of the remainder term (9.5.5) and taking $t = T$ gives (9.4.1) and therefore the dispersive estimate (9.3.6). By the work of reduction of the third section and summed up in lemma 9.18, Theorem 9.3 is therefore demonstrated for the Schrödinger equation.

9.6. The wave equation

In the case of the wave equation, the counterpart of the smoothing estimate without loss outside the trapped set, namely the following L^2- decay of the local energy

$$\|(Au, A\partial_t u)\|_{L^2(\mathbb{R},\dot{H}^\gamma\times\dot{H}^{\gamma-1})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^\gamma}, \tag{9.6.1}$$

where A has micro-support disjoint from $\mathcal{K}$, is obtained using the same commutator argument, writing in the case of the wave equation as

$$0 = \int\int_{\mathbb{R}\times\Omega} \langle u, [\Box, P]u\rangle + \int\int_{\mathbb{R}\times\partial\Omega} \langle Pu, \partial_n u\rangle,$$

where P is any pseudo-differential operator. Notice that the symbol of P at the border, as an operator acting on waves, has been computed in $\{\tau^2 - \eta^2 > 0\}$ by [MRS77]. Our method apply in the exact same way as for the Schrödinger equation.

Once (9.6.1) is obtained, it follows as in [Laf17a] that we can reduce ourselves to prove the Strichartz estimates near the trapped set in logarithmic times, namely

$$\|\mathrm{Op}_h(\phi)u\|_{L^q(\epsilon|\log h|, L^r(\Omega))} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}$$

where $u_{0,1} = \psi(-h^2\Delta)u_{0,1}$ and ϕ is supported in a small neighborhood of $\mathcal{K}$. In order to reduce ourselves at points of the phase-space that remain near a periodic trajectory in logarithmic times, the exact same decomposition as in the third section holds, at the difference that the flow is followed at constant speed one.

Then, the construction of an approximate solution is the same as in [Laf17a], with the adaptations of the N-convex framework presented in the fourth section. In particular, the results of non-degeneracy of the phase and stationary points of [Laf17a] hold, as their proof does not rely on the particular two-convex geometry. Thus, we can perform the same stationary phase argument as in [Laf17a], the difference with the Schrödinger equation been that the phase is now stationary on plain lines due to the constant speed of propagation, and we obtain the good scale in h. Now, the only difference with the conclusion section of [Laf17a] is that we cannot deal with

$$\sum_{J \in \mathcal{I}}$$

as in the two convex case. But we can do it in the exact same way as presented in the fifth section, using the strong hyperbolic setting assumption (9.1.3), in order to deduce the sufficient time decay. Thus the appropriate dispersive estimate for the waves is obtained and the theorem follows.

Part IV.

Scattering for nonlinear equations

10 Scattering for NLS with a potential on the line

10.1. Introduction

We consider the following one dimensional defocusing, non linear Schrödinger equation with a potential

$$i\partial_t u + \Delta u - Vu = u|u|^\alpha,\ u(0) = \varphi \in H^1(\mathbb{R}). \tag{10.1.1}$$

If $V \in L^1$, $-\Delta + V$ is essentially self-adjoint, so by Stone's theorem the linear equation in globally well posed in $L^2(\mathbb{R})$ and $e^{it(-\Delta+V)}$ is an L^2-isometry. Goldberg and Schlag obtained in [GS04] the dispersive estimate

$$\|e^{-it(-\Delta+V)}\psi\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{1}{2}}}\|\psi\|_{L^1}$$

under the assumption that V belongs to $L^1_1(\mathbb{R})$, ie $\int_{-\infty}^{\infty}|V(x)|(1+|x|)dx < \infty$, and that $-\Delta+V$ has no resonance at zero energy. This estimate gives us usual Strichartz estimates described below in the paper, which are sufficient to show that the equation (10.1.1) is locally well posed in $L^2(\mathbb{R})$ by an usual fixed point method. Because of the energy conservation law

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 + \int V|u(t)|^2 + \frac{1}{\alpha+2}\int |u(t)|^{\alpha+2} = E(u(0))$$

this result extends to the global well-posedness of the problem (10.1.1) in $H^1(\mathbb{R})$: for every $\varphi \in H^1(\mathbb{R})$, there exists a unique, global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (10.1.1). Finally, let us recall that the mass $M(u(t)) := \int |u(t)|^2$ is conserved too.

For the mass-supercritical ($\alpha > 4$) homogenous equation

$$i\partial_t u + \Delta u = u|u|^\alpha,\ u(0) = \varphi \in H^1(\mathbb{R}) \tag{10.1.2}$$

it is well known since Nakanishi's paper [Nak99] that the solutions *scatter* in $H^1(\mathbb{R})$, that is, for every solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (10.1.2), there exists a unique couple of data $\psi_\pm \in H^1(\mathbb{R})$ such that

$$\|u(t) - e^{-it\Delta}\psi_\pm\|_{H^1(\mathbb{R})} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

Alternative proofs of this result can be found in [PV09], [CGT09], [FXC11] and [Vis09].

We prove the scattering of solutions of (10.1.1) in dimension one for sufficiently regular, repulsive potential V.

Theorem 10.1. *Let $\alpha > 4$ and $V \in L^1_1(\mathbb{R})$ be such that $V' \in L^1_1(\mathbb{R})$. We suppose moreover that V is positive and repulsive: $V \geq 0$ and $xV' \leq 0$. Then, every solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (10.1.1) with potential V scatters in $H^1(\mathbb{R})$.*

We use the strategy of concentration-compactness/rigidity first introduced by Kenig and Merle in [KM06], and extended to the intercritical case by Holmer and Roudenko in [HR08], Duyckaerts, Holmer and Roudenko in [DHR08]. In the case of a potential, the main difficulty is the lack of translation invariance of the equation. Notice that Hong obtained in [Hon14] the same result in the three dimensional case for the focusing equation. However, his approach cannot be extended to lower dimensions, as it requires endpoint Strichartz estimates which are not available. Banica and Visciglia treated in [BV16] the case of the non linear Schrödinger equation with a Dirac potential on the line, and we follow their approach. The Dirac potential is more singular, but it allows the use of explicit formulas that are not available in the present more general framework.

Remark 10.2. In dimension one or two, assume that V is smooth and compactly supported, and such that $\int V < 0$. Then the operator $-\Delta + V$ has a negative eigenvalue: as a consequence, the hypothesis of positivity of V cannot be relaxed as in dimension three, where [Hon14] only supposes that the potential has a small negative part, and, in the same way, the hypothesis of repulsivity, which is needed for the rigidity, cannot be relaxed to xV' having a small positive part.

Remark 10.3. The hypothesis $V, V' \in L^1$ are needed to show that the operator $A = -\Delta + V$ verifies the hypothesis of the abstract profile decomposition of [BV16], whereas the hypothesis $xV' \in L^1$ and $xV' \leq 0$ are needed in the rigidity part.

Remark 10.4. The same proof holds in dimension two up to the numerology and some changes in the Hölder inequalities used in Propositions 10.11, 10.12, and 10.13 to deal with the fact that $H^1(\mathbb{R}^2)$ is not embedded in $L^\infty(\mathbb{R}^2)$.

Remark 10.5. In the focusing, mass-supercritical case

$$i\partial_t u + \Delta u - Vu + u|u|^\alpha = 0$$

the same arguments could be used to prove the scattering up to the natural threshold given by the ground state associated to the equation, in the spirit of [FXC11].

Notations.

We will denote by V a potential on the line satisfying the hypothesis of theorem 10.1, α will be a real number such that $\alpha > 4$. We set

$$H^1 = H^1(\mathbb{R}),\ C(H^1) = C(\mathbb{R}, H^1(\mathbb{R})),\ L^pL^r = L^p(\mathbb{R}, L^r(\mathbb{R})),\ L^p(I)L^r = L^p(I, L^r(\mathbb{R}))$$

for any interval I of $\mathbb{R}$. We will denote by τ_y the translation operator defined by $\tau_y u = u(\cdot - y)$. Finally, we will use $A \lesssim B$ for inequalities of the type $A \leq CB$ where C is a universal constant.

10.2. Preliminaries

From now on, we will fix the four following Strichartz exponents

$$r = \alpha + 2,\ q = \frac{2\alpha(\alpha+2)}{\alpha^2 - \alpha - 4},\ p = \frac{2\alpha(\alpha+2)}{\alpha+4},\ \gamma = \frac{2\alpha}{\alpha - 2}$$

10.2.1. Strichartz estimates

Recall that we assume all along the paper that V is in $L^1_1(\mathbb{R})$ and non negative. Goldberg and Schlag obtained in particular in [GS04] the dispersive estimate for the Schrödinger operator $-\Delta + V$ under these assumptions.

Indeed, they require the hypothesis of absence of resonances at zero energy. We claim that for $V \geq 0$ this hypothesis is satisfied: by the definition of [GS04], if there is a resonance at zero, the solutions $u_\pm$ of

$$u'' = Vu \tag{10.2.1}$$

such that $u_\pm(x) \to 1$ as $x \to \pm\infty$ have a null Wronskian. Therefore $u_\pm$ are proportional, so they are both non trivial bounded solutions of (10.2.1). But such solutions cannot exist: indeed, if u is such a solution, integrating (10.2.1) one deduces that u' has limits at $\pm\infty$. These limits are both zero otherwise u is not bounded. Now, multiplying (10.2.1) by u, integrating it on $[-R, R]$, and letting R going to infinity, we obtain $\int_{\mathbb{R}} |u'|^2 + V|u|^2 = 0$. Therefore $u = 0$, a contradiction.

Proposition 10.6 (Dispersive estimate, [GS04]). *Let $V \in L^1_1(\mathbb{R})$ be such that $V \geq 0$. Then, for all $\psi \in L^1(\mathbb{R})$, we have*

$$\|e^{-it(-\Delta+V)}\psi\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{1}{2}}}\|\psi\|_{L^1}. \tag{10.2.2}$$

Note that, interpolating the previous dispersive estimate (10.2.2) with the mass conservation law, we obtain immediately for all $a \in [2, \infty]$

$$\|e^{it(-\Delta+V)}\psi\|_{L^a} \lesssim \frac{1}{|t|^{\frac{1}{2}(\frac{1}{a'} - \frac{1}{a})}}\|\psi\|_{L^{a'}}. \tag{10.2.3}$$

Because of (10.2.2), we obtain by the classical $TT^\star$ method (see for example [KT98]) the Strichartz estimates

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^{q_1}L^{r_1}} + \|\int_0^t e^{-i(t-s)(-\Delta+V)}F(s)ds\|_{L^{q_2}L^{r_2}} \lesssim \|\varphi\|_{L^2} + \|F\|_{L^{q_3'}L^{r_3'}} \tag{10.2.4}$$

for all pairs (q_i, r_i) satisfying the admissibility condition in dimension one, that is

$$\frac{2}{q_i} + \frac{1}{r_i} = \frac{1}{2}.$$

We will need moreover the following Strichartz estimates associated to non admissible pairs:

Proposition 10.7 (Strichartz estimates). *For all $\varphi \in H^1$, all $F \in L^{q'}L^{r'}$, all $G \in L^{q'}L^{r'}$ and all $H \in L^{\gamma'}L^1$*

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^pL^r} \lesssim \|\varphi\|_{H^1} \tag{10.2.5}$$

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^\alpha L^\infty} \lesssim \|\varphi\|_{H^1} \tag{10.2.6}$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}F(s)ds\|_{L^\alpha L^\infty} \lesssim \|F\|_{L^{q'}L^{r'}} \tag{10.2.7}$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}G(s)ds\|_{L^pL^r} \lesssim \|G\|_{L^{q'}L^{r'}} \tag{10.2.8}$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}H(s)ds\|_{L^pL^r} \lesssim \|H\|_{L^{\gamma'}L^1}. \tag{10.2.9}$$

Proof. The estimates (10.2.5) − (10.2.8) are exactly the same as (3.1) − (3.4) of [Vis09], with the operator $-\Delta + V$ instead of H_q. As the proof of [Vis09] relies only on the admissible Strichartz estimates (10.2.4) that are given by Proposition 1, the same proof holds here. Finally, (10.2.9) enters on the frame of the non-admissible inhomogeneous Strichartz estimates of Theorem 1.4 of Foschi's paper [Fos05]. □

10.2.2. Perturbative results

We will need the three following classical perturbative results, which follow immediately from the previous Strichartz inequalities:

Proposition 10.8. *Let $u \in C(H^1)$ be a solution of (10.1.1). If $u \in L^pL^r$, then u scatters in H^1.*

Proposition 10.9. *There exists $\epsilon_0 > 0$, such that, for every data $\varphi \in H^1$ such that $\|\varphi\|_{H^1} \leq \epsilon_0$, the corresponding maximal solutions of (10.1.1) and (10.1.2) both scatter in H^1.*

Proof of Propositions 10.8 and 10.9. The proof is the same as for Propositions 3.1 and 3.2 of [BV16], using the Strichartz estimates of our Proposition 10.7 instead of their estimates (3.1), (3.2), (3.3), (3.4). □

Proposition 10.10. *For every $M > 0$ there exists $\epsilon > 0$ and $C > 0$ such that the following occurs. Let $v \in C(H^1) \cap L^pL^r$ be a solution of the following integral equation with source term $e(t,x)$*

$$v(t) = e^{-it(\Delta - V)}\varphi - i\int_0^t e^{-i(t-s)(\Delta-V)}(v(s)|v(s)|^\alpha)ds + e(t)$$

with $\|v\|_{L^pL^r} < M$ and $\|e\|_{L^pL^r} < \epsilon$. Assume moreover that $\varphi_0 \in H^1$ is such that $\|e^{-it(\Delta-V)}\varphi_0\|_{L^pL^r} < \epsilon$. Then, the solution $u \in C(H^1)$ to (10.1.1) with initial condition $\varphi + \varphi_0$ satisfies

$$u \in L^pL^r, \quad \|u - v\|_{L^pL^r} < C.$$

Proof. It is the same as for Proposition 4.7 in [FXC11], using Strichartz estimates (10.2.8) instead of Strichartz-type inequality (4.3) of their paper. □

10.3. Profile decomposition

The aim of this section is to show that we can use the abstract profile decomposition obtained by [BV16], and inspired by [FV13]:

Theorem (Abstract profile decomposition, [BV16])**.** *Let $A : L^2 \supset D(A) \to L^2$ be a self adjoint operator such that:*

- *for some positive constants c, C and for all $u \in D(A)$,*

$$c\|u\|^2_{H^1} \le (Au, u) + \|u\|^2_{L^2} \le C\|u\|^2_{H^1}, \tag{10.3.1}$$

- *let $B : D(A) \times D(A) \ni (u, v) \to (Au, v) + (u, v)_{L^2} - (u, v)_{H^1} \in \mathbb{C}$. Then, as n goes to infinity*

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0 \quad \forall\psi \in H^1 \tag{10.3.2}$$

  as soon as

$$x_n \to \pm\infty, \quad \sup\|h_n\|_{H^1} < \infty$$

  or

$$x_n \to \bar{x} \in \mathbb{R}, \quad h_n \underset{H^1}{\rightharpoonup} 0,$$

- *let $(t_n)_{n\ge1}$, $(x_n)_{n\ge1}$ be sequences of real numbers, and $\bar{t}, \bar{x} \in \mathbb{R}$. Then*

$$|t_n| \to \infty \Longrightarrow \|e^{it_nA}\tau_{x_n}\psi\|_{L^p} \to 0, \quad \forall 2 < p < \infty, \ \forall\psi \in H^1 \tag{10.3.3}$$

$$t_n \to \bar{t}, \ x_n \to \pm\infty \Longrightarrow \forall\psi \in H^1, \exists\varphi \in H^1, \quad \tau_{-x_n}e^{it_nA}\tau_{x_n}\psi \overset{H^1}{\to} \varphi \tag{10.3.4}$$

$$t_n \to \bar{t}, \ x_n \to \bar{x} \Longrightarrow \forall\psi \in H^1, \quad e^{it_nA}\tau_{x_n}\psi \overset{H^1}{\to} e^{i\bar{t}A}\tau_{\bar{x}}\psi. \tag{10.3.5}$$

And let $(u_n)_{n\geq 1}$ *be a bounded sequence in* H^1*. Then, up to a subsequence, the following decomposition holds*

$$u_n = \sum_{j=1}^{J} e^{it_j^n A}\tau_{x_n^j}\psi_j + R_n^J \quad \forall J \in \mathbb{N}$$

where

$$t_j^n \in \mathbb{R},\ x_j^n \in \mathbb{R},\ \psi_j \in H^1$$

are such that

- *for any fixed* j*,*

$$t_j^n = 0\ \forall n, \quad or \quad t_n^j \overset{n\to\infty}{\to} \pm\infty \tag{10.3.6}$$

$$x_j^n = 0\ \forall n, \quad or \quad x_n^j \overset{n\to\infty}{\to} \pm\infty, \tag{10.3.7}$$

- *orthogonality of the parameters:*

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \overset{n\to\infty}{\to} \infty, \quad \forall j \neq k, \tag{10.3.8}$$

- *decay of the remainder:*

$$\forall \epsilon > 0, \exists J \in \mathbb{N}, \quad \limsup_{n\to\infty} \|e^{-itA}R_n^J\|_{L^\infty L^\infty} \leq \epsilon, \tag{10.3.9}$$

- *orthogonality of the Hilbert norm:*

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^2}^2 + \|R_n^J\|_{L^2}^2 + o_n(1), \quad \forall J \in \mathbb{N} \tag{10.3.10}$$

$$\|u_n\|_{H}^2 = \sum_{j=1}^{J} \|\tau_{x_n^j}\psi_j\|_{H}^2 + \|R_n^J\|_{H}^2 + o_n(1), \quad \forall J \in \mathbb{N} \tag{10.3.11}$$

where $(u,v)_H = (Au, v)$*, and*

$$\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \|e^{it_j^n A}\tau_{x_n^j}\psi_j\|_{L^p}^p + \|R_n^J\|_{L^p}^p + o_n(1), \quad \forall 2 < p < \infty, \quad \forall J \in \mathbb{N}. \tag{10.3.12}$$

We will see that the self-adjoint operator $A := -\Delta + V$ verifies the hypothesis of the previous theorem.

Proposition 10.11. *Let* $A := -\Delta + V$*. Then* A *satisfies the assumptions (10.3.1), (10.3.2),(10.3.3),(10.3.4),(10.3.5).*

Proof. **Assumption (10.3.1).** Because V is positive and by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty$,

$$\|u\|_{H^1}^2 \leq (Au, u) + \|u\|_{L^2} = \int |\nabla u|^2 + \int V|u|^2 + \int |u|^2 \leq (1 + \|V\|_{L^1})\|u\|_{H^1}^2$$

and (10.3.1) holds.

Assumption (10.3.2). We have

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) = \int V\tau_{x_n}\psi\overline{\tau_{x_n}h_n}.$$

If $x_n \to \bar{x} \in \mathbb{R}$, $h_n \underset{H^1}{\rightharpoonup} 0$, then $\tau_{x_n}\psi \to \tau_{\bar{x}}\psi$ strongly in L^2 and $V\tau_{x_n}h_n \rightharpoonup 0$ weakly in L^2 (indeed, note that $V \in W^{1,1}(\mathbb{R}) \hookrightarrow L^2$), so $B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0$. Now, let us assume that $x_n \to \pm\infty$ and $\sup \|h_n\|_{H^1} < \infty$. For example assume that $x_n \to +\infty$. $\psi \in H^1(\mathbb{R})$ and therefore decays at infinity: $\epsilon > 0$ been fixed, we can choose $\Lambda > 0$ large enough so that

$$\sup_{|x|\geq\Lambda} |\psi(x)| \leq \epsilon.$$

Because $V \in L^1$, Λ can also be chosen large enough so that

$$\int_{|x|\geq\Lambda} |V| \leq \epsilon.$$

Then, by the Cauchy-Schwarz inequality, and because of the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty$

$$\begin{aligned} |B(\tau_{x_n}\psi, \tau_{x_n}h_n)| &\leq \|h_n\|_{L^\infty} \int |V\tau_{x_n}\psi| \\ &\leq \sup_{j\geq 1} \|h_j\|_{H^1} \left(\int_{|x-x_n|\geq\Lambda} |V\psi(\cdot - x_n)| + \int_{|x-x_n|\leq\Lambda} |V\psi(\cdot - x_n)| \right). \end{aligned}$$

Now, let n_0 be large enough so that for all $n \geq n_0$, $x_n \geq 2\Lambda$. Then, for all $n \geq n_0$

$$|x - x_n| \leq \Lambda \Rightarrow |x| \geq \Lambda$$

and, for all $n \geq n_0$

$$|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \leq M\left(\epsilon\|V\|_{L^1} + \epsilon\|\psi\|_{L^\infty}\right)$$

so (10.3.2) holds.

Assumption (10.3.3). It is an immediate consequence of the dispersive estimate and the translation invariance of the L^p norms. Indeed, because $H_0^1(\mathbb{R}) = H^1(\mathbb{R})$, if $\epsilon > 0$, there exists a C^∞, compactly supported function $\tilde{\psi}$ such that

$$\|\tilde{\psi} - \psi\|_{H^1} \leq \epsilon. \tag{10.3.13}$$

But $\tilde{\psi} \in L^{p'}$, so by the dispersive estimate (10.2.3)

$$\|e^{it_nA}\tau_{x_n}\tilde{\psi}\|_{L^p} \lesssim \frac{1}{|t_n|^{\frac{1}{2}(\frac{1}{p'}-\frac{1}{p})}}\|\tau_{x_n}\tilde{\psi}\|_{L^{p'}} = \frac{1}{|t_n|^{\frac{1}{2}(\frac{1}{p'}-\frac{1}{p})}}\|\tilde{\psi}\|_{L^{p'}} \to 0$$

as $n \to \infty$. Therefore, for n big enough

$$\|e^{it_nA}\tau_{x_n}\tilde{\psi}\|_{L^p} \leq \epsilon. \tag{10.3.14}$$

To achieve the proof, note that $e^{itA}f$ verifies

$$\|e^{itA}f\|_{H^1} \lesssim \|f\|_{H^1}. \tag{10.3.15}$$

Indeed, as V is positive and in L^1, by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty$ we get

$$\|\nabla f\|_{L^2}^2 \leq \|(-\Delta+V)^{\frac{1}{2}}f\|_{L^2}^2 = \int |\nabla u|^2 + \int V|u|^2 \lesssim \|f\|_{H^1}.$$

So, as e^{itA} commute with $(-\Delta+V)^{\frac{1}{2}}$ and is an isometry on L^2,

$$\begin{aligned}\|e^{itA}f\|_{H^1}^2 &\leq \|e^{itA}f\|_{L^2}^2 + \|(-\Delta+V)^{\frac{1}{2}}e^{itA}f\|_{L^2}^2 \\ &= \|e^{itA}f\|_{L^2}^2 + \|e^{itA}(-\Delta+V)^{\frac{1}{2}}f\|_{L^2}^2 \\ &= \|f\|_{L^2}^2 + \|(-\Delta+V)^{\frac{1}{2}}f\|_{L^2}^2 \lesssim \|f\|_{H^1}^2.\end{aligned}$$

Now, because of the Sobolev embedding $H^1 \hookrightarrow L^p$ we obtain using (10.3.13), (10.3.14) and (10.3.15), for n big enough

$$\begin{aligned}\|e^{it_nA}\tau_{x_n}\psi\|_{L^p} &\leq \|e^{it_nA}\tau_{x_n}(\psi-\tilde{\psi})\|_{L^p} + \|e^{it_nA}\tau_{x_n}\tilde{\psi}\|_{L^p} \\ &\lesssim \|e^{it_nA}\tau_{x_n}(\psi-\tilde{\psi})\|_{H^1} + \|e^{it_nA}\tau_{x_n}\tilde{\psi}\|_{L^p} \\ &\lesssim \|\psi-\tilde{\psi}\|_{H^1} + \|e^{it_nA}\tau_{x_n}\tilde{\psi}\|_{L^p} \leq 2\epsilon\end{aligned}$$

which achieves the proof of (10.3.3).

Assumption (10.3.4). We will show that

$$t_n \to \bar{t},\ x_n \to \pm\infty \ \Rightarrow\ \|\tau_{-x_n}e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-i\bar{t}\Delta}\psi\|_{H^1} \to 0$$

and hence (10.3.4) will hold with $\varphi = e^{-i\bar{t}\Delta}\psi$. As τ_{x_n} is an H^1 isometry and commute with $e^{-i\bar{t}\Delta}$, it is sufficient to show that, if $t_n \to \bar{t}$ and $x_n \to \pm\infty$, we have

$$\|e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-i\bar{t}\Delta}\tau_{x_n}\psi\|_{H^1} \to 0.$$

For example, if $x_n \to +\infty$. Let us first remark that, as τ_{x_n} commutes with $e^{-i\bar{t}\Delta}$ and $e^{-it_n\Delta}$, is an H^1 isometry, and because $e^{-it\Delta}\psi \in C(H^1)$

$$\|e^{-i\bar{t}\Delta}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\|_{H^1} = \|e^{-i\bar{t}\Delta}\psi - e^{-it_n\Delta}\psi\|_{H^1} \to 0.$$

Hence, decomposing

$$e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-i\bar{t}\Delta}\tau_{x_n}\psi = \left(e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\right) + \left(e^{-it_n\Delta}\tau_{x_n}\psi - e^{-i\bar{t}\Delta}\tau_{x_n}\psi\right)$$

we see that it is sufficient to show that

$$\|e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\|_{H^1} \to 0. \tag{10.3.16}$$

Note that $e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$i\partial_t u - \Delta u + Vu = Ve^{-it\Delta}\tau_{x_n}\psi.$$

Therefore, by the inhomogeneous Strichartz estimates, as $(4,\infty)$ is admissible in dimension one, and because the translation operator commutes with $e^{-it\Delta}$, we have for n large enough so that $t_n \in (0, \bar{t}+1)$

$$\begin{aligned}
&\|e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\|_{L^2} \\
&\quad \leq \|e^{it(-\Delta+V)}\tau_{x_n}\psi - e^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,\bar{t}+1)L^2} \leq \|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^{\frac{4}{3}}(0,\bar{t}+1)L^1} \\
&\quad = \|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^{\frac{4}{3}}(0,\bar{t}+1)L^1} \leq (\bar{t}+1)^{\frac{3}{4}}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^1}.
\end{aligned}$$

Hence, estimating in the same manner the gradient of these quantities, it is sufficient to obtain (10.3.16) to show that, as n goes to infinity

$$\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)W^{1,1}} \to 0. \tag{10.3.17}$$

Let us fix $\epsilon > 0$. $e^{-it\Delta}\psi \in C([0,\bar{t}+1], H^1)$ and the functions of $H^1(\mathbb{R})$ vanish at infinity, so, using the compactness in time, there exists $\Lambda > 0$ such that

$$\|e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\infty(|x|\geq\Lambda)} \leq \epsilon.$$

On the other hand, as $V \in L^1$, Λ can also be taken large enough so that

$$\int_{|x|\geq\Lambda} |V(x)|dx \leq \epsilon.$$

Let n_0 be large enough so that for all $n \geq n_0$, $x_n \geq 2\Lambda$. Then, for $n \geq n_0$

$$|x + x_n| \leq \Lambda \Rightarrow |x| \geq \Lambda$$

and for all $t \in (0, \bar{t}+1)$ and all $n \geq n_0$ we obtain

$$\begin{aligned}
\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^1} &= \int_{|x+x_n|\geq\Lambda} |V(\cdot + x_n)e^{-it\Delta}\psi| + \int_{|x+x_n|\leq\Lambda} |V(\cdot + x_n)e^{-it\Delta}\psi| \\
&\leq \epsilon\|e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\infty} + \epsilon\|V\|_{L^1} \\
&\leq C(\bar{t}, \psi, V)\epsilon
\end{aligned}$$

thus $\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^1} \to 0$. With the same argument, because $V' \in L^1$, we can show that $\|(\tau_{-x_n}V)'e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^1} \to 0$. To obtain (10.3.17), it only remain to show that

$$\|\tau_{-x_n}V(e^{-it\Delta}\psi)'\|_{L^\infty(0,\bar{t}+1)L^1} \to 0.$$

To this purpose, let $\tilde{\psi}$ be a C^∞, compactly supported function such that (recall that we are in dimension one)

$$\|\psi - \tilde{\psi}\|_{H^1} \leq \epsilon.$$

We have, by the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\tau_{-x_n}V(e^{-it\Delta}\psi)'\|_{L^1} &\leq \|\tau_{-x_n}V(e^{-it\Delta}\tilde{\psi})'\|_{L^1} + \|\tau_{-x_n}V(e^{-it\Delta}(\psi-\tilde{\psi}))'\|_{L^1} \\
&\leq \|\tau_{-x_n}V(e^{-it\Delta}\tilde{\psi})'\|_{L^1} + \|V\|_{L^2}\|(e^{-it\Delta}(\psi-\tilde{\psi}))'\|_{L^2} \\
&\leq \|\tau_{-x_n}V(e^{-it\Delta}\tilde{\psi})'\|_{L^1} + \epsilon\|V\|_{L^2}
\end{aligned}$$

where $V \in L^2$ because of the Sobolev embedding $W^{1,1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$. Then, as $(e^{-it\Delta}\tilde{\psi})' \in H^1$, $\|\tau_{-x_n}V(e^{-it\Delta}\tilde{\psi})'\|_{L^\infty(0,\bar{t}+1)L^1}$ can be estimated as $\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^1}$, so (10.3.17) holds and the proof of (10.3.4) is completed.

Assumption (10.3.5). We decompose

$$e^{it_nA}\tau_{x_n}\psi - e^{i\bar{t}A}\tau_{\bar{x}}\psi = (e^{it_nA}\tau_{x_n}\psi - e^{it_nA}\tau_{\bar{x}}\psi) + (e^{it_nA}\tau_{\bar{x}}\psi - e^{i\bar{t}A}\tau_{\bar{x}}\psi).$$

From one side, using the estimate (10.3.15)

$$\|e^{it_nA}\tau_{x_n}\psi - e^{it_nA}\tau_{\bar{x}}\psi\|_{H^1} \lesssim \|\tau_{x_n}\psi - \tau_{\bar{x}}\psi\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0$$

by the Lebesgue's dominated convergence theorem. From the other side,

$$\|e^{it_nA}\tau_{\bar{x}}\psi - e^{i\bar{t}A}\tau_{\bar{x}}\psi\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0$$

because $e^{i\cdot A}\tau_{\bar{x}}\psi \in C(H^1)$, and the last assumption is verified. □

10.4. Non linear profiles

In this section, we will see that for data which escape to infinity, the solutions of (10.1.1) and (10.1.2) are the same, in the sense given by the three following Propositions.

Propositions 10.12, 10.13 and 10.14 are the analogous of Propositions 3.4 and 3.6 of [BV16]. The non linear Schrödinger equation with a Dirac potential is more singular, but it allows the use of explicit formulas that are not available in the present more general framework.

Proposition 10.12. *Let $\psi \in H^1$, $(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$. Then, up to a subsequence*

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{-it(\Delta-V)}\tau_{x_n}\psi\|_{L^pL^r} \to 0 \tag{10.4.1}$$

as $n \to \infty$.

Proof. Up to a subsequence, we can assume that $x_n \to +\infty$ or $x_n \to -\infty$. Let us assume for example $x_n \to +\infty$.

As a first step, we will show that

$$\sup_{n\in\mathbb{N}} \|e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(T,\infty)L^r} \to 0 \tag{10.4.2}$$

as $T \to \infty$. Pick $\epsilon > 0$. There exists a C^∞, compactly supported function $\tilde{\psi}$ such that

$$\|\tilde{\psi} - \psi\|_{H^1} \leq \epsilon.$$

By Strichartz estimates

$$\|e^{it(-\Delta+V)}(\tau_{x_n}\tilde{\psi} - \tau_{x_n}\psi)\|_{L^pL^r} \lesssim \|\tau_{x_n}\tilde{\psi} - \tau_{x_n}\psi\|_{H^1} = \|\tilde{\psi} - \psi\|_{H^1} \leq \epsilon.$$

From the other side, as $\tau_{x_n}\tilde{\psi} \in L^{r'}$ the dispersive estimate (10.2.3) gives us

$$\|e^{it(-\Delta+V)}\tau_{x_n}\tilde{\psi}\|_{L^r} \lesssim \frac{1}{|t|^{\frac{1}{2}(\frac{1}{r'}-\frac{1}{r})}}\|\tau_{x_n}\tilde{\psi}\|_{L^{r'}} = \frac{1}{|t|^{\frac{1}{2}(1-\frac{2}{r})}}\|\tilde{\psi}\|_{L^{r'}}$$

but $\frac{q}{2}(1-\frac{2}{r}) = \frac{\alpha(\alpha+\frac{3}{2})}{\alpha+4} > 1$ and $t \to \frac{1}{|t|^{\frac{1}{2}(1-\frac{2}{r})}} \in L^q(1,\infty)$. So, there exists $T > 0$ such that

$$\sup_{n\in\mathbb{N}} \|e^{it(-\Delta+V)}\tau_{x_n}\tilde{\psi}\|_{L^p(|t|\geq T)L^r} \leq \epsilon.$$

Taking $\tau_{x_n}\psi = \tau_{x_n}\tilde{\psi} + (\tau_{x_n}\psi - \tau_{x_n}\tilde{\psi})$, we then obtain for $T > 0$ large enough

$$\sup_{n\in\mathbb{N}} \|e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(|t|\geq T)L^r} \lesssim \epsilon$$

and (10.4.2) holds.

To obtain (10.4.1), we are now reduced to show that for $T > 0$ fixed

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(0,T)L^r} \to 0$$

as $n \to \infty$. Let $\epsilon > 0$. $e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$i\partial_t u - \Delta u + Vu = Ve^{-it\Delta}\tau_{x_n}\psi.$$

So, by the inhomogeneous Strichartz estimate (10.2.9)

$$\begin{aligned}
\|e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p_t(0,T)L^r} &\lesssim \|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^{\gamma'}_t(0,T)L^1} \\
&\lesssim T^{\frac{1}{\gamma'}}\|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,T)L^1} \\
&= T^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^1}
\end{aligned}$$

because the translation operator τ_{x_n} commutes with the propagator $e^{-it\Delta}$. But

$$\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^1} \underset{n\to\infty}{\longrightarrow} 0$$

as seen in the proof of Proposition 10.11, point (10.3.4). □

Proposition 10.13. *Let $\psi \in H^1$, $(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$, $U \in C(H^1) \cap L^pL^r$ be the unique solution to (10.1.2) with initial data ψ, and $U_n(t,x) := U(t, x - x_n)$. Then, up to a subsequence*

$$\| \int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right)(s)ds - \int_0^t e^{-i(t-s)(\Delta - V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^pL^r} \to 0 \tag{10.4.3}$$

as $n \to \infty$.

Proof. We follow the same spirit of proof as for Proposition 10.12. We begin to show that

$$\sup_{n\in\mathbb{N}} \| \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty))L^r} \to 0 \tag{10.4.4}$$

as T goes to infinity.

We decompose

$$\begin{aligned}\| \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty))L^r} &\leq \| \int_0^T e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty))L^r} \\ &+ \| \int_T^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty))L^r}\end{aligned}$$

where, by the inhomogeneous Strichartz estimates

$$\| \int_T^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty)L^r)} \leq \|U_n|U_n|^\alpha\|_{L^{q'}([T,\infty)L^{r'})} = \|U|U|^\alpha\|_{L^{q'}([T,\infty)L^{r'})}$$

and, by the Hölder inequality

$$\|U|U|^\alpha\|_{L^{q'}([T,\infty)L^{r'})} \leq \|U\|^{\alpha+1}_{L^p([T,\infty)L^r)} \underset{T\to\infty}{\longrightarrow} 0$$

independently of n. From the other side, by the dispersive estimate (10.2.3)

$$\begin{aligned}&\| \int_0^T e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p[T,\infty)L^r} \\ &\qquad\leq \| \int_0^T \|e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)\|_{L^r} ds\|_{L^p([T,\infty))} \\ &\qquad\lesssim \| \int_0^T (t-s)^{-\frac{1}{2}(1-\frac{2}{r})} \| \left(U_n|U_n|^\alpha\right)(s)\|_{L^{r'}} ds\|_{L^p([T,\infty))} \\ &\qquad= \| \int_0^T (t-s)^{-\frac{1}{2}(1-\frac{2}{r})} \| \left(U|U|^\alpha\right)(s)\|_{L^{r'}} ds\|_{L^p([T,\infty))} \\ &\qquad\qquad\leq \| \int_{\mathbb{R}} |t-s|^{-\frac{1}{2}(1-\frac{2}{r})} \| \left(U|U|^\alpha\right)(s)\|_{L^{r'}} ds\|_{L^p([T,\infty))} \longrightarrow 0\end{aligned}$$

as T goes to infinity. Indeed, note that by the Hardy-Littlewood-Sobolev inequality

$$\| \int_{\mathbb{R}} |t-s|^{-\frac{1}{2}(1-\frac{2}{r})} \| (U|U|^{\alpha})(s)\|_{L^{r'}} ds \|_{L^p} \lesssim \|U|U|^{\alpha}\|_{L^{q'}L^{r'}} \le \|U\|_{L^pL^r}^{\alpha+1} < \infty$$

so (10.4.4) holds. The same estimate is obviously valid for the propagator $e^{-it\Delta}$.

It remains to show that for $T>0$ fixed,

$$\| \int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^{\alpha})\, ds - \int_0^t e^{-i(t-s)(\Delta - V)} (U_n|U_n|^{\alpha})\, ds \|_{L^p(0,T)L^r} \to 0$$

as $n\to\infty$. The difference

$$\int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^{\alpha})\, ds - \int_0^t e^{-i(t-s)(\Delta - V)} (U_n|U_n|^{\alpha})\, ds$$

is the solution of the following linear Schrödinger equation, with zero initial data

$$i\partial_t u - \Delta u + Vu = V \int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^{\alpha})\, ds.$$

As a consequence, by the Strichartz estimate (10.2.9)

$$\begin{aligned}\| \int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^{\alpha})\, ds - \int_0^t e^{-i(t-s)(\Delta - V)} (U_n|U_n|^{\alpha})\, ds \|_{L^p(0,T)L^r} \\ \lesssim \|V \int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^{\alpha})\, ds\|_{L^{\gamma'}(0,T)L^1} \\ \lesssim T^{\frac{1}{\gamma'}} \|(\tau_{-x_n}V) \int_0^t e^{-i(t-s)\Delta} (U|U|^{\alpha})\, ds\|_{L^\infty(0,T)L^1}.\end{aligned}$$

But $\int_0^t e^{-i(t-s)\Delta}(U|U|^{\alpha})ds \in C([0,T],H^1)$ and the functions of $H^1(\mathbb{R})$ vanish at infinity, so there exists $\Lambda>0$ such that

$$\| \int_0^t e^{-i(t-s)\Delta} (U|U|^{\alpha})\, ds\|_{L^\infty(0,T)L^\infty(|x|\ge\Lambda)} \le \epsilon$$

so

$$\|(\tau_{-x_n}V) \int_0^t e^{-i(t-s)\Delta} (U|U|^{\alpha})\, ds\|_{L^\infty(0,T)L^1} \underset{n\to\infty}{\longrightarrow} 0$$

in the same way as in the proof of Proposition 10.11, point (10.3.4) . □

Proposition 10.14. *Let $\psi \in H^1$, $(x_n)_{n\ge1}$, $(t_n)_{n\ge1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n|\to\infty$ and $t_n \to \pm\infty$, U be a solution to (10.1.2) such that*

$$\|U(t) - e^{-it\Delta}\psi\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0$$

and $U_n(t,x) := U(t-t_n, x-x_n)$. Then, up to a subsequence

$$\|e^{-i(t-t_n)\Delta}\tau_{x_n}\psi - e^{-i(t-t_n)(\Delta-V)}\tau_{x_n}\psi\|_{L^pL^r} \to 0 \qquad (10.4.5)$$

and

$$\| \int_0^t e^{-i(t-s)\Delta} (U_n|U_n|^\alpha)\, ds - \int_0^t e^{-i(t-s)(\Delta-V)} (U_n|U_n|^\alpha)\, ds\|_{L^pL^r} \to 0 \tag{10.4.6}$$

as $n \to \infty$.

Proof. The proof is the same as for Proposition 10.12 and Proposition 10.13, decomposing the time interval in $\{|t-t_n| > T\}$ and his complementary. □

Finally, we will need the following Proposition of non linear scattering:

Proposition 10.15. *Let* $\varphi \in H^1$. *Then there exists* $W_\pm \in C(H^1) \cap L^p_{\mathbb{R}\pm} L^r$, *solution of (10.1.1) such that*

$$\|W_\pm(t,\cdot) - e^{-it(\Delta-V)}\varphi\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0 \tag{10.4.7}$$

moreover, if $t_n \to \mp\infty$ *and*

$$\varphi_n = e^{-it_n(\Delta-V)}\varphi, \quad W_{\pm,n}(t) = W_\pm(t-t_n) \tag{10.4.8}$$

then

$$W_{\pm,n}(t) = e^{-it(\Delta-V)}\varphi_n + \int_0^t e^{-i(t-s)(\Delta-V)}(W_{\pm,n}|W_{\pm,n}|^\alpha)(s)ds + f_{\pm,n}(t) \tag{10.4.9}$$

where

$$\|f_{\pm,n}\|_{L^p_{\mathbb{R}\pm}L^r} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{10.4.10}$$

Proof. The same proof as [BV16], Proposition 3.5, holds, as it involves only the analogous Strichartz estimates. □

10.5. Construction of a critical element

We have now all the tools to extract a critical element following the approach of [FXC11]. Let

$$E_c = \sup\left\{E > 0 \mid \forall\varphi \in H^1,\ E(\varphi) < E \Rightarrow \text{the solution of (1.1) with data } \varphi \text{ is in } L^pL^r\right\}$$

we will suppose that the critical energy E_c is finite, and deduce the existence of a solution of (10.1.1) with a relatively compact flow in H^1.

Proposition 10.16. *If* $E_c < \infty$, *then there exists* $\varphi_c \in H^1$, $\varphi_c \neq 0$, *such that the corresponding solution* u_c *of (10.1.1) verifies that* $\{u_c(t),\ t \geq 0\}$ *is relatively compact in* H^1.

Proof. Because of Proposition 10.9, $E_c > 0$. Therefore, if $E_c < \infty$, there exists a sequence φ_n of non-zero elements of H^1, such that, if we denote by $u_n \in C(H^1)$ the corresponding solution of (10.1.1), we have

$$E(\varphi_n) \underset{n\to\infty}{\longrightarrow} E_c$$

and

$$u_n \notin L^q L^r.$$

Thanks to the Proposition 10.11, we can apply the abstract profile decomposition of [BV16] to the H^1-bounded sequence φ_n and the operator $A = -\Delta + V$. Up to a subsequence, φ_n writes, for all $J \in \mathbb{N}$:

$$\varphi_n = \sum_{j=1}^{J} e^{-it_j^n(-\Delta+V)} \tau_{x_j^n} \psi_j + R_n^J.$$

where $t_j^n, x_j^n, \psi_j, R_n^J$ verifies (10.3.6)−(10.3.12). From (10.3.11) and (10.3.12), we have

$$E_c \geq \limsup_{n\to\infty} \sum_{j=1}^{J} E(e^{-it_j^n(-\Delta+V)} \tau_{x_j^n} \psi_j).$$

We show that there is exactly one non trivial profile, that is $J = 1$. By contradiction, assume that $J > 1$. To each profile ψ_j we associate family of non linear profiles $(U_{j,n})_{n\geq 0}$. Let $j \in \{1 \cdots J\}$. We are in exactly one of the following situations:

1. If $(t_j^n, x_j^n) = (0,0)$. By the orthogonality condition, notice that this can happen only for one profile. Because $J > 1$, we have $E(\psi_j) < E_c$, so the solution of (10.1.1) with data ψ_j scatters. If this case happens, let $N \in C(H^1) \cap L^p L^r$ be this solution, otherwise, we set $N = 0$.

2. If $t_j^n = 0$ and $|x_j^n| \to \infty$. Let $U_j \in C(H^1) \cap L^p L^r$ be the unique solution to (10.1.2) with initial data ψ_j. We set $U_{n,j}(x,t) := U(x - x_j^n, t)$.

3. If $x_j^n = 0$ and $t_j^n \to \pm\infty$. By Proposition 10.15, there exists $U_j \in C_{\mathbb{R}_\pm}(H^1) \cap L^p_{\mathbb{R}_\pm} L^r$ a solution to (10.1.1) such that

$$\|U_j(t) - e^{-it(\Delta - V)} \psi_j\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0$$

and verifying (10.4.7), (10.4.8), (10.4.9), (10.4.10). We have

$$E(U_j) = \lim_{n\to\infty} E(e^{-it_j^n(-\Delta+V)} \tau_{x_j^n} \psi_j) < E_c$$

so $U_j \in L^q L^r$. We set $U_{j,n}(t,x) := U_j(t - t_j^n, x)$.

4. If $|x_j^n| \to \infty$ and $t_j^n \to \pm\infty$. Let $U_j \in C(H^1) \cap L^p L^r$ be a solution to (10.1.2) such that

$$\|U_j(t) - e^{-it\Delta} \psi_j\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0$$

We set $U_{j,n}(t,x) := U_j(t - t_j^n, x - x_j^n)$.

Now, let

$$Z_{n,J} := N + \sum_j U_{n,j}.$$

By the results of the non linear profiles section - Propositions 10.12 and 10.13 in situation (2), Proposition 10.14 in situation (3) and Proposition 10.15 in situation (4) -, we have

$$Z_{n,J} = e^{-it(\Delta-V)}(\varphi_n - R_{n,J}) + \int_0^t e^{-i(t-s)(\Delta-V)}(N|N|^\alpha)(s)ds \\ + \sum_j \int_0^t e^{-i(t-s)(\Delta-V)}(U_{j,n}|U_{j,n}|^\alpha)(s)ds + r_{n,J} \quad (10.5.1)$$

with

$$\|r_{n,J}\|_{L^pL^r} \to 0$$

as $n \to \infty$. The decomposition (10.5.1) is the same as obtained in the proof of Proposition 4.1 of [BV16], and we therefore obtain the critical element following their proof, using our perturbative result of Proposition 10.10 instead of their Proposition 3.3, and the Strichartz inequalities of our Proposition 10.7 instead of estimates (3.1), (3.2), (3.3), (3.4) of their paper. □

10.6. Rigidity

In this section, we will show that the critical solution constructed in the previous one assuming the fact that $E_c < \infty$ cannot exist.

We will need the following classical result concerning the compact families of H^1

Proposition 10.17. *Suppose that $\{u(t),\ t \geq 0\}$ is relatively compact in H^1. Then, for any $\epsilon > 0$, there exists $R > 0$ such that*

$$\sup_{t\geq 0} \int_{|x|\geq R} \left(|\nabla u(t,x)|^2 + |u(t,x)|^2 + |u(t,x)|^{\alpha+2}\right) dx \leq \epsilon$$

Proof. Classic, see e.g. [FXC11]. □

Now, we can show the rigidity Proposition needed to end the proof:

Proposition 10.18. *Suppose that $u \in C(H^1)$ is a solution of (10.1.1) such that $\{u(t),\ t \geq 0\}$ is relatively compact in H^1. Then $u = 0$.*

Proof. By a classical elementary computation, we get the following virial identities:

Lemma 10.19. *Let $u \in C(H^1)$ be a solution to (10.1.1) and χ be a compactly supported, regular function. Then*

$$\partial_t \int \chi|u|^2 = 2Im \int \chi' u' \bar{u} \quad (10.6.1)$$

$$\partial_t^2 \int \chi|u|^2 = 4 \int \chi''|u'|^2 + \frac{2\alpha}{\alpha+2} \int \chi''|u|^{\alpha+2} - 2\int \chi' V'|u|^2 - \int \chi^{(4)}|u|^2. \quad (10.6.2)$$

Now, we assume by contradiction that $u \neq 0$. Let $\chi \in C_c^\infty$ be such that $\chi(x) = x^2$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$, set $\chi_R := R^2\chi(\frac{\cdot}{R})$ and

$$z_R(t) = \int \chi_R |u(t)|^2$$

we have, by (10.6.1), the Cauchy-Schwarz inequality and the conservation of energy

$$|z_R'(t)| \leq 2\int |\chi_R'||u'||\bar{u}| \leq CE(u)^{\frac{1}{2}}M(u)^{\frac{1}{2}}R. \tag{10.6.3}$$

Moreover, by (10.6.2)

$$\begin{aligned} z_R''(t) &= 4\int \chi_R''|u'|^2 + \frac{2\alpha}{\alpha+2}\int \chi_R''|u|^{\alpha+2} - 2\int \chi_R' V'|u|^2 - \int \chi_R^{(4)}|u|^2 \\ &\geq 8\int_{|x|\leq R} |u'|^2 + \frac{4\alpha}{\alpha+2}\int_{|x|\leq R}|u|^{\alpha+2} - C\int_{|x|>R}\left(|u|^2 + |u|^{\alpha+2} + |u'|^2\right) \\ &\quad -2\int \chi_R' V'|u|^2 - \int \chi^{(4)}|u|^2 \end{aligned} \tag{10.6.4}$$

but, because of conservation of the mass

$$|\int \chi^{(4)}|u|^2| \leq \frac{C}{R^2}\|u(0)\|_{L^2} \tag{10.6.5}$$

and, because V is repulsive (ie $xV' \leq 0$), using the Cauchy-Schwarz inequality, the Sobolev injection $H^1 \hookrightarrow L^\infty$ and the conservation laws

$$\begin{aligned} -2\int \chi_R' V'|u|^2 &= -2\int_{|x|\leq R} xV'|u|^2 + 2\int_{|x|>R}\chi_R' V'|u|^2 \\ &\geq -C\int_{|x|>R}|xV'||u|^2 \geq -C\|xV'\|_{L^1(|x|>R)}\|u\|_{L^\infty}^2 \\ &\geq -C\|xV'\|_{L^1(|x|>R)}\|u\|_{H^1}^2 \geq -C(u(0))\|xV'\|_{L^1(|x|>R)}. \end{aligned} \tag{10.6.6}$$

Let R_0 be large enough so that

$$\int_{|x|\leq R_0}|u|^{\alpha+2} \geq \frac{1}{2}\int |u|^{\alpha+2} := \delta. \tag{10.6.7}$$

We have $\delta > 0$ because we suppose that u is non zero. For $R \geq R_0$, we obtain combining (10.6.4) with (10.6.5), (10.6.6), and (10.6.7)

$$z_R''(t) \geq C\left(\delta - \int_{|x|>R}\left(|u|^2 + |u|^{\alpha+2} + |u'|^2\right) - \frac{1}{R^2}\|u(0)\|_{L^2} - \|xV'\|_{L^1(|x|>R)}\right). \tag{10.6.8}$$

Because $xV' \in L^1$ and using the compactness hypothesis combined with Proposition 10.17, there exists $R \geq R_0$ large enough so that

$$\int_{|x|>R}\left(|u|^2 + |u|^{\alpha+2} + |u'|^2\right) + \frac{1}{R^2}\|u(0)\|_{L^2} + \|xV'\|_{L^1(|x|>R)} \leq \frac{\delta}{2}$$

then, (10.6.8) gives

$$z_R''(t) \geq \frac{C\delta}{2} > 0.$$

Integrating this last inequality contradicts (10.6.3) as $t \to \infty$. □

We are now in position to end the proof of theorem 10.1 :

Proof of Theorem 10.1. If $E_c < \infty$, then the Proposition 10.16 allows us to extract a critical element $\varphi_c \in H^1$, $\varphi_c \neq 0$, such that the corresponding solution u_c of (10.1.1) verifies that $\{u_c(t),\ t \geq 0\}$ is relatively compact in H^1. By Proposition 10.18, such a critical solution cannot exist, so $E_c = \infty$ and by Proposition 10.8, all the solutions of (10.1.1) scatter in H^1. □

11 Scattering for the nonlinear wave equation outside obstacles

11.1. Introduction

We consider a critical defocusing non linear wave equation in $\mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2)$

$$\begin{cases} \partial_t^2 u - \Delta_D u + u^5 = 0 \\ (u(0), \partial_t u(0)) = (f, g). \end{cases} \tag{11.1.1}$$

Note that the global existence for such an equation in a domain was obtained in [BLP08]. By the finite speed of propagation, their result apply in particular to the exterior of obstacles. Therefore, it is legitimate to wonder what solutions look like in large time, and in particular if the nonlinearity still plays a role. If it is not the case, we say that the solution *scatters*. More precisely, we say that a solution scatters if there exists a solution of the *linear* equation v such that

$$\|u(t) - v(t)\|_{\dot{H}^1(\Omega)} \longrightarrow 0,$$

as t goes to infinity. The scattering in $\mathbb{R}^3$ was shown by Bahouri and Shatah [BS98]. Provided a good set of Strichartz estimates exists for the linear equation, their proof adapts to the case of a finite-border domain if one is able to deal with the arising boundary term. This term can be controlled in particular if one obtain the decay of the local energy near the obstacle (see Section 8.2):

$$\frac{1}{T}\int_0^T \int_{\Omega \cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0 \tag{11.1.2}$$

as T goes to infinity. In the case of the exterior of two strictly convex obstacles, where [Laf17a] gives us the good set of Strichartz estimates, we show that we can obtain this control *everywhere except in the neighborhood of the trapped ray*: more precisely

Theorem 11.1. *Let Θ_1 and Θ_2 be two convex subsets of $\mathbb{R}^3$. Then, there exists a family $(\mathcal{S}(T))_{T\geq 1}$ of open neighborhoods of the trapped ray $\mathcal{R}$ verifying*

$$\mathcal{S}(T) \longrightarrow \mathcal{R} \text{ as } T \longrightarrow +\infty$$

such that any global solution of (11.1.1) in $\Omega := \mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2)$ *verifies, as* T *goes to infinity*

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))\backslash \mathcal{S}(T)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0.$$

This is a first step to show the scattering for any data in this exterior problem. Indeed, it can be seen as a rigidity argument in a concentration-compactness/rigidity approach:

Corollary 11.2. *There is no compact flow solution of (11.1.1) which does not scatter.*

We are precisely following this path in the work in progress [LL].

We now deal with a geometrical situation which is close to the exterior of two convex obstacles, but does not have a trapped ray: the exterior of dog bones. We are actually able to show the scattering outside a class of non star-shaped obstacles containing dog bones with arbitrary thin necks. In order to state this result, let us recall the definition of an illuminated subset - which is a generalization of star-shaped ones - first introduced by [BK74]:

Definition 11.3. A subset $\mathcal{K}$ of $\mathbb{R}^n$ is said to be illuminated by a convex subset $\mathcal{C} \subset \mathbb{R}^n$ if

$$\min_{\partial\mathcal{K}} \nabla\rho\cdot\nu > 0$$

where ρ is the gauge of $\mathcal{C}$ and ν the outward-pointing normal derivative to $\partial\mathcal{K}$.

We are now able to state our result:

Theorem 11.4. *Let* $\mathcal{C} \subset \mathbb{R}^3$ *be the ellipsoid of equation*

$$x^2 + y^2 + \epsilon z^2 = 1, \ 0 < \epsilon \leq 1 \tag{11.1.3}$$

resp.

$$x^2 + \epsilon y^2 + \epsilon z^2 = 1, \ \frac{1+\sqrt{3}}{4} \leq \epsilon \leq 1 \tag{11.1.4}$$

and $\mathcal{K}$ *be a compact subset of* $\mathbb{R}^3$ *illuminated by* $\mathcal{C}$*. Then, any solution of (11.1.1) in* $\Omega = \mathbb{R}^3\backslash\mathcal{K}$ *scatters in* $\dot{H}^1(\Omega)$*.*

Notice that Abou-Shakra obtained in [AS13] the scattering for obstacles illuminated by a deformation of a sphere using a slightly different method, but her result does not permit to handle dog bones with arbitrary thin necks. Our key tool to obtain theorem 11.1 and theorem 11.4 is an identity due to Morawetz [Mor61] in the case of the linear equation, and used here in the spirit of [GV85c]. Such an identity rely on the choice of a good weight function χ which has to be adapted to the geometry and verify a very rigid and poorly understood constraint: $\Delta^2\chi \leq 0$. In the case of theorem 11.4, the natural weight is given by the gauge of the ellipsoid we are dealing with, and does not verify this constraint for arbitrary thin ellipsoids. In order to deal with it, we present a method which permits to bypass this obstruction: noticing that corresponding four dimensional ellipsoids verify the constraint, we extend the solution as the solution of a four dimensional non linear wave equation, show the estimate for such a solution, and then go back to the original, three dimensional solution. We believe that such an argument may be useful in other situations.

11.2. A scattering criterion

Let us consider the following defocusing non linear wave equation in $\mathbb{R}^3\backslash\mathcal{K}$

$$\begin{cases} \partial_t^2 u - \Delta_D u + u^5 = 0 \\ (u(0), \partial_t u(0)) = (f, g). \end{cases}$$

$\mathcal{K}$ will be the reunion of two strictly convex obstacles, or an illuminated obstacle as defined in the introduction, and we are concerned by the scattering problem in both situations. Our main tool will be the following momentum identity, which was first introduced by Morawetz [Mor61] in a similar form to show some decay properties of the linear wave equation:

Lemma 11.5. *Let u be a solution of (NLW) in Ω and $\chi \in C^\infty(\Omega, \mathbb{R})$. Then we have*

$$\partial_t \left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2} \Delta \chi u \partial_t u \right) = \int_\Omega (D^2 \chi \nabla u, \nabla u) - \frac{1}{4} \int_\Omega u^2 \Delta^2 \chi + \frac{3}{2} \int_\Omega |u|^6 \Delta \chi - \frac{1}{2} \int_{\partial\Omega} |\partial_n u|^2 \partial_n \chi. \quad (11.2.1)$$

Proof. The identity can be shown by standard integrations by parts justified by a limiting argument. □

The scattering in $\mathbb{R}^3$ was shown by Bahouri and Shatah [BS98]. Their proof still hold in the case of a domain with boundaries if we are able to control the boundary term appearing in their computations, that is

Lemma 11.6. *Let u be a solution of (NLW) in a finite-border domain Ω of $\mathbb{R}^3$ such that Strichartz estimates (8.1.5) holds. If*

$$\frac{1}{T} \int_0^T \int_{\partial\Omega} |\partial_n u|^2 d\sigma dt \longrightarrow 0,$$

as T goes to infinity, then u scatter in $\dot{H}^1$.

Note that the trace of the normal derivative is not an easy object to deal with, because this trace is a priori not defined in $L^2(\partial\Omega)$ for elements of $\dot{H}^1(\Omega)$. Moreover, even if we can define it for almost every $u(t)$ when u is a solution of (NLW) because of the particular structure of the equation, the application

$$u \in \dot{H}^1 \cap \{\text{value in time } t \text{ of solutions of NLW}\} \longrightarrow \partial_n u \in L^2(\partial\Omega)$$

is in our knowledge not known to be continuous.

For this reason, we prefer to deal with the following criterion, which involve only the local energy of the equation, and that we deduce from the previous one using the momentum identity (11.2.1):

Lemma 11.7. *Let u be a solution of (NLW) in a finite-border domain Ω of $\mathbb{R}^3$ such that Strichartz estimates (8.1.5) holds. There exists $A > 0$, $B(0,A) \supset \partial\Omega$, such that, if*

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0, \tag{11.2.2}$$

as T goes to infinity, then u scatters in $\dot{H}^1$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ be such that $\nabla\chi = -n$ on $\partial\Omega$, supported in $B(0,A)$. Suppose that

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \longrightarrow 0$$

as T goes to infinity. We use lemma 11.5 with the weight χ to get:

$$\partial_t\left(\int_\Omega -\partial_t u \nabla u \nabla\chi - \frac{1}{2}\Delta\chi u \partial_t u\right) = \int_\Omega (D^2\chi\nabla u, \nabla u) - \frac{1}{4}\int_\Omega u^2\Delta^2\chi + \frac{2}{3}\int_\Omega |u|^6\Delta\chi + \frac{1}{2}\int_{\partial\Omega} |\partial_n u|^2 d\sigma.$$

Integrating in time we get

$$\int_0^T \int_{\partial\Omega} |\partial_n u|^2 d\sigma dt \lesssim \int_{\Omega\cap B(0,A)} |\partial_t u\nabla u| + |u\partial_t u| + \int_0^T \int_{\Omega\cap B(0,A)} |u|^6 + |u|^2 + |\nabla u|^2,$$

and using Minkowsky inequality,

$$\begin{aligned}\int_0^T \int_{\partial\Omega} |\partial_n u|^2 d\sigma dt &\lesssim \left(\int_\Omega |\partial_t u|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} + A^{\frac{1}{3}} \left(\int_\Omega |\partial_t u|^2\right)^{\frac{1}{2}} \left(\int_\Omega |u|^6\right)^{\frac{1}{6}} \\ &\quad + \int_0^T \int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) + A^{\frac{2}{3}} \int_0^T \left(\int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}} \\ &\lesssim_A C(E) + \int_0^T \int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) + T^{\frac{2}{3}} \left(\int_0^T \int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{T}\int_0^T \int_{\partial\Omega} |\partial_n u|^2 d\sigma dt &\lesssim_A \frac{C(E)}{T} + \frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) \\ &\quad + \left(\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}} \longrightarrow 0\end{aligned}$$

as $T \to \infty$ and by lemma 11.6 we conclude that u scatter in $\dot{H}^1$. □

Notice that the Morawetz identity (11.2.1) permits to obtain this criterion as soon as one has a weight function χ such that $\nabla\chi \cdot n \geq 0$ on $\partial\Omega$, $D^2\chi$ is positive definite, and $\Delta^2\chi \leq 0$. Constructing such weights will therefore be of key interest in the sequel.

11.3. A partial result in the exterior of two strictly convex obstacles

In the exterior of two strictly convex obstacles, lemma 11.5 seems not to be sufficient to show the scattering criterion (11.2.2) because we are not able to find an appropriate weight function. However, we can choose a weight function which has the right behavior everywhere except in a neighborhood of the trapped ray, and therefore obtain Theorem 11.1.

11.3.1. The particular case of two balls

Proof of Theorem 11.1 in the particular case where $\Theta_{1,2}$ *are two balls.* Without loss of generality, we suppose that Θ_1 is centered in 0. We denote by c the center of Θ_2. We choose the weight

$$\chi(x) := |x| + |x-c|$$

and use lemma 11.5 with weight χ

$$\partial_t\left(\int_\Omega -\partial_t u\nabla u\nabla\chi - \frac{1}{2}\Delta\chi u\partial_t u\right) = \int_\Omega (D^2\chi\nabla u,\nabla u) - \frac{1}{4}\int_\Omega u^2\Delta^2\chi + \frac{2}{3}\int_\Omega |u|^6\Delta\chi - \frac{1}{2}\int_{\partial\Omega}|\partial_n u|^2\nabla\chi\cdot n\, d\sigma.$$

Remark that $-\nabla\chi\cdot n \geq 0$ on $\partial\Theta_1\cup\partial\Theta_2$: indeed, on Θ_1, $-n = \frac{x}{|x|}$ and thus

$$-\nabla\chi\cdot n = 1 + \frac{x-c}{|x-c|}\cdot\frac{x}{|x|} \geq 1 - \left|\frac{x-c}{|x-c|}\right|\left|\frac{x}{|x|}\right| = 0,$$

and the same hold on Θ_2. Moreover, $\Delta^2\chi = 0$. Thus we obtain

$$\partial_t\left(-\int_0^T \partial_t u\nabla u\cdot\nabla\chi + \frac{1}{2}\Delta\chi u\partial_t u\right) \geq \frac{2}{3}\int_\Omega |u|^6\Delta\chi + \int_\Omega (D^2\chi\nabla u,\nabla u) \qquad (11.3.1)$$

Integrating this inequality and controlling the left-hand side using the Hardy inequality

$$\int_\Omega \frac{|f|^2}{|x|^2} \lesssim \int_\Omega |\nabla f|^2 \text{ for } f\in\dot{H}^1_0(\Omega)$$

we get

$$\int_0^T\int_\Omega |u|^6\Delta\chi + (D^2\chi\nabla u,\nabla u)\, dxdt \lesssim E. \qquad (11.3.2)$$

From the one hand, $\Delta\chi \gtrsim \frac{1}{A}$ on $B(0,A)$, thus

$$\int_{\Omega\cap B(0,A)} |u|^6 \lesssim A\int_{\Omega\cap B(0,A)} |u|^6\Delta\chi \lesssim A\int_\Omega |u|^6\Delta\chi,$$

and therefore, by (11.3.2)

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |u|^6 \; dxdt \lesssim \frac{E}{T}. \tag{11.3.3}$$

Now, we would like to estimate the localized kinetic energy using (11.3.2) again. We have

$$D^2\chi = \frac{1}{|x|}\left(\mathrm{Id} - \frac{xx^t}{|x|^2}\right) + \frac{1}{|x-c|}\left(\mathrm{Id} - \frac{(x-c)(x-c)^t}{|x-c|^2}\right).$$

The operators corresponding to the matrices

$$\mathrm{Id} - \frac{xx^t}{|x|^2}, \text{ resp. } \mathrm{Id} - \frac{(x-c)(x-c)^t}{|x-c|^2}$$

are the orthogonal projections on the plane normal to $\frac{x}{|x|}$, resp. to $\frac{x-c}{|x-c|}$. Thus,

$$(D^2\chi\cdot\xi,\xi) = \left(\frac{1}{|x|}+\frac{1}{|x-c|}\right)|\xi|^2 - \frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2 - \frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2. \tag{11.3.4}$$

We choose coordinates (depending of x and c) such that

$$\frac{x}{|x|} = (1,0,0), \; \frac{x-c}{|x-c|} = (\cos\theta, \sin\theta, 0),$$

then we have, if $\xi = \begin{pmatrix}\hat\xi_1 & \hat\xi_2 & \hat\xi_3\end{pmatrix}$ in this set of coordinates

$$\frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2 + \frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2 = \begin{pmatrix}\hat\xi_1 & \hat\xi_2\end{pmatrix}\begin{pmatrix}\frac{1}{|x|}+\frac{\cos^2\theta}{|x-c|} & \frac{\sin\theta\cos\theta}{|x-c|}\\ \frac{\sin\theta\cos\theta}{|x-c|} & \frac{\sin^2\theta}{|x-c|}\end{pmatrix}\begin{pmatrix}\hat\xi_1\\ \hat\xi_2\end{pmatrix}.$$

The largest eigenvalue of this positive quadratic form in $\begin{pmatrix}\hat\xi_1 & \hat\xi_2\end{pmatrix}$ writes

$$\lambda_2 = \frac{1}{2}\left(\frac{1}{|x-c|}+\frac{1}{|x|}+\sqrt{\left(\frac{1}{|x-c|}+\frac{1}{|x|}\right)^2 - 4\frac{\sin^2\theta}{|x||x-c|}}\right)$$

therefore, there exists $\alpha_0 > 0$ small enough and $c > 0$ such that, if $\alpha \leq \alpha_0$, we have, for $x \in \Omega\cap B(0,A)$

$$\sin^2\theta \geq \alpha \implies \lambda_2 \leq \frac{1}{|x-c|}+\frac{1}{|x|} - c\alpha. \tag{11.3.5}$$

On the other hand

$$\frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2 + \frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2 \leq \lambda_2|(\hat\xi_1,\hat\xi_2)|^2 \leq \lambda_2|\xi|^2,$$

thus we get, combining this last inequality with (11.3.4) and (11.3.5), for $x \in \Omega\cap B(0,A)$

$$\sin^2\theta \geq \alpha \implies (D^2\chi\cdot\xi,\xi) \gtrsim \alpha|\xi|^2. \tag{11.3.6}$$

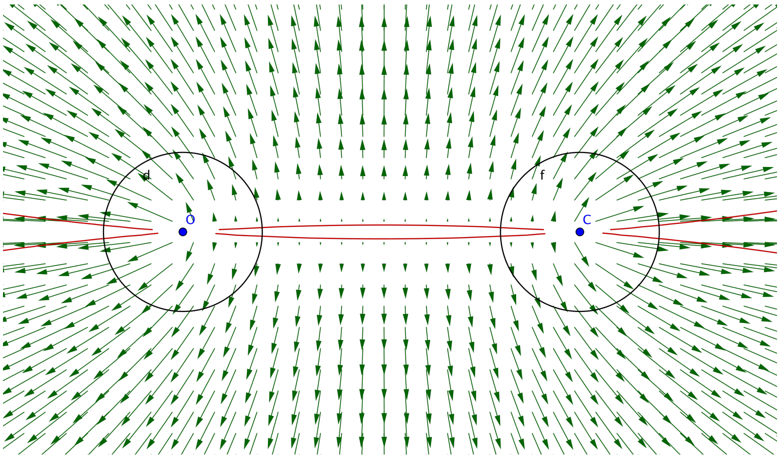

Figure 11.3.1.: $\nabla\chi$ and $V(\alpha)$

Remark that, because θ is the angle between $\frac{x}{|x|}$ and $\frac{x-c}{|x-c|}$

$$\theta = \arccos \frac{x}{|x|} \cdot \frac{x-c}{|x-c|},$$

and let us denote, for $\alpha \leq \alpha_0$

$$V(\alpha) = \Omega \cap B(0,A) \cap \left\{ \sin^2 \left(\arccos \frac{x}{|x|} \cdot \frac{x-c}{|x-c|} \right) \geq \alpha \right\}.$$

Note that $V(\alpha) \to \Omega \cap B(0,A)$ as α goes to zero in the sense that, denoting μ the Lebesgue's measure on $\mathbb{R}^3$

$$\mu\left((\Omega \cap B(0,A)) \backslash V(\alpha) \right) \longrightarrow 0 \tag{11.3.7}$$

as α goes to zero: $V(\alpha)$ is a subset of $\Omega \cap B(0,A)$ excluding a small neighborhood of the line $(0,c)$. We have, on $S(\alpha)$, because of (11.3.6)

$$(D^2\chi \cdot \xi, \xi) \gtrsim \alpha|\xi|^2.$$

Thus we get

$$\int_\Omega (D^2\chi\nabla u, \nabla u) \geq \int_{V(\alpha)} (D^2\chi\nabla u, \nabla u) \gtrsim \alpha \int_{V(\alpha)} |\nabla u|^2$$

and by (11.3.2) we obtain

$$\frac{1}{T}\int_0^T \int_{V(\alpha)} |\nabla u|^2 \, dxdt \lesssim \frac{1}{\alpha}\frac{E}{T}.$$

We take $\alpha = T^{-1/2}$ in order to have

$$\frac{1}{T}\int_0^T \int_{V(\alpha(T))} |\nabla u|^2 \, dxdt \lesssim \frac{E}{\sqrt{T}}. \tag{11.3.8}$$

Choosing $\mathcal{S}(T) := B(0,A)\backslash V(\alpha(T))$, (11.3.3) together with (11.3.8) gives the result. $\square$

11.3.2. The general framework

In the general case of two convex obstacles, we will use a family of weights given by the following lemma:

Lemma 11.8. *Let* $c = (c_1, 0, 0)$*. Then, for* $x \in \partial(\Theta_1 \cup \Theta_2)$*,*

$$\left(\frac{x-c}{|x-c|} + \frac{x+c}{|x+c|}\right) \cdot (-n)(x) = (\textit{something} \geq 0) + O(\frac{1}{c_1^4}).$$

Proof. We recall that we have chosen coordinates such that the trapped ray $\mathcal{R}$ is a segment of the line $\{x_2 = x_3 = 0\}$.

Remark that

$$|x+c| = c_1 + x_1 + \frac{1}{2c_1}|(0, x_2, x_3)|^2 + O(\frac{1}{c_1^2})$$

and therefore

$$\frac{x-c}{|x-c|} + \frac{x+c}{|x+c|} = \frac{1}{|x-c||x+c|}\left(2c_1(0, x_2, x_3) + x\frac{|(0, x_2, x_3)|^2}{c_1} + O(\frac{1}{c_1^2})\right). \quad (11.3.9)$$

Moreover, on $\partial\Theta_i$, $-n$ writes

$$-n = (\frac{x_1}{|x_1|}, 0, 0) + \frac{1}{R_i}(0, x_2, x_3) + O(|(0, x_2, x_3)|^2)$$

where $R_i > 0$ is the ray of curvature of Θ_i in $\Theta_i \cap \mathcal{R}$. Hence

$$\left(2c_1(0, x_2, x_3) + x\frac{|(0, x_2, x_3)|^2}{c_1}\right) \cdot (-n) = \left(\frac{2c_1}{R_i} + \frac{x \cdot n}{c_1}\right)|(0, x_2, x_3)|^2 + O(|(0, x_2, x_3|^2)$$
$$\geq \left(\frac{2c_1}{R_i} - \frac{C}{c_1}\right)|(0, x_2, x_3)|^2 + O(|(0, x_2, x_3|^2)$$

Thus, there exists $\delta \geq 0$ and $D_1 > 0$ such that, for every $c_1 > D_1$ we have

$$|(0, x_2, x_3)| \leq \delta \implies \left(2c_1(0, x_2, x_3) + x\frac{|(0, x_2, x_3)|^2}{c_1}\right) \cdot (-n) \geq 0. \quad (11.3.10)$$

On the other hand, there exists $\epsilon_0 > 0$ such that, for all $x \in \partial(\Theta_1 \cup \Theta_2)$

$$|(0, x_2, x_3)| \geq \delta \implies (0, x_2, x_3) \cdot (-n) \geq \epsilon_0.$$

Hence, if $|(0, x_2, x_3)| \geq \delta$

$$\left(2c_1(0, x_2, x_3) + x\frac{|(0, x_2, x_3)|^2}{c_1}\right) \cdot (-n) \geq 2c_1\epsilon_0 - \frac{C}{c_1},$$

and therefore, there exists $D_2 > 0$ such that, if $c_1 > D_2$

$$|(0, x_2, x_3)| \geq \delta \implies \left(2c_1(0, x_2, x_3) + x\frac{|(0, x_2, x_3)|^2}{c_1}\right) \cdot (-n) \geq 0. \quad (11.3.11)$$

Combining (11.3.9), (11.3.10) and (11.3.11) gives the result. □

We are now in position to prove Theorem 11.1:

Proof of Theorem 11.1. Let u be a solution of (NLW) in $\Omega = \mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2)$ with a relatively compact flow $\{u(t),\ t \geq 0\}$ in $\dot{H}^1$. We will show that u scatter in $\dot{H}^1$. We set $c = (c_1, 0, 0)$, $c_1 > 0$, choose the weight

$$\chi(x) := |x + c| + |x - c|$$

and get:

$$\partial_t \left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2} \Delta \chi u \partial_t u \right) = \int_\Omega (D^2 \chi \nabla u, \nabla u) - \frac{1}{4} \int_\Omega u^2 \Delta^2 \chi + \frac{2}{3} \int_\Omega |u|^6 \Delta \chi - \frac{1}{2} \int_{\partial\Omega} |\partial_n u|^2 \nabla \chi \cdot n \, d\sigma. \quad (11.3.12)$$

Note that (11.3.12) holds in the sense of distributions. In $\mathcal{D}'$, $\nabla\chi$, $D^2\chi$ and $\Delta\chi$ coincides with the classical derivatives and

$$\Delta^2 \chi = -4\pi(\delta_{-c} + \delta_c),$$

therefore (11.3.12) reads in the classical sense

$$\partial_t \left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2} \Delta \chi u \partial_t u \right) = \int_\Omega (D^2 \chi \nabla u, \nabla u) + \pi |u(t, -c)|^2 + \pi |u(t, c)|^2 + \frac{2}{3} \int_\Omega |u|^6 \Delta \chi - \frac{1}{2} \int_{\partial\Omega} |\partial_n u|^2 \nabla \chi \cdot n \, d\sigma. \quad (11.3.13)$$

According to lemma 11.8, we have

$$\nabla \chi \cdot (-n) = (\text{something} \geq 0) + O(\frac{1}{c_1^4}). \quad (11.3.14)$$

Thus, combining (11.3.13) and (11.3.14) with the a-priori estimate

$$\int_0^T \int_{\partial\Omega} |\partial_n u|^2 d\sigma dt \lesssim ET$$

we obtain the inequality:

$$\partial_t \left(- \int_0^T \partial_t u \nabla u \cdot \nabla \chi + \frac{1}{2} \Delta \chi u \partial_t u \right) \geq \frac{2}{3} \int_\Omega |u|^6 \Delta \chi + \int_\Omega (D^2 \chi \nabla u, \nabla u) - C \frac{1}{c_1^4} ET \quad (11.3.15)$$

Integrating this estimate and controlling the left-hand side using the Hardy inequality

$$\int_\Omega \frac{|f|^2}{|x|^2} \lesssim \int_\Omega |\nabla f|^2 \text{ for } f \in \dot{H}_0^1(\Omega)$$

we get

$$\int_0^T \int_\Omega |u|^6 \Delta\chi + (D^2\chi\nabla u, \nabla u)\ dxdt \lesssim E + \frac{1}{c_1^4} ET. \tag{11.3.16}$$

From the one hand,

$$\Delta\chi = \frac{2}{|x-c|} + \frac{2}{|x+c|} \gtrsim \frac{1}{c_1}$$

on $B(0,A)$, thus

$$\int_{\Omega\cap B(0,A)} |u|^6 \lesssim c_1 \int_{\Omega\cap B(0,A)} |u|^6 \Delta\chi \lesssim c_1 \int_\Omega |u|^6 \Delta\chi,$$

and therefore, by (11.3.16)

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |u|^6\ dxdt \lesssim c_1 \frac{E}{T} + \frac{E}{c_1^3}. \tag{11.3.17}$$

Now, we would like to estimate the localized kinetic energy using (11.3.16) again. We have

$$D^2\chi = \frac{1}{|x+c|}\left(\mathrm{Id} - \frac{(x+c)(x+c)^t}{|x+c|^2}\right) + \frac{1}{|x-c|}\left(\mathrm{Id} - \frac{(x-c)(x-c)^t}{|x-c|^2}\right).$$

The operators corresponding to the matrices

$$\mathrm{Id} - \frac{(x+c)(x+c)^t}{|x+c|^2},\ \text{resp. } \mathrm{Id} - \frac{(x-c)(x-c)^t}{|x-c|^2}$$

are the orthogonal projections on the plane normal to $\frac{x+c}{|x+c|}$, resp. to $\frac{x-c}{|x-c|}$. Thus,

$$(D^2\chi\cdot\xi,\xi) = \left(\frac{1}{|x+c|} + \frac{1}{|x-c|}\right)|\xi|^2 - \frac{1}{|x+c|}\left(\xi\cdot\frac{x}{|x+c|}\right)^2 - \frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2. \tag{11.3.18}$$

We choose coordinates (depending of x and c) such that

$$\frac{x+c}{|x+c|} = (1,0,0),\ \frac{x-c}{|x-c|} = (\cos\theta, \sin\theta, 0),$$

then we have, if $\xi = \begin{pmatrix}\hat\xi_1 & \hat\xi_2 & \hat\xi_3\end{pmatrix}$ in this set of coordinates

$$\frac{1}{|x+c|}\left(\xi\cdot\frac{x+c}{|x+c|}\right)^2 + \frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2 = \begin{pmatrix}\hat\xi_1 & \hat\xi_2\end{pmatrix}\begin{pmatrix}\frac{1}{|x|} + \frac{\cos^2\theta}{|x-c|} & \frac{\sin\theta\cos\theta}{|x-c|} \\ \frac{\sin\theta\cos\theta}{|x-c|} & \frac{\sin^2\theta}{|x-c|}\end{pmatrix}\begin{pmatrix}\hat\xi_1 \\ \hat\xi_2\end{pmatrix}.$$

The largest eigenvalue of this positive quadratic form in $\begin{pmatrix}\hat\xi_1 & \hat\xi_2\end{pmatrix}$ writes

$$\lambda_2 = \frac{1}{2}\left(\frac{1}{|x-c|} + \frac{1}{|x+c|} + \sqrt{\left(\frac{1}{|x-c|} + \frac{1}{|x+c|}\right)^2 - 4\frac{\sin^2\theta}{|x+c||x-c|}}\right)$$

therefore, there exists $c > 0$ such that, we have, for $x \in \Omega \cap B(0, A)$ and $\alpha > 0$ small enough

$$\sin^2 \theta \geq \alpha \implies \lambda_2 \leq \frac{1}{|x-c|} + \frac{1}{|x|} - c\frac{\alpha}{c_1}. \tag{11.3.19}$$

On the other hand

$$\frac{1}{|x+c|}\left(\xi \cdot \frac{x}{|x+c|}\right)^2 + \frac{1}{|x-c|}\left(\xi \cdot \frac{x-c}{|x-c|}\right)^2 \leq \lambda_2 |(\hat{\xi}_1, \hat{\xi}_2)|^2 \leq \lambda_2 |\xi|^2,$$

thus we get, combining this last inequality with (11.3.18) and (11.3.19), for $x \in \Omega \cap B(0, A)$

$$\sin^2 \theta \geq \alpha \implies (D^2\chi \cdot \xi, \xi) \gtrsim \frac{\alpha}{c_1}|\xi|^2. \tag{11.3.20}$$

Remark that, because θ is the angle between $\frac{x+c}{|x-c|}$ and $\frac{x-c}{|x-c|}$

$$\theta = \arccos \frac{x+c}{|x+c|} \cdot \frac{x-c}{|x-c|},$$

and let us denote, for $\alpha \leq \alpha_0$

$$S(\alpha, c) = \Omega \cap B(0, A) \cap \left\{ \sin^2 \left(\arccos \frac{x+c}{|x+c|} \cdot \frac{x-c}{|x-c|} \right) \geq \alpha \right\}.$$

We have, on $S(\alpha)$, because of (11.3.20)

$$(D^2\chi \cdot \xi, \xi) \gtrsim \frac{\alpha}{c_1}|\xi|^2.$$

Thus we get

$$\int_\Omega (D^2\chi \nabla u, \nabla u) \geq \int_{S(\alpha)} (D^2\chi \nabla u, \nabla u) \gtrsim \frac{\alpha}{c_1} \int_{S(\alpha)} |\nabla u|^2$$

and by (11.3.16) we obtain

$$\frac{1}{T}\int_0^T \int_{S(\alpha)} |\nabla u|^2 \, dx dt \lesssim \frac{c_1}{\alpha}\frac{E}{T} + \frac{1}{\alpha}\frac{E}{c_1^3}. \tag{11.3.21}$$

We would like to estimate $|(\Omega \cap B(0, A)) \backslash S(\alpha, c)|$. The angle $\theta(x_0)$ between $x + c$ and $x - c$ is maximal when x belongs to the mediator plane of $[-c, c]$. Let x_0 be such a point, such that

$$\sin^2 \theta(x_0) = \alpha,$$

and $d_0 = |x_0|$. Then $(\Omega \cap B(0, A)) \backslash S(\alpha, c)$ is included in a parallelepiped with edges of sizes C, $2d_0$, $2d_0$, therefore

$$|S(\alpha, c)^c| \lesssim d_0^2.$$

From the other hand,

$$\sin(\theta(x_0)/2) = \frac{c_1}{\sqrt{c_1^2 + d_0^2}}, \cos(\theta(x_0)/2) = \frac{d_0}{\sqrt{c_1^2 + d_0^2}}$$

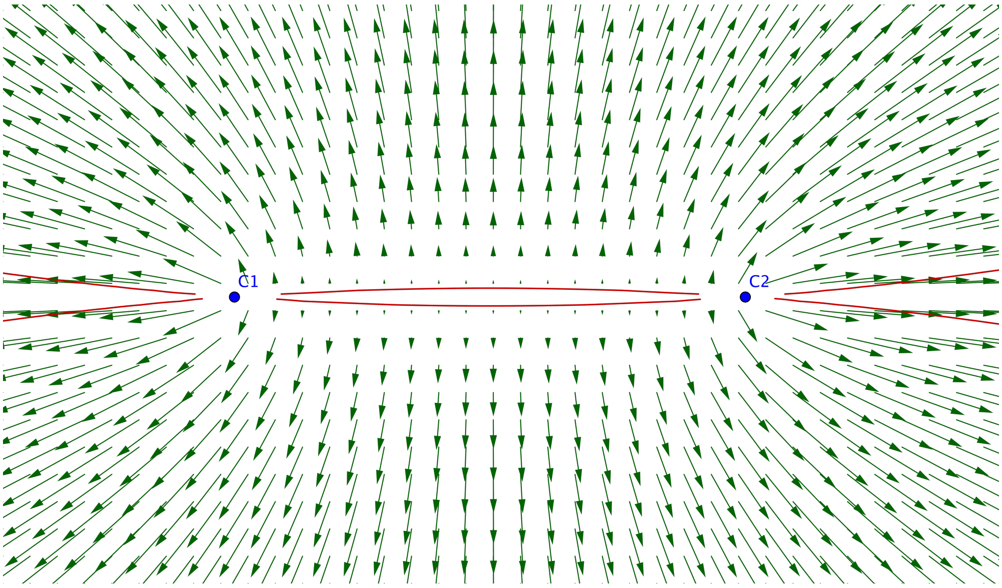

Figure 11.3.2.: $\nabla\chi$ and $S(\alpha)$

and thus

$$\sqrt{\alpha} = \sin\theta(x_0) = \frac{2c_1 d_0}{c_1^2 + d_0^2},$$

which leads to

$$d_0 = \frac{c_1}{\sqrt{\alpha}}(1 - \sqrt{1-\alpha}) = \frac{1}{2}c_1\left(\sqrt{\alpha} + O(\alpha^{3/2})\right)$$

and finally

$$|(\Omega \cap B(0, A))\backslash S(\alpha, c)| \lesssim c_1^2\alpha. \tag{11.3.22}$$

Collecting (11.3.17), (11.3.21), and (11.3.22) we get

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)\cap S(\alpha(T))} |\nabla u(x,t)|^2 + |u(x,t)|^6 \; dxdt \lesssim c_1\frac{E}{T} + \frac{c_1}{\alpha}\frac{E}{T} + \frac{1}{\alpha}\frac{E}{c_1^3}.$$

We take (say) $c_1 = T^{1/5}$ and $\alpha = T^{-1/2}$. Then all the right hand terms go to zero as T goes to infinity, and

$$|(\Omega \cap B(0, A))\backslash S(\alpha, c)| \longrightarrow 0$$

by (11.3.22). □

11.3.3. Proof of Corollary 11.2

In order to deduce Corollary 11.2, we will need

Lemma 11.9. *Let $\{u(t),\ t \geq 0\}$ be a relatively compact family of $\dot{H}^1$, and S_k be a family of subsets of Ω such that the Lebesgue measure of S_k goes to zero as k goes to infinity. Then we have*

$$\sup_{t\geq 0} \int_{S_k} |\nabla u(t,x)|^2 + |u(t,x)|^6 dx \longrightarrow 0$$

as k goes to infinity.

Proof. If it is not the case, there exists a subsequence S_{n_k}, a sequence of times t_k and $\epsilon > 0$ such that

$$\forall k, \int_{S_{n_k}} |\nabla u(t_k)|^2 + |u(t_k)|^6 dx \geq \epsilon.$$

Because $\{u(t)\ t \geq 0\}$ is relatively compact in $\dot{H}^1$, we can suppose, up to extract a subsequence, that, as k goes to infinity

$$u(t_k) \longrightarrow u^\star \in \dot{H}^1 \text{ in } \dot{H}^1.$$

We have

$$\int_{S_{n_k}} |\nabla u(t_k)|^2 + |u(t_k)|^6 dx \leq \int_{S_{n_k}} |\nabla u^\star|^2 + |u^\star|^6 dx + \int_\Omega |\nabla(u^\star - u(t_k))|^2 + \int_\Omega |(u^\star - u(t_k))|^6 dx.$$

By the Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$, both terms are going to zero as k goes to infinity and we obtain a contradiction. □

We can now prove the rigidity corollary:

Proof of Corollary 11.2. Let u be a solution of (NLW) with a relatively compact flow $\{u(t),\ t \geq 0\}$ in $\dot{H}^1$. By lemma 11.9

$$\sup_{t\geq 0} \int_{(\Omega\cap B(0,A))\backslash S(T)} |\nabla u|^2(t,x) dx = \epsilon(|S(T)|),$$

where $\epsilon(h) \longrightarrow 0$ when $h \longrightarrow 0$, and in particular

$$\frac{1}{T}\int_0^T \int_{(\Omega\cap B(0,A))\backslash S(T)} |\nabla u|^2(t,x) dx = \epsilon(|S(T)|). \tag{11.3.23}$$

Thus, by theorem 11.1, u verifies the criterion (11.2.2) and thus scatters in $\dot{H}^1$. □

11.4. Obstacles illuminated by an ellipsoid

Motivated by the above result, we are interested by the scattering problem in non trapping geometries close to the exterior of two convex obstacles, such as dog bones with arbitrary thin neck. Theorem 11.4, which we will prove now, gives in particular the scattering in such settings. More precisely, it permits to handle obstacles illuminated by arbitrary cigar-shaped ellipsoids and a certain class of flat-shaped ones.

In order to show such a result using the Morawetz identity (11.2.1) to obtain the criterion of lemma 11.7, it is natural to choose the gauge of the ellipsoid we are dealing with as the weight function. The next lemma gives us the range of ellipsoids for which such a weight verify the bilaplacian constraint:

Lemma 11.10. *Let $n \geq 2$ and*

$$\rho(x) = \sqrt{x_1^2 + \cdots + x_k^2 + \epsilon(x_{k+1}^2 + \cdots + x_n^2)}.$$

Then,

$$\Delta^2 \rho \leq 0, \ \forall \epsilon \in [\epsilon_0, 1]$$

with

$$\epsilon_0 = \begin{cases} 0 & \text{if } k \geq 3, \\ \frac{1}{n} + \frac{\sqrt{2(n-2)(n-1)}}{n(n-2)} & \text{if } k = 2, \\ \frac{4}{n+1} & \text{if } k = 1. \end{cases}$$

Proof. An elementary computation gives

$$\Delta^2 \rho(x) = \frac{A(\epsilon)}{\rho^3} + \frac{B(\epsilon)(x_{k+1}^2 + \cdots + x_n^2)}{\rho^5} + \frac{C(\epsilon)(x_{k+1}^2 + \cdots + x_n^2)^2}{\rho^7}$$

where

$$A(\epsilon) = -(n-k+2)(n-k)\epsilon^2 - 2(n-k)(k-3)\epsilon - (k-1)(k-3),$$
$$B(\epsilon) = 6\epsilon((n-k+2)\epsilon^2 + [(2k-n)-5]\epsilon - k + 3),$$
$$C(\epsilon) = -15\epsilon^2(\epsilon - 1)^2.$$

In all cases, $C(\epsilon) \leq 0$.

If $k \geq 3$, it is clear that $A(\epsilon) \leq 0$. Let us denote $\tilde{B}(\epsilon) = \frac{B(\epsilon)}{6\epsilon} = (n-k+2)\epsilon^2 + [(2k-n)-5]\epsilon - k + 3$. Then $\tilde{B}(0) = -(k-3) \leq 0$, $\tilde{B}(1) = 0$, so $B(\epsilon) \leq 0$ for $\epsilon \in [0,1]$ and therefore $\Delta^2 \rho \leq 0$ for $\epsilon \in [0,1]$.

If $k = 2$, the roots of $A(\epsilon) = -n(n-2)\epsilon^2 + 2(n-2)\epsilon + 1$ are $\epsilon_{1,2} = \frac{1}{n} \pm \frac{\sqrt{2(n-2)(n-1)}}{n(n-2)}$ and the roots of $\tilde{B}(\epsilon) = n\epsilon^2 - (n+1)\epsilon + 1$ are $\frac{1}{n}$ and 1, so $\Delta^2 \rho \leq 0$ for $\epsilon \in [\frac{1}{n} + \frac{\sqrt{2(n-2)(n-1)}}{n(n-2)}, 1]$.

If $k = 1$, $A(\epsilon) = -(n+1)(n-1)\epsilon^2 + 4(n-1)\epsilon$ is non-positive if and only if $\epsilon \geq \frac{4}{n+1}$, and the roots of $\tilde{B}(\epsilon) = (n+1)\epsilon^2 - (n+3)\epsilon + 2$ are $\frac{2}{n+1}$ and 1, so $\Delta^2 \rho \leq 0$ for all ϵ in $[\frac{4}{n+1}, 1]$. □

Notice that in dimension three, this weight cannot be explicitly used as its bilaplacian is not non positive – $\epsilon_0 > 0$ in the previous lemma. Hence, to derive the control (11.2.2), we will extend u as a solution of a four dimensional non linear wave equation, get the control for this extended solution, and manage to go back to u.

To this purpose, we need the following existence result for the four dimensional problem:

Lemma 11.11. *Let Ω be a smooth domain of $\mathbb{R}^4$ with compact boundary, and $(u_0, u_1) \in \dot{H}^{\frac{7}{4}}(\Omega) \times H^{\frac{3}{4}}(\Omega)$. Then, there exists a unique global solution of*

$$\begin{cases}(\partial_t^2 - \Delta)u + u^5 = 0 & \text{in } \mathbb{R}\times\Omega \\ u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|\mathbb{R}\times\partial\Omega} = 0\end{cases} \tag{11.4.1}$$

satisfying

$$u \in C(\mathbb{R}, \dot{H}^{\frac{7}{4}}(\Omega) \cap L^6(\Omega)) \cap C^1(\mathbb{R}, H^{\frac{3}{4}}(\Omega)) \cap L^{48}(\mathbb{R}, L^6(\Omega)).$$

Proof. Let $0 < T < 1$. By the work of [BSS09b], Theorem 1.1, applied to the admissible triple $(p = 48, q = 6, \gamma = \frac{7}{4})$ in dimension 4, if u is solution of

$$\begin{cases}(\partial_t^2 - \Delta)u = F & \text{in } (0,T)\times\Omega \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g, \quad u_{|\mathbb{R}\times\partial\Omega} = 0\end{cases} \tag{11.4.2}$$

then the following Strichartz estimate holds

$$\|u\|_{L^{48}((0,T),L^6(\Omega))} \leq C\left(\|f\|_{\dot{H}^{\frac{7}{4}}(\Omega)} + \|g\|_{H^{\frac{3}{4}}(\Omega)} + \|F\|_{L^1((0,T),L^2(\Omega))}\right).$$

Using this estimate, we obtain the local existence following a classical fixed point method, in the space

$$X_T = C^0(\mathbb{R}, \dot{H}^{\frac{7}{4}}(\Omega) \cap L^6(\Omega)) \cap C^1(\mathbb{R}, H^{\frac{3}{4}}(\Omega)) \cap L^{48}(\mathbb{R}, L^6(\Omega))$$

for T sufficiently small depending of $\|u_0\|_{\dot{H}^{\frac{7}{4}}(\Omega)}$ and $\|u_1\|_{H^{\frac{3}{4}}(\Omega)}$. The global existence result follows using the energy conservation law. □

We are now in position to show:

Proposition 11.12. *Let $\mathcal{C} \subset \mathbb{R}^3$ be the ellipsoid of equation (11.1.3), resp. (11.1.4), $\mathcal{K}$ be a compact subset of $\mathbb{R}^3$ illuminated by $\mathcal{C}$ and $\Omega := \mathbb{R}^3 \backslash \mathcal{K}$. Let $u \in C(\mathbb{R}, \dot{H}^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$ be the global solution of (11.1.1) in Ω. Then*

$$\frac{1}{T}\int_0^T \int_{\Omega\cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \leq \frac{1}{\ln T} C(E(u)).$$

Proof. Let $\delta > 0$. There exists u_0^δ, u_1^δ, smooth functions vanishing on $\partial\Omega$ such that

$$\|u_0 - u_0^\delta\|_{\dot{H}^1(\Omega)} + \|u_1 - u_1^\delta\|_{L^2(\Omega)} \leq \delta.$$

We denote by $u^\delta \in C^0(\mathbb{R}, \dot{H}^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$ the solution of (11.1.1) in Ω with data (u_0^δ, u_1^δ).

Let $T > 0$ and $\phi \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ on $[-1, 1]$ and $\phi = 0$ on $[-2, 2]^c$. We take $\chi_T = \phi(\frac{\cdot}{2T})$ and, for $(x, z) \in \Omega \times \mathbb{R}$

$$v_0^\delta(x,z) = u_0^\delta(x)\chi_T(z)\chi_T(|x|),$$
$$v_1^\delta(x,z) = u_1^\delta(z)\chi_T(z)\chi_T(|x|).$$

Let us denote by ρ the gauge of ellipsoid we are dealing with, consider

$$\tilde{\mathcal{K}} = \mathcal{K} \times [-4T, 4T],\ \tilde{\Omega} = \mathbb{R}^4 \backslash \tilde{\mathcal{K}}$$
$$\tilde{\rho}(x,z) = \sqrt{\rho(x)^2 + z^2},$$

and $v^\delta \in C(\mathbb{R}, H^{\frac{7}{4}}(\tilde{\Omega})) \cap C^1(\mathbb{R}, H^{\frac{3}{4}}(\tilde{\Omega}))$ the solution of the four dimensional equation

$$\begin{cases} (\partial_t^2 - \Delta)v^\delta + (v^\delta)^5 = 0 \quad \text{in } \mathbb{R} \times \Omega \\ v^\delta_{|t=0} = v_0^\delta, \quad \partial_t v^\delta_{|t=0} = v_1^\delta, \quad v^\delta_{|\mathbb{R}\times\partial\Omega} = 0 \end{cases} \tag{11.4.3}$$

given by lemma 11.11.

Notice that, by the finite speed of propagation,

$$v^\delta(x,z,t) = u^\delta(x,t), \tag{11.4.4}$$
$$\forall x \in \Omega \cap B(0, 2T-t),\ \forall t \in [0, 2T[,\ \forall z \in [-2T+t, 2T-t].$$

We denote by n the outward pointing normal vector to $\partial\mathcal{K}$ and $\tilde{n} = (n, 0)$ the outward pointing normal vector to $\partial\tilde{\mathcal{K}}$. The momentum derivation (11.2.1) applied to v^δ gives

$$\frac{d}{dt}\left(-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla\tilde{\rho} - \frac{1}{2}\int_{\tilde{\Omega}} \Delta\tilde{\rho} v^\delta \partial_t v^\delta\right) = \int_{\tilde{\Omega}} (D^2\tilde{\rho}\nabla v^\delta, \nabla v^\delta) + \frac{1}{4}\Delta\tilde{\rho}|v^\delta|^6 - \Delta^2\tilde{\rho}|v^\delta|^2$$
$$+ \frac{1}{2}\int_{\partial\tilde{V}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}. \tag{11.4.5}$$

By lemma 11.10, $\Delta^2\tilde{\rho} \leq 0$. Moreover, as $\tilde{\rho}$ is convex, $(D^2\tilde{\rho}\nabla v^\delta, \nabla v^\delta) \geq 0$ and $\Delta\tilde{\rho} \geq 0$. Therefore, integrating (11.4.5) we obtain

$$\frac{1}{2}\int_0^T \int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dt \leq \left[-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla\tilde{\rho} - \frac{1}{2}\int_{\tilde{\Omega}} \Delta\tilde{\rho} v^\delta \partial_t v^\delta\right]_0^T. \tag{11.4.6}$$

Let us take $T > 0$ large enough so that $\partial V \subset \Omega \cap B(0,T)$. Then, for $t \in [0,T]$,

$$\int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\tilde{\sigma} = \int_{\mathbb{R}}\int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz \geq \int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz.$$

But, by finite speed of propagation (11.4.4),

$$\int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz = \int_{-T}^T \int_{\partial K} |\partial_n u^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz,$$

so we obtain, as $\partial_n \rho \geq C$ by the definition of an illuminated subset

$$\int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\tilde{\sigma} \geq \int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} u^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz$$
$$= \int_{-T}^T \int_{\partial K} |\partial_n u^\delta|^2 \frac{\rho}{\sqrt{\rho^2 + z^2}} \partial_n \rho\, d\sigma dz \gtrsim \int_{\partial K} |\partial_n u^\delta|^2 \int_{-T}^T \frac{1}{\sqrt{1+z^2}}\, dz d\sigma,$$

and the integration of the right hand side gives

$$\int_{\partial K} |\partial_n u^\delta|^2 \, d\sigma \lesssim \frac{1}{\ln T} \int_{\partial \tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}} \tilde{\rho} \, d\tilde{\sigma}. \tag{11.4.7}$$

Moreover, as $\nabla \tilde{\rho}$ is bounded and $0 \leq \Delta \tilde{\rho} \leq \frac{C}{|(x,z)|}$, by the Cauchy-Schwarz inequality and the Hardy inequality

$$\int_\Omega \frac{|f|^2}{|x|^2} \lesssim \int_\Omega |\nabla f|^2 \text{ for } f \in \dot{H}^1_0(\Omega)$$

we obtain

$$\left| \left[-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla \tilde{\rho} - \frac{1}{2} \int_{\tilde{\Omega}} \Delta \tilde{\rho} v^\delta \partial_t v^\delta \right]_0^T \right| \lesssim E(v^\delta), \tag{11.4.8}$$

and this last inequality combined to (11.4.6) and (11.4.7) gives

$$\int_0^T \int_{\partial \Omega} (\partial_n u^\delta)^2 \, d\sigma dt \lesssim \frac{1}{\ln T} E(v^\delta). \tag{11.4.9}$$

It remains to estimate the energy of v^δ. We have

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla v_0^\delta|^2 &= \int_{\tilde{\Omega}} |\nabla u_0^\delta(x) \chi_T(z) \chi_T(|x|) + u_0^\delta(x) \chi_T(z) \chi_T'(|x|) \cdot \frac{x}{|x|}|^2 \, dxdz \\
&\quad + \int_{\tilde{\Omega}} |u_0^\delta(x)|^2 \chi_T'(z)^2 \chi_T(|x|)^2 \, dxdz \\
&\lesssim \int_{\tilde{\Omega}} |\nabla u_0^\delta(x)|^2 \chi_T(z)^2 \chi_T(|x|)^2 \, dxdz + \int_{\tilde{\Omega}} |u_0^\delta(x)|^2 \chi(z)^2 \chi'(|x|)^2 \, dxdz \\
&\quad + \int_{\tilde{\Omega}} |u_0^\delta(x)|^2 \chi_T'(z)^2 \chi_T(|x|)^2 \, dxdz,
\end{aligned}$$

so, by the Hölder inequality

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla v_0^\delta|^2 &\lesssim \int_\Omega |\nabla u_0^\delta(x)|^2 dx \int_{\mathbb{R}} \chi_T(z)^2 \, dz + \left(\int_\Omega |u_0^\delta(x)|^6 dx \right)^{\frac{1}{3}} \left(\int_\Omega \chi_T'(|x|)^3 dx \right)^{\frac{2}{3}} \int_{\mathbb{R}} \chi_T(z)^2 dz \\
&\quad + \left(\int_\Omega |u_0^\delta(x)|^6 dx \right)^{\frac{1}{3}} \left(\int_\Omega \chi_T(|x|)^3 dx \right)^{\frac{2}{3}} \int_{\mathbb{R}} \chi_T'(z)^2 dz \\
&\lesssim \int_\Omega |\nabla u_0^\delta|^2 \int_{-4T}^{4T} \|\phi\|_\infty^2 + \left(\int_\Omega |u_0^\delta|^6 \right)^{\frac{1}{3}} \left(\int_{B(0,4T)} \left(\frac{1}{2T} \|\phi'\|_{L^\infty} \right)^3 \right)^{\frac{2}{3}} \int_{-4T}^{4T} \|\phi\|_{L^\infty}^2 \\
&\quad + \left(\int_\Omega |u_0^\delta|^6 \right)^{\frac{1}{3}} \left(\int_{B(0,4T)} \|\phi\|_{L^\infty}^3 \right)^{\frac{2}{3}} \int_{-4T}^{4T} \frac{1}{4T^2} \|\phi'\|_{L^\infty}^2 \\
&\lesssim T \int_\Omega |\nabla u_0^\delta|^2 + T \left(\int_\Omega |u_0^\delta|^6 \right)^{\frac{1}{3}} + T \left(\int_\Omega |u_0^\delta|^6 \right)^{\frac{1}{3}} \\
&\lesssim TC(E(u^\delta)).
\end{aligned}$$

Moreover,

$$\int_{\tilde{\Omega}} |v_0^\delta|^6 = \int_\Omega \int_{\mathbb{R}} |u_0^\delta(x)|^6 \chi_T(z)^6 \chi_T(|x|)^6 dxdz \leq 4T \int_\Omega |u_0^\delta|^6,$$

and

$$\int_{\tilde{\Omega}} |v_1^\delta|^2 = \int_\Omega \int_{\mathbb{R}} |u_1^\delta(x)|^2 \chi_T(z)^2 \chi_T(|x|)^2 dxdz \leq 4T \int_\Omega |u_1^\delta|^2,$$

therefore

$$E(v^\delta) \lesssim TC(E(u^\delta)),$$

and by (11.4.8) we obtain

$$\int_0^T \int_{\partial K} |\partial_n u^\delta|^2 \, d\sigma dt \lesssim \frac{T}{\ln T} C(E(u^\delta)). \tag{11.4.10}$$

Notice that we can not pass to the limit directly in this expression because as mentioned before, the application

$$u \in \dot{H}^1 \cap \{\text{value in time } t \text{ of solutions of NLW}\} \longrightarrow \partial_n u \in L^2(\partial\Omega)$$

is not known to be continuous. But, notice that using lemma 11.6 with the weight $\chi = |x|^2$ gives in particular

$$\int_0^T \int_{\Omega \cap B(0,A)} |\nabla u^\delta(x,t)|^2 + |u^\delta(x,t)|^6 \, dxdt \lesssim \int_0^T \int_{\partial K} |\partial_n u^\delta|^2 \, d\sigma dt.$$

Therefore, combining the last inequality with (11.4.10) we get

$$\frac{1}{T} \int_0^T \int_{\Omega \cap B(0,A)} |\nabla u^\delta(x,t)|^2 + |u^\delta(x,t)|^6 \, dxdt \lesssim \frac{1}{\ln T} C(E(u^\delta)),$$

and we can let δ go to zero in this expression: as $u_0^\delta \underset{\delta \to 0}{\longrightarrow} u_0$ in $\dot{H}^1$ and $u_1^\delta \underset{\delta \to 0}{\longrightarrow} u_1$ in L^2, we obtain using the energy conservation law that $C(E(u^\delta)) \underset{\delta \to 0}{\longrightarrow} C(E(u))$, and, because the problem (11.1.1) is well posed, the left hand side goes as δ goes to zero to

$$\frac{1}{T} \int_0^T \int_{\Omega \cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt,$$

and the proposition holds. □

Finally, we can conclude:

Proof of Theorem 11.4. The above proposition combined with the scattering criterion of lemma 11.7 gives immediately the result. □

12 Scattering for NLS with a sum of two repulsive potentials

12.1. Introduction

We are concerned by the following defocusing non-linear Schrödinger equation with a potential

$$i\partial_t u + \Delta u - Vu = u|u|^{\alpha},\ u(0) = \varphi \in H^1. \tag{12.1.1}$$

in arbitrary spatial dimension $d \geq 1$. Once good dispersive properties of the linear flow, such as Strichartz estimates described below in the paper, are established, the local well-posedness of (12.1.1) follows by usual fixed point arguments. Because of the energy conservation law,

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 + \int V|u(t)|^2 + \frac{1}{\alpha+2}\int |u(t)|^{\alpha+2} = E(u(0))$$

this result extends to global well-posedness. Thus, it is natural to investigate the asymptotic behavior of solutions of (12.1.1).

It is well-known since Nakanishi's paper [Nak99] that for $V = 0$, in the intercritical regime

$$\frac{4}{d} < \alpha < \frac{4}{d-1}, \tag{12.1.2}$$

the solutions *scatter* in $H^1(\mathbb{R}^d)$, that is, for every solution $u \in C(\mathbb{R},H^1(\mathbb{R}^d))$ of (12.1.1), there exists a unique couple of data $\psi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\|u(t) - e^{-it\Delta}\psi_\pm\|_{H^1(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

The inhomogeneous setting $V \neq 0$ was investigated more recently, for example in [BV16], [Laf16], [Hon14], [FV18]. However, all these scattering results rely on a non-trapping assumption, namely, that the potential is *repulsive:*

$$x \cdot \nabla V \leq 0,$$

or, as in [Car16], that his non-repulsive part is sufficiently small. The aim of this paper is to establish a first - as far as we know - scattering result in a trapping situation. More precisely, we are interested in one of the simplest of such frameworks, that is, in the case where V is the sum of two positive, repulsive potentials with convex level sets, which is the potential-analog of the homogeneous problem outside two strictly convex obstacles.

Let V_1 and V_2 be two positive and strictly repulsive potentials, that is, there exists a_1 and a_2 in $\mathbb{R}^d$ such that

$$(x - a_{1,2}) \cdot \nabla V_{1,2} < 0.$$

Moreover, we suppose that the level surfaces of V_1 and V_2 are strictly convex. Let

$$V := V_1 + V_2.$$

We will suppose that V follows the following decay assumption

$$V, \nabla V \in L^{\delta}(\mathbb{R}^d, (1 + |x|^{\beta})dx), \;\; \beta > \frac{2}{3} \tag{12.1.3}$$

where

$$\delta = \begin{cases} 1 & \text{if } d = 1, \\ \frac{d}{2} & \text{if } d > 1. \end{cases}$$

This assumption is the (improved) multi-dimensional analog of the decay assumption arising in [Laf16]. Finally, we will assume that the dispersive pointwise estimate

$$\|e^{it(-\Delta+V)}\|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{|t|^{d/2}} \tag{12.1.4}$$

holds. Note that, as remarked in [Laf16], this last assumption is automatically verified using Goldberg and Schlag's result [GS04] under the non-negativity and decay assumptions with $\beta \geq 1$ in dimension $d = 1$, $\beta \geq 2$ in dimension $d = 3$. Our main result reads

Theorem 12.1. *Assume that $d \geq 3$. Let V_1 and V_2 be two non-negative, strictly repulsive potentials with strictly convex level surfaces. Assume moreover that $V = V_1 + V_2$ verify the decay assumption (12.1.3), and the dispersive estimate (12.1.4). Then, in the intercritical regime (12.1.2), every solution of (12.1.1) with potential $V = V_1 + V_2$ scatters in $H^1(\mathbb{R}^d)$.*

As in the afforementioned papers, we use the strategy of concentration-compactness and rigidity first introduced by Kenig and Merle in [KM06]: assuming that there exists a finite energy above which solutions does not scatter, one constructs a compact-flow solution and eliminates it. Notice that in the case of a repulsive potential, this last rigidity part is immediate by classical Morawetz estimates. It will be here the main difficulty to overcome and the novelty of this note. After some preliminaries, we construct a critical solution in the second section, following [Laf16] and generalizing it to any spatial dimension. In the last section, we eliminate it using a Morawetz multiplier that almost vanish on the trapped trajectory $[a_1 a_2]$.

Remark 12.2. We assume that $d \neq 2$ because our proof rely on endpoint Strichartz estimates that are not true in dimension two, and the convexity assumption we make on the potentials have no sense in the one dimensional case.

Remark 12.3. The first two sections of this paper generalize in particular the one-dimensional result of [Laf16], to any spatial dimension $d \geq 3$.

Remark 12.4. This framework is the potential-analog of the homogeneous problem outside two strictly convex obstacles. This is the subject of a work in progress [LL]. A rigidity argument in the particular case of two balls can be found in [Laf17a].

Remark 12.5. It is straightforward from the last section that the result is still valid for an arbitrary finite sum of convex repulsive potentials $V = V_1 + \cdots + V_N$ for which the centers a_k are aligned. However, we present the proof for two potentials in the seek of simplicity.

12.2. Preliminaries

12.2.1. Useful exponents

From now on, we will fix the three following Strichartz exponents

$$r = \alpha + 2, \; q = \frac{2\alpha(\alpha+2)}{d\alpha^2 - (d-2)\alpha - 4}, \; p = \frac{2(\alpha+2)}{4-(d-2)\alpha}.$$

Moreover, let η be the conjugate of the critical exponent $2^\star$:

$$\frac{1}{2^\star} + \frac{1}{\eta} = 1. \tag{12.2.1}$$

Notice, for the sequel, the following two identities

$$\frac{1}{\delta} + \frac{1}{2^\star} = \frac{1}{\eta}, \tag{12.2.2}$$

and

$$\frac{1}{\delta} + \frac{2}{2^\star} = 1. \tag{12.2.3}$$

Finally, let γ be such that (γ, η') follows the admissibility condition of Theorem 1.4 of Foshi's inhomogeneous Strichartz estimates [Fos05]. Note that, in the intercritical regime (12.1.2), all these exponents are well defined and larger than one.

12.2.2. Strichartz estimates

Let us recall that $e^{-it(-\Delta+V)}$ verifies the pointwise dispersive estimates (12.1.4), by [GS04] in dimensions $d = 1, 3$ for $\beta \geq 1$ (resp $\beta \geq 2$), or by assumption in other cases. Interpolating it with the mass conservation law, we obtain immediately for all $a \in [2, \infty]$

$$\|e^{it(-\Delta+V)}\psi\|_{L^a} \lesssim \frac{1}{|t|^{\frac{d}{2}(\frac{1}{a'}-\frac{1}{a})}} \|\psi\|_{L^{a'}}. \tag{12.2.4}$$

Moreover, it leads by the classical $TT^\star$ method (see for example [KT98]) to the Strichartz estimates

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^{q_1}L^{r_1}} + \|\int_0^t e^{-i(t-s)(-\Delta+V)}F(s)ds\|_{L^{q_2}L^{r_2}} \lesssim \|\varphi\|_{L^2} + \|F\|_{L^{q_3'}L^{r_3'}} \quad (12.2.5)$$

for all pairs (q_i, r_i) satisfying the admissibility condition

$$\frac{2}{q_i} + \frac{d}{r_i} = \frac{d}{2}, \ (q_i, r_i, d) \neq (2, \infty, 2).$$

We will use moreover the following Strichartz estimates associated to non admissible pairs:

Proposition 12.6 (Strichartz estimates)**.** *For all $\varphi \in H^1$, all $F \in L^{q'}L^{r'}$, all $G \in L^{q'}L^{r'}$ and all $H \in L^{\gamma'}L^{\eta}$*

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^pL^r} \lesssim \|\varphi\|_{H^1} \quad (12.2.6)$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}F(s)ds\|_{L^\alpha L^\infty} \lesssim \|F\|_{L^{q'}L^{r'}} \quad (12.2.7)$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}G(s)ds\|_{L^pL^r} \lesssim \|G\|_{L^{q'}L^{r'}} \quad (12.2.8)$$

$$\|\int_0^t e^{-i(t-s)(-\Delta+V)}H(s)ds\|_{L^pL^r} \lesssim \|H\|_{L^{\gamma'}L^{\eta}}. \quad (12.2.9)$$

Proof. The estimate (12.2.6) follows from admissible Strichartz estimate

$$\|e^{-it(-\Delta+V)}\varphi\|_{L^pL^{\frac{2dp}{dp-4}}} \lesssim \|\varphi\|_{L^2}$$

together with a Sobolev embedding. The estimate (12.2.8) is contained in Lemma 2.1 of [CW92]. Finally, (12.2.7) and (12.2.9) enters on the frame of the non-admissible inhomogeneous Strichartz estimates of Theorem 1.4 of Foschi's paper [Fos05]. □

12.2.3. Perturbative results

The three following classical perturbative results, follow immediately from the previous Strichartz inequalities with exact same proof as in [Laf16].

Proposition 12.7. *Let $u \in C(H^1)$ be a solution of (12.1.1). If $u \in L^pL^r$, then u scatters in H^1.*

Proposition 12.8. *There exists $\epsilon_0 > 0$, such that, for every data $\varphi \in H^1$ such that $\|\varphi\|_{H^1} \leq \epsilon_0$, the corresponding maximal solutions of (12.1.1) and (10.1.2) both scatter in H^1.*

Proposition 12.9. *For every $M > 0$ there exists $\epsilon > 0$ and $C > 0$ such that the following occurs. Let $v \in C(H^1) \cap L^pL^r$ be a solution of the following integral equation with source term $e(t, x)$*

$$v(t) = e^{-it(\Delta - V)}\varphi - i\int_0^t e^{-i(t-s)(\Delta - V)}(v(s)|v(s)|^\alpha)ds + e(t)$$

with $\|v\|_{L^pL^r} < M$ and $\|e\|_{L^pL^r} < \epsilon$. Assume moreover that $\varphi_0 \in H^1$ is such that $\|e^{-it(\Delta-V)}\varphi_0\|_{L^pL^r} < \epsilon$. Then, the solution $u \in C(H^1)$ to (12.1.1) with initial condition $\varphi + \varphi_0$ satisfies

$$u \in L^pL^r, \quad \|u - v\|_{L^pL^r} < C.$$

12.3. Construction of a critical solution

The aim of this section is to extend the construction of a critical element of [Laf16] to any dimension $d \neq 2$ – no repulsivity assumption is used in this first part of this work. This previous paper itself follows [BV16] which deals with a Dirac potential, which is more singular but for which explicit formulas are at hand. More precisely, let

$$E_c = \sup\left\{E > 0 \mid \forall \varphi \in H^1,\ E(\varphi) < E \Rightarrow \text{the sol. of (1.1) with data } \varphi \text{ is in } L^pL^r\right\}.$$

We will prove

Theorem 12.10. *If $E_c < \infty$, then there exists $\varphi_c \in H^1$, $\varphi_c \neq 0$, such that the corresponding solution u_c of (12.1.1) has a relatively compact flow $\{u_c(t),\ t \geq 0\}$ in H^1.*

As this theorem was obtained in [Laf16] in the one dimensional case, we assume all along this section that $d \geq 3$.

12.3.1. Profile decomposition

We first show, with the same method as in [Laf16], extended to any dimension, that we can use the abstract profile decomposition obtained by [BV16]:

Theorem (Attract profile decomposition, [BV16])**.** *Let $A : L^2 \supset D(A) \to L^2$ be a self adjoint operator such that:*

- *for some positive constants c, C and for all $u \in D(A)$,*

$$c\|u\|_{H^1}^2 \leq (Au, u) + \|u\|_{L^2}^2 \leq C\|u\|_{H^1}^2, \qquad (12.3.1)$$

- *let $B : D(A) \times D(A) \ni (u, v) \to (Au, v) + (u, v)_{L^2} - (u, v)_{H^1} \in \mathbb{C}$. Then, as n goes to infinity*

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0 \quad \forall \psi \in H^1 \qquad (12.3.2)$$

as soon as

$$x_n \to \pm\infty, \quad \sup \|h_n\|_{H^1} < \infty$$

or

$$x_n \to \bar{x} \in \mathbb{R}, \quad h_n \underset{H^1}{\rightharpoonup} 0,$$

- *let $(t_n)_{n\geq 1}$, $(x_n)_{n\geq 1}$ be sequences of real numbers, and $\bar{t}, \bar{x} \in \mathbb{R}$. Then*

$$|t_n| \to \infty \Longrightarrow \|e^{it_nA}\tau_{x_n}\psi\|_{L^p} \to 0, \quad \forall 2 < p < \infty, \ \forall \psi \in H^1 \tag{12.3.3}$$

$$t_n \to \bar{t}, \ x_n \to \pm\infty \Longrightarrow \forall \psi \in H^1, \exists \varphi \in H^1, \quad \tau_{-x_n}e^{it_nA}\tau_{x_n}\psi \overset{H^1}{\to} \varphi \tag{12.3.4}$$

$$t_n \to \bar{t}, \ x_n \to \bar{x} \Longrightarrow \forall \psi \in H^1, \quad e^{it_nA}\tau_{x_n}\psi \overset{H^1}{\to} e^{i\bar{t}A}\tau_{\bar{x}}\psi. \tag{12.3.5}$$

And let $(u_n)_{n\geq 1}$ be a bounded sequence in H^1. Then, up to a subsequence, the following decomposition holds

$$u_n = \sum_{j=1}^{J} e^{it_j^nA}\tau_{x_n^j}\psi_j + R_n^J \quad \forall J \in \mathbb{N}$$

where

$$t_j^n \in \mathbb{R}, \ x_j^n \in \mathbb{R}, \ \psi_j \in H^1$$

are such that

- *for any fixed j,*

$$t_j^n = 0 \ \forall n, \quad \text{or} \quad t_n^j \overset{n\to\infty}{\to} \pm\infty \tag{12.3.6}$$

$$x_j^n = 0 \ \forall n, \quad \text{or} \quad x_n^j \overset{n\to\infty}{\to} \pm\infty, \tag{12.3.7}$$

- *orthogonality of the parameters:*

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \overset{n\to\infty}{\to} \infty, \quad \forall j \neq k, \tag{12.3.8}$$

- *decay of the remainder:*

$$\forall \epsilon > 0, \exists J \in \mathbb{N}, \quad \limsup_{n\to\infty} \|e^{-itA}R_n^J\|_{L^\infty L^\infty} \leq \epsilon, \tag{12.3.9}$$

- *orthogonality of the Hilbert norm:*

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^2}^2 + \|R_n^J\|_{L^2}^2 + o_n(1), \quad \forall J \in \mathbb{N} \tag{12.3.10}$$

$$\|u_n\|_H^2 = \sum_{j=1}^{J} \|\tau_{x_n^j}\psi_j\|_H^2 + \|R_n^J\|_H^2 + o_n(1), \quad \forall J \in \mathbb{N} \tag{12.3.11}$$

where $(u, v)_H = (Au, v)$, and

$$\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \|e^{it_j^nA}\tau_{x_n^j}\psi_j\|_{L^p}^p + \|R_n^J\|_{L^p}^p + o_n(1), \quad \forall 2 < p < 2^\star, \quad \forall J \in \mathbb{N}. \tag{12.3.12}$$

Let us show that the self-adjoint operator $A := -\Delta + V$ verifies the hypothesis of the previous theorem.

Proposition 12.11. *Let $A := -\Delta + V$. Then A satisfies the assumptions (12.3.1), (12.3.2),(12.3.3),(12.3.4),(12.3.5).*

Proof. **Assumption (12.3.1).** Because V is non-negative, by the Hölder inequality - let us recall that δ verifies (12.2.3) - and the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$,

$$\|u\|_{H^1}^2 \leq (Au, u) + \|u\|_{L^2} = \int |\nabla u|^2 + \int V|u|^2 + \int |u|^2 \leq \|u\|_{H^1}^2 + \|V\|_{L^\delta} \|u\|_{L^{2^\star}}^2 \leq (1 + C_{\text{Sobolev}} \|V\|_{L^\delta}) \|u\|_{H^1}^2.$$

and (12.3.1) holds.

Assumption (12.3.2). We have

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) = \int V \tau_{x_n}\psi \overline{\tau_{x_n}h_n}.$$

Assume that $x_n \to \bar{x} \in \mathbb{R}$ and $h_n \underset{H^1}{\rightharpoonup} 0$. Notice that B can also be written

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) = \int (\tau_{-x_n}V)\psi\overline{h_n}.$$

By Sobolev embedding, $h_n \rightharpoonup 0$ weakly in $L^{2^\star}$. Moreover, $\tau_{-x_n}V \to \tau_{-\bar{x}}V$ strongly in L^δ. Therefore, because $\psi \in L^{2^\star}$ by Sobolev embedding again, it follows from (12.2.3) that $B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0$.

Now, let us assume that

$$x_n \to +\infty, \ \sup \|h_n\|_{H^1} < \infty.$$

We fix $\epsilon > 0$. By the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$, we can choose $\Lambda > 0$ large enough so that

$$\|\psi\|_{L^{2^\star}(|x|\geq\Lambda)} \leq \epsilon. \tag{12.3.13}$$

Because $V \in L^\delta$, Λ can also be chosen large enough so that

$$\|V\|_{L^\delta(|x|\geq\Lambda)} \leq \epsilon. \tag{12.3.14}$$

Then, by the Hölder inequality - recall that η is defined in (12.2.1) as the conjugate of $2^\star$ -, by Sobolev embedding and the Minkowski inequality

$$|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \leq \|h_n\|_{L^{2^\star}} \|V\tau_{x_n}\|_{L^\eta} \lesssim \sup_{j\geq 1} \|h_j\|_{H^1} \left(\|V\psi(\cdot - x_n)\|_{L^\eta(|x-x_n|\geq\Lambda)} + \|V\psi(\cdot - x_n)\|_{L^\eta(|x-x_n|\leq\Lambda)} \right).$$

Thus, by the Hölder inequality again, using this time (12.2.2), we have

$$|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \lesssim \|V\|_{L^\delta} \|\psi \mathbf{1}_{|x|\geq\Lambda}\|_{L^{2^\star}} + \|V\mathbf{1}_{|x-x_n|\leq\Lambda}\|_{L^\delta} \|\psi\|_{L^{2^\star}}. \tag{12.3.15}$$

Now, let n_0 be large enough so that for all $n \geq n_0$, $x_n \geq 2\Lambda$. Then, for all $n \geq n_0$

$$|x - x_n| \leq \Lambda \Rightarrow |x| \geq \Lambda$$

and, for all $n \geq n_0$ we get by (12.3.13), (12.3.14), (12.3.15)

$$|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \lesssim (\epsilon\|V\|_{L^\delta} + \epsilon\|\psi\|_{L^{2^\star}})$$

so (12.3.2) holds.

Assumption (12.3.3). The same proof as in [Laf16] holds: it is an immediate consequence of the pointwise dispersive estimate (12.2.4) and the translation invariance of the L^p norms. Notice that the estimate

$$\|e^{itA}f\|_{H^1} \lesssim \|f\|_{H^1}, \tag{12.3.16}$$

which is useful to close the density argument of this previous paper, generalizes to dimensions $d \geq 2$ because, as V is positive and in L^δ, by the Hölder inequality together with the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$ we get

$$\|\nabla f\|_{L^2}^2 \leq \|(-\Delta + V)^{\frac{1}{2}} f\|_{L^2}^2 = \int |\nabla u|^2 + \int V|u|^2 \leq \|f\|_{H^1}^2 + \|V\|_{L^\delta}\|u\|_{L^{2^\star}}^2 \lesssim \|f\|_{H^1}^2.$$

from which (12.3.16) follows because e^{itA} commute with $(-\Delta + V)^{\frac{1}{2}}$ and is an isometry on L^2.

Assumption (12.3.4). We will show that

$$t_n \to \bar{t},\ x_n \to +\infty \ \Rightarrow\ \|\tau_{-x_n} e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-i\bar{t}\Delta}\psi\|_{H^1} \to 0$$

and (12.3.4) will hold with $\varphi = e^{-i\bar{t}\Delta}\psi$. As remarked in [Laf16], it is sufficient to show that

$$\|e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\|_{H^1} \to 0. \tag{12.3.17}$$

Notice $e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$i\partial_t u - \Delta u + Vu = Ve^{-it\Delta}\tau_{x_n}\psi.$$

Therefore, by the inhomogeneous Strichartz estimates, as $(2, 2^\star)$ is admissible in dimension $d \geq 3$ with dual exponent $(2, \eta)$, and because the translation operator commutes with $e^{-it\Delta}$, we have for n large enough so that $t_n \in (0, \bar{t}+1)$

$$\begin{aligned}
&\|e^{it_n(-\Delta+V)}\tau_{x_n}\psi - e^{-it_n\Delta}\tau_{x_n}\psi\|_{L^2} \\
&\quad \leq \|e^{it(-\Delta+V)}\tau_{x_n}\psi - e^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,\bar{t}+1)L^2} \leq \|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^2(0,\bar{t}+1)L^\eta} \\
&\qquad = \|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^2(0,\bar{t}+1)L^\eta} \leq (\bar{t}+1)^{1/2}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\eta}.
\end{aligned}$$

Hence, estimating in the same manner the gradient of these quantities, it is sufficient to obtain (12.3.17) to show that, as n goes to infinity

$$\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)W^{1,\eta}} \to 0. \tag{12.3.18}$$

Let us fix $\epsilon > 0$. By Sobolev embedding in $L^{2^\star}$, because $e^{-it\Delta}\psi \in C([0,\bar{t}+1], H^1)$ and using the compactness in time, there exists $\Lambda > 0$ such that

$$\|e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^{2^\star}(|x|\geq\Lambda)} \leq \epsilon. \tag{12.3.19}$$

On the other hand, as $V \in L^\delta$, Λ can also be taken large enough so that

$$\|V\|_{L^\delta(|x|\geq\Lambda)} \leq \epsilon. \tag{12.3.20}$$

Let n_0 be large enough so that for all $n \geq n_0$, $x_n \geq 2\Lambda$. Then, for $n \geq n_0$

$$|x + x_n| \leq \Lambda \Rightarrow |x| \geq \Lambda$$

and for all $t \in (0, \bar{t}+1)$ and all $n \geq n_0$ we obtain, by Minkowski inequality, Hölder inequality together with (12.3.19) and (12.3.20), and Sobolev embedding

$$\begin{aligned}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\eta} &\leq \|V(\cdot+x_n)e^{-it\Delta}\psi\|_{L^\eta(|x+x_n|\geq\Lambda)} + \|V(\cdot+x_n)e^{-it\Delta}\psi\|_{L^\eta(|x+x_n|\leq\Lambda)} \\ &\leq \epsilon\|e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^{2^\star}} + \epsilon\|V\|_{L^\delta} \lesssim \epsilon(\|e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)H^1} + \|V\|_{L^\delta}).\end{aligned}$$

thus

$$\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\eta} \to 0.$$

With the same argument, because $\nabla V \in L^\delta$, we have

$$\|\nabla(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\eta} \to 0.$$

Hence, to obtain (12.3.18), it only remains to show that

$$\|\tau_{-x_n}V\nabla(e^{-it\Delta}\psi)\|_{L^\infty(0,\bar{t}+1)L^\eta} \to 0. \tag{12.3.21}$$

To this purpose, let $\tilde{\psi}$ be a C^∞, compactly supported function such that

$$\|\psi - \tilde{\psi}\|_{H^1} \leq \epsilon.$$

Notice that, by (12.2.1) we have

$$\frac{1}{\eta} = \frac{1}{2} + \frac{1}{d},$$

hence, by Minkowski and Hölder inequality,

$$\begin{aligned}\|\tau_{-x_n}V\nabla(e^{-it\Delta}\psi)\|_{L^\eta} &\leq \|\tau_{-x_n}V\nabla(e^{-it\Delta}\tilde{\psi})\|_{L^\eta} + \|\tau_{-x_n}V\nabla(e^{-it\Delta}(\psi-\tilde{\psi}))\|_{L^\eta} \\ &\leq \|\tau_{-x_n}V\nabla(e^{-it\Delta}\tilde{\psi})\|_{L^\eta} + \|V\|_{L^d}\|\nabla(e^{-it\Delta}(\psi-\tilde{\psi}))\|_{L^2} \\ &\leq \|\tau_{-x_n}V\nabla(e^{-it\Delta}\tilde{\psi})\|_{L^\eta} + \epsilon\|V\|_{L^d},\end{aligned} \tag{12.3.22}$$

where $V \in L^d$ because of the (critical) Sobolev embedding $W^{1,\delta}(\mathbb{R}^3) \hookrightarrow L^d(\mathbb{R}^3)$ - recall that $\delta = d/2$ for $d \geq 2$.

Then, because $\nabla(e^{-it\Delta}\tilde{\psi}) \in H^1$,

$$\|\tau_{-x_n}V\nabla(e^{-it\Delta}\tilde{\psi})\|_{L^\infty(0,\bar{t}+1)L^\eta}$$

can be estimated as $\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,\bar{t}+1)L^\eta}$, hence (12.3.21) follows from (12.3.22) and the assumption is verified.

Assumption (12.3.5). It is a consequence of (12.3.16), the Lebesgue's dominated convergence theorem and the continuity of $t \in \mathbb{R} \longrightarrow e^{itA}\tau_{\bar{x}}\psi \in H^1$ with the exact same proof as in [Laf16]. □

12.3.2. Non linear profiles

Similarly to [Laf16], we will now see that for a data which escapes to infinity, the solutions of (12.1.1) are the same as those of the homogeneous equation ($V = 0$), in the sense given by the three next Propositions:

Proposition 12.12. *Let $\psi \in H^1$, $(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$. Then, up to a subsequence*

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{-it(\Delta-V)}\tau_{x_n}\psi\|_{L^pL^r} \to 0 \tag{12.3.23}$$

as $n \to \infty$.

Proof. We assume for example $x_n \to +\infty$.

By the dispersive estimate and a density argument, the same proof as in [Laf16] gives

$$\sup_{n\in\mathbb{N}} \|e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(T,\infty)L^r} \to 0 \tag{12.3.24}$$

as $T \to \infty$. We are therefore reduced to show that for $T > 0$ fixed

$$\|e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p(0,T)L^r} \to 0$$

as $n \to \infty$. Let us pick $\epsilon > 0$. The difference $e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$i\partial_t u - \Delta u + Vu = Ve^{-it\Delta}\tau_{x_n}\psi.$$

So, by the inhomogeneous Strichartz estimate (12.2.9)

$$\begin{aligned}\|e^{-it\Delta}\tau_{x_n}\psi - e^{it(-\Delta+V)}\tau_{x_n}\psi\|_{L^p_t(0,T)L^r} &\lesssim \|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^{\gamma'}_t(0,T)L^\eta} \\ &\lesssim T^{\frac{1}{\gamma'}}\|Ve^{-it\Delta}\tau_{x_n}\psi\|_{L^\infty(0,T)L^\eta} \\ &= T^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^\eta}\end{aligned}$$

because the translation operator τ_{x_n} commutes with the propagator $e^{-it\Delta}$. But

$$\|(\tau_{-x_n}V)e^{-it\Delta}\psi\|_{L^\infty(0,T)L^\eta} \underset{n\to\infty}{\longrightarrow} 0$$

as seen in the proof of Proposition 12.11, point (12.3.4). □

Proposition 12.13. *Let $\psi \in H^1$, $(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$, $U \in C(H^1)\cap L^pL^r$ be the unique solution to the homogeneous equation*

$$i\partial_t u + \Delta u = u|u|^\alpha$$

with initial data ψ, and $U_n(t,x) := U(t, x - x_n)$. Then, up to a subsequence

$$\|\int_0^t e^{-i(t-s)\Delta}\left(U_n|U_n|^\alpha\right)(s)ds - \int_0^t e^{-i(t-s)(\Delta-V)}\left(U_n|U_n|^\alpha\right)(s)ds\|_{L^pL^r} \to 0 \tag{12.3.25}$$

as $n \to \infty$.

Proof. In the exact same way as in [Laf16], inhomogeneous Strichartz estimates, and the pointwise dispersive estimate together with Hardy-Littlewood-Sobolev inequality leads

$$\sup_{n\in\mathbb{N}} \| \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)ds\|_{L^p([T,\infty))L^r} \to 0 \tag{12.3.26}$$

as T goes to infinity. Thus; it remains to show that for $T>0$ fixed,

$$\| \int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right) ds - \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right) ds\|_{L^p(0,T)L^r} \to 0$$

as $n\to\infty$. The difference

$$\int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right) ds - \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right) ds$$

is the solution of the following linear Schrödinger equation, with zero initial data

$$i\partial_t u - \Delta u + Vu = V\int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right) ds.$$

Hence, by the Strichartz estimate (12.2.9)

$$\begin{aligned} &\| \int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right) ds - \int_0^t e^{-i(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right) ds\|_{L^p(0,T)L^r} \\ &\qquad \lesssim \|V\int_0^t e^{-i(t-s)\Delta} \left(U_n|U_n|^\alpha\right) ds\|_{L^{\gamma'}(0,T)L^\eta} \\ &\qquad\qquad \lesssim T^{\frac{1}{\gamma'}} \|(\tau_{-x_n}V)\int_0^t e^{-i(t-s)\Delta} \left(U|U|^\alpha\right) ds\|_{L^\infty(0,T)L^\eta}. \end{aligned}$$

But $\int_0^t e^{-i(t-s)\Delta}(U|U|^\alpha)ds \in C([0,T],H^1)$, so by Sobolev embedding in $L^{2^\star}$ and compactness in time there exists $\Lambda>0$ such that

$$\| \int_0^t e^{-i(t-s)\Delta} \left(U|U|^\alpha\right) ds\|_{L^\infty(0,T)L^{2^\star}(|x|\geq\Lambda)} \leq \epsilon$$

therefore

$$\|(\tau_{-x_n}V)\int_0^t e^{-i(t-s)\Delta} \left(U|U|^\alpha\right) ds\|_{L^\infty(0,T)L^\eta} \underset{n\to\infty}{\longrightarrow} 0$$

in the same way as in the proof of Proposition 12.11, point (12.3.4) . □

Proposition 12.14. *Let* $\psi\in H^1$, $(x_n)_{n\geq1}$, $(t_n)_{n\geq1}\in\mathbb{R}^{\mathbb{N}}$ *be such that* $|x_n|\to\infty$ *and* $t_n\to\pm\infty$, U *be a solution to the homogeneous equation such that*

$$\|U(t)-e^{-it\Delta}\psi\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0$$

and $U_n(t,x) := U(t-t_n, x-x_n)$. *Then, up to a subsequence*

$$\|e^{-i(t-t_n)\Delta}\tau_{x_n}\psi - e^{-i(t-t_n)(\Delta-V)}\tau_{x_n}\psi\|_{L^pL^r} \to 0 \tag{12.3.27}$$

and

$$\|\int_0^t e^{-i(t-s)\Delta}\left(U_n|U_n|^\alpha\right) ds - \int_0^t e^{-i(t-s)(\Delta-V)}\left(U_n|U_n|^\alpha\right) ds\|_{L^pL^r} \to 0 \tag{12.3.28}$$

as $n \to \infty$.

Proof. The proof is the same as for Proposition 12.12 and Proposition 12.13, decomposing the time interval in $\{|t-t_n| > T\}$ and his complementary. □

Finally, we will need the following Proposition of non linear scattering:

Proposition 12.15. *Let* $\varphi \in H^1$. *Then there exists* $W_\pm \in C(H^1) \cap L^p_{\mathbb{R}^\pm} L^r$, *solution of (12.1.1) such that*

$$\|W_\pm(t,\cdot) - e^{-it(\Delta-V)}\varphi\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0 \tag{12.3.29}$$

moreover, if $t_n \to \mp\infty$ *and*

$$\varphi_n = e^{-it_n(\Delta-V)}\varphi, \quad W_{\pm,n}(t) = W_\pm(t-t_n) \tag{12.3.30}$$

then

$$W_{\pm,n}(t) = e^{-it(\Delta-V)}\varphi_n + \int_0^t e^{-i(t-s)(\Delta-V)}(W_{\pm,n}|W_{\pm,n}|^\alpha)(s)ds + f_{\pm,n}(t) \tag{12.3.31}$$

where

$$\|f_{\pm,n}\|_{L^p_{\mathbb{R}^\pm}L^r} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{12.3.32}$$

Proof. The same proof as [BV16], Proposition 3.5, holds, as it involves only the analogous Strichartz estimates. □

12.3.3. Conclusion

The theorem 12.10 is now a consequence of the linear profile decomposition together with the nonlinear profiles results of Propositions 12.12, 12.13, 12.14, 12.15, perturbative result of Proposition 12.9 and Strichartz inequalities of Proposition 12.6, in the exact same way as in [Laf16], Section 5.

12.4. Extinction of the critical solution

The aim of this section is to prove the following rigidity theorem

Theorem 12.16. *There is no non trivial compact-flow solution of (12.1.1).*

Our key tool will be the following virial computation:

Lemma 12.17. *Let $u \in C(H^1)$ be a solution of (12.1.1) and χ be a regular function. Then*

$$\partial_t \int \chi |u|^2 = 2Im \int \nabla\chi \cdot \nabla u \bar{u} \tag{12.4.1}$$

$$\partial_t^2 \int \chi |u|^2 = 4 \int (D^2\chi \nabla u, \nabla u) + \frac{2}{\alpha+2} \int \Delta\chi |u|^{\alpha+2} - 2 \int \nabla\chi \cdot \nabla V |u|^2 - \int \Delta^2 \chi |u|^2. \tag{12.4.2}$$

In the case of a repulsive potential, taking the weight $\chi = |x|^2$ gives the result by a classical argument, as all the terms, and in particular

$$\int \nabla\chi \cdot \nabla V |u|^2 \tag{12.4.3}$$

have the right sign. However, with a non-repulsive potential, this straightforward choose of weight does not permits to conclude because (12.4.3) is no more signed.

In our framework of the sum of two convex repulsive potentials, we are able to construct a family of weights that has the right behavior and for which the non-negative part of (12.4.3) can be made small enough. The idea is to construct a weight such that $\nabla\chi$ is almost orthogonal to the line $(a_1 a_2)$.

It is given by the following lemma. The convexs $\mathcal{C}_1$ and $\mathcal{C}_2$ will be level surfaces of V_1 and V_2.

Lemma 12.18. *Let $R > 0$ and $\mathcal{C}_1$, $\mathcal{C}_2$ be two strictly convex of $\mathbb{R}^d$ contained in $B(0,R)$, such that the trapped ray $\mathcal{R}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ is a segment of the line $\{x_2 = \cdots = x_d = 0\}$. Let $c = (c_1, 0, \ldots, 0)$. We assume moreover that R is a continuous function of c_1 such that $R(c_1)/c_1 \to 0$ as $c_1 \to +\infty$. Then, for $x \in \partial(\mathcal{C}_1 \cup \mathcal{C}_2)$,*

$$\left(\frac{x-c}{|x-c|} + \frac{x+c}{|x+c|} \right) \cdot (-n)(x) \geq O(\frac{R^2}{c_1^4}).$$

Proof. For $x \in \mathbb{R}^d$, let us denote $x = (x_1, \tilde{x})$ with $\tilde{x} \in \mathbb{R}^{d-1}$. Remark that

$$|x+c| = c_1 + x_1 + \frac{1}{2c_1}|\tilde{x}|^2 + O(\frac{R^2}{c_1^2})$$

and therefore

$$\frac{x-c}{|x-c|} + \frac{x+c}{|x+c|} = \frac{1}{|x-c||x+c|} \left(2c_1(0,\tilde{x}) + x\frac{|\tilde{x}|^2}{c_1} + O(\frac{R^2}{c_1^2}) \right). \tag{12.4.4}$$

Moreover, on $\partial\mathcal{C}_i$, $-n$ is near $\mathcal{R}$ of the form

$$-n = (\pm\frac{x_1}{|x_1|}, 0, \cdots, 0) + (0, \lambda_2 x_2, \cdots, \lambda_d x_d) + O(|\tilde{x}|^2)$$

with $\lambda_k > 0$. And thus

$$\left(2c_1(0,\tilde{x}) + x\frac{|\tilde{x}|^2}{c_1}\right)\cdot(-n) \geq \left(2c_1 \min\lambda_k - \frac{C}{c_1}\right)|\tilde{x}|^2 + O(|\tilde{x}|^2)$$

Hence, there exists $\rho \geq 0$ and $D_1 > 0$ such that, for every $c_1 > D_1$ we have

$$|\tilde{x}| \leq \rho \implies \left(2c_1(0,\tilde{x}) + x\frac{|\tilde{x}|^2}{c_1}\right)\cdot(-n) \geq 0. \tag{12.4.5}$$

On the other hand, there exists $\epsilon_0 > 0$ such that, for all $x \in \partial(C_1 \cup C_2)$

$$|\tilde{x}| \geq \rho \implies (0,\tilde{x})\cdot(-n) \geq \epsilon_0.$$

Hence, if $|\tilde{x}| \geq \delta$

$$\left(2c_1(0,\tilde{x}) + x\frac{|\tilde{x}|^2}{c_1}\right)\cdot(-n) \geq 2c_1\epsilon_0 - \frac{C}{c_1},$$

and therefore, there exists $D_2 > 0$ such that, if $c_1 > D_2$

$$|\tilde{x}| \geq \rho \implies \left(2c_1(0,\tilde{x}) + x\frac{|\tilde{x}|^2}{c_1}\right)\cdot(-n) \geq 0. \tag{12.4.6}$$

Combining (12.4.4), (12.4.5) and (12.4.6), and noticing that

$$|x-c||x+c| \gtrsim c_1^2$$

by the hypothesis $R/c_1 \to 0$, gives the result. □

We are now in position to prove the rigidity theorem:

Proof of theorem 12.16. By contradiction, let $u \neq 0$ be a solution of (12.1.1) with a relatively compact flow $\{u(t),\, t{\in}\mathbb{R}\}$ in H^1 .

We choose a system of coordinates such that $(a_1a_2) = \{x_2 = \cdots = x_d = 0\}$. Let $c_1 > 0$ and $c = (c_1, 0, \ldots, 0)$. We take

$$\chi(x) := |x-c| + |x+c|,$$

and denote

$$z(t) = \int \chi|u|^2.$$

By (12.4.1), the Cauchy-Schwarz inequality and the conservation of mass and energy

$$|z'(t)| \leq \sqrt{CE(u)M(u)}. \tag{12.4.7}$$

Moreover, by (12.4.2)

$$z''(t) = 4\int (D^2\chi\nabla u, \nabla u) + \frac{2}{\alpha+2}\int \Delta\chi |u|^{\alpha+2} - 2\int \nabla\chi\cdot\nabla V|u|^2 - \int \Delta^2\chi|u|^2$$
$$\gtrsim \frac{1}{c_1}\int_{B(0,A)} |u|^{\alpha+2} - \int \nabla\chi\cdot\nabla V|u|^2.$$

Because $u\neq 0$ and $\{u(t),\ t\in\mathbb{R}\}$ is by assumption relatively compact in H^1, there exists $\mu>0$ and $A>0$ such that

$$\sup_{t\in\mathbb{R}}\int_{B(0,A)} |u|^{\alpha+2} \geq \mu.$$

Therefore we have

$$z''(t) \gtrsim \frac{1}{c_1}\mu - \int \nabla\chi\cdot\nabla V|u|^2. \tag{12.4.8}$$

Let $R>0$. Note that, by Hölder inequality and because $\nabla\chi$ is bounded and $|x|^\beta\nabla V\in L^\delta$

$$\Big|\int_{|x|\geq R} \nabla\chi\cdot\nabla V|u|^2\Big| \leq \|\nabla V\|_{L^\delta(|x|\geq R)}\|u\|^2_{L^{2^\star}(|x|\geq R)}$$
$$\leq \frac{1}{R^\beta}\||x|^\beta\nabla V\|_{L^\delta}\|u\|^2_{L^{2^\star}(|x|\geq R)}, \tag{12.4.9}$$

but note that, because $\{u(t),\ t\in\mathbb{R}\}$ is relatively compact in H^1 and by Sobolev embedding in $L^{2^\star}$,

$$\sup_t \|u\|_{L^{2^\star}(|x|\geq R)} = \epsilon(R), \tag{12.4.10}$$

where $\epsilon(R)\longrightarrow 0$ when $R\longrightarrow\infty$ and thus, using (12.4.8), (12.4.9) and (12.4.10)

$$z''(t) \gtrsim \mu/c_1 - \int_{B(0,R)} \nabla\chi\cdot\nabla V|u|^2 + \frac{1}{R^\beta}\epsilon(R). \tag{12.4.11}$$

Now, there exists $a,b\in\mathbb{R}$ such that

$$B(0,R)\subset \bigcup_{s\in[a,b]}\{V_1=s\}\subset B(0,2R),$$

and

$$B(0,R)\subset \bigcup_{s\in[a,b]}\{V_2=s\}\subset B(0,2R).$$

Thus, if $x\in B(0,R)$, there exists $s_{1,2}\in[a,b]$ such that $x\in\{V_1=s_1\}$ and $x\in\{V_2=s_2\}$. Let us denote $\mathcal{C}_{1,2}=\{V_{1,2}=s_{1,2}\}$. Then, by lemma 12.18 applied to $\mathcal{C}_1$ and $\mathcal{C}_2$, we get

$$-\nabla\chi\cdot\frac{\nabla V_{1,2}}{\|\nabla V_{1,2}\|}(x) \geq O(\frac{R^2}{c_1^4}).$$

Thus, by Hölder inequality, Sobolev embedding and conservation of energy

$$\Big|\int_{|x|\leq R, -\nabla\chi\cdot\nabla V(x)<0} -\nabla\chi\cdot\nabla V|u|^2\Big| \lesssim \frac{R^2}{c_1^4}\int|\nabla V||u|^2$$
$$\leq \frac{R^2}{c_1^4}\|\nabla V\|_{L^\delta}\|u\|_{L^{2^\star}}^2 \leq \frac{R^2}{c_1^4}\|\nabla V\|_{L^\delta}\|u\|_{H^1}^2 \lesssim \frac{R^2}{c_1^4}E(u_0)^2. \quad (12.4.12)$$

Hence, (12.4.11) together with (12.4.12) gives

$$z''(t) \gtrsim \frac{\mu}{c_1} + O(\frac{R^2}{c_1^4}) + \frac{1}{R^\beta}\epsilon(R).$$

Let us take $R = c_1^\nu$. Then we get

$$z''(t) \geq \frac{1}{c_1}(\mu + O(c_1^{2\nu-3}) + c_1^{1-\beta\nu}\epsilon(c_1^\nu)).$$

Thus, taking

$$\nu = \frac{1}{\beta}$$

and assuming

$$2\nu - 3 < 0 \iff \beta > \frac{2}{3}$$

we get, if we fix c_1 large enough

$$z''(t) \geq \frac{\mu}{2c_1},$$

and (12.4.7) is contradicted. □

And our main result follows:

Proof of theorem 12.1. If $E_c < \infty$, then theorem 12.10 allows us to extract a critical element $\varphi_c \in H^1$, $\varphi_c \neq 0$, such that the corresponding solution u_c of (12.1.1) verifies that $\{u_c(t),\ t\geq 0\}$ is relatively compact in H^1. By theorem 12.16, such a solution cannot exist, so $E_c = \infty$ and by Proposition 12.7, all the solutions of (12.1.1) scatter in H^1. □

Bibliography

[Ana08] Nalini Anantharaman, *Entropy and the localization of eigenfunctions*, Ann. of Math. (2) **168** (2008), no. 2, 435–475. MR 2434883

[Ant08] Ramona Anton, *Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains*, Bull. Soc. Math. France **136** (2008), no. 1, 27–65. MR 2415335

[ANV04] Ryuichi Ashino, Michihiro Nagase, and Rémi Vaillancourt, *Pseudodifferential operators in $L^p(\mathbb{R}^n)$ spaces*, Cubo **6** (2004), no. 3, 91–129. MR 2124828

[AS13] Farah Abou Shakra, *Asymptotics of the critical nonlinear wave equation for a class of non-star-shaped obstacles*, J. Hyperbolic Differ. Equ. **10** (2013), no. 3, 495–522. MR 3104079

[AS14] ———, *Asymptotics of wave models for non star-shaped geometries*, Discrete Contin. Dyn. Syst. Ser. S **7** (2014), no. 2, 347–362. MR 3109476

[B77] Pierre H. Bérard, *On the wave equation on a compact Riemannian manifold without conjugate points*, Math. Z. **155** (1977), no. 3, 249–276. MR 0455055

[BG99] Hajer Bahouri and Patrick Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), no. 1, 131–175. MR 1705001

[BGH10] Nicolas Burq, Colin Guillarmou, and Andrew Hassell, *Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics*, Geom. Funct. Anal. **20** (2010), no. 3, 627–656. MR 2720226 (2012f:58068)

[BGT04a] N. Burq, P. Gérard, and N. Tzvetkov, *On nonlinear Schrödinger equations in exterior domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 3, 295–318. MR 2068304 (2005g:35264)

[BGT04b] N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605. MR 2058384

[BK74] Clifford O. Bloom and Nicholas D. Kazarinoff, *Local energy decay for a class of nonstar-shaped bodies*, Arch. Rational Mech. Anal. **55** (1974), 73–85. MR 0347237 (49 #11957)

[BLP08] Nicolas Burq, Gilles Lebeau, and Fabrice Planchon, *Global existence for energy critical waves in 3-D domains*, J. Amer. Math. Soc. **21** (2008), no. 3, 831–845. MR 2393429

[Bou11] Jean-Marc Bouclet, *Strichartz estimates on asymptotically hyperbolic manifolds*, Anal. PDE **4** (2011), no. 1, 1–84. MR 2783305

[Bri10] David Brink, *Hölder continuity of roots of complex and p-adic polynomials*, Communications in Algebra (2010).

[BS98] Hajer Bahouri and Jalal Shatah, *Decay estimates for the critical semilinear wave equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), no. 6, 783–789. MR 1650958

[BSS08] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge, *On Strichartz estimates for Schrödinger operators in compact manifolds with boundary*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 247–256. MR 2350410

[BSS09a] ———, *Strichartz estimates for the wave equation on manifolds with boundary*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 5, 1817–1829. MR 2566711

[BSS09b] ———, *Strichartz estimates for the wave equation on manifolds with boundary*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 5, 1817–1829. MR 2566711 (2010k:35500)

[BT07] Jean-Marc Bouclet and Nikolay Tzvetkov, *Strichartz estimates for long range perturbations*, Amer. J. Math. **129** (2007), no. 6, 1565–1609. MR 2369889

[Bur93] Nicolas Burq, *Contrôle de l'équation des plaques en présence d'obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) (1993), no. 55, 126. MR 1254820

[Bur98] ———, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), no. 1, 1–29. MR 1618254 (99j:35119)

[Bur03] N. Burq, *Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: "Global Strichartz estimates for nontrapping perturbations of the Laplacian" [Comm. Partial Differential Equation* **25** *(2000), no. 11-12 2171–2183; MR1789924 (2001j:35180)]*, Comm. Partial Differential Equations **28** (2003), no. 9-10, 1675–1683. MR 2001179

[Bur04] ———, *Smoothing effect for Schrödinger boundary value problems*, Duke Math. J. **123** (2004), no. 2, 403–427. MR 2066943 (2006e:35026)

[BV16] Valeria Banica and Nicola Visciglia, *Scattering for NLS with a delta potential*, J. Differential Equations **260** (2016), no. 5, 4410–4439. MR 3437592

[Car16] Rémi Carles, *On semi-classical limit of nonlinear quantum scattering*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 711–756. MR 3503830

[CGT09] J. Colliander, M. Grillakis, and N. Tzirakis, *Tensor products and correlation estimates with applications to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **62** (2009), no. 7, 920–968. MR 2527809 (2010c:35175)

[CV02] F. Cardoso and G. Vodev, *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II*, Ann. Henri Poincaré **3** (2002), no. 4, 673–691. MR 1933365

[CW92] Thierry Cazenave and Fred B. Weissler, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Comm. Math. Phys. **147** (1992), no. 1, 75–100. MR 1171761 (93d:35150)

[CW13] Hans Christianson and Jared Wunsch, *Local smoothing for the Schrödinger equation with a prescribed loss*, Amer. J. Math. **135** (2013), no. 6, 1601–1632. MR 3145005

[Dat09] Kiril Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets*, Comm. Math. Phys. **286** (2009), no. 3, 837–850. MR 2472019

[DHR08] Thomas Duyckaerts, Justin Holmer, and Svetlana Roudenko, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*, Math. Res. Lett. **15** (2008), no. 6, 1233–1250. MR 2470397 (2010e:35257)

[DV12] Kiril Datchev and András Vasy, *Semiclassical resolvent estimates at trapped sets*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2379–2384 (2013). MR 3060761

[DV13] ______, *Propagation through trapped sets and semiclassical resolvent estimates*, Microlocal methods in mathematical physics and global analysis, Trends Math., Birkhäuser/Springer, Basel, 2013, pp. 7–10. MR 3307787

[Fos05] Damiano Foschi, *Inhomogeneous Strichartz estimates*, J. Hyperbolic Differ. Equ. **2** (2005), no. 1, 1–24. MR 2134950 (2006a:35043)

[FV13] Luca Fanelli and Nicola Visciglia, *The lack of compactness in the Sobolev-Strichartz inequalities*, J. Math. Pures Appl. (9) **99** (2013), no. 3, 309–320. MR 3017992

[FV18] Luigi Forcella and Nicola Visciglia, *Double scattering channels for* 1*D NLS in the energy space and its generalization to higher dimensions*, J. Differential Equations **264** (2018), no. 2, 929–958. MR 3720834

[FXC11] DaoYuan Fang, Jian Xie, and Thierry Cazenave, *Scattering for the focusing energy-subcritical nonlinear Schrödinger equation*, Sci. China Math. **54** (2011), no. 10, 2037–2062. MR 2838120

[G88] C. Gérard, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) (1988), no. 31, 146. MR 998698

[Gri90] Manoussos G. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. (2) **132** (1990), no. 3, 485–509. MR 1078267 (92c:35080)

[GS04] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178. MR 2096737 (2005g:81339)

[GV85a] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z. **189** (1985), no. 4, 487–505. MR 786279

[GV85b] ———, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J. Math. Pures Appl. (9) **64** (1985), no. 4, 363–401. MR 839728

[GV85c] ———, *Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **43** (1985), no. 4, 399–442. MR 824083

[GV87] ———, *Conformal invariance and time decay for nonlinear wave equations. I, II*, Ann. Inst. H. Poincaré Phys. Théor. **47** (1987), no. 3, 221–261, 263–276. MR 921307

[GV95] ———, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** (1995), no. 1, 50–68. MR 1351643

[Hon14] Hong, *Scattering for a nonlinear schrödinger equation with a potential*, Preprint (2014).

[HR08] Justin Holmer and Svetlana Roudenko, *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys. **282** (2008), no. 2, 435–467. MR 2421484 (2009h:35403)

[HTW06] Andrew Hassell, Terence Tao, and Jared Wunsch, *Sharp Strichartz estimates on nontrapping asymptotically conic manifolds*, Amer. J. Math. **128** (2006), no. 4, 963–1024. MR 2251591

[Ika82] Mitsuru Ikawa, *Decay of solutions of the wave equation in the exterior of two convex obstacles*, Osaka J. Math. **19** (1982), no. 3, 459–509. MR 676233

[Ika88] ———, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier (Grenoble) **38** (1988), no. 2, 113–146. MR 949013

[IL17] Oana Ivanovici and Gilles Lebeau, *Dispersion for the wave and the Schrödinger equations outside strictly convex obstacles and counterexamples*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 7, 774–779. MR 3673052

[ILLP] Oana Ivanovici, Richard Lascar, Gilles Lebeau, and Fabrice Planchon, *Dispersion for the wave equation inside strictly convex domains II: the general case*, Preprint.

[ILP14a] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon, *Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case*, Ann. of Math. (2) **180** (2014), no. 1, 323–380. MR 3194817

[ILP14b] ———, *Estimations de Strichartz pour les ondes dans le modèle de Friedlander en dimension 3*, Séminaire Laurent Schwartz—Équations aux Dérivées Partielles et Applications. Année 2013–2014, Ed. Éc. Polytech., Palaiseau, 2014, pp. Exp. No. III, 12. MR 3445472

[IP] Oana Ivanovici and Fabrice Planchon, *Square function and heat flow estimates on domains.*

[Iva10] Oana Ivanovici, *On the Schrödinger equation outside strictly convex obstacles*, Anal. PDE **3** (2010), no. 3, 261–293. MR 2672795 (2011j:58037)

[Iva12] ———, *Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1357–1388. MR 2966654

[Kap89] L. V. Kapitanskiĭ, *Some generalizations of the Strichartz-Brenner inequality*, Algebra i Analiz **1** (1989), no. 3, 127–159. MR 1015129

[KM93] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268. MR 1231427

[KM06] Carlos E. Kenig and Frank Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, 645–675. MR 2257393 (2007g:35232)

[KT98] Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980. MR 1646048

[Laf16] D. Lafontaine, *Scattering for NLS with a potential on the line*, Asymptot. Anal. **100** (2016), no. 1-2, 21–39. MR 3570874

[Laf17a] ———, *About the wave equation outside two strictly convex obstacles*, Preprint, https://arxiv.org/abs/1711.09734 (2017).

[Laf17b] ______, *Strichartz estimates without loss outside two strictly convex obstacles*, Preprint, https://arxiv.org/abs/1709.03836 (2017).

[Laf18a] ______, *Scattering for nls with a sum of two repulsive potentials*, preprint to appear (2018).

[Laf18b] ______, *Strichartz estimates without loss outside many strictly convex obstacles*, preprint to appear (2018).

[Leb92] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9) **71** (1992), no. 3, 267–291. MR 1172452

[LL] D. Lafontaine and C. Laurent, *Scattering for critical waves outside two strictly convex obstacles*, Work in progress.

[LP89] Peter D. Lax and Ralph S. Phillips, *Scattering theory*, second ed., Pure and Applied Mathematics, vol. 26, Academic Press, Inc., Boston, MA, 1989, With appendices by Cathleen S. Morawetz and Georg Schmidt. MR 1037774

[LS95] Hans Lindblad and Christopher D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), no. 2, 357–426. MR 1335386

[Lun44] R. K. Luneberg, *Mathematical Theory of Optics*, Brown University, Advanced Instruction and Research in Mechanics, Providence, R. I., 1944. MR 0011035

[Mal64] Bernard Malgrange, *The preparation theorem for differentiable functions*, Differential Analysis, Bombay Colloq.,Oxford Univ. Press, London (1964), 203–208. MR 182695

[Met04] Jason L. Metcalfe, *Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle*, Trans. Amer. Math. Soc. **356** (2004), no. 12, 4839–4855. MR 2084401

[Mor61] Cathleen S. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math. **14** (1961), 561–568. MR 0132908

[MRS77] Cathleen S. Morawetz, James V. Ralston, and Walter A. Strauss, *Decay of solutions of the wave equation outside nontrapping obstacles*, Comm. Pure Appl. Math. **30** (1977), no. 4, 447–508. MR 0509770

[MS78] R. B. Melrose and J. Sjöstrand, *Singularities of boundary value problems. I*, Comm. Pure Appl. Math. **31** (1978), no. 5, 593–617. MR 0492794 (58 #11859)

[MS82] ______, *Singularities of boundary value problems. II*, Comm. Pure Appl. Math. **35** (1982), no. 2, 129–168. MR 644020 (83h:35120)

[MS98] S. J. Montgomery-Smith, *Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations*, Duke Math. J. **91** (1998), no. 2, 393–408. MR 1600602

[MSS93] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. **6** (1993), no. 1, 65–130. MR 1168960

[MT87] Richard B. Melrose and Michael E. Taylor, *Boundary problems for the wave equations with grazing and gliding rays.*

[Nak99] Kenji Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2, J. Funct. Anal. **169** (1999), no. 1, 201–225. MR 1726753 (2000m:35141)

[NZ06] Stéphane Nonnenmacher and Maciej Zworski, *Quantum decay rates in chaotic scattering*, Séminaire: Équations aux Dérivées Partielles. 2005–2006, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2006, pp. Exp. No. XXII, 8. MR 2276087

[PV09] Fabrice Planchon and Luis Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 261–290. MR 2518079 (2010b:35441)

[Smi98] Hart F. Smith, *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 3, 797–835. MR 1644105

[SS94] Jalal Shatah and Michael Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, Internat. Math. Res. Notices (1994), no. 7, 303ff., approx. 7 pp. (electronic). MR 1283026 (95e:35132)

[SS95] Hart F. Smith and Christopher D. Sogge, *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. **8** (1995), no. 4, 879–916. MR 1308407

[SS00] ———, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations **25** (2000), no. 11-12, 2171–2183. MR 1789924

[SS07a] ———, *On the L^p norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198** (2007), no. 1, 107–153. MR 2316270

[SS07b] ———, *On the L^p norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198** (2007), no. 1, 107–153. MR 2316270

[ST02] Gigliola Staffilani and Daniel Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Partial Differential Equations **27** (2002), no. 7-8, 1337–1372. MR 1924470

[Str77] Robert S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714. MR 0512086

[Str88] Michael Struwe, *Globally regular solutions to the u^5 Klein-Gordon equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), no. 3, 495–513 (1989). MR 1015805 (90j:35142)

[Tat02] Daniel Tataru, *Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III*, J. Amer. Math. Soc. **15** (2002), no. 2, 419–442. MR 1887639

[TZ00] Siu-Hung Tang and Maciej Zworski, *Resonance expansions of scattered waves*, Comm. Pure Appl. Math. **53** (2000), no. 10, 1305–1334. MR 1768812

[Vis09] Nicola Visciglia, *On the decay of solutions to a class of defocusing NLS*, Math. Res. Lett. **16** (2009), no. 5, 919–926. MR 2576708 (2011a:35501)

[Vn88] B. R. Vaĭ nberg, *The asymptotic behavior as $t \to \infty$ of the solutions of exterior mixed problems for hyperbolic equations, and quasiclassics*, Current problems in mathematics. Fundamental directions. Vol. 34 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988, pp. 57–92, 248. MR 1066956

[VZ00] András Vasy and Maciej Zworski, *Semiclassical estimates in asymptotically Euclidean scattering*, Comm. Math. Phys. **212** (2000), no. 1, 205–217. MR 1764368 (2002b:58047)

[Zwo12] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012. MR 2952218