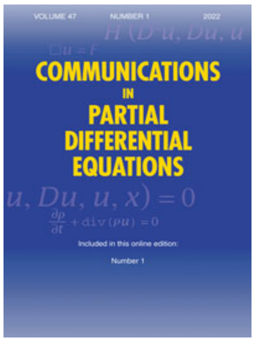

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About the wave equation outside two strictly convex obstacles

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ABSTRACT
We prove global Strichartz estimates without loss for the wave equation outside two strictly convex obstacles, following the road-map previously introduced by the author for the Schrödinger equation. Moreover, we show a first step towards the large data scattering for the critical non-linear equation associated to this geometrical setting, and prove the scattering for a class of non-trapping obstacles close to the two convex framework.

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1. Introduction

Let (M, g) be a Riemannian manifold of dimension d. We are interested in the linear wave equation on M

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 \\ (u(0), \partial_t u(0)) = (f, g). \end{cases} \tag{1.1}$$

where Δ_g designs the Laplace-Beltrami operator. In order to study the perturbative theory and the nonlinear problems associated with this equation, it is crucial to estimate the size and the decay of the solutions. Such estimates are the so called *Strichartz estimates*

$$||u||_{L^p(0,T)L^q(\Omega)} \leq C_T\left(||u_0||_{\dot{H}^\gamma} + ||u_1||_{\dot{H}^{\gamma-1}}\right), \tag{1.2}$$

where (p, q) has to follow the admissibility condition given by the scaling of the equation

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \tag{1.3}$$

and

$$\frac{1}{p} \leq \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right). \tag{1.4}$$

We say that the estimates hold with a loss of order $\lambda > 0$ if they hold for (p, q) satisfying the scaling condition (1.3), and

$$\frac{1}{p} \leq \left(\frac{d-1}{2} - \lambda\right)\left(\frac{1}{2} - \frac{1}{q}\right).$$

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Strichartz estimates were first introduced and established in Strichartz [1] for the $p=q$ case in $\mathbb{R}^d$, then extended to all exponents in [2,3], and [4]. As usual, the variable coefficient case is more difficult. In the case of a manifold without boundary, the finite speed of propagation shows that it suffices to obtain the estimates in local coordinates to obtain local Strichartz estimates. Such estimates were obtained by [5–7], and [8]. The estimates outside one convex obstacle were obtained by Smith and Sogge [9], following the parametrix construction of Melrose and Taylor. Local estimates on a general domain were first proved by Nicolas Burq, Gilles Lebeau and Fabrice Planchon [10] for certain ranges of (p, q), using spectral estimates of Smith and Sogge [11]. The range of indices was then extended by Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge [12]. This range cannot recover all indices satisfying (1.4): [13] showed indeed that a loss have to occur if some concavity is met. Recently, Oana Ivanovici, Richard Lascar, Gilles Lebeau, and Fabrice Planchon [14] proved in a model case local Strichartz estimates inside a convex domain with a loss close to the sharpest one. Their result is extended in Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon [15] to the wave equation inside a general strictly convex domain.

Phenomenons such as closed geodesics can be obstacles to the establishment of global estimates. Under a non-trapping assumption, Smith [16] proved in the odd dimensional case that local estimates can be extended to global ones. This result was extended to the even dimensions independently by Burq [17] and Metcalfe [18].

However, Nicolas Burq, Colin Guillarmou, and Andrew Hassell [19] showed that Strichartz estimates without loss for the Schrödinger equation hold for an asymptotically Euclidian manifold without boundary for which the trapped set is sufficiently small and exhibit an hyperbolic dynamic.

Going in the same direction for the problem with boundaries, we recently showed in [20] global Strichartz estimates without loss for the Schrödinger equation outside two convex obstacles. The aim of this paper is to extend this result to the wave equation. More precisely, we prove

Theorem 1. *Let Θ_1 and Θ_2 be two compact, strictly convex subsets of $\mathbb{R}^d$, u be a solution of (1.1) in $\Omega = \mathbb{R}^d \backslash (\Theta_1 \cup \Theta_2)$ and (p, q, γ) verifying (1.3) and (1.4). Then*

$$||u||_{L^p(\mathbb{R}, L^q)} \leq C\big(||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}\big). \tag{1.5}$$

The crucial remark of Smith and Sogge [16] is that local Strichartz estimates combined with the exponential decay of the energy permits to obtain global Strichartz estimates. For the exterior of one convex obstacle in odd dimension, this decay holds and global Strichartz estimates without loss are obtained. In even dimension, such an exponential decay does not hold anymore. However, Burq [17] remarked that it can be replaced by weaker estimates of L^2- integrability of the local energy

$$||(\chi u, \chi \partial_t u)||_{L^2(\mathbb{R}, L^2 \times H^{-1})} \lesssim ||u_0||_{\dot{H}^\gamma} + ||u_1||_{\dot{H}^{\gamma-1}}, \tag{1.6}$$

where χ is any compactly supported function, and such an estimate for the complementary of a convex obstacle is a direct consequence of well-known resolvent estimates.

But in the case of the exterior of two convex obstacles, (1.6) does not hold anymore: a logarithmic loss occurs due to the trapped geodesic and we only have

$$||(\chi u, \chi \partial_t u)||_{L^2(\mathbb{R}, L^2 \times H^{-1})} \lesssim |\log h| \big(||u_0||_{\dot{H}^\gamma} + ||u_1||_{\dot{H}^{\gamma-1}}\big). \tag{1.7}$$

for data supported in frequencies $\sim h^{-1}$. The L^2-integrability of the local energy is the waves-analogue of the *smoothing effect* for the Schrödinger equation, for which a loss occurs in the same way. Nicolas Burq, Colin Guillarmou, and Andrew Hassell [19] remarked that such a loss can be compensated if we show Strichartz estimates in logarithmic times and we followed this idea in [20]. We follow here the same road-map and show that this logarithmic loss can be compensated if we show Strichartz estimates in logarithmic times in the neighbourhood of the trapped ray

$$||\chi u||_{L^p(0,|\log h|)L^q} \lesssim ||u_0||_{L^2} + ||u_1||_{H^{-1}}.$$

Then, we reduce again the problem, to data which micro-locally contains only points of the tangent space which do not escape a given neighbourhood of the periodic ray after logarithmic times. Finally, we construct an approximate solution for such data, inspired by [21–23], and we show that this approximation gives the desired estimate.

Note that a large part of the construction we are doing here is similar to the one we did in [20] for the Schrödinger equation, and we will extensively use results of this previous paper. On the one hand, the wave equation enjoys an exact speed of propagation whereas the Schödinger flow propagates at a speed proportional to the frequency, hence the proofs of the results based on this phenomenon for the Schrödinger flow will hold for the wave flow with simpler proofs. On the other hand, the phases of the approximate solution we are building stationate now in whole lines, instead of points, and it is a little more subtle to close the final argument.

1.1. Application

As an application, we consider a critical defocusing non-linear wave equation in $\mathbb{R}^3\backslash(\Theta_1 \cup \Theta_2)$

$$\begin{cases} \partial_t^2 u - \Delta_D u + u^5 = 0 \\ (u(0), \partial_t u(0)) = (f, g). \end{cases} \tag{1.8}$$

Note that the global existence for such an equation in a domain was obtained in Nicolas Burq, Gilles Lebeau and Fabrice Planchon [10]. By the finite speed of propagation, their result apply in particular to the exterior of obstacles. Therefore, it is legitimate to wonder what solutions look like in large time, and in particular if the non-linearity still plays a role. If it is not the case, we say that the solution *scatters*. More precisely, we say that a solution scatters if there exists a solution of the *linear* equation v such that

$$||u(t) - v(t)||_{\dot{H}^1(\Omega)} \to 0,$$

as t goes to infinity. The scattering in $\mathbb{R}^3$ was shown by Bahouri and Shatah [24]. Provided a good set of Strichartz estimates exists for the linear equation, their proof adapts to the case of a finite-border domain if one is able to deal with the arising boundary term. This term can be controlled in particular if one obtain the decay of the local energy near the obstacle (see Section 5):

$$\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)}|\nabla u(x,t)|^2+|u(x,t)|^6\,dxdt\to 0 \tag{1.9}$$

as T goes to infinity. In the case of the exterior of two balls, where Theorem 1 gives us the good set of Strichartz estimates, we show that we can obtain this control *everywhere except in the neighbourhood of the trapped ray*: more precisely

Theorem 2. *Let Θ_1 and Θ_2 be two disjoint balls of $\mathbb{R}^3$. Then, there exists a family $(\mathcal{S}(T))_{T\geq 1}$ of open neighbourhoods of the trapped ray $\mathcal{R}$ verifying*

$$\mathcal{S}(T)\to\mathcal{R}\text{ as }T\to+\infty$$

such that any global solution of (1.8) in $\Omega:=\mathbb{R}^3\backslash(\Theta_1\cup\Theta_2)$ verifies, as T goes to infinity

$$\frac{1}{T}\int_0^T\int_{(\Omega\cap B(0,A))\backslash\mathcal{S}(T)}|\nabla u(x,t)|^2+|u(x,t)|^6\,dxdt\to 0.$$

This is a first step to show the scattering for any data in this exterior problem. We are precisely following this path in the work in progress [25], where this step is extended to the exterior of two arbitrary convex obstacles and used to show the scattering in this framework.

We now deal with a geometrical situation which is close to the exterior of two convex obstacles, but does not have a trapped ray: the exterior of dog bones. We are actually able to show the scattering outside a class of non-star-shaped obstacles containing dog bones with arbitrary thin necks. In order to state this result, let us recall the definition of an illuminated subset – which is a generalisation of star-shaped ones – first introduced by Bloom and Kazarinoff [26]:

Definition 1. A subset $\mathcal{K}$ of $\mathbb{R}^d$ is said to be illuminated by a convex subset $\mathcal{C}\subset\mathbb{R}^d$ if

$$\min_{\partial\mathcal{K}}\nabla\rho\cdot\nu>0$$

where ρ is the gauge of $\mathcal{C}$ and ν the outward-pointing normal derivative to $\partial\mathcal{K}$.

We are now able to state our result:

Theorem 3. *Let $\mathcal{C}\subset\mathbb{R}^3$ be the ellipsoïd of equation*

$$x^2+y^2+\epsilon z^2=1,\ 0<\epsilon\leq 1 \tag{1.10}$$

resp.

$$x^2+\epsilon y^2+\epsilon z^2=1,\ \frac{1+\sqrt{3}}{4}\leq\epsilon\leq 1 \tag{1.11}$$

and $\mathcal{K}$ be a compact subset of $\mathbb{R}^3$ illuminated by $\mathcal{C}$. Then, any solution of (1.8) in $\Omega=\mathbb{R}^3\backslash\mathcal{K}$ scatters in $\dot{H}^1(\Omega)$.

Notice that Abou-Shakra obtained in [27] the scattering for obstacles illuminated by a deformation of a sphere using a slightly different method, but her result does not permit to handle dog bones with arbitrary thin necks. Our key tool to obtain Theorem 2 and Theorem 3 is an identity due to Morawetz [28] in the case of the linear equation, and used here in the spirit of Ginibre and Velo [29]. Such an identity rely on the choice of a good weight function χ which has to be adapted to the geometry and verify a very rigid and poorly understood constraint: $\Delta^2\chi \leq 0$. In the case of Theorem 3, the natural weight is given by the gauge of the ellipsoïd we are dealing with, and does not verify this constraint for arbitrary thin ellipsoïds. In order to deal with it, we present a method which permits to bypass this obstruction: noticing that corresponding four dimensional ellipsoïds verify the constraint, we extend the solution as the solution of a four dimensional non-linear wave equation, show the estimate for such a solution, and then go back to the original, three dimensional solution. We believe that such an argument may be useful in other situations.

2. Reduction of the problem

2.1. Estimates of L^2-integrability of the local energy

We first show the following two estimates of the L^2- integrability of the local energy, that we will need in the sequel. There are the analogs of the smoothing estimates for the Schrödinger flow, and were introduced by Burq [17] in the non-trapping case. The first one is an estimate without loss away of the trapped ray. The second one holds in the whole exterior domain, but with a logarithmic loss.

Proposition 1 (Global L^2-integrability with no loss away of the trapped ray). *Let $\chi \in C_0^\infty$ be supported outside a small enough neighbourhood of the trapped ray. Then, if u is the solution of (1.1) with data (f, g):*

$$||(\chi u, \chi\partial_t u)||_{L^2(\mathbb{R}, \dot{H}^\gamma\times\dot{H}^{\gamma-1})} \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}} \tag{2.1}$$

Proof. By [17, Section 2], it suffices to show the following resolvent estimate in order to obtain (2.1):

$$||\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi||_{L^2\to L^2} \lesssim \frac{1}{1+\sqrt{|\lambda|}}.$$

In the spirit of [20], let K be a non-trapping obstacle such that K and $\Theta_1 \cup \Theta_2$ coincide in the support of χ. In particular, $\Delta_\Omega = \Delta_{\mathbb{R}^d\setminus K}$ on the support of χ. As, moreover, the resolvent estimate is well-known in the non-trapping case (see Vasy and Zworski [30] and Melrose and Sjöstrand [31,32] for the high frequencies part, Burq [33] for the low frequencies), we have

$$||\chi(-\Delta_\Omega - (\lambda \pm i\epsilon))^{-1}\chi||_{L^2\to L^2} = ||\chi(-\Delta_{\mathbb{R}^d\setminus K} - (\lambda \pm i\epsilon))^{-1}\chi||_{L^2\to L^2} \lesssim \frac{1}{1+\sqrt{|\lambda|}},$$

and the Proposition is shown. □

Proposition 2 (Global L^2-integrability with logarithmic loss). *Let $\chi \in C_0^\infty$ be supported near the trapped ray. Then, if f, g verifies $\psi(-h^2\Delta)f = f$, $\psi(-h^2\Delta)g = g$ and u is the solution of (1.1) with data (f, g):*

$$||(\chi u, \chi\partial_t u)||_{L^2(\mathbb{R}, \dot{H}^\gamma\times\dot{H}^{\gamma-1})} \lesssim |\log h|^{1/2}\left(||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}\right) \tag{2.2}$$

Proof. Denote

$$\dot{H}^{\gamma,-} = D((-\Delta_D)^{s/2}\log(2I-\Delta)^{-1/2}),$$
$$H^{\gamma,-} = D((I-\Delta_D)^{s/2}\log(2I-\Delta)^{-1/2}),$$

by $\dot{H}^{-\gamma,+}$ and $H^{-\gamma,+}$ their dual, and

$$\mathcal{H}^{\gamma,-} = \dot{H}^{\gamma,-} \times \dot{H}^{\gamma-1,-},\ \mathcal{H}^{-\gamma,+} = \dot{H}^{-\gamma,+} \times \dot{H}^{-(\gamma-1),+}.$$

Finally, let us denote

$$A = i\begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}.$$

We will show the estimate

$$||(\chi u, \chi\partial_t u)||_{L^2(\mathbb{R}, \dot{H}^{\gamma,-}\times\dot{H}^{\gamma-1,-})} \lesssim ||f||_{\dot{H}^{\gamma}} + ||g||_{\dot{H}^{\gamma-1}}. \tag{2.3}$$

By a classical $TT^\star$ argument (see, for example, Burq [17] and [34]), (2.3) is a direct consequence of the following Proposition

Proposition 3. *The resolvent $\chi(A-(x-i\epsilon)^{-1})\chi$ is uniformly bounded in*

$$\mathcal{H}^{-s,+} \to \mathcal{H}^{s,-}$$

for $x \in \mathbb{R}$ and $0 < \epsilon < 1$.

Which, in turn, we will obtain as a consequence of the following resolvent estimate obtained by Burq [34]:

$$||\chi(-\Delta_D - (\lambda \pm i\epsilon))^{-1}\chi||_{L^2\to L^2} \lesssim \frac{2\log(2+|\lambda|)}{1+\sqrt{|\lambda|}}. \tag{2.4}$$

As

$$(A-z)^{-1} = \begin{pmatrix} -z(\Delta+z^2)^{-1} & i(\Delta+z^2)^{-1} \\ i\Delta(\Delta+z^2)^{-1} & -z(\Delta+z^2)^{-1} \end{pmatrix},$$

we have to show that the following norms are uniformly bounded, for all $s \in \mathbb{R}$

$$\begin{aligned} &||\chi(1+|z|)(\Delta+z^2)^{-1}\chi||_{H^{-s,+}\to H^{s,-}}, \\ &||\chi(\Delta+z^2)^{-1}\chi||_{H^{-s,+}\to H^{s+1,-}}, \\ &||\chi\Delta(\Delta+z^2)^{-1}\chi||_{H^{-s,+}\to H^{s-1,-}}. \end{aligned}$$

With the same arguments as Burq [17], theses bounds are all consequences of the first one for $s = 0$, that is of

$$||\chi(1+|z|)(\Delta+z^2)^{-1}\chi||_{\mathcal{H}^{0,+}\to\mathcal{H}^{0,-}}. \tag{2.5}$$

To show (2.5), we follow [34], Section 4. Let

$$u = (1+|z|)(\Delta+z^2)^{-1}\chi f.$$

For $\Psi \in C_0^\infty(-1/2, 2)$ equal to one close to 1, we decompose

$$u = \Psi\left(-\frac{\Delta}{z^2}\right)u + \left(1 - \Psi\left(-\frac{\Delta}{z^2}\right)\right)u.$$

On the one hand,

$$\|\left(1-\Psi\left(-\frac{\Delta}{z^2}\right)\right)u\|_{L^2}\lesssim\|\left(1-\Psi\left(-\frac{\Delta}{z^2}\right)\right)\chi f\|_{L^2}.$$

On the other hand, as

$$\Psi\left(-\frac{\Delta}{z^2}\right)u=(1+|z|)(\Delta+z^2)^{-1}\Psi\left(-\frac{\Delta}{z^2}\right)\chi f,$$

we have from (2.4)

$$||\chi\Psi\left(-\frac{\Delta}{z^2}\right)u||_{L^2}\lesssim\log(2+z^2)||\Psi\left(-\frac{\Delta}{z^2}\right)\chi f||_{L^2},$$

and thus

$$\log(2+z^2)^{-1/2}||\chi\Psi\left(-\frac{\Delta}{z^2}\right)u||_{L^2}\lesssim\log(2+z^2)^{1/2}||\Psi\left(-\frac{\Delta}{z^2}\right)\chi f||_{L^2}.$$

Finally, like in Burq [34], the localisation in frequencies allows us to replace the weights in z by the $H^{0,\pm}$ norms, and we get (2.5). $\square$

2.2. Reduction to logarithmic times near the trapped ray

The aim of this section is to show that the following Proposition implies Theorem 1

Proposition 4. *There exists $\epsilon>0$ and a small neighbourhood D of the trapped ray, such that, for all $\chi\in C_0^\infty$ supported in D, if f, g are such that $\psi(-h^2\Delta)f=f$, $\psi(-h^2\Delta)g=g$ and u is the solution of (1.1) with data (f, g):*

$$||\chi u||_{L^p(0,\epsilon|\log h|)L^q}\lesssim||f||_{\dot H^\gamma}+||g||_{\dot H^{\gamma-1}}.$$

Thus, we will assume the previous Proposition and show Theorem 1. As the value of $\epsilon>0$ does not play any role, we assume here that $\epsilon=1$.

In the spirit of [20], let $\chi_{\text{obst}},\chi_{\text{ray}}\in C_0^\infty$ be such that $\chi_{\text{obst}}=1$ in a neighbourhood of $\Theta_1\cup\Theta_2\cup\mathcal{R}$, and $\chi_{\text{ray}}\in C_0^\infty$ such that $\chi_{\text{ray}}=1$ in a neighbourhood of $\mathcal{R}$. We decompose u as the sum

$$u=(1-\chi_{\text{obst}})u+\chi_{\text{obst}}(1-\chi_{\text{ray}})u+\chi_{\text{obst}}\chi_{\text{ray}}u \tag{2.6}$$

2.2.1. The first term: away from the trapped ray and the obstacles

Let $v=(1-\chi_{\text{obst}})u$. Then v verifies

$$(\partial_t^2-\Delta_D)v=-[\Delta_D,\chi_{\text{obst}}]u,$$
$$(v(0),\partial_t v(0))=((1-\chi_{\text{obst}})f,(1-\chi_{\text{obst}})g).$$

As v is supported away from the obstacle, it solves a problem in the full space and we can replace the Laplacian in $(\partial_t^2-\Delta_D)$ by $\Delta_{\mathbb{R}^d}$. Therefore, by the Duhamel formula

$$\begin{aligned} v(t) = {} & \cos\left(t\sqrt{-\Delta_{\mathbb{R}^d}}\right)(1-\chi_{\text{obst}})f + \frac{\sin t\sqrt{-\Delta_{\mathbb{R}^d}}}{\sqrt{-\Delta_{\mathbb{R}^d}}}(1-\chi_{\text{obst}})g \\ & - \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta_{\mathbb{R}^d}}\right)}{\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi]u(s)ds. \end{aligned} \tag{2.7}$$

The first two terms are handled thanks to the Strichartz estimates for the waves in $\mathbb{R}^d$:

$$\begin{aligned} & || \cos\left(t\sqrt{-\Delta_{\mathbb{R}^d}}\right)(1-\chi_{\text{obst}})f + \frac{\sin t\sqrt{-\Delta_{\mathbb{R}^d}}}{\sqrt{-\Delta_{\mathbb{R}^d}}}(1-\chi_{\text{obst}})g||_{L^p(\mathbb{R}, L^q)} \\ & \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}. \end{aligned} \tag{2.8}$$

And by Christ-Kiselev lemma, cutting the sinus in half wave operators and the Strichartz estimates in the full space again

$$\begin{aligned} & || \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta_{\mathbb{R}^d}}\right)}{\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{L^pL^q} \\ & \lesssim || \int_{\mathbb{R}} \frac{\sin\left((t-s)\sqrt{-\Delta_{\mathbb{R}^d}}\right)}{\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{L^pL^q} \\ & \lesssim || \frac{e^{-it\sqrt{-\Delta_{\mathbb{R}^d}}}}{\sqrt{-\Delta_{\mathbb{R}^d}}} \int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{L^pL^q} \\ & \lesssim || \int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{\dot{H}^{\gamma-1}}. \end{aligned} \tag{2.9}$$

Now, thanks to the dual version of the L^2 estimate (2.1) in $\mathbb{R}^d$ (Proposition 2 replacing Ω by $\mathbb{R}^d$, which enjoys no trapped geodesic) we get

$$\begin{aligned} & || \int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^d}}}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{\dot{H}^{\gamma-1}} \\ & = || \int_{\mathbb{R}} e^{is\sqrt{-\Delta_{\mathbb{R}^d}}}\tilde{\chi}[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{\dot{H}^{\gamma-1}} \lesssim ||[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{L^2\dot{H}^{\gamma-1}}, \end{aligned} \tag{2.10}$$

where $\tilde{\chi} = 1$ on the support of $\nabla\chi_{\text{obst}}$. But, using the L^2 estimate away from the trapped ray (2.1), because $\nabla\chi_{\text{obst}}$ is supported away from the trapped ray:

$$\begin{aligned} & ||[\Delta_D, \chi_{\text{obst}}]u(s)ds||_{L^2\dot{H}^{\gamma-1}} \lesssim ||f||_{\dot{H}^{\gamma-1}} + ||g||_{\dot{H}^{\gamma-2}} + ||\nabla f||_{\dot{H}^{\gamma-1}} + ||\nabla g||_{\dot{H}^{\gamma-2}} \\ & \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}. \end{aligned} \tag{2.11}$$

Collecting (2.7), (2.8), (2.9), (2.10) and (2.11) we conclude that

$$||(1-\chi_{\text{obst}})u||_{L^pL^q} \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}. \tag{2.12}$$

2.2.2. The second term: away from the trapped ray and near the obstacles

Let us now deal with

$$w := \chi_{\text{obst}}(1-\chi_{\text{ray}})u.$$

We denote here $\chi := \chi_{\text{obst}}(1-\chi_{\text{ray}})$ and consider $\varphi \in C_0^\infty((-1,1))$ satisfying $\varphi \geq 0$, $\varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s-j) = 1$. We decompose

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi(t-j)\chi u =: \sum_{j\in\mathbb{Z}} u_j.$$

Because χ is supported away from the trapped ray, using the L^2 estimate away from the trapped ray (2.1) combined with the local Strichartz estimates in time 1 for each u_j allows us to recover the estimate in the full space, with the exact same proof as Burq [17], the only difference been using (2.1) instead of his the L^2 estimate for non-trapping geometries and we get:

$$||\chi_{\text{obst}}(1-\chi_{\text{ray}})u||_{L^p(\mathbb{R},L^q)} \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}. \tag{2.13}$$

2.2.3. The third term: near the trapped ray

We will denote here $\chi = \chi_{\text{obst}}\chi_{\text{ray}}$. We will cut u in time intervals of length $|\log h|$. Consider $\varphi \in C_0^\infty((-1,1))$ satisfying $\varphi \geq 0, \varphi(0) = 1$ and $\sum_{j\in\mathbb{Z}} \varphi(s-j) = 1$. We decompose

$$\chi u = \sum_{j\in\mathbb{Z}} \varphi\left(\frac{t}{|\log h|} - j\right)\chi u =: \sum_{j\in\mathbb{Z}} u_j.$$

The u_j satisfy the equation

$$(\partial_t^2 - \Delta)u_j = F_j + G_j$$

where

$$F_j = |\log h|^{-2}\varphi''\left(\frac{t}{|\log h|} - j\right)\chi u + 2|\log h|^{-1}\varphi'\left(\frac{t}{|\log h|} - j\right)\chi\partial_t u, \tag{2.14}$$

$$G_j = -\varphi\left(\frac{t}{|\log h|} - j\right)[\Delta, \chi]u. \tag{2.15}$$

We denote

$$v_j(t) = \int_{(j-1)|\log h|}^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} F_j(s)ds,$$
$$w_j(t) = \int_{(j-1)|\log h|}^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} G_j(s)ds,$$

in such a way that $u_j = v_j + w_j$. By the L^2-global integrability estimate near the trapped ray (2.2) and (2.14) we get

$$\sum_{j\in\mathbb{Z}} |||\log h|F_j||^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim |\log h|\left(||u_0||^2_{\dot{H}^\gamma} + ||u_1||^2_{\dot{H}^{\gamma-1}}\right),$$

and therefore

$$\sum_{j\in\mathbb{Z}} ||F_j||^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim |\log h|^{-1}\left(||u_0||^2_{\dot{H}^\gamma} + ||u_1||^2_{\dot{H}^{\gamma-1}}\right). \tag{2.16}$$

On the other hand, by the Strichartz estimate on logarithmic interval

$$||v_j||_{L^pL^q} \lesssim ||F_j||_{L^1\dot{H}^{\gamma-1}}. \tag{2.17}$$

But, as F_j is support on a time interval of size proportional to $|\log h|$, by the Cauchy-Schwarz inequality we get

$$||F_j||_{L^1\dot{H}^{\gamma-1}} \lesssim |\log h|^{1/2}||F_j||_{L^2\dot{H}^{\gamma-1}}. \tag{2.18}$$

Therefore, by (2.16), (2.17), and (2.18)

$$\sum_{j\in\mathbb{Z}} ||v_j||^2_{L^pL^q} \lesssim \left(||u_0||^2_{\dot{H}^\gamma} + ||u_1||^2_{\dot{H}^{\gamma-1}}\right). \tag{2.19}$$

Now, let us deal with w_j. Let us define

$$\tilde{w}_j^{\pm} = e^{-it\sqrt{-\Delta}} \int_{(j-1)|\log h|}^{(j+1)|\log h|} \frac{e^{is\sqrt{-\Delta}}}{\sqrt{-\Delta}} G_j(s)ds.$$

Decomposing the sinus operator in half wave operators and make use of the Christ-Kiselev lemma allows us to estimate the norm of $\tilde{w}_j^{\pm}$ instead of these of w_j. By the Strichartz estimates on logarithmic interval we get

$$||\tilde{w}_j^{\pm}||_{L^pL^q} \leq || \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} G_j(s)ds||_{\dot{H}^{\gamma-1}}.$$

Now, remark that $[\Delta, \chi]$ is supported away from the periodic ray. Let $\tilde{\chi}$ be equal to 1 in the support of $\nabla\chi$ and vanishing on the trapped ray. By the dual version of the L^2-global integrability estimate outside the trapped ray (2.1) we get

$$\begin{aligned} ||\tilde{w}_j^{\pm}||_{L^pL^q} &\leq || \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} G_j(s)ds||_{\dot{H}^{\gamma-1}} \\ &= || \int_{(j-1)|\log h|}^{(j+1)|\log h|} e^{is\sqrt{-\Delta}} \tilde{\chi} G_j(s)ds||_{\dot{H}^{\gamma-1}} \lesssim ||G_j||_{L^2\dot{H}^{\gamma-1}}. \end{aligned}$$

And now, by the L^2-global integrability estimate outside the trapped ray (2.1) itself:

$$\sum_{j\in\mathbb{Z}} ||G_j||^2_{L^2(\mathbb{R},\dot{H}^{\gamma-1})} \lesssim \left(||u_0||^2_{\dot{H}^\gamma} + ||u_1||^2_{\dot{H}^{\gamma-1}}\right).$$

Therefore, we get:

$$\sum_{j\in\mathbb{Z}} ||w_j||^2_{L^pL^q} \lesssim \left(||u_0||^2_{\dot{H}^\gamma} + ||u_1||^2_{\dot{H}^{\gamma-1}}\right). \tag{2.20}$$

Thus, combining (2.19) and (2.20) we conclude thanks to the embedding $l^2(\mathbb{Z}), \to l^p(\mathbb{Z})$ (we recall that $p \geq 2$):

$$||\chi u||_{L^pL^q} \sim \left(\sum_{j\in\mathbb{Z}} ||u_j||^p_{L^pL^q}\right)^{\frac{1}{p}} \lesssim \left(\sum_{j\in\mathbb{Z}} ||u_j||^2_{L^pL^q}\right)^{\frac{1}{2}} \lesssim ||u_0||_{\dot{H}^\gamma} + ||u_1||_{\dot{H}^{\gamma-1}}.$$

Combining this last estimate with 2.12 and 2.13 we conclude that

$$||u||_{L^pL^q} \lesssim ||u_0||_{\dot{H}^\gamma} + ||u_1||_{\dot{H}^{\gamma-1}}.$$

Finally, standard techniques permit to remove the frequencies cut-off. Therefore, Proposition 4 implies our main Theorem.

2.3. Reduction to the trapped set

Let D be an open-neighbourhood of the trapped ray, chosen to be a cylinder with the trapped ray for axis. We define the trapped set of D in time T

Definition 2. $(x, \xi) \in T^\star\Omega$ belongs to the trapped set of D in time T, denoted $\mathcal{T}_T(D)$, if and only if one of the rays starting from $\left(x, \frac{\xi}{|\xi|}\right)$ and $\left(x, -\frac{\xi}{|\xi|}\right)$ belongs to D after time T.

Note that the only differences with the definition of [20] are that the rays are all followed at speed one instead of $|\xi| \in [\alpha_0, \beta_0]$. Therefore, with the same proofs, we get

Lemma 1. *For all bicharacteristic γ starting from D with speed one, we have*

$$d(\gamma(t), \mathcal{T}_T(D)^c) > 0 \,\forall t \in [-T-1, -T]$$

and

Lemma 2. *For all $D, \tilde{D}$, there exists $T^\star > 0$, $c > 0$ such that for all $T \geq 0$:*

$$d(\mathcal{T}_{T-T^\star}(D)^c, \mathcal{T}_T(D)) \geq e^{-cT}, \tag{2.21}$$

and, if $D \subset \tilde{D}$

$$d(\mathcal{T}_T(\tilde{D})^c, \mathcal{T}_T(D)) \geq \frac{1}{4} e^{-cT} d(\tilde{D}^c, D). \tag{2.22}$$

We say that $f \in L^2$ is microlocally supported in $U \subset T^\star\Omega$, if for all $a \in C^\infty(T^\star\Omega)$ such that $a = 1$ in U we have $\mathrm{Op}(a)f = f$. Using the same time translations as in [20] combined with the finite speed of propagation, the following Proposition implies our main Theorem:

Proposition 5. *There exists $\epsilon > 0$ and a small neighbourhood D of the trapped ray, such that, for all $\chi \in C_0^\infty$ supported in D, if f, g are such that $\psi(-h^2\Delta)f = f$, $\psi(-h^2\Delta)g = g$, are microlocally supported in $\mathcal{T}_{\epsilon|\log h|}(D)$ and spatially in D and away from $\partial(\Theta_1 \cup \Theta_2)$, and u is the solution of (1.1) with data (f, g), we have:*

$$||\chi u||_{L^p(0,\epsilon|\log h|)L^q} \lesssim ||f||_{\dot{H}^\gamma} + ||g||_{\dot{H}^{\gamma-1}}.$$

The rest of the paper is thus devoted to prove Proposition 5.

3 Construction of an approximate solution

3.1. The phase functions

We recall here the definition of the phase functions we used in [20] following the works of Ikawa [22,23] and Burq [21]. We call $\varphi : \mathcal{U} \to \mathbb{R}$ a phase function on the open set $\mathcal{U} \subset \mathbb{R}^d$ if φ is C^∞ on $\mathcal{U}$ and verifies $|\nabla\varphi| = 1$. We say that φ verifies (P) on $\partial\Theta_1$ if

(1) the principal curvatures of the level surfaces of φ with respect to $-\nabla\varphi$ are non-negative in every point of $\mathcal{U}$,

(2) we have
$$\Theta_2 \subset \{y + \tau\nabla\varphi(x) \text{ s.t. } \tau \geq 0, \mathrm{y} \in \mathcal{U} \cap \partial\Theta_1, \nabla\varphi(\mathrm{y}) \cdot \mathrm{n}(\mathrm{y}) \geq 0\},$$

(3) for all $A \in \mathbb{R}$, the set $\{\varphi \leq A\}$ is empty or convex,

and we say that φ verifies (P) on $\partial\Theta_2$ if the role of Θ_1 and Θ_2 are exchanged in the above.

In addition, we recall the definition of $X^i(x,p)$ and $\Xi^i(x,p)$ from [20]: $X^i(x,p)$ is the i-th point of intersection of the reflected ray starting at x in the direction p with the obstacles, and $\Xi^i(x,p)$ is the direction of the reflected ray starting at x in the direction p after i reflections.

Let $\delta_1 \geq 0$ and φ be a phase function. We set
$$\Gamma_p(\varphi) = \{x \in \partial\Theta_p \text{ s.t.} - n(x) \cdot \nabla\varphi(x) \geq \delta_1\},$$
$$\mathcal{U}_p(\varphi) = \bigcup_{X^1(x,\nabla\varphi(x)) \in \Gamma_p(\varphi)} \{X^1(x,\nabla\varphi(x)) + \tau\Xi(x,\nabla\varphi(x)),\ \tau \geq 0\}.$$

Then, there exists $\delta_1 \geq 0$ such that, if φ is a phase function verifying (P) on $\partial\Theta_p$, we can define the phase φ_j reflected on the obstacle Θ_j on the open set $\mathcal{U}_j(\varphi)$, verifying (P) on $\partial\Theta_j$, by the following relation, for $X^1(x,\nabla\varphi(x)) \in \Gamma_p(\varphi)$:
$$\varphi_j(X^1(x,\nabla\varphi) + \tau\Xi^1(x,\nabla\varphi)) = \varphi(X^1(x,\nabla\varphi)) + \tau.$$

We call a finite sequence $J = (j_1, \dots, j_n)$, $j_i \in \{1,2\}$ with $j_i \neq j_{i+1}$ a story of reflections, and will denote $\mathcal{I}$ the set of all the stories of reflection. By induction, we can define the phases φ_J for any $J \in \mathcal{I}$, on the sets $\mathcal{U}_J(\varphi)$.

For $f \in C^\infty(\mathcal{U})$ and $m \in \mathbb{N}$, let
$$|f|_m(\mathcal{U}) = \max_{(a_i)\in(\mathcal{S}^2)^m} \sup_{\mathcal{U}} |(a_1 \cdot \nabla) \cdots (a_m \cdot \nabla) f|.$$

The following estimate due to [21–23]:

Proposition 6. *For every $m \geq 0$ we have*
$$|\nabla\varphi_J|_m \leq C_m |\nabla\varphi|_m.$$

Moreover, according to Burq [21]:

Proposition 7. *There exists $M>0$ such that, for each $(i,j)\in\{1,2\}^2$, there exists open sets containing the trapped ray $\mathcal{U}_{i,j}$ such that, if $J=\{i,...,j\}$ verifies $|J|\geq M$, and φ verifies (P), φ_J can be defined in $\mathcal{U}_{i,j}$.*

We set

$$\hat{\mathcal{U}}_\infty=\mathcal{U}_{11}\cap\mathcal{U}_{12}\cap\mathcal{U}_{21}\cap\mathcal{U}_{22},$$

and $\mathcal{U}_\infty\subset\hat{\mathcal{U}}_\infty$ to be an open cylinder having for axis the periodic trajectory and contained in $\hat{\mathcal{U}}_\infty$. It will be shrunk in the sequel if necessary. Finally, we recall the following estimate concerning the derivatives with respect to ξ of the phases built beginning with $\varphi=(x-y)\cdot\frac{\xi}{|\xi|}$ we obtained in [20]:

Proposition 8. *Let $\varphi(x)=(x-y)\cdot\frac{\xi}{|\xi|}$. We denote $\varphi_J(x,\xi)$ the reflected phase we build beginning with φ. Then, for all multi-indices α, β there exists a constant $D_{\alpha,\beta}>0$ such that the following estimate holds on $\mathcal{U}_\infty$:*

$$|D_\xi^\alpha D_x^\beta\nabla\varphi_J|\leq D_{\alpha,\beta}^{|J|}.$$

3.2. The microlocal cut-off

According to Section 2, we are reduced to show Proposition 5. By Lemma 2, we can construct a small shrinking of $\mathcal{U}_\infty$, $\tilde{\mathcal{U}}_\infty\subset\mathcal{U}_\infty$, and $\tilde{q}_{\epsilon,h}\in C^\infty(T^\star\Omega)$ such that $\tilde{q}_{\epsilon,h}=1$ in an open neighbourhood of $\mathcal{T}_{2\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$, $\tilde{q}_{\epsilon,h}=0$ outside $\mathcal{T}_{2\epsilon|\log h|}(\mathcal{U}_\infty)$ in such a way that, for all multi-indexes α,

$$|\partial_\alpha\tilde{q}_{\epsilon,h}|\lesssim h^{-2|\alpha|c\epsilon}. \tag{3.1}$$

It suffices to show Strichartz estimates in time $\epsilon|\log h|$ for data microlocally supported in $\mathcal{T}_{\epsilon|\log h|}(\tilde{\mathcal{U}}_\infty)$ and spatially supported in $\tilde{\mathcal{U}}_\infty$ and away from a small neighbourhood $\mathcal{V}$ of $\partial(\Theta_1\cup\Theta_2)$. Let $\chi_0\in C^\infty$ such that $\chi_0=0$ near $\partial(\Theta_1\cup\Theta_2)$ and $\chi_0=1$ outside $\mathcal{V}$. For such functions, $\chi_0\mathrm{Op}(\tilde{q}_{\epsilon,h})f=f$, thus it suffices to show

$$||\chi e^{-it\sqrt{-\Delta}}\chi_0\mathrm{Op}(\tilde{q}_{\epsilon,h})f||_{L^p(0,\epsilon|\log h|)L^q}\lesssim||f||_{\dot{H}^\gamma}, \tag{3.2}$$

for all $\chi\in C^\infty$ supported in $\tilde{\mathcal{U}}_\infty$. We will show the strongest estimate:

$$||e^{-it\sqrt{-\Delta}}\chi_0\mathrm{Op}(\tilde{q}_{\epsilon,h})f||_{L^p(0,\epsilon|\log h|)L^q}\lesssim||f||_{\dot{H}^\gamma}. \tag{3.3}$$

by the $TT^\star$ method – see, for example, [4] – it suffices to show the dispersive estimate, for $0\leq t\leq\epsilon|\log h|$:

$$||Q_{\epsilon,h}^\star e^{-it\sqrt{-\Delta}}Q_{\epsilon,h}||_{L^1\to L^\infty}\lesssim\frac{1}{h^{\frac{d+1}{2}}t^{\frac{d-1}{2}}}. \tag{3.4}$$

where

$$Q_{\epsilon,h}:=\psi(-h^2\Delta)\chi_0\mathrm{Op}(\tilde{q}_{\epsilon,h}).$$

The symbol associated with $Q_{\epsilon,h}$ admits the development (see, for example, Zworski [35])

$$\sum_{k=0}^{N} \frac{(ih)^k}{k!} \langle D_\xi, D_y \rangle^k (\tilde{q}_{\epsilon,h}(x,\xi)\chi_0(y)\psi(\eta))_{\eta=\xi, y=x} + O_S(h^{N+1}),$$

hence we introduce as in [20], for $N \geq 1$

$$q_{\epsilon,h,N}(x,\xi) := \sum_{k=0}^{N} \frac{(ih)^k}{k!} \langle D_\xi, D_y \rangle^k (\tilde{q}_{\epsilon,h}(x,\xi)\chi_0(y)\psi(\eta))_{\eta=\xi, y=x}.$$

Then, to show (3.4), it suffices to show

$$||\mathrm{Op}(q_{\epsilon,h,N})^\star e^{-it\sqrt{-\Delta}} \mathrm{Op}(q_{\epsilon,h,N})||_{L^1 \to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}} \tag{3.5}$$

for N large enough. Note that, in particular,

$$\mathrm{Supp} q_{\epsilon,h,N} \subset \mathcal{T}_{2\epsilon|\log h|}(\mathcal{U}_\infty) \cap \mathcal{U}_\infty \times \{|\xi| \in [\alpha_0, \beta_0]\} \tag{3.6}$$

and $q_{\epsilon,h,N}$ is spatially supported outside a small neighbourhood of $\partial(\Theta_1 \cup \Theta_2)$ not depending of ϵ, h, N.

It suffices to obtain

$$||\mathrm{Op}(q_{\epsilon,h,N})^\star \cos\left(t\sqrt{-\Delta}\right) \mathrm{Op}(q_{\epsilon,h,N})||_{L^1 \to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}},$$
$$||\mathrm{Op}(q_{\epsilon,h,N})^\star \sin\left(t\sqrt{-\Delta}\right) \mathrm{Op}(q_{\epsilon,h,N})||_{L^1 \to L^\infty} \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}.$$

We will deal for example with the cosinus part, the sinus is handled in the same way. We set

$$\delta^y_{\epsilon,h,N}(x) = \frac{1}{(2\pi h)^d} \int e^{-i(x-y)\cdot\xi/h} q_{\epsilon,T,N}(x,\xi) d\xi,$$

in order to have, for $u \in L^2$

$$\left(\mathrm{Op}(q_{\epsilon,h,N})u\right)(x) = \int \delta^y_{\epsilon,h,N}(x) u(y) dy.$$

Notice that

$$\mathrm{Op}(q_{\epsilon,h,N})^\star \cos\left(t\sqrt{-\Delta}\right) \mathrm{Op}(q_{\epsilon,h,N}) u(x) = \int \mathrm{Op}(q_{\epsilon,h,N})^\star \cos\left(t\sqrt{-\Delta}\right) \delta^y_{\epsilon,T,N}(x) u(y) dy,$$

thus, to show (3.5), it suffices to study $\delta^y_{\epsilon,h,N}$ and to show that, for N large enough

$$|\mathrm{Op}(q_{\epsilon,h,N})^\star \cos\left(t\sqrt{-\Delta}\right) \delta^y_{\epsilon,h,N}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \text{ for } 0 \leq t \leq \epsilon|\log h|.$$

Let $\mathcal{V}_1$ be a small neighbourhood of $\partial(\Theta_1 \cup \Theta_2)$ on which $q_{\epsilon,h,N}$ is vanishing and $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_0 = 1$ on $\mathcal{U}_\infty \cap \mathcal{V}_1^c$. We choose χ_+ to be supported on $\mathrm{Conv}(\Theta_1 \cup \Theta_2) \backslash (\Theta_1 \cup \Theta_2)$ and away from a small enough neighbourhood of $\partial(\Theta_1 \cup \Theta_2)$, Conv denoting the convex hull. Note that in particular, $\mathrm{Op}(q_{\epsilon,h,N})^\star = \mathrm{Op}(q_{\epsilon,h,N})^\star \chi_+$. The symbol of $\mathrm{Op}(q_{\epsilon,T,N})^\star$ enjoys the development

$$q_{\epsilon,h,N}{}^\star(x,\xi) = e^{ih\langle D_x, D_\xi \rangle} q_{\epsilon,h,N}.$$

Thus, by (3.1), taking $\epsilon > 0$ small enough, we have $|q^{\star(\alpha)}_{\epsilon,T,N}| \lesssim 1$ for all $|\alpha| \le d+1$. Moreover, $q^{\star(\alpha)}_{\epsilon,T,N}$ is compactly supported in frequencies. Therefore, by Ryuichi Ashino, Michihiro Nagase, and Rémi Vaillancourt [36], Section 4, $\mathrm{Op}(q_{\epsilon,T,N})$ is bounded on $L^\infty \to L^\infty$ independently of h. Therefore, we only have to show, for all $0 \le T \le \epsilon |\log h|$

$$|\chi_0 \cos\left(t\sqrt{-\Delta}\right)\delta^y_{\epsilon,h,N}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \text{ for } 0 \le t \le \epsilon|\log h| \tag{3.7}$$

for N large enough.

In order to do so, we will construct a parametrix in time $0 \le t \le \epsilon|\log h|$ for the wave equation with data $(\delta^y_{\epsilon,h,N}, 0)$. The first step will be to construct an approximate solution of wave equation with data

$$(e^{-i(x-y)\cdot\xi/h} q_{\epsilon,h,N}(x,\xi), 0)$$

where $\xi \in \mathbb{R}^d, \xi \in \mathrm{Supp} q_{\epsilon,h,N}$ is fixed and considered as a parameter.

3.3. Approximation of the solution

3.3.1. The Neumann sum

We look for the solution w of

$$\begin{cases} \partial_t^2 w - \Delta w & = 0 \text{ in } \Omega \\ w(t=0)(x) & = e^{-i(x-y)\cdot\xi/h} q(x,\xi) \\ \partial_t w(t=0)(x) & = 0 \\ w_{|\partial\Omega} & = 0 \end{cases}$$

as the Neumann series

$$w = \sum_{J \in \mathcal{I}} (-1)^{|J|} w^J$$

where

$$\begin{cases} \partial_t^2 w^\emptyset - \Delta w^\emptyset & = 0 \text{ in } \mathbb{R}^d \\ w(t=0)(x) & = e^{-i(x-y)\cdot\xi/h} q(x,\xi) \\ \partial_t w(t=0)(x) & = 0 \end{cases} \tag{3.8}$$

and, for $J \neq \emptyset$, $J = (j_1, \ldots, j_n)$, $J' = (j_1, \ldots, j_{n-1})$

$$\begin{cases} \partial_t^2 w^J - \Delta w^J & = 0 \text{ in } \mathbb{R}^d \backslash \Theta_{j_n} \\ w(t=0), \partial_t w(t=0) & = 0 \\ w^J_{|\partial\Theta_{j_n}} & = w^{J'}_{|\partial\Theta_{j_n}}. \end{cases} \tag{3.9}$$

Let us denote

$$\varphi_J^+(x,\xi) = \varphi_J(x,\xi) \text{ and } \varphi_J^-(x,\xi) = \varphi_J(x,-\xi),$$

that is, φ_J^+ is the reflected phase constructed with $\varphi(x,\xi) = (x-y)\cdot\frac{\xi}{|\xi|}$ and φ_J^- is the reflected phase constructed with $\varphi(x,-\xi) = -(x-y)\cdot\frac{\xi}{|\xi|}$. We look for w^J as the sum of the two series

$$\begin{aligned} w^J &= w^{J,+} + w^{J,-} \\ &= \sum_{k\geq 0} w_k^{J,+}(x,t)e^{-i(\varphi_J^+|\xi|-t|\xi|)/h}(-i\frac{h}{|\xi|})^k + \sum_{k\geq 0} w_k^{J,-}(x,t)e^{i(\varphi_J^-|\xi|-t|\xi|)/h}(i\frac{h}{|\xi|})^k. \end{aligned}$$

If $w_0^{J,\pm}$ solves the transport equations

$$(2\partial_t + \nabla\varphi_J^\pm \cdot \nabla + \Delta\varphi_J)w_0^{J,\pm} = 0,$$

and for $k \geq 1$

$$(2\partial_t + \nabla\varphi_J^\pm \cdot \nabla + \Delta\varphi_J)w_k^{J,\pm} = -\Box w_{k-1}^{J,\pm},$$

with, for $J = \emptyset$

$$\begin{aligned} w_0^{\emptyset,+}(x,0) = w_0^{\emptyset,-}(x,0) &= \frac{1}{2}q(x,\xi), \\ w_k^{\emptyset,\pm}(x,0) &= 0, \ \forall k \geq 1, \\ \partial_t w_k^{\emptyset,+}(x,0) + \partial_t w_k^{\emptyset,-}(x,0) &= 0, \ \forall k \geq 0, \end{aligned}$$

and, for $|J| \geq 1$

$$\begin{aligned} w^{J,\pm}_{k|\partial\Theta_{j_n}} &= w^{J',\pm}_{k|\partial\Theta_{j_n}}, \\ \partial_t w_k^{J,\pm}(x,0) = w_k^{J,\pm}(x,0) &= 0, \end{aligned}$$

then w^J solves (3.8), (3.9).

Solving the transport equations for $J = \emptyset$ gives immediately

$$\begin{aligned} w_0^{\emptyset,+} &= \frac{1}{2}q\left(x - t\frac{\xi}{|\xi|}, \xi\right), \\ w_0^{\emptyset,-} &= \frac{1}{2}q\left(x + t\frac{\xi}{|\xi|}, \xi\right), \\ w_k^{\emptyset,\pm} &= -\int_0^t \Box w_{k-1}^\pm(x \mp (s-t)\frac{\xi}{|\xi|}, s)ds \quad k \geq 1. \end{aligned}$$

3.3.2. Reflections on the obstacles

Now, we would like to reflect $w^{\emptyset,\pm}$ on the obstacle. To this purpose, starting from the phases $\varphi(x,\xi) = \frac{(x-y)\cdot\xi}{|\xi|}$ and $\varphi(x,-\xi) = -\frac{(x-y)\cdot\xi}{|\xi|}$ we would like to define the reflected phases as explained in the first subsection.

We decompose the set of the stories of reflections as

$$\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$$

where $\mathcal{I}_1$ are all stories beginning with a reflection on Θ_1, that is of the form $(1,\ldots)$, and $\mathcal{I}_2$ beginning with a reflection on Θ_2, that is of the form $(2,\ldots)$. Let e be a unit vector with the same direction as $\mathcal{R}$. We take e oriented from Θ_1 to Θ_2. For $\frac{\xi}{|\xi|}$ in a small enough neighbourhood V of $\{e, -e\}$ we have

(1) if $\xi \cdot e > 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_1 and $-\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_2
(2) if $\xi \cdot e < 0$, then $\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_2 and $-\frac{(x-y)\cdot\xi}{|\xi|}$ verifies (P) on Θ_1

Remark that

- In situation (1), the support of $w^{\emptyset,+}$ never cross Θ_1 and the support of $w^{\emptyset,-}$ never cross Θ_2 in any time,
- in situation (2), the support of $w^{\emptyset,+}$ never cross Θ_2 and the support of $w^{\emptyset,-}$ never cross Θ_1 in any time.

We set

- In situation (1): $(\mathcal{I}_+,\mathcal{I}_-) := (\mathcal{I}_2,\mathcal{I}_1)$,
- in situation (2): $(\mathcal{I}_+,\mathcal{I}_-) := (\mathcal{I}_1,\mathcal{I}_2)$.

Then, (3.9) is satisfied for $w^{\pm}$ for all $J \in \mathcal{I}_{\mp}$: indeed, because the support of $w^{\emptyset,\pm}$ never cross $\Theta_{i_\mp}$, we have for all time $0 = w^{\emptyset,\pm}_{|\partial\Theta_{i_\mp}} = w^{\{i_\mp\}}_{|\partial\Theta_{i_\mp}}$, and so on. Thus, we are reduced to construct the $w^{J,\pm}$'s for $J \in \mathcal{I}_{\pm}$. In the same way as in [20], shrinking $\mathcal{U}_\infty$ if necessary, all the phases we will be dealing with are well defined according to the previous remarks.

Then, in the exact same way as [20], we solve the transport equations along the rays:

Proposition 9. *We denote by $\hat{X}_{-t}(x,\nabla\varphi_J^{\pm})$ the backward spatial component of the flow starting from $(x,\nabla\varphi_J^{\pm})$, defined in the same way as $X_{-t}(x,\nabla\varphi_J^{\pm})$, at the difference that we ignore the first obstacle encountered if it's not Θ_{j_n}, and we ignore the obstacles after $|J|$ reflections. Moreover, for $J = (j_1 = i_1, ..., j_n) \in \mathcal{I}_{i_1}$, denote by*

$$J(x,\xi,t) = \begin{cases} (j_1,...,j_k) & \text{if } \hat{X}_{-t}(x,\nabla\varphi_J^{\pm}) \text{ has been reflected } n-k \text{ times,} \\ \emptyset & \text{if } \hat{X}_{-t}(x,\nabla\varphi_J^{\pm}) \text{ has been reflected } n \text{ times.} \end{cases}$$

Then, the w_k^J's are given by, for $t \geq 0$ and $x \in \mathcal{U}_J(\varphi)$

$$w_0^{J,\pm}(x,t) = \Lambda\varphi_J^{\pm}(x,\xi)q(\hat{X}_{-t}(x,\nabla\varphi_J^{\pm}),\xi)$$

where

$$\Lambda\varphi_J^{\pm}(x,\xi) = \frac{G\varphi_J^{\pm}(x)}{G\varphi_J^{\pm}(X^{-1}(x,\nabla\varphi_J^{\pm}))} \times \cdots \times \frac{G\varphi^{\pm}(X^{-|J|-1}(x,\nabla\varphi_J^{\pm}))}{G\varphi^{\pm}(X^{-|J|}(x,\nabla\varphi_J^{\pm}))},$$

and, for $k \geq 1$, and $x \in \mathcal{U}_J(\varphi)$

$$w_k^{J,\pm}(x,t) = \int_0^t g_{\varphi_J}(x,(t-s),\xi)\Box w_{k-1}^{J(x,\xi,\pm(t-s))}(\hat{X}_{-(t-s)}(x,\nabla\varphi_J^{\pm}),s)ds$$

where

$$g_{\varphi_J}^{\pm}(x,\xi,t) = \frac{G\varphi_J^{\pm}(x)}{G\varphi_J^{\pm}(X^{-1}(x,\nabla\varphi_J^{\pm}))} \times \cdots \times \frac{G\varphi^{\pm}_{J(x,\xi,t)}(X^{-|J(x,\xi,t)|-1)}(x,\nabla\varphi_J^{\pm}))}{G\varphi^{\pm}_{J(x,\xi,t)}(\hat{X}_{-t}(x,\nabla\varphi_J^{\pm}))}.$$

With the exact same proofs as in [20], Section 4.3.2, following the rays at speed one instead of speed $|\xi| \in [\alpha_0,\beta_0]$, $w^{J,\pm}$ verify the following properties:

Proposition 10. *We have*

(1) $w_k^J(x,t) \neq 0 \Rightarrow (\hat{X}_{-t}(x, \nabla\varphi_J), \xi) \in \text{Supp}q$,
(2) *for* $x \notin \mathcal{U}_J(\varphi)$ *and* $0 \le t \le \epsilon|\log h|$, $w_k^{J,\pm}(x,t) = 0$,
(3) *there exists* $c_1, c_2 > 0$ *such that for every* $J \in \mathcal{I}$, *the support of* $w_k^{J,\pm}(x,\cdot)$ *is included in* $\{c_1|J| \le t\}$ *for any* x, *and for* $x \in \text{Supp}\chi_0$ *it is included in* $\{c_1|J| \le t \le c_2(|J|+1)\}$,
(4) *in times* $0 \le t \le \epsilon|\log h|$, $\text{Supp}\chi_0 \cap \text{Supp}w_k^{J,\pm}(\cdot,t)$ *is included in* $\mathcal{U}_\infty$.

Moreover, Proposition 6 and Proposition 8 combined with Proposition 9 and (3.1) gives immediately:

Proposition 11. *The following bounds hold on* $\mathcal{U}_\infty$

$$|D_\xi^\alpha w_k^{J,\pm}| \lesssim C_\alpha^{|J|} h^{-(2k+|\alpha|)c\epsilon}.$$

3.4. Decay estimates

We recall the principal result who permits to estimate the decay of the reflected solutions, namely the convergence of the product of the Gaussian curvatures $\Lambda\varphi_J$ obtained by Ikawa [22,23] and Burq [21]. In the present framework of two obstacles, it writes, from [21, Proposition 3.23]:

Proposition 12. *Let* $0 < \lambda < 1$ *be the product of the two eigenvalues lesser than one of the Poincaré map associated with the periodic trajectory. Then, there exists* $0 < \alpha < 1$, *and for* $I = (1,2)$ *and* $I = (2,1)$, *for every* $l \in \{\{1\},\{2\},\emptyset\}$, *there exists a* C^∞ *function* $a_{I,l}$ *defined in* $\mathcal{U}_\infty$, *such that, for all* $J = (\underbrace{I, \dots, I}_{r\ \text{times}}, l)$, *we have*

$$\sup_{\mathcal{U}_\infty} |\Lambda\varphi_J - \lambda^r a_{I,l}|_m \le C_m \lambda^r \alpha^{|J|}.$$

Combined with the explicit expressions of Proposition 9 and (3.1), this result gives as in [20] the following decay:

Proposition 13. *We following bounds hold on* $\mathcal{U}_\infty$:

$$|w_k^{J,\pm}|_m \le C_k \lambda^{|J|} h^{-(2k+m)c\epsilon}.$$

Moreover, on the whole space, $|w_k^J|_m \le C_k h^{-(2k+m)c\epsilon}$.

3.5. Critical points of the phase

We need to study the critical points of the phase in order to be able to perform a stationary phase argument on the solution we are building. At the difference of [20], the phases here stationate in whole directions. Therefore, we will perform a stationary phase on each sphere $\mathcal{S}^{d-1}(0,s)$. To this purpose, we need

Proposition 14. *Let us denote*

$$\mathcal{S}_J^{\pm}(x,\xi,t)=\varphi_J^{\pm}(x,\xi)|\xi|-t|\xi|.$$

Then, there exists $\eta>0$ such that for all $|J|\geq 1$,

$$\begin{aligned}d(X^{-|J|}(x,\nabla\varphi_J^{\pm}(x,\xi)),y)\leq\eta \text{ and } w^{J,\pm}(x,t,\xi)\neq 0\\ \Rightarrow D_\xi\mathcal{S}_J^{\pm}(x,\xi,t)\neq 0.\end{aligned}\tag{3.10}$$

Moreover, as soon as $d(X^{-|J|}(x,\nabla\varphi_J(x,\xi)),y)\geq\eta$, for all $s>0$ and x there exists a unique $s_J(x,s)\in\mathcal{S}^{d-1}(0,s)$ such that, for all $t\geq 0$

$$w^{J,\pm}(x,\cdot,s_J^{\pm}(x,s))\neq 0 \text{ and } D_{\mathcal{S}^{d-1}(0,s)}\mathcal{S}_J^{\pm}(x,s_J^{\pm}(x,s),t)=0.\tag{3.11}$$

Furthermore, if $d(X^{-|J|}(x,\nabla\varphi_J(x,\xi)),y)\geq\eta$ and $s\in[\alpha_0,\beta_0]$,

$$\det D^2_{\mathcal{S}^{d-1}(0,s)}\mathcal{S}^{\pm}{}_J(x,s_J^{\pm}(x,s),t)\geq c>0.\tag{3.12}$$

Proof. For the seek of readability, we denote $\mathcal{S}_J=\mathcal{S}_J^{+}$, $\varphi_J:=\varphi_J^{+}$, $w^J=w^{J,+}$ and we make the proof for the positive part of the wave, $w^{J,+}$: for $w^{J,-}$, the proof is the same.

In the same way as in [20], we obtain, differentiating $|\nabla\varphi_J(x,\xi)|\xi||^2=|\xi|^2$ with respect to ξ and integrating the transport equation obtained along the rays up to the first phase:

$$D_\xi\mathcal{S}_J^{\pm}(x,\xi,t)=X^{-|J|}(x,\nabla\varphi_J(x,\xi))-y-(t-l_J(x,\xi))\frac{\xi}{|\xi|},\tag{3.13}$$

where $l_J(x,\xi)$ is given by

$$\begin{aligned}l_J(x,\xi)&=d(x,X^{-1}(x,\nabla\varphi_J(x,\xi)))+d(X^{-1}(x,\nabla\varphi_J(x,\xi)),X^{-2}(x,\nabla\varphi_J(x,\xi)))\\&\quad+\cdots+d(X^{-|J|-1}(x,\nabla\varphi_J(x,\xi)),X^{-|J|}(x,\nabla\varphi_J(x,\xi))).\end{aligned}$$

Note that, by Proposition 10, (1), $w^{J,\pm}(x,\xi,t)\neq 0$ implies that, because q is supported away of the boundary, for $|J|\geq 1$

$$t-l_J(x,\xi)\geq\delta_0>0$$

and thus, we get (3.10). Moreover, we deduce that

$$D_{\mathcal{S}^{d-1}(0,s)}\mathcal{S}_J(x,\xi)=X^{-|J|}(x,\nabla\varphi_J^{\pm}(x,\xi))-y-\left(\left(X^{-|J|}(x,\nabla\varphi_J(x,\xi))-y\right)\cdot\frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|}.$$

Thus, if ξ is such that $D_{\mathcal{S}^{d-1}(0,s)}\mathcal{S}_J(x,\xi)=0$,

$$X^{-|J|}(x,\nabla\varphi_J(x,\xi))-y//\frac{\xi}{|\xi|},$$

hence $\frac{\xi}{|\xi|}$ is a direction allowing reaching the point x from the point y following the story of reflection J. Note that there are a priori two such vectors on $\mathcal{S}^{d-1}(0,s)$: one and its opposite, but because $w^{J,+}=0$ for $J\in\mathcal{I}_-$, we have $w^J(x,t,\xi)=0$ for one of them. We thus get (3.11). Note that the critical point ξ such that $w^J(x,t,\xi)\neq 0$ is the one verifying

$$\left(X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - y\right) \cdot \frac{\xi}{|\xi|} > 0. \tag{3.14}$$

The Hessian in $\mathbb{R}^d$ of $\mathcal{S}_J$ is derivated like in [20], differentiating $|\nabla\varphi_J(x,\xi)|\xi||^2 = |\xi|^2$ with respect to ξ and integrating the transport equation obtained along the rays once again:

$$D^2_\xi S_J^\mp(x,\xi,t) = \frac{l_J(x,\xi)}{|\xi|} Id - \left(\frac{Id}{|\xi|} - \frac{\xi\xi^t}{|\xi|^3}\right)t$$
$$- \sum_{k=1}^{d} \int_0^{\frac{l}{|\xi|}} D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)),\xi)\left(D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi)\right)^t,$$

where we denoted $J^{(s)} := J(\cdot, s/2)$ and $\psi_J(x,\xi) := |\xi|\varphi_J(x,\xi)$. We would like to deduce an expression of $D^2_{\mathcal{S}^{d-1}(0,s)} S_J(x,\xi,t)$ for $\xi = s_J(x,t)$. To this purpose, we recall that

Lemma 3. *Let $g : \mathbb{R}^d \to \mathbb{R}^m$ be a submersion in 0 and $M := g^{-1}(0)$. Moreover, let $f : \mathbb{R}^d \to \mathbb{R}$ and F be its restriction to M. We suppose that F has a critical point in $a \in M$. Then, the Hessian of F in a is the restriction of*

$$d^2 f_a - \lambda \circ d^2 g_a$$

to $T_a M$, where λ is the Lagrange multiplier of f with respect to g in a, that is the unique linear form $\lambda \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ such that $df_a = \lambda \circ dg_a$.

Here, we can take $g(\xi) := |\xi|^2 - s^2$. Then $Dg(\xi) = 2\xi$. The Lagrange multiplier of $\mathcal{S}_J(x,\cdot)$ with respect to g in $\xi := s_J(x,s)$ is the unique $\lambda \in \mathbb{R}$ such that

$$D_\xi \mathcal{S}_J(x,t,\xi) = 2\lambda\xi.$$

Therefore, according to (3.13)

$$2\lambda = \left(X^{-|J|}(x,\nabla\varphi_J(x,\xi)) - y\right) \cdot \frac{\xi}{|\xi|^2} - (t - l_J(x,\xi))\frac{1}{|\xi|}.$$

On the other hand, by Lemma 3

$$D^2_{\mathcal{S}^{d-1}(0,s)} S_J(x,\xi,t) = \left(D^2_\xi S_J(x,\xi,t) - 2\lambda \mathrm{Id}\right)_{|T_\xi \mathcal{S}^{n-1}(0,s)}.$$

Thus, $D^2_{\mathcal{S}^{d-1}(0,s)} S_J(x,\xi,t)$ is the restriction to $T_\xi\mathcal{S}^{d-1}(0,s)$ of

$$-(X^{-|J|}(x,\nabla\varphi_J^\mp(x,\xi)) - y) \cdot \frac{\xi}{|\xi|^2} Id + \frac{\xi\xi^t}{|\xi|^3} t$$
$$- \sum_{k=1}^{d} \int_0^{\frac{l}{|\xi|}} D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi)\left(D_\xi\partial_{x_k}\psi_{J^{(s)}}(X_{-s}(x,\nabla\varphi_J(x,\xi)),\xi)\right)^t.$$

But, the quadratic form $\xi\xi^t$ vanishes in $T_\xi\mathcal{S}^{d-1}$. Therefore, $D^2_{\mathcal{S}^{d-1}(0,s)} S_J(x,\xi,t)$ is the restriction to $T_\xi\mathcal{S}^{d-1}(0,s)$ of

$$- (X^{-|J|}(x, \nabla\varphi_J(x,\xi)) - y) \cdot \frac{\xi}{|\xi|^2} \mathrm{Id}$$
$$- \sum_{k=1}^{d} \int_0^{\frac{t}{|\xi|}} D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi) \big(D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi) \big)^t.$$

And as the matrices

$$D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi) \big(D_\xi \partial_{x_k} \psi_{J^{(s)}}(X_{-s}(x, \nabla\varphi_J(x,\xi)), \xi) \big)^t$$

are positives, and according to (3.14), we get the last part of the statement. □

4. Proof of the main result

Let $K \geq 0$. By the previous section, the function

$$(x,t) \to \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J \in \mathcal{I}} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi$$

satisfies the approximate equation

$$\partial_t^2 u - \Delta u = (\mp h)^K \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J \in \mathcal{I}} \int \Box w_{K-1}^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} |\xi|^{-K} d\xi$$

with data

$$(u(0), \partial_t u(0)) = (\delta^y_{\epsilon,h,N}, 0).$$

By the Duhamel formula, the difference from the actual solution, that is from $\cos\big(t\sqrt{-\Delta}\big)\delta^y$, is bounded in H^m norm by

$$C \times h^{K-d} \times |t| \sup_{s \in [0,t],\, \xi \in [\alpha_0, \beta_0]} \sum_{\pm} \sum_{J \in \mathcal{I}} ||\Box w_{K-1}^{J,\pm}(\cdot, s, \xi) e^{\mp i(\varphi_J^{\pm}(\cdot,\xi)|\xi| - t|\xi|)/h}||_{H^m}.$$

So, for $0 \leq t \leq \epsilon |\log h|$

$$\big(\cos\big(t\sqrt{-\Delta}\big)\delta^y\big)(x) = S_K(x,t) + R_K(x,t) \tag{4.1}$$

with

$$S_K(x,t) = \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J \in \mathcal{I}} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi \tag{4.2}$$

and

$$||R_K||_{H^m} \lesssim |\log h| h^{K-d} \sup_{s,\xi} \sum_{\pm} \sum_{J \in \mathcal{I}} ||\Box w_{K-1}^{J,\pm}(\cdot, s, \xi) e^{\mp i(\varphi_J^{\pm}(\cdot,\xi)|\xi| - t|\xi|)/h}||_{H^m}. \tag{4.3}$$

4.1. The reminder

We first deal with the reminder term R_K. Let us denote

$$W^{J,\pm}_{K-1}(x,s,\xi)=\Box w^{J,\pm}_{K-1}(\cdot,s,\xi)e^{\mp i(\varphi^\pm_J(\cdot,\xi)|\xi|-t|\xi|)/h}$$

Notice that, by construction of the w_k's, w^J_k is supported in a set of diameter $(C+t)$. Therefore, using Proposition 13, Proposition 6 and the derivative of a product we get:

$$\sum_{J\in\mathcal{I}}||\partial^m W^{J,\pm}_{K-1}||_{L^2}\lesssim C_K(1+t)^{\frac{d}{2}}\sum_{J\in\mathcal{I}}||\partial^m W^{J,\pm}_{K-1}||_{L^\infty}\lesssim C_K(1+t)^{\frac{d}{2}}h^{-m}\sum_{J\in\mathcal{I}}h^{-(2K+m)c\epsilon}$$

and thus, by the Sobolev embedding $H^d, \to L^\infty$ and (4.3)

$$||R_K||_{L^\infty}\lesssim C_K|\log h|h^{K-2d}(1+t)^{\frac{d}{2}}h^{-(2K+d)c\epsilon}\left|\left\{J\in\mathcal{I},\mathrm{s.t}w^{\mathrm{J}}_{\mathrm{K-1}}\neq 0\right\}\right|.$$

By Proposition 10 we get in the same way as in [20], due to the finite speed of propagation:

$$\left|\left\{J\in\mathcal{I},\mathrm{s.t}w^J_{K-1}\neq 0\right\}\right|\lesssim(1+t), \tag{4.4}$$

and therefore

$$||R_K||_{L^\infty}\lesssim C_K|\log h|h^{K(1-2c\epsilon)-d(2+c\epsilon)}(1+t)^{\frac{d}{2}+1} \tag{4.5}$$

Thus, for $0\le t\le\epsilon|\log h|$

$$||R_K||_{L^\infty}\lesssim C_K|\log h|^{\frac{d}{2}+1}h^{K(1-2c\epsilon)-d(2+c\epsilon)}\le C_K h^{K(1-2c\epsilon)-d(1+c\epsilon)-1}.$$

We take $\epsilon>0$ small enough so that $2c\epsilon\le\frac{1}{2}$ and we get

$$||R_K||_{L^\infty}\le C_K h^{\frac{K}{2}-3d-1}.$$

Let us now fix K large enough so that

$$\frac{K}{2}-3d-1\ge-\frac{d+1}{2}+1.$$

Then, as $0\le t\le\epsilon|\log h|$ is equivalent to $h\le e^{-\frac{t}{\epsilon}}$, we obtain

$$||R_K||_{L^\infty}\le C_K h^{-\frac{d+1}{2}}e^{-\frac{t}{\epsilon}}. \tag{4.6}$$

for $0\le t\le\epsilon|\log h|$.

4.2. The free wave J=$\emptyset$

Let us denote

$$S^\emptyset_K(x,t)=\frac{1}{(2\pi h)^d}\sum_{\pm}\int\sum_{k=0}^K(\mp ih/|\xi|)^k w^{\emptyset,\pm}_k(x,t,\xi)e^{-i((x-y)\cdot\xi\mp t|\xi|)/h}d\xi$$

the free part of the wave, and

$$S^r_K=\frac{1}{(2\pi h)^d}\sum_{\pm}\sum_{|J|\ge 1}\int\sum_{k=0}^K w^{J,\pm}_k(x,t,\xi)e^{\mp i(\varphi^\pm_J(x,\xi)|\xi|-t|\xi|)/h}(\mp ih/|\xi|)^k d\xi,$$

the reflected waves, in such a way that

$$S_K = S_K^{\emptyset} + S_K^r.$$

Note that $S_K^{\emptyset}$ is simply the approximate expression of the solution of the wave equation with data $(\delta^y, 0)$, in the free space:

$$S_K^{\emptyset}(x,t) = \left(\cos\left(t\sqrt{-\Delta_0}\right)\delta^y\right)(x) + R_K^{\emptyset}(x,t)$$

where Δ_0 denote the Laplacian in the free space and by the Duhamel formula, for $0 \le t \le \epsilon|\log h|$

$$||R_K^{\emptyset}||_{H^m} \lesssim h^{K-d}|\log h| \sup_{s,\xi} ||\Box w^{\emptyset}(\cdot, s, \xi)||_{H^m}$$

The usual dispersive estimate for the waves in the free space gives, by the frequencies localisation of δ^y

$$|\cos\left(t\sqrt{-\Delta_0}\right)\delta^y| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}$$

and thus dealing with $R_K^{\emptyset}$ as we did for R_K we get

$$|S_K^{\emptyset}| \lesssim \frac{1}{h^{\frac{d+1}{2}} t^{\frac{d-1}{2}}}, \text{ for } 0 \le t \le \epsilon|\log h|. \tag{4.7}$$

4.3. The reflected waves $|J| \ge 1$

According to Proposition 14, the parts

$$d(X^{-|J|}(x, \nabla\varphi_J^{\pm}), y) \le \eta$$

enjoys a rapid decay and we thus have

$$S_K^r = \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{|J|\ge 1} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} 1_{d(X^{-|J|}(x,\nabla\varphi_J^{\pm}),y)\ge\eta} (\mp ih/|\xi|)^k d\xi$$
$$+ \sum_{|J|\ge 1} O(h^{\infty}).$$

Note that, using (4.4), the sum over J is finite at fixed t, hence the $O(h^{\infty})$ part does not contribute. We write the remaining part of S_K^r as, illegitimately omitting $1_{d(X^{-|J|}(x,\nabla\varphi_J^{\pm}),y)\ge\eta}$ for the seek of clarity:

$$S_K^r(x,t) = \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J\in\mathcal{I}} \int \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi$$
$$= \frac{1}{(2\pi h)^d} \sum_{\pm} \sum_{J\in\mathcal{I}} \int_{s=\alpha_0}^{\beta_0} \int_{|\xi|=s} \sum_{k=0}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/s)^k d\xi ds.$$

Let $k_0 \ge 1$ to be fixed later depending only on the dimension. According to Proposition 14, we can perform a stationary phase on each sphere $\{|\xi| = s\}$, for each term of the sum (Tex translation failed), up to order h^{k_0}. We obtain, as the sphere is of dimension $d-1$, for $t \ge 0$,

$$\begin{aligned} S_K^r(x,t) &= \frac{1}{(2\pi h)^d} h^{\frac{d-1}{2}} \sum_{\pm, J\in\mathcal{I}} \sum_{k=0}^{k_0} \int_{\alpha_0}^{\beta_0} e^{\mp i(\varphi_J^{\pm}(x,s_J(t,x))s - ts)/h} (\mp ih/s)^k \tilde{w}_k^{J,\pm}(t,x)ds \\ &+ \frac{1}{(2\pi h)^d} h^{\frac{d-1}{2}} \sum_{\pm, J\in\mathcal{I}} \int_{\alpha_0}^{\beta_0} R_{\text{st.ph.}}^{J,\pm}(x,t,s)ds \\ &+ \frac{1}{(2\pi h)^d} \sum_{\pm, J\in\mathcal{I}} \int \sum_{k=k_0+1}^{K} w_k^{J,\pm}(x,t,\xi) e^{\mp i(\varphi_J^{\pm}(x,\xi)|\xi| - t|\xi|)/h} (\mp ih/|\xi|)^k d\xi, \end{aligned} \tag{4.8}$$

(with the convention $\sum_{k_0+1}^{K} = 0$ if $K < k_0+1$) where for $0 \le k \le k_0$, the term $\tilde{w}_k^{J,\pm}$ is a linear combination of

$$D_\xi^{2k} w_0^{J,\pm}(t,x,s_J(t,x,s)), D_\xi^{2(k-1)} w_1^{J,\pm}(t,x,s_J(t,x,s)), \ldots, w_k^{J,\pm}(t,x,s_J(t,x,s)),$$

where $s_J(t,x,s)$ is the stationary point of the phase on the hypersphere $\{|\xi| = s\}$, and $R_{\text{st.ph.}}^{J,\pm}$ is the reminder involved in the stationary phase, verifying (see, for example, [35, Section 3.5])

$$|R_{\text{st.ph.}}^{J,\pm}| \lesssim h^{k_0+1} \sum_{\substack{0 \le k \le k_0 \\ |\alpha| \le M(k_0,d)}} \sup |D_\xi^\alpha w_k^{J,\pm}(x,\cdot,t)|, \tag{4.9}$$

where $M(k_0,d)$ depends only on k_0 and on the dimension d, hence only on d. Then the proof proceeds as in [20, Section 5]. Indeed, as by Proposition 10, $\mathrm{Supp}\chi_0 \cap \mathrm{Supp} w_k^{J,\pm}$ is included in $\mathcal{U}_\infty$ in times $0 \le t \le \epsilon|\log h|$, we can use the decay estimates of Proposition 13 and Proposition 11 to obtain respectively

$$|w_k^{J,\pm}| \le C_k \lambda^{|J|} h^{-2k c\epsilon}, \forall\, 0 \le t \le \epsilon|\log h|,\ x \in \mathrm{Supp}\chi_0, \tag{4.10}$$

and

$$|D_\xi^\alpha w_k^{J,\pm}| \le C_\alpha^{|J|} h^{(-2k+|\alpha|)c\epsilon}, \forall\, |\alpha| \ge 1,\ 0 \le t \le \epsilon|\log h|,\ x \in \mathrm{Supp}\chi_0, \tag{4.11}$$

where we recall that $0 < \lambda < 1$. Let us now denote

$$K' := \max(k_0, K, M(k_0,d)),$$

depending only on the dimension d. By Proposition 10 again, $w_k^{J,\pm}(x,\cdot)$ is supported in $\{c_1|J| \le t \le c_2(|J|+1)\}$ for $x \in \mathrm{Supp}\chi_0$, hence using (4.10), for $0 \le t \le \epsilon|\log h|$ and $x \in \mathrm{Supp}\chi_0$ we get

$$\sum_{J\in\mathcal{I}} |w_k^{J,\pm}| \le C_k h^{-2kc\epsilon} \sum_{J \text{ s.t.} w_k^J \neq 0} \lambda^{|J|} \le C_k h^{-2kc\epsilon} \sum_{r \ge \frac{t}{c_2}} \lambda^{r-1} \lesssim C_k h^{-2kc\epsilon} e^{-\mu t},$$

for some $\mu > 0$ depending only on λ and on c_2. We take $\epsilon > 0$ small enough so that $2K'c\epsilon \le \frac{1}{2}$ to get, for $0 \le t \le \epsilon|\log h|$ and $x \in \mathrm{supp}\chi_0$

$$\sum_{J\in\mathcal{I}} |w_0^{J,\pm}| \lesssim C_0 e^{-\mu t}, \text{ and } \sum_{J\in\mathcal{I}} |w_k^{J,\pm}| \lesssim C_k h^{-\frac{1}{2}} e^{-\mu t} \text{ for } 1 \le k \le K'. \tag{4.12}$$

In the same way, using (4.11) together with Proposition 10, for $0 \le t \le \epsilon|\log h|$, $x \in \mathrm{Supp}\chi_0$, $0 \le k \le K'$, and $|\alpha| \le 2K'$, we get for some $\nu > 0$ (depending only on the constants C_α for $|\alpha| \le 2K'$ and on c_1, hence only on d and the geometry of the obstacles)

$$\begin{aligned}\sum_{J\in\mathcal{I}}|D_\xi^\alpha w_k^{J,\pm}| &\le h^{-(2k+|\alpha|)c\epsilon}\sum_{J \text{ s.t. } w_k^J\neq 0} C_\alpha^{|J|}\\
&\le h^{-(2k+|\alpha|)c\epsilon}|\{J, \text{s.t. } w_k^{J,\pm}\neq 0\}|C_\alpha^{\frac{t}{c_1}} \lesssim h^{-(2k+|\alpha|)c\epsilon}(1+t)C_\alpha^{\frac{t}{c_1}}\\
&\le (1+t)h^{-4K'c\epsilon}e^{\nu t} = (1+t)h^{-4K'c\epsilon}e^{\frac{3}{2}\nu t}e^{-\frac{1}{2}\nu t}\\
&\le (1+\epsilon|\log h|)h^{-4K'c\epsilon}h^{-\frac{3}{2}\nu\epsilon}e^{-\frac{1}{2}\nu t},\end{aligned}$$

where the bound on the second line is obtained similarly to (4.4) (the implicit constant arising depends a-priori on k, but we can take it uniform with respect to $0\le k\le K'$), we used the fact that $|\alpha|\le 2K'$ on the third line, and that $0\le t\le \epsilon|\log h|$ on the last line. Taking $\epsilon>0$ small enough so that $\left(4K'c+\frac{3}{2}\nu\right)\epsilon\le\frac{1}{3}$, we thus obtain, for $0\le t\le \epsilon|\log h|$, $x\in \mathrm{Supp}\chi_0$, $0\le k\le K'$ and $|\alpha|\le 2K'$

$$\sum_{J\in\mathcal{I}}|D_\xi^\alpha w_k^{J,\pm}|\lesssim h^{-\frac{1}{2}}e^{-\frac{1}{2}\nu t}. \tag{4.13}$$

Therefore, combining (4.12) and (4.13) with (4.8) and (4.9) we obtain, taking $k_0 := \lceil\frac{d-1}{2}\rceil$, and recalling that $\tilde{w}_k^{J,\pm}$ is a linear combination of $D_\xi^{2k}w_0^{J,\pm}(t,x,s_J(t,x,s))$, $D_\xi^{2(k-1)}w_1^{J,\pm}(t,x,s_J(t,x,s))$, ..., $w_k^{J,\pm}(t,x,s_J(t,x,s))$ (and in particular $\tilde{w}_0^{J,\pm}=w_0^{J,\pm}$),

$$|\chi_0 S_K^r(x,t)|\le Ch^{-\frac{d+1}{2}}e^{-\rho t}\quad \text{for } 0\le t\le\epsilon|\log h|, \tag{4.14}$$

for some $C>0$ and $\rho>0$ depending only of α_0,β_0,d and of the geometry of the obstacles.

4.4. Conclusion

Thus, collecting (4.1), (4.6), (4.7) and (4.14) we get

$$|\chi_0\cos\left(t\sqrt{-\Delta}\right)\delta_{\epsilon,h,N}^y|\lesssim\frac{1}{h^{\frac{d+1}{2}}t^{\frac{d-1}{2}}},\quad \text{for } 0\le t\le\epsilon|\log h|.$$

That is (3.7). Thus, Theorem 1 is proved by the work of reduction of the previous sections.

5. The non-linear problem

Let us now consider the following defocusing non-linear wave equation in $\mathbb{R}^3\backslash\mathcal{K}$

$$\begin{cases}\partial_t^2 u-\Delta_D u+u^5=0\\ (u(0),\partial_t u(0))=(f,g).\end{cases}$$

$\mathcal{K}$ will be the reunion of two balls, or an illuminated obstacle as defined in the introduction, and we are concerned by the scattering problem in both situations. Our main tool will be the following momentum identity, which was first introduced by Morawetz [28] in a similar form to show some decay properties of the linear wave equation:

Lemma 4. *Let u be a solution of (NLW) in Ω and $\chi \in C^\infty(\Omega, \mathbb{R})$. Then we have*

$$\begin{aligned}\partial_t\left(\int_\Omega -\partial_t u \nabla u \nabla \chi - \frac{1}{2}\Delta\chi u \partial_t u\right) = \int_\Omega (D^2\chi\nabla u, \nabla u) - \frac{1}{4}\int_\Omega u^2\Delta^2\chi \\ + \frac{3}{2}\int_\Omega |u|^6\Delta\chi - \frac{1}{2}\int_{\partial\Omega}|\partial_n u|^2\partial_n\chi.\end{aligned} \tag{5.1}$$

Proof. The identity can be shown by standard integrations by parts justified by a limiting argument. □

5.1. A scattering criterion

The scattering in $\mathbb{R}^3$ was shown by Bahouri and Shatah [24]. Their proof still hold in the case of a domain with boundaries if we are able to control the boundary term appearing in their computations, that is

Lemma 5. *Let u be a solution of (NLW) in a finite-border domain Ω of $\mathbb{R}^3$ such that Strichartz estimates (1.5) holds. If*

$$\frac{1}{T}\int_0^T\int_{\partial\Omega}|\partial_n u|^2 d\sigma dt \to 0,$$

as T goes to infinity, then u scatter in $\dot{H}^1$.

Note that the trace of the normal derivative is not an easy object to deal with, because this trace is a priori not defined in $L^2(\partial\Omega)$ for elements of $\dot{H}^1(\Omega)$. Moreover, even if we can define it for almost every $u(t)$ when u is a solution of (NLW) because of the particular structure of the equation, the application

$$u \in \dot{H}^1 \cap \{\text{value in time } t \text{ of solutions of NLW}\} \to \partial_n u \in L^2(\partial\Omega)$$

is in our knowledge not known to be continuous.

For this reason, we prefer to deal with the following criterion, which involve only the local energy of the equation, and that we deduce from the previous one using the momentum identity (5.1):

Lemma 6. *Let u be a solution of (NLW) in a finite-border domain Ω of $\mathbb{R}^3$ such that Strichartz estimates (1.5) holds. There exists $A > 0$, $B(0, A) \supset \partial\Omega$, such that, if*

$$\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)}|\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \to 0, \tag{5.2}$$

as T goes to infinity, then u scatters in $\dot{H}^1$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ be such that $\nabla\chi = -n$ on $\partial\Omega$, supported in $B(0, A)$. Suppose that

$$\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)}|\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt \to 0$$

as T goes to infinity. We use Lemma 4 with the weight χ to get:

$$\partial_t\left(\int_\Omega -\partial_t u\nabla u\nabla\chi - \frac{1}{2}\Delta\chi u\partial_t u\right) = \int_\Omega (D^2\chi\nabla u, \nabla u) - \frac{1}{4}\int_\Omega u^2\Delta^2\chi + \frac{2}{3}\int_\Omega |u|^6\Delta\chi + \frac{1}{2}\int_{\partial\Omega} |\partial_n u|^2 d\sigma.$$

Integrating in time we get

$$\int_0^T\int_{\partial\Omega} |\partial_n u|^2 d\sigma dt \lesssim \int_{\Omega\cap B(0,A)} |\partial_t u\nabla u| + |u\partial_t u| + \int_0^T\int_{\Omega\cap B(0,A)} |u|^6 + |u|^2 + |\nabla u|^2,$$

and using Minkowsky inequality,

$$\begin{aligned}\int_0^T\int_{\partial\Omega} |\partial_n u|^2 d\sigma dt &\lesssim \left(\int_\Omega |\partial_t u|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} + A^{\frac{1}{3}}\left(\int_\Omega |\partial_t u|^2\right)^{\frac{1}{2}}\left(\int_\Omega |u|^6\right)^{\frac{1}{6}} \\ &\quad + \int_0^T\int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) + A^{\frac{2}{3}}\int_0^T\left(\int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}} \\ &\lesssim_A C(E) + \int_0^T\int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) + T^{\frac{2}{3}}\left(\int_0^T\int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{\partial\Omega} |\partial_n u|^2 d\sigma dt &\lesssim_A \frac{C(E)}{T} + \frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)} (|u|^6 + |\nabla u|^2) \\ &\quad + \left(\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)} |u|^6\right)^{\frac{1}{3}} \longrightarrow 0\end{aligned}$$

as $T\to\infty$ and by Lemma 5 we conclude that u scatter in $\dot{H}^1$. □

Notice that the Morawetz identity (5.1) permits to obtain this criterion as soon as one has a weight function χ such that $\nabla\chi\cdot n \geq 0$ on $\partial\Omega$, $D^2\chi$ is positive definite, and $\Delta^2\chi \leq 0$. Constructing such weights will therefore be of key interest in the sequel.

5.2. A partial result in the exterior of two balls

In the exterior of two balls, Lemma 4 seems not to be sufficient to show the scattering criterion (5.2) because we are not able to find an appropriate weight function. However, we can choose a weight function which has the right behaviour everywhere except in a neighbourhood of the trapped ray, and therefore obtain Theorem 2, which is a first step towards the scattering for all data: it is extended to the exterior of two convex obstacles and used to show the scattering in this framework in the work in progress [25].

Proof of Theorem 2. Without loss of generality, we suppose that Θ_1 is centred in 0. We denote by c the centre of Θ_2. We choose the weight

$$\chi(x) := |x| + |x-c|$$

and use Lemma 4 with weight χ

$$\partial_t\left(\int_\Omega -\partial_t u\nabla u\nabla\chi-\frac{1}{2}\Delta\chi u\partial_t u\right)=\int_\Omega(D^2\chi\nabla u,\nabla u)-\frac{1}{4}\int_\Omega u^2\Delta^2\chi$$
$$+\frac{2}{3}\int_\Omega|u|^6\Delta\chi-\frac{1}{2}\int_{\partial\Omega}|\partial_n u|^2\nabla\chi\cdot n\,d\sigma.$$

Remark that $-\nabla\chi\cdot n\geq 0$ on $\partial\Theta_1\cup\partial\Theta_2$: indeed, on Θ_1, $-n=\frac{x}{|x|}$ and thus

$$-\nabla\chi\cdot n=1+\frac{x-c}{|x-c|}\cdot\frac{x}{|x|}\geq 1-\left|\frac{x-c}{|x-c|}\right|\left|\frac{x}{|x|}\right|=0,$$

and the same hold on Θ_2. Moreover, $\Delta^2\chi=0$. Thus we obtain

$$\partial_t\left(-\int_0^T\partial_t u\nabla u\cdot\nabla\chi+\frac{1}{2}\Delta\chi u\partial_t u\right)\geq\frac{2}{3}\int_\Omega|u|^6\Delta\chi+\int_\Omega(D^2\chi\nabla u,\nabla u)\tag{5.3}$$

Integrating this inequality and controlling the left-hand side using the Hardy inequality

$$\int_\Omega\frac{|f|^2}{|x|^2}\lesssim\int_\Omega|\nabla f|^2\text{ for }f\in\dot{H}^1_0(\Omega)$$

we get

$$\int_0^T\int_\Omega|u|^6\Delta\chi+(D^2\chi\nabla u,\nabla u)\,dxdt\lesssim E.\tag{5.4}$$

From the one hand, $\Delta\chi\gtrsim\frac{1}{A}$ on $B(0,A)$, thus

$$\int_{\Omega\cap B(0,A)}|u|^6\lesssim A\int_{\Omega\cap B(0,A)}|u|^6\Delta\chi\lesssim A\int_\Omega|u|^6\Delta\chi,$$

and therefore, by (5.4)

$$\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)}|u|^6\,dxdt\lesssim\frac{E}{T}.\tag{5.5}$$

Now, we would like to estimate the localised kinetic energy using (5.4) again. We have

$$D^2\chi=\frac{1}{|x|}\left(\mathrm{Id}-\frac{xx^t}{|x|^2}\right)+\frac{1}{|x-c|}\left(\mathrm{Id}-\frac{(x-c)(x-c)^t}{|x-c|^2}\right).$$

The operators corresponding to the matrices

$$\mathrm{Id}-\frac{xx^t}{|x|^2},\text{resp.}\mathrm{Id}-\frac{(x-c)(x-c)^t}{|x-c|^2}$$

are the orthogonal projections on the plane normal to $\frac{x}{|x|}$, resp. to $\frac{x-c}{|x-c|}$. Thus,

$$(D^2\chi\cdot\xi,\xi)=\left(\frac{1}{|x|}+\frac{1}{|x-c|}\right)|\xi|^2-\frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2-\frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2.\tag{5.6}$$

We choose coordinates (depending of x and c) such that

$$\frac{x}{|x|}=(1,0,0),\ \frac{x-c}{|x-c|}=(\cos\theta,\ \sin\theta,0),$$

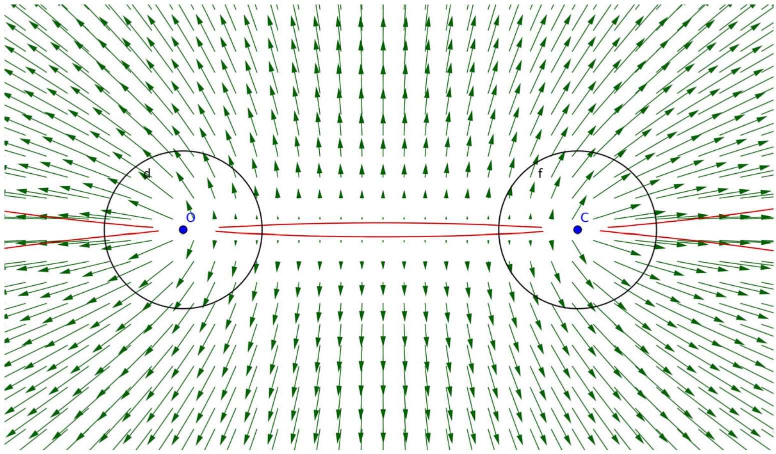

Figure 1. $\nabla\chi$ and $V(\alpha)$.

then we have, if $\xi = (\hat{\xi}_1 \quad \hat{\xi}_2 \quad \hat{\xi}_3)$ in this set of coordinates

$$\frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2+\frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2=\begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}\begin{pmatrix}\frac{1}{|x|}+\frac{\cos^2\theta}{|x-c|} & \frac{\sin\theta\cos\theta}{|x-c|}\\ \frac{\sin\theta\cos\theta}{|x-c|} & \frac{\sin^2\theta}{|x-c|}\end{pmatrix}\begin{pmatrix}\hat{\xi}_1\\ \hat{\xi}_2\end{pmatrix}.$$

The largest eigenvalue of this positive quadratic form in $(\hat{\xi}_1 \quad \hat{\xi}_2)$ writes

$$\lambda_2=\frac{1}{2}\left(\frac{1}{|x-c|}+\frac{1}{|x|}+\sqrt{\left(\frac{1}{|x-c|}+\frac{1}{|x|}\right)^2-4\frac{\sin^2\theta}{|x||x-c|}}\right)$$

therefore, there exists $\alpha_0>0$ small enough and $c>0$ such that, if $\alpha\le\alpha_0$, we have, for $x\in\Omega\cap B(0,A)$

$$\sin^2\theta\ge\alpha\Rightarrow\lambda_2\le\frac{1}{|x-c|}+\frac{1}{|x|}-c\alpha. \tag{5.7}$$

On the other hand

$$\frac{1}{|x|}\left(\xi\cdot\frac{x}{|x|}\right)^2+\frac{1}{|x-c|}\left(\xi\cdot\frac{x-c}{|x-c|}\right)^2\le\lambda_2|(\hat{\xi}_1,\hat{\xi}_2)|^2\le\lambda_2|\xi|^2,$$

thus we get, combining this last inequality with (5.6) and (5.7), for $x\in\Omega\cap B(0,A)$

$$\sin^2\theta\ge\alpha\Rightarrow(D^2\chi\cdot\xi,\xi)\gtrsim\alpha|\xi|^2. \tag{5.8}$$

Remark that, because θ is the angle between $\frac{x}{|x|}$ and $\frac{x-c}{|x-c|}$ (Figure 1)

$$\theta=\arccos\frac{x}{|x|}\cdot\frac{x-c}{|x-c|},$$

and let us denote, for $\alpha\le\alpha_0$

$$V(\alpha) = \Omega \cap B(0,A) \cap \left\{ \sin^2 \left(\arccos \frac{x}{|x|} \cdot \frac{x-c}{|x-c|} \right) \geq \alpha \right\}.$$

Note that $V(\alpha) \to \Omega \cap B(0,A)$ as α goes to zero in the sense that, denoting μ the Lebesgue's measure on $\mathbb{R}^3$

$$\mu((\Omega \cap B(0,A)) \backslash V(\alpha)) \to 0 \tag{5.9}$$

as α goes to zero: $V(\alpha)$ is a subset of $\Omega \cap B(0,A)$ excluding a small neighbourhood of the line $(0,c)$. We have, on $S(\alpha)$, because of (5.8)

$$(D^2\chi \cdot \xi, \xi) \gtrsim \alpha |\xi|^2.$$

Thus we get

$$\int_\Omega (D^2 \chi \nabla u, \nabla u) \geq \int_{V(\alpha)} (D^2 \chi \nabla u, \nabla u) \gtrsim \alpha \int_{V(\alpha)} |\nabla u|^2$$

and by (5.4) we obtain

$$\frac{1}{T} \int_0^T \int_{V(\alpha)} |\nabla u|^2 \, dxdt \lesssim \frac{1}{\alpha} \frac{E}{T}.$$

We take $\alpha = T^{-1/2}$ in order to have

$$\frac{1}{T} \int_0^T \int_{V(\alpha(T))} |\nabla u|^2 \, dxdt \lesssim \frac{E}{\sqrt{T}}. \tag{5.10}$$

Choosing $\mathcal{S}(T) := B(0,A) \backslash V(\alpha(T))$, (5.5) together with (5.10) gives the result. □

5.3. Obstacles illuminated by an ellipsoïd

Motivated by the above result, we are interested by the scattering problem in non-trapping geometries close to the exterior of two convex obstacles, such as dog bones with arbitrary thin neck. Theorem 3, which we will prove now, gives in particular the scattering in such settings. More precisely, it permits to handle obstacles illuminated by arbitrary cigar-shaped ellipsoïds and a certain class of flat-shaped ones.

In order to show such a result using the Morawetz identity (5.1) to obtain the criterion of Lemma 6, it is natural to choose the gauge of the ellipsoïd we are dealing with as the weight function. The next Lemma gives us the range of ellipsoïds for which such a weight verify the bilaplacian constraint:

Lemma 7. *Let $n \geq 2$ and*

$$\rho(x) = \sqrt{x_1^2 + \cdots + x_k^2 + \epsilon(x_{k+1}^2 + \cdots + x_n^2)}.$$

Then,

$$\Delta^2 \rho \leq 0, \, \forall \epsilon \in [\epsilon_0, 1]$$

with

$$\epsilon_0 = \begin{cases} 0 & \text{if } k \geq 3, \\ \dfrac{1}{n} + \dfrac{\sqrt{2(n-2)(n-1)}}{n(n-2)} & \text{if } k = 2, \\ \dfrac{4}{n+1} & \text{if } k = 1. \end{cases}$$

Proof. An elementary computation gives

$$\Delta^2 \rho(x) = \frac{A(\epsilon)}{\rho^3} + \frac{B(\epsilon)(x_{k+1}^2 + \cdots + x_n^2)}{\rho^5} + \frac{C(\epsilon)(x_{k+1}^2 + \cdots + x_n^2)^2}{\rho^7}$$

where

$$\begin{aligned} A(\epsilon) &= -(n-k+2)(n-k)\epsilon^2 - 2(n-k)(k-3)\epsilon - (k-1)(k-3), \\ B(\epsilon) &= 6\epsilon((n-k+2)\epsilon^2 + [(2k-n)-5]\epsilon - k + 3), \\ C(\epsilon) &= -15\epsilon^2(\epsilon-1)^2. \end{aligned}$$

In all cases, $C(\epsilon) \leq 0$.

If $k \geq 3$, it is clear that $A(\epsilon) \leq 0$. Let us denote $\tilde{B}(\epsilon) = \frac{B(\epsilon)}{6\epsilon} = (n-k+2)\epsilon^2 + [(2k-n)-5]\epsilon - k + 3$. Then $\tilde{B}(0) = -(k-3) \leq 0$, $\tilde{B}(1) = 0$, so $B(\epsilon) \leq 0$ for $\epsilon \in [0,1]$ and therefore $\Delta^2\rho \leq 0$ for $\epsilon \in [0,1]$.

If $k=2$, the roots of $A(\epsilon) = -n(n-2)\epsilon^2 + 2(n-2)\epsilon + 1$ are $\epsilon_{1,2} = \frac{1}{n} \pm \frac{\sqrt{2(n-2)(n-1)}}{n(n-2)}$ and the roots of $\tilde{B}(\epsilon) = n\epsilon^2 - (n+1)\epsilon + 1$ are $\frac{1}{n}$ and 1, so $\Delta^2\rho \leq 0$ for $\epsilon \in [\frac{1}{n} + \frac{\sqrt{2(n-2)(n-1)}}{n(n-2)}, 1]$.

If $k=1$, $A(\epsilon) = -(n+1)(n-1)\epsilon^2 + 4(n-1)\epsilon$ is non-positive if and only if $\epsilon \geq \frac{4}{n+1}$, and the roots of $\tilde{B}(\epsilon) = (n+1)\epsilon^2 - (n+3)\epsilon + 2$ are $\frac{2}{n+1}$ and 1, so $\Delta^2\rho \leq 0$ for all ϵ in $[\frac{4}{n+1}, 1]$. □

Notice that in dimension three, this weight cannot be explicitly used as his bilaplacian is not non positive. Hence, to derive the control (5.2), we will extend u as a solution of a four dimensional non-linear wave equation, get the control for this extended solution, and manage to go back to u.

To this purpose, we need the following existence result for the four dimensional problem:

Lemma 8. *Let Ω be a smooth domain of $\mathbb{R}^4$ with compact boundary, and $(u_0, u_1) \in \dot{H}^{\frac{7}{4}}(\Omega) \times H^{\frac{3}{4}}(\Omega)$. Then, there exists a unique global solution of*

$$\begin{cases} (\partial_t^2 - \Delta)u + u^5 = 0 & \text{in } \mathbb{R} \times \Omega \\ u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|\mathbb{R}\times\partial\Omega} = 0 \end{cases} \tag{5.11}$$

satisfying

$$u \in C(\mathbb{R}, \dot{H}^{\frac{7}{4}}(\Omega) \cap L^6(\Omega)) \cap C^1(\mathbb{R}, H^{\frac{3}{4}}(\Omega)) \cap L^{48}(\mathbb{R}, L^6(\Omega)).$$

Proof. Let $0 < T < 1$. By the work of Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge [12], Theorem 1.1, applied to the admissible triple $\left(p = 48, q = 6, \gamma = \frac{7}{4}\right)$ in dimension 4, if u is solution of

$$\begin{cases} (\partial_t^2 - \Delta)u = F \quad \text{in } (0,T)\times\Omega \\ u_{|t=0} = f, \quad \partial_t u_{|t=0} = g, \quad u_{|\mathbb{R}\times\partial\Omega} = 0 \end{cases} \tag{5.12}$$

then the following Strichartz estimate holds

$$||u||_{L^{48}((0,T),L^6(\Omega))} \le C\Big(||f||_{\dot{H}^{\frac{7}{4}}(\Omega)} + ||g||_{H^{\frac{3}{4}}(\Omega)} + ||F||_{L^1((0,T),L^2(\Omega))}\Big).$$

Using this estimate, we obtain the local existence following a classical fixed point method, in the space

$$X_T = C^0(\mathbb{R}, \dot{H}^{\frac{7}{4}}(\Omega)\cap L^6(\Omega)) \cap C^1(\mathbb{R}, H^{\frac{3}{4}}(\Omega)) \cap L^{48}(\mathbb{R}, L^6(\Omega))$$

for T sufficiently small depending of $||u_0||_{\dot{H}^{\frac{7}{4}}(\Omega)}$ and $||u_1||_{H^{\frac{3}{4}}(\Omega)}$. The global existence result follows using the energy conservation law. □

We are now in position to show:

Proposition 15. *Let $\mathcal{C}\subset\mathbb{R}^3$ be the ellipsoïd of equation (1.10), resp. (1.11), $\mathcal{K}$ be a compact subset of $\mathbb{R}^3$ illuminated by $\mathcal{C}$ and $\Omega := \mathbb{R}^3\backslash\mathcal{K}$. Let $u\in C(\mathbb{R},\dot{H}^1(\Omega))\cap C^1(\mathbb{R},L^2(\Omega))$ be the global solution of (1.8) in Ω. Then*

$$\frac{1}{T}\int_0^T\int_{\Omega\cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6\,dxdt \le \frac{1}{\ln T}C(E(u)).$$

Proof. Let $\delta > 0$. There exists u_0^δ, u_1^δ, smooth functions vanishing on $\partial\Omega$ such that

$$||u_0 - u_0^\delta||_{\dot{H}^1(\Omega)} + ||u_1 - u_1^\delta||_{L^2(\Omega)} \le \delta.$$

We denote by $u^\delta\in C^0(\mathbb{R},\dot{H}^1(\Omega))\cap C^1(\mathbb{R},L^2(\Omega))$ the solution of (1.8) in Ω with data (u_0^δ, u_1^δ).

Let $T>0$ and $\phi\in C_c^\infty(\mathbb{R})$ be such that $0\le\phi\le 1$, $\phi = 1$ on $[-1,1]$ and $\phi = 0$ on $[-2,2]^c$. We take $\chi_T = \phi(\frac{\cdot}{2T})$ and, for $(x,z)\in\Omega\times\mathbb{R}$

$$v_0^\delta(x,z) = u_0^\delta(x)\chi_T(z)\chi_T(|x|),$$
$$v_1^\delta(x,z) = u_1^\delta(z)\chi_T(z)\chi_T(|x|).$$

Let us denote by ρ the gauge of ellipsoïd we are dealing with, consider

$$\tilde{\mathcal{K}} = \mathcal{K}\times[-4T,4T],\ \tilde{\Omega} = \mathbb{R}^4\backslash\tilde{\mathcal{K}}$$
$$\tilde{\rho}(x,z) = \sqrt{\rho(x)^2 + z^2},$$

and $v^\delta\in C(\mathbb{R},H^{\frac{7}{4}}(\tilde{\Omega}))\cap C^1(\mathbb{R},H^{\frac{3}{4}}(\tilde{\Omega}))$ the solution of the four dimensional equation

$$\begin{cases} (\partial_t^2 - \Delta)v^\delta + (v^\delta)^5 = 0 \quad \text{in } \mathbb{R}\times\Omega \\ v^\delta_{|t=0} = v_0^\delta, \quad \partial_t v^\delta_{|t=0} = v_1^\delta, \quad v^\delta_{|\mathbb{R}\times\partial\Omega} = 0 \end{cases} \tag{5.13}$$

given by Lemma 8.

Notice that, by the finite speed of propagation,

$$v^\delta(x,z,t) = u^\delta(x,t),$$
$$\forall x\in\Omega\cap B(0,2T-t),\ \forall t\in[0,2T[,\ \forall z\in[-2T+t,2T-t]. \tag{5.14}$$

We denote by n the outward pointing normal vector to $\partial\mathcal{K}$ and $\tilde{n} = (n, 0)$ the outward pointing normal vector to $\partial\tilde{\mathcal{K}}$. The momentum derivation (5.1) applied to v^δ gives

$$\frac{d}{dt}\left(-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla\tilde{\rho} - \frac{1}{2}\int_{\tilde{\Omega}} \Delta\tilde{\rho} v^\delta \partial_t v^\delta\right) = \int_{\tilde{\Omega}} (D^2\tilde{\rho}\nabla v^\delta, \nabla v^\delta) + \frac{1}{4}\Delta\tilde{\rho}|v^\delta|^6 - \Delta^2\tilde{\rho}|v^\delta|^2 + \frac{1}{2}\int_{\partial\tilde{V}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}. \tag{5.15}$$

By Lemma 7, $\Delta^2\tilde{\rho} \le 0$. Moreover, as $\tilde{\rho}$ is convex, $(D^2\tilde{\rho}\nabla v^\delta, \nabla v^\delta) \ge 0$ and $\Delta\tilde{\rho} \ge 0$. Therefore, integrating (5.15) we obtain

$$\frac{1}{2}\int_0^T \int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dt \le \left[-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla\tilde{\rho} - \frac{1}{2}\int_{\tilde{\Omega}} \Delta\tilde{\rho} v^\delta \partial_t v^\delta\right]_0^T. \tag{5.16}$$

Let us take $T > 0$ large enough so that $\partial V \subset \Omega \cap B(0, T)$. Then, for $t \in [0, T]$,

$$\int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\tilde{\sigma} = \int_{\mathbb{R}}\int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\sigma dz \ge \int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\sigma dz.$$

But, by finite speed of propagation (5.14),

$$\int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\sigma dz = \int_{-T}^T \int_{\partial K} |\partial_n u^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\sigma dz,$$

so we obtain, as $\partial_n\rho \ge C$ by the definition of an illuminated subset

$$\begin{aligned}\int_{\partial\tilde{K}} |\partial_{\tilde{n}}\, v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\tilde{\sigma} &\ge \int_{-T}^T \int_{\partial K} |\partial_{\tilde{n}} u^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\, d\sigma dz \\ &= \int_{-T}^T \int_{\partial K} |\partial_n u^\delta|^2 \frac{\rho}{\sqrt{\rho^2 + z^2}} \partial_n\rho\ \ d\sigma dz \gtrsim \int_{\partial K} |\partial_n u^\delta|^2 \int_{-T}^T \frac{1}{\sqrt{1 + z^2}}\ \ dz d\sigma,\end{aligned}$$

and the integration of the right hand side gives

$$\int_{\partial K} |\partial_n u^\delta|^2\, d\sigma \lesssim \frac{1}{\ln T}\int_{\partial\tilde{K}} |\partial_{\tilde{n}} v^\delta|^2 \partial_{\tilde{n}}\tilde{\rho}\ \ d\tilde{\sigma}. \tag{5.17}$$

Moreover, as $\nabla\tilde{\rho}$ is bounded and $0 \le \Delta\tilde{\rho} \le \frac{C}{|(x,z)|}$, by the Cauchy-Schwarz inequality and the Hardy inequality

$$\int_\Omega \frac{|f|^2}{|x|^2} \lesssim \int_\Omega |\nabla f|^2 \text{ for } f \in \dot{H}_0^1(\Omega)$$

we obtain

$$\left|\left[-\int_{\tilde{\Omega}} \partial_t v^\delta \nabla v^\delta \cdot \nabla\tilde{\rho} - \frac{1}{2}\int_{\tilde{\Omega}} \Delta\tilde{\rho} v^\delta \partial_t v^\delta\right]_0^T\right| \lesssim E(v^\delta), \tag{5.18}$$

and this last inequality combined to (5.16) and (5.17) gives

$$\int_0^T \int_{\partial\Omega} (\partial_n u^\delta)^2\, d\sigma dt \lesssim \frac{1}{\ln T} E(v^\delta). \tag{5.19}$$

It remains to estimate the energy of v^δ. We have

$$\int_{\tilde\Omega} |\nabla v_0^\delta|^2 = \int_{\tilde\Omega} |\nabla u_0^\delta(x)\chi_T(z)\chi_T(|x|) + u_0^\delta(x)\chi_T(z)\chi_T'(|x|)\cdot\frac{x}{|x|}|^2 \ dxdz$$
$$+\int_{\tilde\Omega} |u_0^\delta(x)|^2\chi_T'(z)^2\chi_T(|x|)^2 \ dxdz \lesssim \int_{\tilde\Omega} |\nabla u_0^\delta(x)|^2\chi_T(z)^2\chi_T(|x|)^2 \ dxdz$$
$$+\int_{\tilde\Omega} |u_0^\delta(x)|^2\chi(z)^2\chi\prime(|x|)^2 \ dxdz + \int_{\tilde\Omega} |u_0^\delta(x)|^2\chi_T'(z)^2\chi_T(|x|)^2 \ dxdz,$$

so, by the Hölder inequality

$$\int_{\tilde\Omega} |\nabla v_0^\delta|^2 \lesssim \int_\Omega |\nabla u_0^\delta(x)|^2 dx \int_{\mathbb{R}} \chi_T(z)^2 \ dz + \left(\int_\Omega |u_0^\delta(x)|^6 dx\right)^{\frac{1}{3}}\left(\int_\Omega \chi_T{}'(|x|)^3 dx\right)^{\frac{2}{3}}\int_{\mathbb{R}} \chi_T(z)^2 dz$$
$$+\left(\int_\Omega |u_0^\delta(x)|^6 dx\right)^{\frac{1}{3}}\left(\int_\Omega \chi_T(|x|)^3 dx\right)^{\frac{2}{3}}\int_{\mathbb{R}} \chi_T{}'(z)^2 dz \lesssim \int_\Omega |\nabla u_0^\delta|^2 \int_{-4T}^{4T} ||\phi||_\infty^2$$
$$+\left(\int_\Omega |u_0^\delta|^6\right)^{\frac{1}{3}}\left(\int_{B(0,4T)}\left(\frac{1}{2T}||\phi'||_{L^\infty}\right)^3\right)^{\frac{2}{3}}\int_{-4T}^{4T} ||\phi||_{L^\infty}^2$$
$$+\left(\int_\Omega |u_0^\delta|^6\right)^{\frac{1}{3}}\left(\int_{B(0,4T)} ||\phi||_{L^\infty}^3\right)^{\frac{2}{3}}\int_{-4T}^{4T}\frac{1}{4T^2}\Big||\phi'||_{L^\infty}^2 \lesssim T\int_\Omega |\nabla u_0^\delta|^2$$
$$+T\left(\int_\Omega |u_0^\delta|^6\right)^{\frac{1}{3}} + T\left(\int_\Omega |u_0^\delta|^6\right)^{\frac{1}{3}} \lesssim TC(E(u^\delta)).$$

Moreover,

$$\int_{\tilde\Omega} |v_0^\delta|^6 = \int_\Omega\int_{\mathbb{R}} |u_0^\delta(x)|^6\chi_T(z)^6\chi_T(|x|)^6 dxdz \le 4T\int_\Omega |u_0^\delta|^6,$$

and

$$\int_{\tilde\Omega} |v_1^\delta|^2 = \int_\Omega\int_{\mathbb{R}} |u_1^\delta(x)|^2\chi_T(z)^2\chi_T(|x|)^2 dxdz \le 4T\int_\Omega |u_1^\delta|^2,$$

therefore

$$E(v^\delta) \lesssim TC(E(u^\delta)),$$

and by (5.18) we obtain

$$\int_0^T\int_{\partial K} |\partial_n u^\delta|^2 \ d\sigma dt \lesssim \frac{T}{\ln T} C(E(u^\delta)). \tag{5.20}$$

Notice that we cannot pass to the limit directly in this expression because as mentioned before, the application

$$u \in \dot H^1 \cap \{\text{value in time } t \text{ of solutions of NLW}\} \to \partial_n u \in L^2(\partial\Omega)$$

is not known to be continuous. But, notice that using Lemma 5 with the weight $\chi = |x|^2$ gives in particular

$$\int_0^T \int_{\Omega \cap B(0,A)} |\nabla u^\delta(x,t)|^2 + |u^\delta(x,t)|^6 \, dxdt \lesssim \int_0^T \int_{\partial K} |\partial_n u^\delta|^2 \, d\sigma dt.$$

Therefore, combining the last inequality with (5.20) we get

$$\frac{1}{T}\int_0^T \int_{\Omega \cap B(0,A)} |\nabla u^\delta(x,t)|^2 + |u^\delta(x,t)|^6 \, dxdt \lesssim \frac{1}{\ln T} C(E(u^\delta)),$$

and we can let δ go to zero in this expression: as $u_0^\delta \rightarrow_{\delta\to 0} u_0$ in $\dot{H}^1$ and $u_1^\delta \rightarrow_{\delta \to 0} u_1$ in L^2, we obtain using the energy conservation law that $C(E(u^\delta)) \rightarrow_{\delta\to 0} C(E(u))$, and, because the problem (1.8) is well posed, the left hand side goes as δ goes to zero to

$$\frac{1}{T}\int_0^T \int_{\Omega \cap B(0,A)} |\nabla u(x,t)|^2 + |u(x,t)|^6 \, dxdt,$$

and the Proposition holds. □

Finally, we can conclude:

Proof of Theorem 3. The above Proposition combined with the scattering criterion of Lemma 6 gives immediately the result. □

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