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Dismantlings in graphs

and a relation with

evasiveness conjecture for simplicial complexes

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Boolean functions and query complexity

Let F a given (=known) boolean function

$$F : \{0,1\}^n \to \{0,1\}$$

For every *unknown* $\sigma = (x_1, \ldots, x_n) \in \{0,1\}^n$, we want to know the value of $F(\sigma)$, the only questions possible being « what is the value of x_i ? »

Our goal is to ask as few questions as possible and $D(F)$ is the minimum number of questions which permits to know the value of $F(\sigma)$ for every σ.

If F is not constant (trivial case : $D(F) = 0$), we have

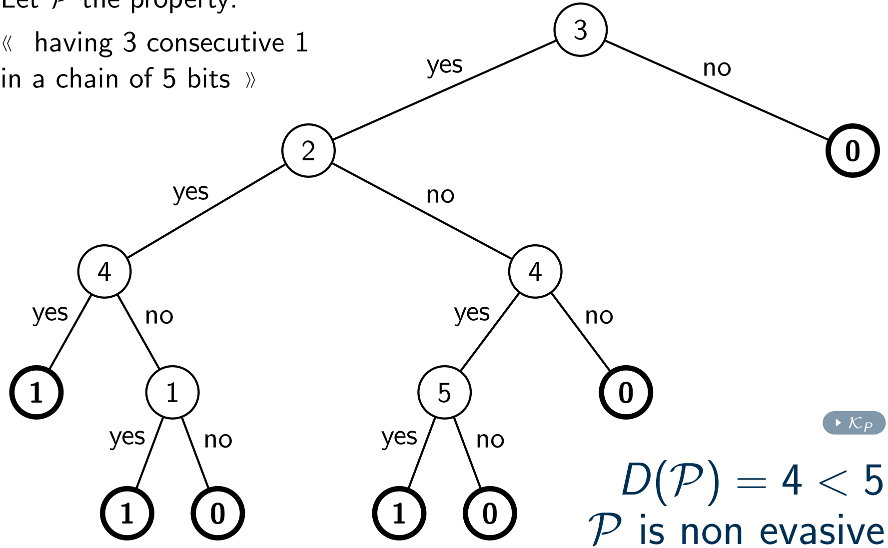
$$1 \leq D(F) \leq n$$

The function F is said evasive if $D(F) = n$ (maximal « complexity ») ; it means that there is at least one σ for which there is no strategy which permits to know $F(\sigma)$ in less than n questions.

Decision tree (example)

Let $\mathcal{P}$ the property:

« having 3 consecutive 1 in a chain of 5 bits »



$\mathcal{K}_P$

$$D(\mathcal{P}) = 4 < 5$$

$\mathcal{P}$ is non evasive

Boolean functions and evasiveness, examples

- « $F(x) = x_1 + x_2 + \ldots + x_n \mod 2$» is evasive.
- $(n = N^2)$: « $f\left((x_{ij})_{1 \leq i,j \leq N}\right) = \bigwedge_i \bigvee_j x_{ij}$» is evasive
- « having at least k 1 » is evasive if, and only if, $1 \leq k \leq n$
- For $n \geq 3$, the property « to have three consecutive 1 » is evasive if, and only if, $n \equiv 0$ or $n \equiv 3$ modulo 4

Motivation for evasiveness : graph properties

Let $\mathcal{V}$ a set of k elements. If $n = \binom{k}{2} = \frac{k(k-1)}{2}$, every $\mathcal{E} \subset 2^{[n]}$ represents a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ ($x_i = 1$ denotes the presence of the edge x_i).

A *graph property* $\mathcal{P}$ is a set of graphs such that

$$(S, A) \simeq (S, A') \Longrightarrow A, A' \in \mathcal{P} \;\text{ ou }\; A, A' \notin \mathcal{P}$$

or may be seen as a boolean function

$$F_{\mathcal{P}} : [n] = \{x_{ij},\, 1 \leq i < j \leq k\} \longrightarrow \{0, 1\}$$

such that, for all permutation $\sigma \in \mathcal{S}_k$:

$$F_{\mathcal{P}}\big((x_{ij})_{1 \leq i < j \leq k}\big) = F_{\mathcal{P}}\big((x_{\sigma(i)\sigma(j)})_{1 \leq i < j \leq k}\big)$$

DEFINITION: $\mathcal{P}$ is evasive $\Longleftrightarrow$ $F_{\mathcal{P}}$ is evasive

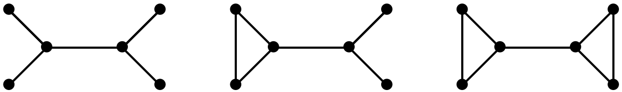
Many graph properties are evasive, but not all...

Evasive Proprerties

- being planar (with $n \geq 5$ vertices...)
- having at most j edges with $j < \binom{k}{2}$
- being acyclic
- being connected

Non evasive properties

- ($k = 6$) to be one of the three following graphs :



- to be a tournament with k vertices with a source ($c(P) \leq 3k - 4$)
- to be a *scorpion-graph* with $k \geq 11$ vertices ($c(P) \leq 6k - 13$)

Conjectures about hereditary properties

A graph property $\mathcal{P}$ is *monotone increasing* if it is preserved by addition of edges, i.e. $G = (S, A)$ verifies $\mathcal{P} \implies G - e := (S, A - \{e\})$ verifies $\mathcal{P}$ (and *monotone decreasing* if it is preserved by deletion of edges).

Aanderaa-Karp-Rosenberg conjecture

Any *monotone* and non trivial graph property is evasive.

$F : \{0,1\}^n \to \{0,1\}$ is *weakly symmetric* if there is a subgroup Γ of S_n transitive on $\{1, 2, \ldots, n\}$ such that F is Γ-invariant, i.e. for all $g \in \Gamma$ and all $(x_i)_{1 \le i \le n} \in \{0,1\}^n$, $F\big((x_i)_{1 \le i \le n}\big) = F\big((x_{g(i)})_{1 \le i \le n}\big)$.

Generalized AKR conjecture others versions

Any *monotone*, non trivial and weakly symmetric boolean function is evasive.

Simplicial complexes

An abstract simplicial complex $K = (V(K), \Sigma(K))$ is given by :

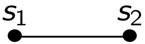
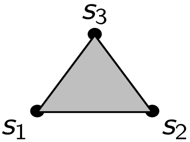
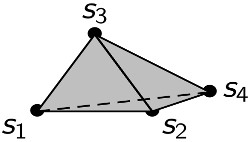
- $V(K)$, a set of vertices
- $\Sigma(K) \subset 2^{V(K)}$ a set of simplices such that

$$\tau \subset \sigma \text{ and } \sigma \in \Sigma(K) \Rightarrow \tau \in \Sigma(K)$$

DÉFINITIONS/NOTATIONS :

- If $\tau \subset \sigma \in \Sigma(K)$ and $\tau \neq \sigma$, one says that τ is a *face* of σ.

- $|K|$: *geometric realization* of K (if $\#V(K) = n$, $|K| \subset \mathbf{R}^n$).

	abstract simplices	geometrical simplices
0-simplex	$\{s_1\}$	
1-simplex	$\{s_1, s_2\}$	
2-simplex	$\{s_1, s_2, s_3\}$	
3-simplex	$\{s_1, s_2, s_3, s_4\}$	

From monotone boolean functions to simplicial complexes

For $S \subset [n] := \{1, 2, \ldots, n\}$, $\chi^S \in 2^{[n]}$ is its characteristic function (defined by $\chi_S(i) = 1$ if and only if $i \in S$).

A *monotone decreasing* boolean function F defines a simplicial complex

$$\mathcal{K}_F := \{S \subset [n],\ F(\chi_S) = 1\}$$

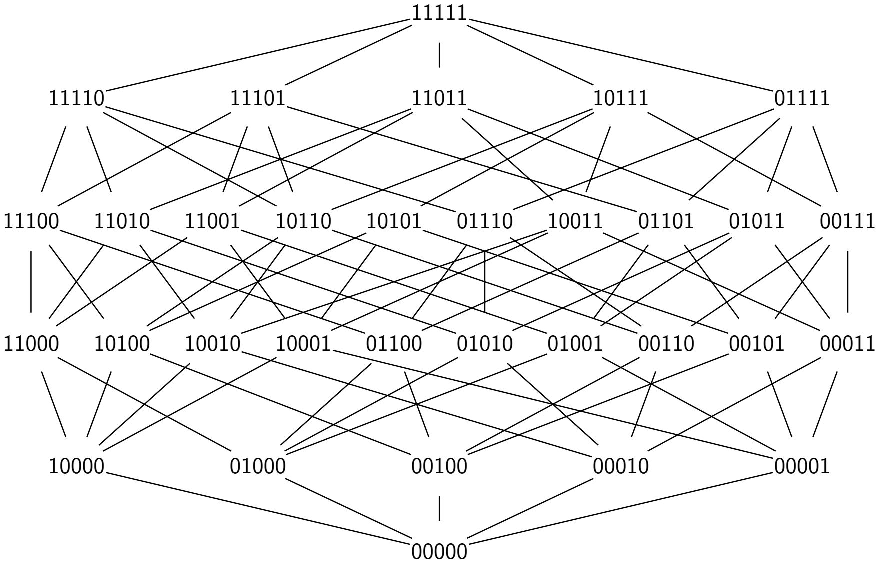
(and a *monotone increasing* boolean function F defines a simplicial complex $\mathcal{K}_{\overline{F}}$ where $\overline{F} = 1 - F$).

Reciprocally, a simplicial complex K defines a monotone (decreasing) boolean function F_K such that $K = \mathcal{K}_{F_K}$ (and also $F_{\mathcal{K}_F} = F$ if F is monotone decreasing) and

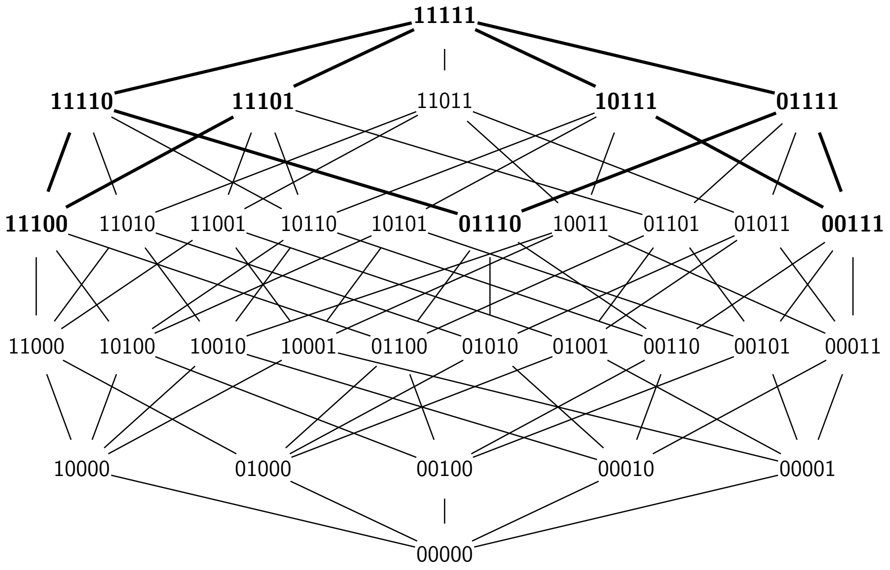
$K = \mathcal{K}_F$ is said evasive if, and only if, F is evasive

Note that : F trivial $\Longleftrightarrow$ K is a simplex

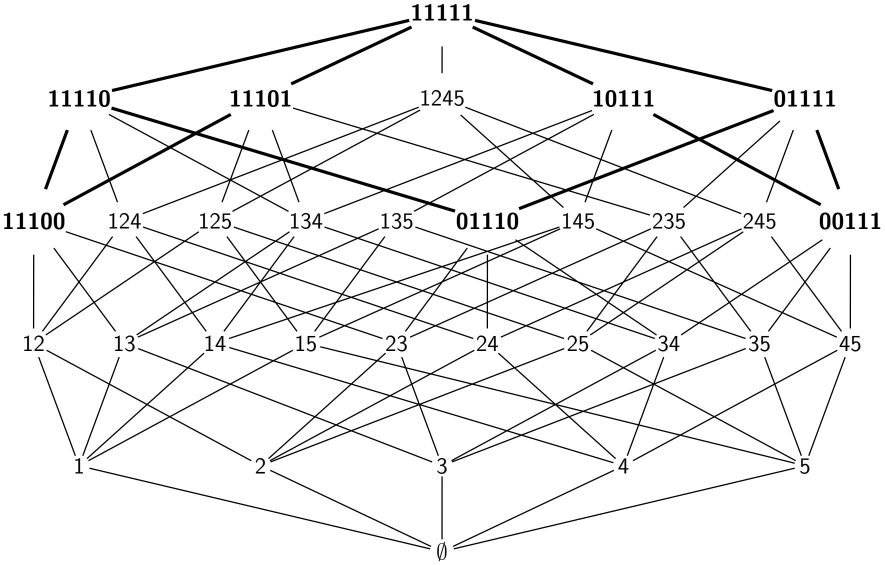
Example with F associated to « (at least) 3 consecutive 1 »



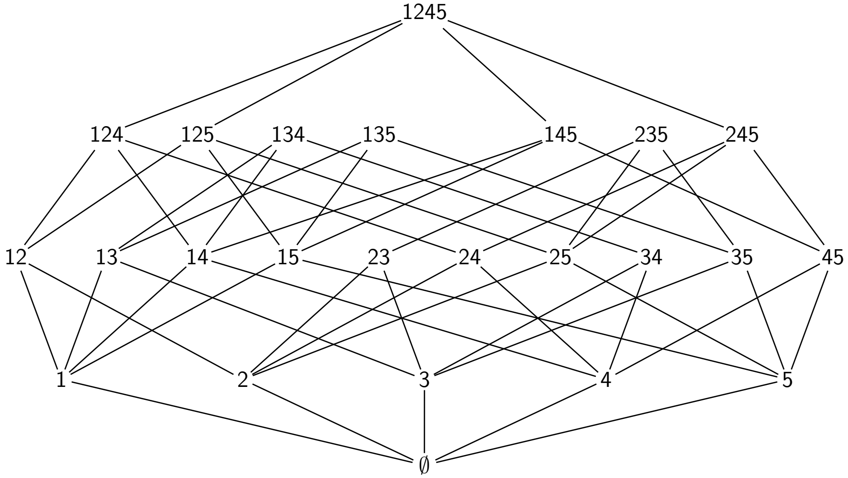
Example with F associated to « (at least) 3 consecutive 1 »



Example with F associated to « (at least) 3 consecutive 1 »

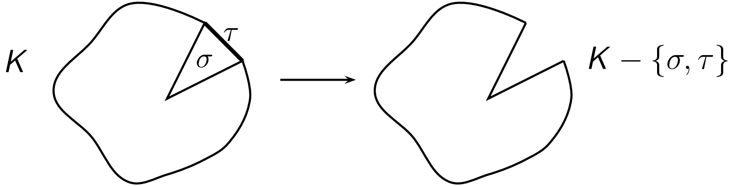


(face poset of the) simplicial complex $\mathcal{K}_F$ obtained for « F : 3 consecutive 1 » :



Collapsibility

If τ is a maximal face of σ and is not a strict face of another simplex, one says that τ is free face and that $\{\sigma, \tau\}$, is a collapsible pair.

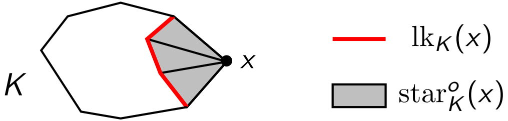


- The deletion of a collapsible pair is an elementary simplicial collapse
- Notation : $K \searrow^{sc} K - \{\sigma, \tau\}$
- A collapse $K \searrow^{sc} L$ is a succession of elementary collapses transforming K in L
- K is said collapsible if $K \searrow^{sc} pt$ where pt is a simplicial complex reduced to a point

A criterion for collapsibility

Let x a vertex of the simplicial complex K

- $\mathrm{lk}_K(x) = \{\sigma \in K,\ \{x\} \cup \sigma \in K \text{ et } x \notin \sigma\}$
- $\mathrm{del}_K(x) = \{\sigma \in K,\ x \notin \sigma\}$



Theorem : For every vertex x of K,

$$\mathrm{lk}_K(x) \ \text{ et } \ \mathrm{del}_K(x) \ \text{ collapsible} \quad \Longrightarrow \quad K \ \text{ collapsible}$$

non evasive $\Longrightarrow$ collapsible

Theorem

A simplicial complex K is non evasive if, and only if, it has a vertex x such that $\mathrm{lk}_K(x)$ and $\mathrm{del}_K(x)$ are non evasive.

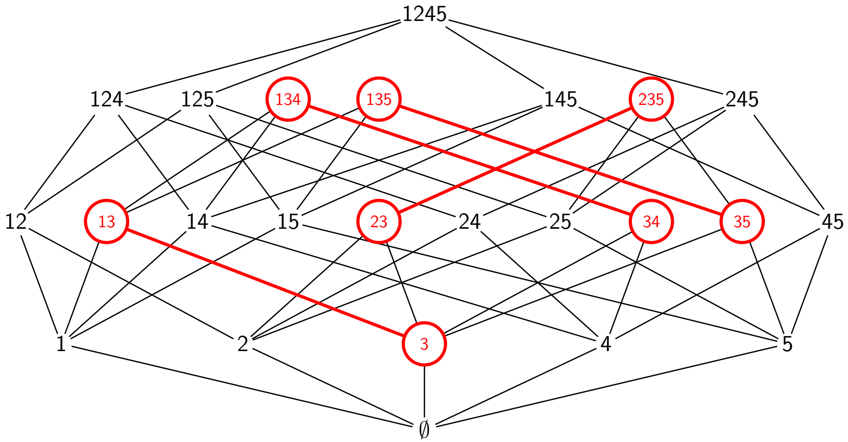
Theorem (Kahn, Saks, Sturtevant, 1984) :

If K is a non evasive simplicial complex, then K is collapsible.

Simplicial collapsing of $\mathcal{K}_F$:

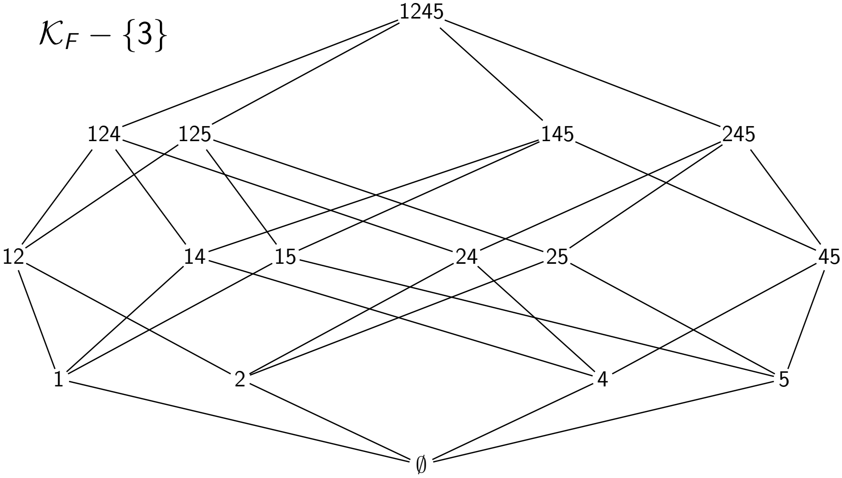
decision tree

$$\mathcal{K}_F \searrow^{sc} \mathcal{K}_1 := \mathcal{K}_F - \{23, 235\}$$
$$\searrow^{sc} \mathcal{K}_2 := \mathcal{K}_1 - \{34, 134\}$$
$$\searrow^{sc} \mathcal{K}_2 - \{34, 134\} - \{3, 13\} = K - \{3\}$$



Simplicial collapsing of $\mathcal{K}_F$: $\mathcal{K}_F \searrow^{sc} \mathcal{K}_F - \{3\} \searrow^{sc} pt$

$\mathcal{K}_F - \{3\}$



vertex-collapsibilities

Let K a simplicial complex and x a vertex (i.e. a 0-simplex) of K

- x is 0-collapsible if $\mathrm{lk}_K(x)$ is a cone
  $Coll_0(K)$ is the set of 0-collapsible vertices of K.
- K is strong collapsible if it is a reducible to a single vertex by successive deletions of 0-collapsible vertices
  $Coll_0$ is the set of strong collapsible finite complexes.
- For $k > 0$ integer, x is k-collapsible if $\mathrm{lk}_K(x) \in Coll_{k-1}$
- A complex K is k-collapsible if it is reducible to a vertex by successive deletions of k-collapsible vertices
  $Coll_k$ is the set of strong collapsible finite complexes.

Proposition (J. Barmak, G. Minian, 2009)

$NE = \bigcup_k Coll_k$ (where NE is the set of non evasive simplicial complexes)

Evasiveness conjecture

A simplicial complex K is said vertex-homogeneous if $\mathrm{Aut}(K)$, the group of simplicial automorphisms of K, acts transitively on the vertices of K.

We get the following reformulations of the generalized AKR conjecture:

Generalized AKR conjecture, version 2

If K is a non evasive and vertex homogeneous simplicial complex, then K is a simplex.

Generalized AKR conjecture, version 3

If $K \in Coll_k$ for some integer $k \geq 0$ and if K is vertex homogeneous, then K is a simplex.

▸ AKR versions

Topological considerations

The topological space X is said *contractible* il there is a continuous map $H : X \times [0,1] \to X$ such that, for all x in X, $H(x,0) = x$ and $H(x,1) = x_0$ for some point x_0 of X.

Theorem K collapsible $\Longrightarrow$ K contractible

Brouwer theorem
Let K a simplicial complex and $\varphi : |K| \to |K|$ continue.

$$|K| \text{ contractile } \Longrightarrow \text{fix}(\varphi) \neq \emptyset$$

where $\mathrm{Fix}(\varphi) := \{x \in |K|, \varphi(x) = x\}$ is the set of fixed points of φ.

NOTE : Every simplicial $f : K \to K$ (i.e. $f(\sigma)$ is a simplex for any simplex σ), induces $\varphi := |f| : |K| \to |K|$. Nevertheless

$$|K| \text{ contractible } \not\Longrightarrow \text{Fix}(f) \neq \emptyset$$

Group actions and fixed points

Let K a vertex homogeneous complexe simplicial for the action of a finite group Γ.

Proposition $|K|^\Gamma \neq \emptyset \implies K$ is a simplex

Theorem (Oliver, 1975)

If 1) $|K|$ is contractible
2) Γ has a normal subgroup H which is a p-group
3) Γ/H is cyclic

then, $|K|^\Gamma \neq \emptyset$.

Application : with $\#S = p^r$, p prime ; $S = \mathbf{F}_{p^r}$.

- $f_{a,b} : S \to S$, $x \mapsto ax + b$
- $\Gamma := \{f_{a,b} \,;\, a, b \in \mathbf{F}_{p^r},\, a \neq 0\}$ acts transitively on S
- $H := \{f_{1,b} \,;\, b \in \mathbf{F}_{p^r}\} \lhd \Gamma$ et $\#H = p^r$
- $\Gamma/H \cong (\mathbf{F}_{p^r})^*$ is cyclic

Evasiveness conjecture is proved when $n = p^r$

Theorem (Kahn, Saks, Sturtevant, 1984) :

Every non trivial monotone graph property on graphs with p^r vertices with p prime and $r \in \mathbf{N}^*$ is evasive.

Idea of the proof

- non evasive graph property $\mathcal{P} \Longrightarrow K_{\mathcal{P}}$ contractible simplicial complex
- graph property $\Longrightarrow K_{\mathcal{P}}$ invariant for the action of $\Gamma := \{f_{a,b}\,;\ a, b \in \mathbf{F}_{p^r},\ a \neq 0\}$
- Oliver theorem $\Longrightarrow |K|^{\Gamma} \neq \emptyset$
- Γ is transitive on $V(K_{\mathcal{P}}) \Longrightarrow K_{\mathcal{P}}$ is a simplex (i.e. $\mathcal{P}$ is a trivial graph property)

0-dismantlability (« classical » dismantlability)

Graphs $G = (V(G), E(G))$ are finite.

A vertex $a \in V(G)$ is called 0-dismantlable if there is another vertex $b \in V(G)$ such that every neighbour of a is also a neighbour of b :

$$N_G[a] \subset N_G[b]$$

Then, we say that G is 0-dismantlable on $G - a$; notation : $G \searrow^0 G - a$.

A graph G is called 0-dismantlable if $V(G) = \{x_1, x_2, \ldots, x_n\}$ with

$$G = G_1 \searrow^0 G_2 \searrow^0 G_3 \ldots \searrow^0 G_i \searrow^0 G_{i-1} \searrow^0 \ldots \searrow^0 G_n = \{x_n\}$$

where G_i is the subgraph of G induced by $\{x_i, x_{i+1}, \ldots, x_n\}$.

Theorem (Quilliot 1978, Nowakowski, Winkler 1983, ...)
Let G be a *reflexive* finite graph

G is cop-win $\Longleftrightarrow$ G is dismantlable $\Longleftrightarrow$ G is contractible

▸ C-strategy

k-dismantlability

Inductively, for k integer ≥ 1 : a vertex $a \in V(G)$ is called k-dismantlable if its open neighbourhood is a $(k$-$1)$-dismantlable graph. Then, we say that G is k-dismantlable on $G - a$; notation : $G \searrow^k G - a$.

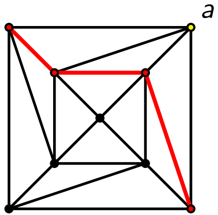
A graph G is called k-dismantlable if $V(G) = \{x_1, x_2, \ldots, x_n\}$ with

$$G = G_1 \searrow^k G_2 \searrow^k G_3 \ldots \searrow^k G_i \searrow^k G_{i-1} \searrow^k \ldots \searrow^k G_n = \{x_n\}$$

where $G_i := G[x_i, x_{i+1}, \ldots, x_n]$.

NOTATIONS : $D_k := \{k\text{-dismantlable graphs}\}$

$\mathfrak{Q}_2$



a is 1-dismantlable

G is 1-dismantlable
and minimal in $D_1 \setminus D_0$
(in number of vertices)

a strict hierachy

Theorem (E. F.; B. Jouve)

The sequence $(D_k)_{k\geq 1}$ is strictly increasing and $D_\infty := \bigcup_{k\geq 0} D_k \subsetneq D_{coll}$:

$$D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots\ldots \subsetneq D_k \subsetneq D_{k+1} \subsetneq \ldots\ldots \subsetneq D_{coll}$$

where D_{coll} is the set of graphs whose clique complex is collapsible.

proof :

- For $k \geq 0$, $\mathfrak{Q}_{k+1} \in D_k \setminus D_{k-1}$ (cubions)
- For $n \geq 7$, $\Uparrow_n \in D_{coll} \setminus D_\infty := \bigcup_{k\geq 0} D_k$

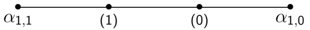
The cubions $\mathfrak{Q}_n$, $n \in \mathbf{N}$

Definition of the *n-Cubion* $\mathfrak{Q}_n$

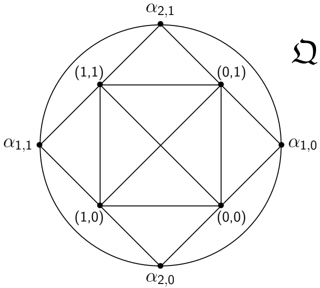
$V(\mathfrak{Q}_n) = \{\alpha_{i,\epsilon}, i = 1, \cdots, n \text{ and } \epsilon = 0, 1\} \cup \{x = (x_1, \cdots, x_n), x_i = 0, 1\}$ and $E(\mathfrak{Q}_n)$ defined by:

- $\forall i \neq j, \alpha_{i,\epsilon} \sim \alpha_{j,\epsilon'}$
- $\forall x \neq x', x \sim x'$
- $\forall i \in [n], \alpha_{i,1} \sim (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$ and

$\alpha_{i,0} \sim (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$

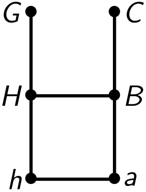


$\mathfrak{Q}_1$



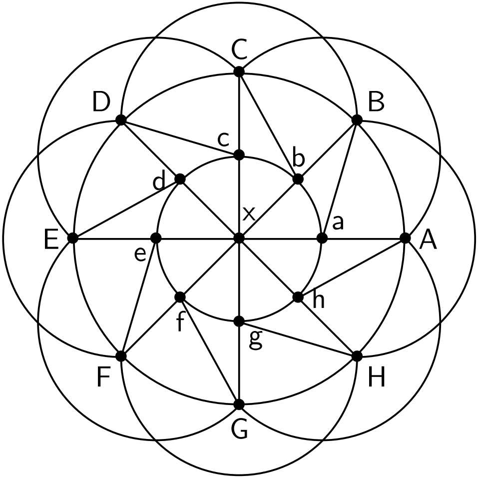
$\mathfrak{Q}_2$

$\mathbb{T}_8 \in D_{coll} \setminus D_\infty$



$N\mathbb{T}_8(A)$

and



big-top $\mathbb{T}_8$

$$\Delta(\mathbb{T}_8) \searrow\searrow \Delta(\mathbb{T}_8)) - \{HB, HAB\}$$

Back to evasiveness

a weak evasiveness conjecture

Let X a graph

$$\left.\begin{array}{c} \exists\, k \in \mathbf{N},\ X \in D_k \\ X \textit{ vertex transitive} \end{array}\right\} \Longrightarrow X \textit{ complete}$$

This conjecture may be seen as the particular case of evasiveness conjecture restricted to flag complexes (or clique complexes):

First direction of results

Let X a graph and k an integer ≥ 0.

Let $\mathcal{C}_k(X)$ denote the set of $(k+1)$-subsets of $V(X)$ which induce a complete subgraph of X (e.g. $\mathcal{C}_0(X) = V(X)$ and $\mathcal{C}_1(X) = E(X)$).

X will be called `k-transitive` if $\mathrm{Aut}(X)$ acts transitively on $\mathcal{C}_k(X)$, i.e.:

$$\forall\, (\{a_0, a_1, a_2, \ldots, a_k\}, \{b_0, b_1, b_2, \ldots, b_k\}) \in \mathcal{C}_k(X) \times \mathcal{C}_k(X),$$

$$\exists\, \varphi \in \mathrm{Aut}(X)\ s.t.\ \varphi(a_u) = b_u, \ \text{ for all } u \in \{0, 1, 2, \ldots, k\}$$

Exemples : Johnson graphs $J(v, k, i)$; for $i = 0$: Kneser graphs

Theorem 1 (E. F.; B. Jouve)

Let X a finite graph and k an integer ≥ 0.
If $X \in D_k$ and X is j-transitive for all $j \in \{0, 1, 2, \ldots, k\}$, then X is a complete graph.

Second direction of results

Theorem 2
If a Cayley graph $X = Cay(\mathbf{Z}/n\mathbf{Z}, S)$ is k-dismantlable for some integer $k \geq 0$, then X is a complete graph.
In particular, if a vertex transitive graph with a prime number of vertices is k-dismantlable for some integer $k \geq 0$, then it is a complete graph.

proof :

- By non evasiveness, $|\Delta X|^{\Gamma} \neq \emptyset$ (where $|\Delta X|^{\Gamma}$ the set of fixed points of $|\Delta X|$ under the action of Γ).
- By vertex transitivity, $V(X)$ is the unique orbit.
- So, X is a complete.

Thanks for your attention !

《 cop and rob 》 game

- Player 1 (the cop) chooses a vertex
- Then, player 2 (the robber) chooses a vertex
- Then, cop and rob move to an adjacent vertex alternatively (first cop, next rob) ; and so on...
- The cop wins if he 《catches》 the robber (they are on the same vertex)

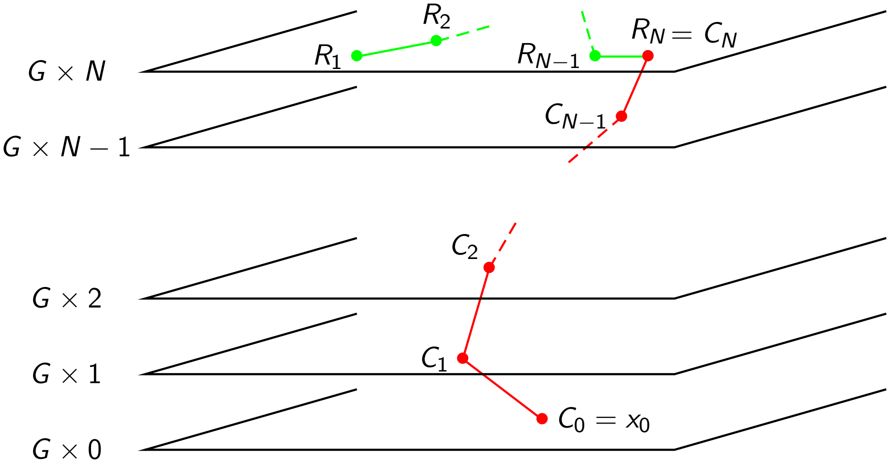
Theorem (Quilliot 1978, Nowakowski, Winkler 1983, ...)

Let G be a reflexive finite graph

G is cop-win $\Longleftrightarrow$ G is dismantlable $\Longleftrightarrow$ G is contractible

$\mathcal{H} : G \times I_N \ \rightarrow\ G$ from $\mathcal{H}_0 = C_{x_0}$ to $\mathcal{H}_N = 1_G$ gives a **winning strategy** :

$$\boxed{\begin{array}{cccccc} & R_1 & R_2 & \dots & R_{N-1} & R_N \\ C_0 & C_1 & C_2 & \dots & C_{N-1} & C_N \end{array}}$$ with $C_i = \mathcal{H}_i(R_i)$, for $i = 1, \dots, N$.



dismantlable graph