The profinite theory of rational languages

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LIP, ENS Lyon

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The 3 reasons I am here...

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1 - Topology: metric space, limits of sequences of words...

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2 - Languages: classes of rational languages...

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3 - Toulouse...

Profinite theory

$\rightarrow$ a topological approach for the study of rational languages.

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A^*: set of words
$\cup\!\!\mid$
L: rational language

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$$\begin{array}{ccc} A^*\text{: set of words} & \dashrightarrow & (A^*, d)\text{: metric space} \\ \cup\!\mid & & \cup\!\mid \\ L\text{: rational language} & & L\text{: subset satisfying some topological properties} \end{array}$$

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- automata
- logic
- rational expressions
- monoids

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Finite monoids and rational languages

A few things to know about monoids...

Monoid: a set with an associative operation and a neutral element.

Idempotent: $e^2 = e$

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A monoid M recognises a language L if there is a morphism $\varphi : A^* \to M$ and $P \subseteq M$ s.t. $L = \varphi^{-1}(P)$.

$$\begin{tikzcd} A^* \arrow[r, "\varphi"] \arrow[d, phantom, "\cup\!\mid"] & M \arrow[d, phantom, "\cup\!\mid"] \\ \varphi^{-1}(P) = L \arrow[r, dashed] & P \end{tikzcd}$$

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A language is rational iff it is recognised by a finite monoid.

Examples and syntactic monoid

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Example: $L = \{w \in A^* \mid |w| \text{ is even}\}$

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Example: $L = \{w \in A^* \mid |w| \text{ is even}\}$ $\quad (\mathbb{Z}/2\mathbb{Z}, +)$ - $\varphi(A) = \{1\}$

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Syntactic monoid: the smallest monoid recognising L.
$=$ Monoid of transitions of a minimal deterministic automaton.

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Example 3: $u \in A^*$, $n \in \mathbb{N}$ - separate $u^{n!}$ and $u^{(n+1)!}$?
$x \in M$ then $x^{|M|!} = x^{(|M|+1)!}$ = the idempotent power of x in M
$\implies \varphi(u)^{|M|!} = \varphi(u)^{(|M|+1)!}$

$u^{n!}$ and $u^{(n+1)!}$ cannot be separated by a monoid of size less than n

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d is an ultrametric distance:

- $d(u,v) = 0$ iff $u = v$
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The words $u^{n!}$ and $u^{(n+1)!}$ are closer and closer...

Profinite monoid

Definition

Profinite monoid $\widehat{A^*}$:
completion of A^* with respect to the distance d.

- Monoid if u and v sequences of words, $(u.v)_n = u_n v_n$
- Metric space
- A^* dense subset
- Compact

V.I.P. words (very important profinite words)

Idempotent power

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Zero (Reilly-Zhang 2000, Almeida-Volkov 2003)

$|A| \geqslant 2$

$u_0, u_1, \ldots$ an enumeration of the words of A^*

$v_0 = u_0, \quad v_{n+1} = (v_n u_{n+1} v_n)^{(n+1)!}$

$$\rho_A = \lim_{n\to\infty} v_n$$

Profinite monoid and rational languages

Universal property

M a finite monoid.

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$\boxed{\text{A language } L \text{ is rational iff } \overline{L} \text{ is open and closed in } \widehat{A^*}.}$

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$\longrightarrow$ For all morphisms $\varphi : A^* \to M$ (finite monoid):
if M has a zero then $\widehat{\varphi}(\rho_A) = 0$.

Study of classes of rational languages

Birkhoff variety of monoids: class of monoids closed under:

- direct product
- submonoid
- quotient N quotient of M: $M \xrightarrow{\varphi} N$ with φ a surjective morphism.

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Varieties of finite monoids $\longleftrightarrow$ varieties of rational languages [Eilenberg]

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$\to$ Equations for pseudovarieties? Profinite equations! [Reiterman]

Classes of rational languages

- Lattice (union, intersection)
- Boolean algebra (lattice, complement)
- Lattice closed under quotient
- Boolean algebra closed under quotient

quotient : $u^{-1}Lv^{-1} = \{w \mid uwv \in L\}$

Equations

Definition

Given two profinite words u, v, a rational language L satisfies

$$u \to v$$

if $u \in \overline{L}$ implies $v \in \overline{L}$

$a, b \in A$

Equation $ab \to aba$

$\{L \subseteq A^* \mid ab \notin L\} \cup \{L \subseteq A^* \mid ab, aba \in L\}$

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Given two profinite words u, v, a rational language L satisfies

$$u \leqslant v$$

if for all $w, w' \in A^*, wuw' \in \bar{L}$ implies $wvw' \in \bar{L}$

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Equation $ab = aba$

$\{L \subseteq A^* \mid \text{for all } w, w',\ wabw' \in L \text{ iff } wabaw' \in L\}$

Characterisation by equations on profinite words

Theorem [Gehrke, Grigorieff, Pin 2008]

Classes of rational languages

- Lattice (union, intersection): $\rightarrow$
- Boolean algebra (lattice, complement): $\leftrightarrow$
- Lattice closed under quotient: $\leqslant$
- Boolean algebra closed under quotient: $=$

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Examples

A alphabet

Example 1: Commutative languages
A language L is commutative if for all $u \in L$, $com(u) \subseteq L$.

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$\{\rho_A u = u\rho_A = \rho_A \mid u \in A^*\}$

Generalised star-height problem

Rational expressions:

- 1, $a \in A$,
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$\rightarrow$ Star-height 0 [Schützenberger, McNaughton-Papert]
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$\rightarrow$ Star-height 1
Example: $(aa)^*$ - Is there a nontrivial identity for this class ?

Equations for u^* (joint work with Charles Paperman)

$$P_u = \bigcup_{p \text{ prefix of } u} u^* p \quad \text{and} \quad S_u = \bigcup_{s \text{ suffix of } u} s u^*$$

$$x^\omega y^\omega = 0 \text{ for } x, y \in A^* \text{ such that } xy \neq yx \qquad (E_1)$$

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$$x^\ell \leftrightarrow x^{\omega+\ell} \text{ for } x \in A^*, \ \ell > 0 \qquad (E_5)$$

$$x \to x^\ell \text{ for } x \in A^*, \ \ell > 0 \qquad (E_6)$$

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$\boxed{\text{DECIDABLE}}$ Lattice closed under quotients

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The Boolean algebra

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An example:

$$(a^2)^* - (a^6)^* = (a^6)^* a^2 \cup (a^6)^* a^4$$

The Boolean algebra

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Equivalence relation over the integers

$r \equiv_m s$ if and only if $\gcd(r, m) = \gcd(s, m)$
$(u^m)^* u^r \subseteq L$ if and only if $(u^m)^* u^s \subseteq L$

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$x^\alpha \leftrightarrow x^\beta$ for α and β representing sequences of integers $(km + r)_k$ and $(km + s)_k$ with $r \equiv_m s$...

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$x^{\alpha} \leftrightarrow x^{\beta}$ for α and β profinite numbers in $\widehat{\mathbb{N}} = \widehat{\{a\}^*}$
satisfying some specific conditions...

The Boolean algebra

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Γ is the set of all the pairs of profinite numbers $(dz^{\mathcal{P}}, dpz^{\mathcal{P}})$ s.t.:

- $\mathcal{P}$ is a cofinite sequence of prime numbers $\{p_1, p_2, \ldots\}$
- $z^{\mathcal{P}} = \lim_n (p_1 p_2 \ldots p_n)^{n!}$
- $p \in \mathcal{P}$
- if q divides d then $q \notin \mathcal{P}$

$$x^\alpha \leftrightarrow x^\beta \text{ for all } (\alpha, \beta) \in \Gamma \qquad (E_7)$$

Conclusion

Topology ✓

Languages ✓

Thank you for your attention