The (poly)topologies of provability logic

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Topologie et Langages 2016, Toulouse

Gödel-Löb logic

Language:

$$p \qquad \neg\varphi \qquad \varphi \wedge \psi \qquad \Box\varphi$$

Axioms:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

Second incompleteness theorem:

$$\Box\Diamond\top \to \Box\bot$$

Arithmetical interpretation

An arithmetical interpretation assigns a formula p^* in the language of arithmetic to each propositional variable p.

- $p \mapsto p^*$
- $\Box\varphi \mapsto \exists x\, \mathtt{Proof}_{\mathsf{PA}}(x, \ulcorner\varphi^*\urcorner)$

Theorem (Solovay)

If GL $\vdash \varphi$ *if and only if, for every arithmetical interpretation* $*$*,* PA $\vdash \varphi^*$*.*

Relational semantics

Kripke models:

- Frames: Well-founded partial orders $\langle W, < \rangle$
- Valuations: $[\![\varphi]\!] \subseteq \mathcal{P}(W)$,

$$w \in [\![\Box\varphi]\!] \Leftrightarrow \forall v < w, v \in [\![\varphi]\!]$$

Theorem

GL *is sound for* $\langle W, < \rangle$ *if and only if* $<$ *is well-founded.*

Further, GL *is complete for the class of well-founded frames and enjoys the finite model property.*

Topological semantics:

- GL-spaces: scattered topological spaces $\langle X, \mathcal{T} \rangle$
  Scattered: Every non-empty subset contains an isolated point.
- Valuations: dA is the set of limit (or accumulation) points of A.

$$[\![\Diamond\varphi]\!] = d\,[\![\varphi]\!]\,.$$

GL is also sound and complete for this interpretation.

Some scattered spaces

- A finite partial order $\langle W, < \rangle$ with the downset topology
- An ordinal ξ with the initial segment topology
- An ordinal ξ with the order topology

Non-scattered:

- The real line
- The rational numbers
- The Cantor set

Ordinal numbers

Ordinals serve as canonical representatives of well-orders.

Well-order: Structure $\langle A, \preccurlyeq \rangle$ such that

- A is any set,
- $\preccurlyeq$ is a linear order on A, and
- if $B \subseteq A$ is non-empty, then it has a $\preccurlyeq$-minimal element.

The class Ord of ordinals is itself well-ordered:

$$\xi \leq \zeta \Leftrightarrow \xi \subseteq \zeta.$$

Examples:

- Every interval $[0, n)$ is an ordinal for $n \in \mathbb{N}$.
- The set of natural numbers can itself be seen as the first infinite ordinal, and is denoted ω.

Ordinal topologies

Intervals on ordinals are defined in the usual way, e.g.

$$[\alpha, \beta) = \{\xi : \alpha \leq \xi < \beta\}.$$

- Initial topologies: Topology $\mathcal{I}_0$ on an ordinal Θ generated by sets of the form $[0, \alpha)$.
- Interval topologies: Topology $\mathcal{I}_1$ on an ordinal Θ generated by sets of the form $[0, \alpha)$ and (α, β).

Ordinal recursion

There are three kinds of ordinals ξ:

1. $\xi = 0$ (the empty well-order)
2. $\xi = \zeta + 1$ (successor ordinals)
3. $\xi = \bigcup_{\zeta<\xi} \zeta$ (limit ordinals).

We can use this to define addition recursively:

1. $\xi + 0 = \xi$
2. $\xi + (\zeta + 1) = (\xi + \zeta) + 1$
3. $\xi + \lambda = \bigcup_{\eta<\lambda}(\xi + \eta)$ if λ is a limit.

Ordinal arithmetic

Other arithmetical operations can be generalized similarly.

Multiplication:

1. $\xi \cdot 0 = 0$
2. $\xi \cdot (\zeta + 1) = (\xi \cdot \zeta) + \zeta$
3. $\xi \cdot \lambda = \bigcup_{\eta<\lambda}(\xi \cdot \eta)$ if λ is a limit.

Exponentiation:

1. $\xi^0 = 1$
2. $\xi^{\zeta+1} = \xi^\zeta \cdot \xi$
3. $\xi^\lambda = \bigcup_{\eta<\lambda} \xi^\eta$ if λ is a limit.

Iterated derived sets

Recall that if $\langle X, \mathcal{T} \rangle$ is any topological space and $A \subseteq X$, dA denotes the set of limit points of A.

If ξ is an ordinal, define $d^{\xi}A$ recursively by:

1. $d^0 A = A$
2. $d^{\zeta+1} A = dd^{\zeta} A$
3. $d^{\lambda} A = \bigcap_{\zeta<\lambda} d^{\zeta} A$ (λ a limit).

Ranks on a scattered space

Theorem

The following are equivalent:

- *$\langle X, \mathcal{T} \rangle$ is scattered*
- *there exists an ordinal Λ such that $d^{\Lambda} X = \varnothing$.*

Let $\mathfrak{X} = \langle X, \mathcal{T} \rangle$ be a scattered space.

- Define $\rho(x)$ to be the least ordinal such that $x \notin d^{\rho(x)+1} X$.
- Define $\rho(\mathfrak{X})$ to be the least ordinal such that $d^{\rho(\mathfrak{X})} X = \varnothing$.

Fact: The rank on $\langle \Theta, \mathcal{I}_0 \rangle$ is the identity.

Cantor normal forms

Theorem

Every ordinal $\xi > 0$ can be uniquely written in the form

$$\xi = \omega^{\alpha_0} + \ldots + \omega^{\alpha_n}$$

with the α_i's non-increasing.

Define $\ell\xi = \alpha_n$ (the last exponent or least logarithm of ξ).

CNFs allow us to write many ordinals using $0, \omega, +$ and exponentiation, up to the ordinal

$$\varepsilon_0 = \bigcup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}_{n}.$$

Ranks on the interval topology

Theorem

If $\langle \Theta, \mathcal{I}_1 \rangle$ *is an ordinal with the interval topology, then* $\rho(\theta) = \ell\theta$ *for all* $\theta < \Theta$*.*

Henceforth:

- ρ_0 is the rank with respect to $\mathcal{I}_0$
- ρ_1 is the rank with respect to $\mathcal{I}_1$.

Completeness

Observation:

- The initial topology validates

$$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p).$$

- Any space of rank $n < \omega$ validates $\Box^{n+1}\bot$.
- The first ordinal with infinite ρ_1 is ω^ω.

Theorem (Abashidze, Blass)

If $\Theta \geq \omega^\omega$, then GL *is complete for $\langle \Theta, \mathcal{I}_1 \rangle$.*

Polymodal Gödel-Löb

GLP: Contains one modality $[n]$ for each $n < \omega$.

Axioms:

$$[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi) \qquad (n < \omega)$$

$$[n]([n]\varphi \to \varphi) \to [n]\varphi \qquad (n < \omega)$$

$$[n]\varphi \to [m]\varphi \qquad (n < m < \omega)$$

$$\langle n \rangle\varphi \to [m]\langle n \rangle\varphi \qquad (n < m < \omega)$$

(Possible) arithmetical interpretation:

$[n]\varphi \equiv$ "φ is provable using n instances of the ω-rule".

Introduced by Japaridze in 1988.

Kripke semantics

Frames:

$$\langle W, \langle <_n \rangle_{n<\omega} \rangle$$

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

Valid iff $<_n$ is well-founded

$[n]\varphi \to [n+1]\varphi$:

Valid iff $w <_{n+1} v \Rightarrow w <_n v$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$:

Valid iff

$$v <_n w \text{ and } u <_{n+1} w \Rightarrow v <_n u$$

Even GLP_2 has no non-trivial Kripke models.

Topological semantics

Spaces:

$$\mathfrak{X} = \langle X, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle$$

Write d_n for the limit point operator on $\mathcal{T}_n$.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$: Valid iff $\mathcal{T}_n$ is scattered

$[n]\varphi \to [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$: Valid iff

$$A \subseteq X \Rightarrow d_n A \in \mathcal{T}_{n+1}$$

Canonical ordinal spaces

For a topological space $\langle X, \mathcal{T} \rangle$, define $\mathcal{T}^+$ to be the least topology containing

$$\mathcal{T} \cup \{dA : A \subseteq X\}.$$

Denote the join of topologies by $\bigsqcup$.

The canonical polytopology on Θ is given by

1. $\mathcal{T}_0 = \mathcal{I}_1$
2. $\mathcal{T}_{\xi+1} = \mathcal{T}_\xi^+$
3. $\mathcal{T}_\lambda = \bigsqcup_{\xi<\lambda} \mathcal{T}_\xi$ for λ a limit.

Independence results

Blass: It is consistent with ZFC that GLP_2 is incomplete for the class of canonical ordinal spaces

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for this class

Bagaria, Beklemishev For all $n > 1$ it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.

Icard topologies

Icard defined a structure

$$\mathfrak{I} = \langle \varepsilon_0, \langle \mathcal{I}_n \rangle_{n<\omega} \rangle.$$

Generalized intervals:

$$(\alpha, \beta)_n = \{\vartheta : \alpha < \ell^n \vartheta < \beta\}.$$

- $\mathcal{I}_0$ is generated by intervals of the form $[0, \beta)$
- $\mathcal{I}_{n+1}$ is generated by sets of the form $(\alpha, \beta)_m$ for $m \leq n$

Topological conditions

Icard's model does not satisfy all frame conditions either.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

$\mathcal{I}_n$ is scattered since $\mathcal{I}_0$ is.

$[n]\varphi \to [n+1]\varphi$: $\mathcal{I}_{n+1}$ is always a refinement of $\mathcal{I}_n$.

$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: The point

$$\omega^\omega = \lim_{n\to\omega} \omega^n$$

should be isolated in $\mathcal{I}_2$.

Provability ambiances

Ambiance:

$$\mathfrak{X} = \langle X, \mathcal{A}, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle,$$

where:

- $\mathcal{T}_n$ is scattered
- $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$
- $\mathcal{A} \subseteq \mathcal{P}(X)$ is such that
  - $\varnothing \in \mathcal{A}$
  - $\mathcal{A}$ is closed under finite unions, complements and d_n
  - $A \in \mathcal{A} \Rightarrow d_n A \in \mathcal{T}_{n+1}$

Models: Ambiances with a valuation such that $[\![\varphi]\!] \in \mathcal{A}$ for all φ.

The simple ambiance

A subset of Θ is simple if it is of the form

$$\bigcup_{i<n} \bigcap_{j<m_i} (\alpha_{ij}, \beta_{ij})_{k_{ij}}.$$

The family of simple sets is denoted $\mathcal{S}$.

Theorem

If Θ is any ordinal then

$$\langle \Theta, \mathcal{S}, \langle \mathcal{I}_n \rangle_{n<\omega} \rangle$$

is a provability ambiance.

The closed fragment

The variable-free fragment of GLP is denoted GLP^0 (the only atom is $\bot$).

Beklemishev: GLP^0 may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

Theorem (Icard)

GLP^0 *is complete for the class of simple ambiances.*

Lime topologies

If $\mathcal{T} \subseteq \mathcal{S}$ are two scattered topologies on X, we say that $\mathcal{S}$ is:

- a rank-preserving extension if $\rho_{\mathcal{S}} = \rho_{\mathcal{T}}$
- a limit extension if it is rank-preserving and

$$Id \colon \langle X, \mathcal{T} \rangle \to \langle X, \mathcal{S} \rangle$$

  is only discontinuous on points of limit rank
- a lime topology if it is a LImit, Maximal Extension.

Zorn's lemma: Lime extensions always exist.

Beklemishev-Gabelaia spaces

A polytopology $\langle \Theta, \langle \mathcal{T}_n \rangle \rangle$ is a Beklemishev-Gabelaia space if $\mathcal{T}_0$ is a lime of $\mathcal{I}_1$ and for every n, $\mathcal{T}_{n+1}$ is a lime of $\mathcal{T}_n^+$.

Theorem

Given any BG-space $\langle \Theta, \langle \mathcal{T}_n \rangle \rangle$ and any $n < \omega$, $\mathcal{T}_n$ is a lime of $\mathcal{I}_{n+1}$.

Theorem (Beklemishev, Gabelaia)

GLP *is complete for the class of BG-spaces based on ε_0.*

Idyllic ambiances

An ambiance $\mathfrak{X} = \langle \Theta, \mathcal{A}, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle$ is idyllic if

- $\mathcal{T}_n = \mathcal{I}_{n+1}$ for all n, and
- there is a BG polytopology on Θ with derived set operators d_n such that
$$d_n \upharpoonright \mathcal{A} = d_{\mathcal{I}_{n+1}} \upharpoonright \mathcal{A}.$$

Theorem (DFD)

GLP *is complete for the class of idyllic ambiances.*

Transfinite Gödel-Löb

Λ is an arbitrary ordinal.

GLP_Λ: One modality $[\lambda]$ for each ordinal $\lambda < \Lambda$.

Axioms:

$$
\begin{aligned}
[\xi](\varphi \to \psi) &\to ([\xi]\varphi \to [\xi]\psi) && (\xi < \Lambda) \\
[\xi]([\xi]\varphi \to \varphi) &\to [\xi]\varphi && (\xi < \Lambda) \\
[\xi]\varphi &\to [\zeta]\varphi && (\xi < \zeta < \Lambda) \\
\langle\xi\rangle\varphi &\to [\zeta]\langle\xi\rangle\varphi && (\xi < \zeta < \Lambda)
\end{aligned}
$$

DFD, Joosten: Proof-theoretic interpretations using iterated ω-rules in second-order arithmetic.

Can we generalize Icard topologies?

Icard topologies are generated by intervals

$$\{\xi : \alpha < \ell^n \xi < \beta\}.$$

We could define $\mathcal{I}_\lambda$ if we had transfinite iterations of ℓ.

These should satisfy:

- $\ell^0 = \text{id}$
- $\ell^1 = \ell$
- $\ell^{\xi+\zeta} = \ell^\zeta \circ \ell^\xi$
- ℓ^ξ is always initial.

Initial functions map initial segments to initial segments.

Cohyperations

Definition:

The cohyperation of an initial function f is the unique family of initial functions $\langle f^\xi \rangle_{\xi \in \mathrm{On}}$ such that

- $f^1 = f$
- $f^{\xi+\zeta} = f^\zeta \circ f^\xi$
- f^ξ is always initial
- f^ξ is pointwise maximal among all such families of functions.

Theorem (DFD, Joosten)

Every initial function admits a unique cohyperation.

We define $\langle \ell^\xi \rangle_{\xi \in \mathrm{On}}$ to be the cohyperation of ℓ and call it the hyperlogarithm.

Generalized Icard topologies

We can now define

$$\mathfrak{I}_{\Lambda}^{\Theta} = \langle \Theta, \langle \mathcal{I}_\lambda \rangle_{\lambda < \Lambda} \rangle.$$

Generlized intervals:

$$(\alpha, \beta)_\xi = \{\vartheta : \alpha < \ell^\xi \vartheta < \beta\}.$$

$\mathcal{I}_{1+\lambda}$ is generated by intervals of the form $(\alpha, \beta)_\xi$ for $\xi < \lambda$.

Original Icard space: $\mathfrak{I}_{\omega}^{\varepsilon_0}$

Hyperations

The hyperation of a normal function f is the unique family of normal functions $\langle f^\xi \rangle_{\xi \in \mathrm{On}}$ such that

- $f^1 = f$
- $f^{\xi+\zeta} = f^\xi \circ f^\zeta$
- f^ξ is always normal
- f^ξ is pointwise minimal among all such families of functions.

Normal: Strictly increasing and continuous.

Theorem (DFD, Joosten)

Every normal function admits a unique hyperation.

Computing hyperations

Let $\varphi(\alpha) = \omega^\alpha$ and $e(\alpha) = -1 + \omega^\alpha$.

- $\varphi^3(0) = e^2(1) = \omega^\omega$
- $\varphi^3(1) = e^3(1) = \omega^{\omega^\omega}$
- $\varphi^{\omega^\xi} = \varphi_\xi$ (Veblen functions)
- $\varphi^\omega(0) = e^\omega(1) = \varepsilon_0$
- $\varphi^{\Gamma_0}(0) = e^{\Gamma_0}(1) = \Gamma_0$

Completeness

Theorem (DFD, Joosten)

GLP^0_Λ *is complete for* $\mathfrak{I}^\Theta_\Lambda$ *if and only if* $\Theta > e^\Lambda 1$.

Theorem (DFD)

If Λ *is countable, then* GLP_Λ *is complete for the set of idyllic ambiances over any* $\Theta > e^{1+\Lambda}1$.

Theorem (Aguilera, DFD)

If Λ *is arbitrary, then* GL *is complete for* $\langle \Theta, T_\lambda \rangle$*, provided* $\Theta > e^{1+\Lambda}1$.

Concluding remarks

- Provability logics give rise to an unexpected link between formal theories and point-set topology.
- The study of this link has led to new constructions in proof theory, topology and set theory.
- Many open questions remain (e.g., completeness for canonical ordinal topologies).

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Thank you!