Garside groups and some of their properties

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June 15, 2016

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The defining relations in G and in M generated by X

$$x_1^2 = x_2^2 \qquad x_3^2 = x_4^2$$
$$x_1x_2 = x_3x_4 \qquad x_1x_3 = x_4x_2$$
$$x_2x_4 = x_3x_1 \qquad x_2x_1 = x_4x_3$$

Definition of left divisor

Let M be a monoid and let X, Y be elements in M.

Left divisor

X is a *left divisor* of Y if there is an element T in M such that $Y = XT$.

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$x_1x_2 = x_3x_4$	$x_1x_3 = x_4x_2$
$x_2x_4 = x_3x_1$	$x_2x_1 = x_4x_3$

Definition of Right least common multiple

Right least common multiple - Right lcm

The element Z in M is the *right lcm* of X and Y if:

- X and Y are both left divisors of Z.
- If X and Y are both left divisors of W, then Z is a left divisor of W.

Definition of Right least common multiple

Example 1: Right lcm

The element X_1^2 is the right lcm of X_1 and X_2. Why?
Since in M, $X_1^2 = X_2^2$ and:

- X_1 and X_2 are both left divisors of X_1^2.
- X_1^2 is of minimal length amongst all right common multiples of X_1 and X_2.

Definition of Right least common multiple

Example 2: Right lcm

Let $M = Mon\langle a, b | ab = ba, a^2 = b^2 \rangle$. Then a and b don't have a right lcm !!

Definition of Complement at right

Complement at right of X and Y

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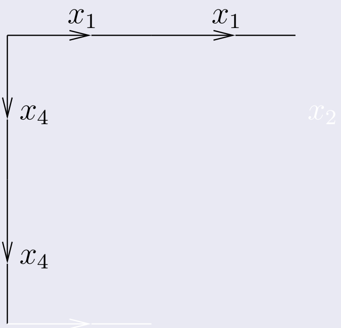
Example 2: Complement at right [Picantin]

Let $M = \mathrm{Mon}\langle X, Y \mid XYYXYXYYX = YXYYXY \rangle$. M is a Garside monoid and $X \setminus Y$ is $YYXYXYYX$.

Right reversing method

lcm of x_1^2 and x_4^2



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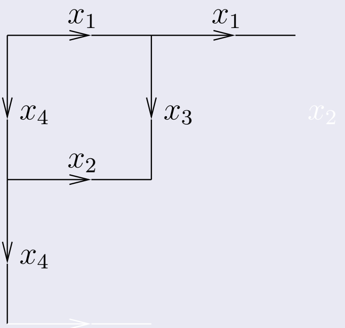

In M

$x_1x_3 = x_4x_2$

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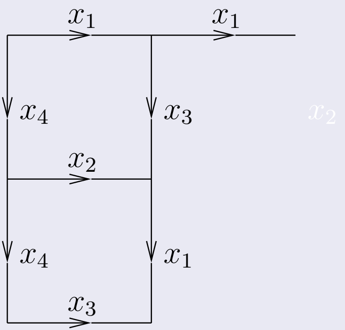

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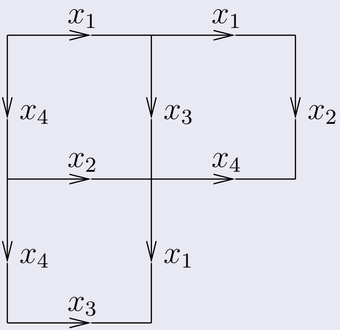

In M

$$
\begin{aligned}
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\end{aligned}
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$$\begin{tikzcd}
\bullet \arrow[r, "x_1"] \arrow[d, "x_4"'] & \bullet \arrow[r, "x_1"] \arrow[d, "x_3"] & \bullet \arrow[d, "x_2"] \\
\bullet \arrow[r, "x_2"] \arrow[d, "x_4"'] & \bullet \arrow[r, "x_4"] \arrow[d, "x_1"] & \bullet \arrow[d, "x_2"] \\
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The lcm is:

$x_1^2x_2^2 = x_1^4 =$
$x_4^2x_3^2 = x_4^4$

$x_1^2 \setminus x_4^2 = x_2^2$

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Since in M, $X_1^4 = X_2^4 = X_3^4 = X_4^4 = \ldots$

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A Garside group is the group of fractions of a Garside monoid.

A criteria for recognizing Garside monoids

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- *M is left and right cancellative.*
- *Any two elements in M with a right common multiple admit a right lcm.*
- *M has a finite generating set S closed under complement, that is if $X, Y \in S$ then the complement $X \setminus Y$ is in S.*

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- G has finite homological dimension [P.Dehornoy and Y.Lafont 2003][R.Charney, J. Meier and K. Whittlesey 2004]

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Examples of Garside groups

- Braid groups [Garside]
- Artin groups of finite type [Deligne, Brieskorn-Saito]
- Torus link groups [Picantin]

Some questions about the Garside groups

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Are all the Garside groups linear groups?

question raised by M.Elder.

The quantum Yang-Baxter equation - QYBE

Let $R : V \otimes V \to V \otimes V$ be a linear operator, where V is a vector space.
The QYBE is the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ of linear transformations on $V \otimes V \otimes V$, where R^{ij} means R acting on the $i-$th and $j-$th components.

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A set-theoretical solution (X, S) of this equation [Drinfeld]

- V is a vector space spanned by a set X.
- R is the linear operator induced by a mapping $S : X \times X \to X \times X$.

Properties of a solution (X, S)

Let $X = \{x_1, ..., x_n\}$ and let S be defined in the following way: $S(i,j) = (g_i(j), f_j(i))$, where $f_i, g_i : X \to X$.

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- (X, S) is non-degenerate $\Leftrightarrow$ f_i and g_i are bijective, $1 \leq i \leq n$.

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- (X, S) is involutive $\Leftrightarrow$ $g_{g_i(j)} f_j(i) = i$ and $f_{f_j(i)} g_i(j) = j$, $1 \leq i, j \leq n$.
- (X, S) is braided $\Leftrightarrow$ $g_i g_j = g_{g_i(j)} g_{f_j(i)}$ and $f_j f_i = f_{f_j(i)} f_{g_i(j)}$ and $f_{g_{f_j(i)}(k)} g_i(j) = g_{f_{g_j(k)}(i)} f_k(j)$, $1 \leq i, j, k \leq n$.

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There are exactly $\frac{n(n-1)}{2}$ *defining relations.*

The example

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The functions that define S

$f_1 = g_1 = f_3 = g_3 = (1, 2, 3, 4)$
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The defining relations in G and in M

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The correspondence between QYBE groups and Garside groups

Theorem (F.C. 2009)

Let (X, S) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

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Assume that $\mathrm{Mon}\langle X \mid R\rangle$ is a **Garside monoid** such that:
- the cardinality of R is $n(n-1)/2$
- each side of a relation in R has length 2.
- if the word $x_i x_j$ appears in R, then it appears only once.

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Then $G = \mathrm{Gp}\langle X \mid R\rangle$ is the structure group of a non-degenerate, involutive and braided solution (X, S), with $\mid X \mid = n$.

Do Coxeter-like quotient groups exist for Garside groups? (1)

The original Coxeter group

There exits a short exact sequence: $1 \to P_n \to B_n \to S_n \to 1$

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There exits a bijection between the elements in the finite quotient group (S_n or finite Coxeter) and the set $\mathrm{Div}(\Delta)$ in B_n or finite-type Artin group.

Do Coxeter-like quotient groups exist for Garside groups? (2)

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Dehornoy's extension 2014: condition (C) can be relaxed

QYBE groups with condition (C) admit Coxeter-like quotient groups

Theorem (F.C and E.Godelle 2013)

Let (X, S) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, S) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$
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- *N is a normal free abelian group of rank n*
- *There exists a bijection between W and $\mathrm{Div}(\Delta)$*
- *W is a finite group of order 2^n*

QYBE groups with condition (C) admit Coxeter-like quotient groups

Theorem (F.C and E.Godelle 2013)

Let (X,S) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X,S) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$
satisfying

- *N is a normal free abelian group of rank n*
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What is condition (C)?

Let $x_i, x_j \in X$. If $S(i,j) = (i,j)$, then $f_i f_j = g_i g_j = Id_X$.

A remark about: QYBE groups admit Coxeter-like quotient groups

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T.Gateva-Ivanova and M. Van den Bergh show G is a Bieberbach group (i.e $G \leq \mathrm{Iso}(\mathbb{R}^n)$).
E.Jespers and J.Okninski call W a IYB group, but there is no connection between W and $\mathrm{Div}(\Delta)$.

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
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Some more definitions

- A left order $\prec$ in a countable group G is *recurrent* if for every $g \in G$ and every finite increasing sequence $h_1 \prec h_2 \prec \ldots \prec h_r$ with $h_i \in G$, there exists $n_i \to \infty$ such that $\forall i,\ h_1 g^{n_i} \prec h_2 g^{n_i} \prec \ldots \prec h_r g^{n_i}$.

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- The set $LO(G)$ cannot be countably infinite (P. Linnell). If G is a countable left-orderable group, $LO(G)$ is either finite, or homeomorphic to the Cantor set, or homeomorphic to a subspace of the Cantor space with isolated points.

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Recurrent left-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$Diffuse $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

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Question from book *Ordering braids*
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  - with all left orders Conradian .
- There exist Garside groups that do not satisfy the unique product property (example of E. Jespers and I. Okninski).

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- B_n satisfy the zero divisor conjecture, as they are left-orderable (P. Dehornoy).

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The end

Thank you!