UNIVALENT UNIVERSES
FOR ELEGANT MODELS OF HOMOTOPY TYPES

by

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Abstract. — We construct a univalent universe in the sense of Voevodsky in some suitable model categories for homotopy types (obtained from Grothendieck's theory of test categories). In practice, this means for instance that, appart from the homotopy theory of simplicial sets, intensional type theory with the univalent axiom can be interpreted in the homotopy theory of cubical sets (with connections or not), or of Joyal's cellular sets.

We recall that any right proper cofibrantly generated model category structure on a (pre)sheaf topos whose cofibrations are exactly the monomorphims is a type theoretic model category in the sense of Shulman [**Shu12**, Definition 2.12]. This means that, up to coherence issues which are solved by Kapulkin, Lumsdaine, and Voevodsky [**KLV12**] and by Lumsdaine and Warren [**LW**], we can interpret Martin-Löf intensional type theory in such a model category. The purpose of these notes is to prove the existence of univalent universes in suitable model categories for (local systems of) homotopy types, (such as simplicial sets or cubical sets): presheaves over a local test category in the sense of Grothendieck which is also elegant in the sense of Bergner and Rezk. We give two constructions. The first one uses Voevodsky's construction in the setting of simplicial sets as well as Shulman's extension to simplicial presheaves over elegant Reedy categories. This has the advantage of giving a rather short proof, but the disadvantage of giving a non-explicit construction. The second one consists to develop the theory of minimal fibrations in the context of presheaves over an Eilenberg-Zilber Reedy category (which is a slightly more restrictive notion than the one of elegant Reedy category), following classical approaches (as in [**GZ67**] for instance), and then to check how robust is Voevodsky's proof. The latter point of view is much more general and also gives another proof of Shulman's construction of univalent universes, at least in the case of simplicial presheaves on Eilenberg-Zilber Reedy categories.

The preparation of these notes started after discussions I had with Thierry Coquand about his joint work with Marc Bezem and Simon Huber on cubical sets [**BCH14**]. His kind invitation to give a talk at the Institut Henri Poincaré in Paris gave the decisive impulse to turn these into actual mathematics.

Contents

1. First construction: reduction to the case of simplicial sets

We write Δ for the category of simplices (the full subcategory of the category of small category whose objects are the non-empty finite totally ordered sets). For an integer $n \geq 0$, we write Δ_n for the presheaf represented by the totally ordered set $\{0,\dots,n\}$. If A is a small category, we write $\widehat{A}$ for the category of presheaves of sets over A, and $s\widehat{A} = \widehat{A \times \Delta}$ for the category of simplicial presheaves over A. We will consider $\widehat{A}$ as a full subcategory of $s\widehat{A}$ (by considering sets as constant presehaves on Δ). Given a cardinal κ, a morphism of presheaves $X \to Y$ over a small category A will be said to have κ-small fibers if, for any object a of A and any section of Y over a, the fiber product $a \times_Y X$, seen as a presheaf over A/a, is κ-accessible.

Let A be an elegant Reedy category (see [**BR13**, Definition 3.2]). Let us consider the *locally constant model structure* on the category of simplicial presheaves over A, that is the left Bousfield localization of the injective model structure on the category of simplicial presheaves by maps of the form $f \times 1_{\Delta_n} : a \times \Delta_n \to b \times \Delta_n$ for any map $f : a \to b$ in A and any integer $n \geq 0$. The fibrant objects of the locally constant model structure are thus the injectively fibrant simplicial presheaves X such that, for any map $a \to b$ in A, the induced morphism $X_b \to X_a$ is a simplicial weak homotopy equivalence. Note that the locally constant model structure is right proper (this follows from [**Cis06**, Theorem 4.4.30 and Corollary 6.4.27], for instance), and is thus a type theoretic model category. We let κ an inaccessible cardinal larger than the cardinal of the set of arrows of A.

Proposition 1.1. — *There exists a univalent universe $\pi : \overline{U} \to U$ in $s\widehat{A}$ which classifies fibrations with κ-small fibers in the locally constant model structure.*

Proof. — Let $p : \overline{V} \to V$ a univalent universe in $s\widehat{A}$ with respect to the injective model structure, classifying fibrations with κ-small fibers (this exists by virtue of a result of Shulman; see [**Shu13**, Theorem 5.6]). Then, as the $(\infty,1)$-category associated to the locally constant model structure is an ∞-topos (because it corresponds to the $(\infty,1)$-category of functors from the ∞-groupoid of A to the $(\infty,1)$-category of ∞-groupoids), one can apply a general result of Rezk [**Lur09**, Theorem 1.6.6.8], and get the existence of a univalent universe up to homotopy with respect to the locally constant model structure; see [**GK12**, Proposition 6.10]. In other words, one can find a univalent fibration between fibrant objects $q : \overline{W} \to W$ of the locally constant model structure such that, for any fibration $f : X \to Y$ of the locally constant model structure, there exists an homotopy pullback square in $s\widehat{A}$ of the following form.

$$\begin{array}{ccc} X & \longrightarrow & \overline{W} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \longrightarrow & W \end{array}$$

Let us choose a (strict) pullback square

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\overline{w}} & \overline{V} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{w} & V \end{array}$$

and choose a factorization of the map w into a trivial cofibration : $W \to U$ followed by a fibration $u : U \to V$ (with respect to the injective model structure). Then U is fibrant for the locally constant model structure. Moreover, if we put $\overline{U} = U \times_V \overline{V}$, then the map $\overline{W} \to \overline{U}$ is a trivial cofibration of the injective model structure (because the latter is right proper) and thus $\overline{U}$ is also fibrant with respect to the locally constant model structure. The projection $\pi : \overline{U} \to U$ being a fibration of the injective model structure between fibrant objects of the locally constant model structure, it is a fibration for the locally constant model structure. By construction, the projection π is a univalent universe up to homomotopy. It remains to prove that any fibration of the locally constant model structure can be obtained as a strict pullback of π. Let $f : X \to Y$ be a fibration of the locally constant model structure (with κ-small fibers). Then the map f can be obtained as a strict pullback of the universe p.

$$\begin{array}{ccc} X & \xrightarrow{x} & \overline{V} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{y} & V \end{array}$$

The classifying map $y : Y \to V$ can be lifted to U up to simplicial homotopy. Indeed, there exists an homotopy pullback square in the locally constant model structure of the following form.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \overline{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\lambda} & U \end{array}$$

This square is also an homotopy pullback square in the injective model structure: the comparison map $X \to Y \times_U \overline{U}$ is a weak equivalence between fibrant and cofibrant objects over Y with respect to the model structure on $s\widehat{A}/Y$ induced by the locally constant model structure, and is thus a simplicial homotopy equivalence (in particular, a weak equivalence of the injective model structure), and the map π being a fibration of the injective model structure (which is right proper), this proves our assertion. Therefore, we obtain the homotopy pullback square below in the injective model structure.

$$\begin{array}{ccc} X & \longrightarrow & \overline{V} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{u\lambda} & V \end{array}$$

The fibration p being univalent, there exists a map $h : \Delta_1 \times Y \to V$ such that $h|_{\{0\}\times Y} = y$ and $h|_{\{1\}\times Y} = u\lambda$. As the map u is a fibration of the injective model structure, the commutative square

$$\begin{array}{ccc} \{1\} \times Y & \xrightarrow{\lambda} & U \\ \downarrow & & \downarrow{\scriptstyle u} \\ \Delta_1 \times Y & \xrightarrow{h} & V \end{array}$$

admits a lift $l : \Delta_1 \times Y \to U$. If we define $y' : Y \to U$ as $y' = l|_{\{0\}\times Y}$, then we have $uy' = h|_{\{0\}\times Y} = y$. In other words, we then obtain a strict pullback square

$$\begin{array}{ccc} X & \xrightarrow{x'} & \overline{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{y'} & U \end{array}$$

which shows that π is a univalent universe for the locally constant model structure in the strict sense. □

1.2. — Assume now that A is a local test category in the sense of Grothendieck; see [**Mal05**, Definition 1.5.2 and Theorem 1.5.6]. Then the category $\widehat{A}$ is endowed with a proper combinatorial model structure with the monomorphisms as cofibrations, while the weak equivalences are the maps $X \to Y$ which induce a simplicial weak homotopy equivalence $N\,A/X \to N\,A/Y$ (where A/X denotes the category of elements of the presheaf X, while N is the nerve functor); see [**Cis06**, Corollary 4.2.18 and Theorem 4.4.30]. We will refer to this model structure (which is in particular type theoretic) as the *Grothendieck model structure*. Moreover, the inclusion functor $\widehat{A} \to s\widehat{A}$ is then a left Quillen equivalence with the locally constant model structure, with right adjoint the evaluation at zero functor. If, moreover, the category A is elegant, the universe $\pi : \overline{U} \to U$ obtained in Proposition 1.1 induces a fibration between fibrant objects $\pi_0 : \overline{U}_0 \to U_0$ in $\widehat{A}$.

Theorem 1.3. — *The fibration $\pi_0 : \overline{U}_0 \to U_0$ is a univalent universe in $\widehat{A}$ which classifies fibrations with κ-small fibers.*

Proof. — Remark that the category Δ is an example of a (local) test category and that the Grothendieck model structure on $\widehat{\Delta}$ coincides with the usual model structure (whose fibrant objects are the Kan complexes). Let $D : \Delta \to \widehat{A}$ be a normalized cosimplicial resolution in the sense of [**Cis06**, Definition 2.3.12] (this always exists:

see [**Cis06**, 2.3.13] for a canonical example). Note that, in particular, D_0 is the terminal presheaf. By virtue of [**Cis06**, Proposition 2.3.27, Corollaries 4.4.10 and 6.4.27], we then have a left Quillen equivalence from the locally constant model structure to the Grothendieck model structure

$$Real_D : s\widehat{A} \to \widehat{A}$$

with right adjoint

$$Sing_D : \widehat{A} \to s\widehat{A}$$

given by the formula

$$Sing_D(X)_n = Hom(D_n, X)$$

(where *Hom* denotes the internal Hom of $\widehat{A}$ with respect to the cartesian product). Let $f : X \to Y$ be a fibration of $\widehat{A}$ with κ-small fibers. Then, the map $Sing_D(f)$ being a fibration of the locally constant model structure with κ-small fibers, there exists a (strict) pullback square of the following form.

$$\begin{array}{ccc} Sing_D(X) & \longrightarrow & \overline{U} \\ {\scriptstyle Sing_D(f)}\downarrow & & \downarrow{\scriptstyle \pi} \\ Sing_D(Y) & \longrightarrow & U \end{array}$$

Evaluating at zero thus gives the (strict) pullback below.

$$\begin{array}{ccc} X & \longrightarrow & \overline{U}_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ Y & \longrightarrow & U_0 \end{array}$$

As evaluating at zero is a right Quillen equivalence, the map π_0 is a univalent universe up to homotopy (because the fibration π has this property), and what precedes thus proves that it is a univalent universe in the strict sense. □

Corollary 1.4. — *For any elegant local test category* A, *the Grothendieck model structure on the category of presheaves of sets* $\widehat{A}$ *supports a model of intensional type theory with dependent sums and products, identity types, and as many univalent universes as there are inaccessible cardinals greater than the set of arrows of* A.

The most well known example of elegant local test categories is provided by the category of simplices Δ, the Grothendieck model structure on $\widehat{\Delta}$ being then the standard model structure of simplicial sets. The preceding corollary is in this case Voevodsky's theorem that univalence holds in simplicial sets; see [**KLV12**]. As the proof of this corollary relies on Voevodsky's results, the interesting examples for us are of course the other elegant (local) test categories. We give a few examples below.

Example 1.5. — The category of cubical sets supports a Grothendieck model structure. Indeed, cubical sets are presheaves on the category $\square$ which is defined as follows. Let $\overline{\square}$ be the full subcategory of the category of sets whose objects are the sets $\square_n = \{0,1\}^n$, $n \geq 0$. This is a (strict) symmetric monoidal category (with cartesian product as tensor product). The category $\square$ is defined as the smallest monoidal subcategory of $\overline{\square}$ generated by the maps

$$\delta^0 : \square_0 \to \square_1$$
$$\delta^1 : \square_0 \to \square_1$$
$$\sigma : \square_1 \to \square_0$$

where δ^e is the constant map with value e. For $n \geq 1$, $1 \leq i \leq n$ and $e = 0,1$, one defines a map

$$\delta_n^{i,e} = 1_{\square_{i-1}} \otimes \delta^e \otimes 1_{\square_{n-i}} : \square_{n-1} \to \square_n\,,$$

and for $n \geq 0$ and $1 \leq i \leq n$, a map

$$\sigma_n^i = 1_{\square_{i-1}} \otimes \sigma \otimes 1_{\square_{n+1-i}} : \square_{n+1} \to \square_n\,.$$

The category $\square$ is an elegant test category; see [**Cis06**, Cor. 8.4.13, Prop. 8.1.24 and 8.4.17].

Moreover, the Grothendieck model structure on $\widehat{\square}$ admits the following explicit description. We will consider the Yoneda embedding as an inclusion, and thus write $\square_n$ for the presheaf represented by $\square_n$ for each $n \geq 0$. The boundary of $\square_n$ is defined as the union of the images of the maps $\delta_n^{i,e}$ for $1 \leq i \leq n$ and $e = 0,1$, and will be denoted by $\partial\square_n$. We also define, for $1 \leq i \leq n$ and $e = 0,1$, the presheaf $\sqcap_n^{i,e}$ as the union in $\square_n$ of the images of the maps $\delta_n^{j,\varepsilon}$ for $(j,\varepsilon) \neq (i,e)$. Then the generating cofibrations (trivial cofibrations) of the Grothendieck model structure on $\widehat{\square}$ are the inclusions

$$\partial\square_n \to \square_n\,,\ n \geq 0 \quad (\sqcap_n^{i,e} \to \square_n\,,\ n \geq 1\,,\ 1 \leq i \leq n\,,\ e = 0,1, \text{respectively.})$$

In other words, the fibrations are precisely the cubical Kan fibrations; see [**Cis06**, Theorem 8.4.38].

Example 1.6. — Cubical sets with connections also give an example of a category of presheaves over an elegant test category. The category $\square^c$ is defined as the smallest monoidal subcategory of $\overline{\square}$ generated by the maps δ^0, δ^1 and σ as above, as well as by the map

$$\gamma : \square_2 = \square_1 \otimes \square_1 \to \square_1$$

given by $\gamma(x,y) = \sup\{x,y\}$. Cubical sets with connections are presheaves of sets on the category $\square^c$. The category $\square^c$ is an elegant test category for the same reasons as for $\square$ (applying [**Cis06**, Prop. 8.1.24 and 8.4.12] for instance). But in fact, as was proved by Maltsiniotis [**Mal09**, Prop. 3.3], it has a better property: it is a strict test category, which means that the weak equivalences of the Grothendieck model category structure on the category of cubical sets with connections are closed under finite products (while this property is known to fail for cubical sets without connections). We define the inclusions

$$\partial\square_n \to \square_n \quad \text{and} \quad \sqcap_n^{i,e} \to \square_n$$

in the same way as for cubical sets above. Note that the category of cubical sets with connections has a natural (non symmetric) closed monoidal structure induced by the monoidal structure on $\square^c$ (using Day convolution). The following theorem is a rather direct consequence of its analog for cubical sets, but we state and prove it explicitely for the convenience of the reader and for future reference.

Theorem 1.7. — *The Grothendieck model category structure on the category of cubical sets with connections is a proper cofibrantly generated monoidal model category with generating cofibrations*

$$\partial\square_n \longrightarrow \square_n \quad , \quad n \geq 0 \, ,$$

and generating trivial cofibrations

$$\sqcap_n^{i,e} \longrightarrow \square_n \quad , \quad n \geq 1 \, , \; 1 \leq i \leq n \, , \; e = 0,1 \, .$$

Furthermore, the class of weak equivalences is closed under finite products.

Proof. — As the category $\square^c$ is elegant, it is clear that the boundary inclusions $\partial\square_n \longrightarrow \square_n$ generate the class of monomorphisms. We have an obvious inclusion functor $u : \square \to \square^c$ which provides an adjunction

$$u_! : \widehat{\square} \rightleftarrows \widehat{\square}^c : u^*$$

where $u_!$ is the left Kan extension of u. The functor $u_!$ is symmetric monoidal and sends $\partial\square_n$ as well as $\sqcap_n^{i,\varepsilon}$ to their versions with connections (it is sufficient to prove this for $\partial\square_n$, which follows from [**Cis06**, Lemma 8.4.21]). Therefore, the functor $u_!$ preserves monomorphisms, and, as any map between representable cubical sets with connections is a weak equivalence, using the last assertion of [**Cis06**, 8.4.27], we see immediately that the functor $u_!$ is a left Quillen functor. In particular, the inclusions $\sqcap_n^{i,\varepsilon} \longrightarrow \square_n$ are trivial cofibrations of cubical sets with connections. The theorem now follows straight away from [**Cis06**, Lemma 8.4.37] which allows to apply [**Cis06**, Lemma 8.2.17 and Theorem 8.2.18] (the last assertion is due to the fact, that $\square^c$ is a strict test category). □

Example 1.8. — For $1 \leq n \leq \omega$, one can consider Joyal's category Θ_n, which is to strict n-categories what Δ is to categories (in particular, $\Theta_1 = \Delta$). The category Θ_n can be thought of as the full subcategory of the category of strict n-categories whose objects are the strict n-categories freely generated on finite pasting schemes of dimension $\leq n$ (Joyal also gave a description of Θ_n^{op} as the category of n-disks). We know from [**BR13**, Cor. 4.5] that Θ_n is elegant, and from [**CM11**, Examples 5.8 and 5.12] that Θ_n is a strict test category. Therefore, the category of presheaves of sets on Θ_n carries a Grothendieck model category structure in which the class of weak equivalences is closed under finite products.

2. Minimal fibrations

2.1. — In this section, we fix once and for all an *Eilenberg-Zilber category* A. This means that A is a Reedy category with the following properties.

(EZ1) Any map in A_- has a section in A.
(EZ2) If two maps in A_- have the same set of sections, then they are equal.

Such a Reedy category is elegant; see [**BR13**, Proposition 4.2]. We also consider given a model category structure on $\widehat{A}$, whose cofibrations precisely are the monomorphisms. Given a representable presheaf a, we denote by $\partial a \to a$ the boundary inclusion (which means that ∂a is the maximal proper subobject of a). These inclusions form a generating set for the class of cofibrations. We also choose an interval I such that the projection $I \times X \to X$ is a weak equivalence for any presheaf X on A (e.g. we can take for I the subobject classifier of the topos $\widehat{A}$). We write $\partial I = \{0\} \amalg \{1\} \subset I$ for the inclusion of the two end-points of I.

2.2. — Let $h : I \times X \to Y$ be an homotopy. For $e = 0, 1$, we write h_e for the composite

$$X = \{e\} \times X \to I \times X \xrightarrow{h} Y .$$

Given a subobject $S \subset X$, we say that h is *constant on* S if the restriction $h|_{I\times S}$ factors through the second projection $I \times S \to S$.

Given a presheaf X, a *section of* X is a map $x : a \to X$ with a a representable presheaf. The *boundary* of such a section x is the map

$$\partial x : \partial a \to a \xrightarrow{x} X .$$

Definition 2.3. — Let X be an object of $\widehat{A}$.

Two sections $x, y : a \to X$ are ∂-*equivalent* if the following conditions are satisfied.

(i) These have the same boundaries: $\partial x = \partial y$.

(ii) There exists an homotopy $h : I \times a \to X$ which is constant on ∂a, and such that $h_0 = x$ and $h_1 = y$.

We write $x \simeq y$ whenever x and y are ∂-equivalent.

A *minimal complex* is a fibrant object S such that, for any two sections $x, y : a \to S$, if x and y are ∂-equivalent, then $x = y$.

A *minimal model* of X is a trivial cofibration $S \to X$ with S a minimal complex.

Proposition 2.4. — *Let X be a fibrant object. The ∂-equivalence relation is an equivalence relation.*

Proof. — This can be proved directly (exercise). Here is a fancy argument. The interval I defines an enrichment of the category $\widehat{A}$ over the category of cubical sets: given two presheaves E and F over A, the cubical set Map(E, F) is defined by

$$\mathrm{Map}(E, F)_n = \mathrm{Hom}_{\widehat{A}}(E \times I^n, F)$$

for $n \geq 0$, with I^n the cartesian product of n copies of I. The functor Map(E, −) is a right Quillen functor to the Grothendieck model category structure on the category of cubical sets (see Example 1.6). Therefore, given a section $x : a \to X$, we can form the following pullback square.

$$\begin{array}{ccc} X(\partial x) & \longrightarrow & \mathrm{Map}(a, X) \\ \downarrow & & \downarrow \\ e & \xrightarrow{\partial x} & \mathrm{Map}(\partial a, X) \end{array}$$

(where e denotes the terminal cubical set). The section x can be seen as a global section of the cubical Kan complex $X(\partial x)$. The relation of cubical homotopy is an equivalence relation on the set of points of $X(\partial x)$. Finally, note that, if $y : a \to X$ is another section such that $\partial x = \partial y$, then $X(\partial x) = X(\partial y)$, from which we deduce right away this proposition. □

Proposition 2.5. — *Let $\varepsilon \in \{0,1\}$, and consider a fibrant object X, together with two maps $h, k : I \times a \to X$, with a representable, such that the restrictions of k and h coincide on $I \times \partial a \cup \{1-\varepsilon\} \times a$. Then the sections h_ε and k_ε are ∂-equivalent.*

Proof. — Put $x = h_\varepsilon$, $y = k_\varepsilon$, and $z = h_{1-\varepsilon} = k_{1-\varepsilon}$. We will use the cubical mapping spaces as in the proof of the preceding proposition. We can think of h and k as two fillings in the commutative square

$$\begin{array}{ccc} \{1-\varepsilon\} & \xrightarrow{z} & \mathrm{Map}(a, X) \\ \big\downarrow & \overset{h \text{ or } k}{\nearrow} & \big\downarrow \\ \square_1 & \xrightarrow{\xi} & \mathrm{Map}(\partial a, X) \end{array}$$

in which ξ correspond to the restriction of h (or k) to $I \times \partial a$. As maps in the homotopy category of pointed cubical sets over $\mathrm{Map}(\partial a, X)$, we must have $h = k$. Therefore, there exists a map $H : \square_2 \to \mathrm{Map}(a, X)$ with the following properties. The restriction of H to $\{0\} \otimes \square_1 = \square_1$ (resp. to $\{1\} \otimes \square_1 = \square_1$) is h (resp. k), the restriction to $\square_1 \otimes \{1-\varepsilon\}$ is constant with value z, and the following diagram commutes.

$$\begin{array}{ccc} \square_2 & \xrightarrow{H} & \mathrm{Map}(a, X) \\ {\scriptstyle 1\otimes\sigma}\big\downarrow & & \big\downarrow \\ \square_1 & \xrightarrow{\xi} & \mathrm{Map}(\partial a, X) \end{array}$$

The restriction of H to $\square \otimes \{\varepsilon\}$ defines a map $l : I \times a \to X$ which is constant on ∂a and such that $l_0 = x$ and $l_1 = y$. □

Lemma 2.6. — *Let X be a fibrant object, and $x_0, x_1 : a \to X$ two degenerate sections. If x_0 and x_1 are ∂-equivalent, then they are equal.*

Proof. — For $\varepsilon = 0, 1$, there is a unique couple $(p_\varepsilon, y_\varepsilon)$, where $p_\varepsilon : a \to b_\varepsilon$ is a split epimorphism in A and $y_\varepsilon : b_\varepsilon \to X$ is a non-degenerate section of X such that $x_\varepsilon = y_\varepsilon p_\varepsilon$. Let us choose a section s_ε of p_ε. As x_0 and x_1 are degenerate and since $\partial x_0 = \partial x_1$, we have $x_0 s_0 = x_1 s_0$ and $x_0 s_1 = x_1 s_1$. On the other hand, we have $y_\varepsilon = x_\varepsilon s_\varepsilon$. We thus have the equalities $y_0 = y_1 p_1 s_0$ and $y_1 = y_0 p_0 s_1$. These imply that the maps $p_\varepsilon s_{1-\varepsilon} : b_{1-\varepsilon} \to b_\varepsilon$ are in A_+ and that b_0 and b_1 have the same dimension. This means that $p_\varepsilon s_{1-\varepsilon}$ is the identity for $\varepsilon = 0, 1$. In other words, we have $b_0 = b_1$ and $y_0 = y_1$, and we also have proven that p_0 and p_1 have the same sections, whence are equal. □

Theorem 2.7. — *Any fibrant object has a minimal model.*

Proof. — Let X be a fibrant object. We choose a representant of each ∂-equivalence class. A section of X which is a chosen representative of its ∂-equivalence class will be called *selected*. By virtue of Lemma 2.6, we may assume that any degenerate section X is selected. Let E be the set of subobjects S of X such that any section of S is selected Then E is not empty: the image of any selected section of the 0-skeleton of X is an element of E. By Zorn's lemma, we can choose a maximal element S of E (with respect to inclusion). Remark that any selected section $x : a \to X$ whose boundary ∂x factors through S must belong to S. Indeed, if x is degenerate, then it factors through ∂a hence through S. Otherwise, let us consider $S' = S \cup \mathrm{Im}(x)$. A non-degenerate section of S' must either factor through S or be precisely equal to x. In any case, such a section must be selected, and the maximality of S implies that $S = S'$.

We will prove that S is a retract of X and that the inclusion $S \to X$ is an I-homotopy equivalence. This will prove that S is fibrant and thus a minimal model of X. Let us write $i : S \to X$ for the inclusion map. Consider triples (T, h, p), where T is a subobject of X which contains S, $p : T \to S$ is a retraction (i.e. the restriction of p to S is the identity), and $h : I \times T \to X$ is a map which is constant on S, and such that h_0 is the inclusion map $T \to X$, while $h_1 = ip$. Such triples are ordered in the obvious way: $(T, h, p) \leq (T', h', p')$ if $T \subset T'$, with $h'|_{I\times T} = h$ and $p'_T = p$. By Zorn's lemma, we can choose a maximal triple (T, h, p). To finish the proof, it is sufficient to prove that $T = X$. In other words, it is sufficient to prove that any non-degenerate section of X belongs to T. Let $x : a \to X$ be a non-degenerate section which does not belong to T. Assume that the dimension of a is minimal for this property. Then ∂x must factor through T, so that, if we define T' to be the union of T and of the image of x in X, then we have a bicartesian square of the following form.

$$\begin{array}{ccc} \partial a & \xrightarrow{\partial x} & T \\ \downarrow & & \downarrow \\ a & \xrightarrow{x} & T' \end{array}$$

We have a commutative square

$$\begin{array}{ccc} \{0\} \times \partial a & \longrightarrow & I \times \partial a \\ \downarrow & & \downarrow {\scriptstyle h(1\times\partial x)} \\ \{0\} \times a & \xrightarrow{x} & X \end{array}$$

If we put $u = (h(1 \times \partial x), x) : I \times \partial a \cup \{0\} \times a \to X$, we can choose a map $H : I \times a \to X$ such that $H_0 = x$, while $H|_{I\times\partial a} = h(1 \times \partial x)$. If we write $y_0 = H_1$, as h_1 factors through S, we see that the boundary ∂y_0 must factor through S. Let y be the selected section ∂-equivalent to y_0. We choose an homotopy $K : I \times a \to X$ which is constant on ∂a and such that $K_0 = y_0$ and $K_1 = y$. By an easy path lifting argument (composing the homotopies H and K), we see that we may choose H such that $y = y_0$. Note that, as y is selected with boundary in S, we must have y in S. We obtain the commutative diagram

$$\begin{array}{ccc} I\times\partial a & \xrightarrow{1_I\times\partial x} & I\times T \\ \downarrow & & \downarrow{\scriptstyle h} \\ I\times a & \xrightarrow{H} & X \end{array}$$

so that, identifying $I\times T'$ with $I\times a\amalg_{I\times\partial a} I\times T$, we define $h' = (H,h) : I\times T' \to X$. Similarly, the commutative diagram

$$\begin{array}{ccc} \partial a & \xrightarrow{\partial x} & T \\ \downarrow & & \downarrow{\scriptstyle p} \\ a & \xrightarrow{y} & S \end{array}$$

defines a map $p' = (y,p) : T' = a\amalg_{\partial a} T \to X$. It is clear that the triple (T', h', p') extends (T, h, p), which leads to a contradiction. □

Proposition 2.8. — *Let X be a fibrant object and $i : S \to X$ a minimal resolution of X. Consider a map $r : X \to S$ such that $ri = 1_S$ (such a map always exists because i is a trivial cofibration with fibrant domain). Then the map r is a trivial fibration.*

Proof. — There exists a map $h : I\times X \to X$ which is constant on S and such that $h_0 = ir$ and $h_1 = 1_X$: we can see i as a trivial cofibation between cofibrant and fibrant objects in the model category of objects under S, and r is then an inverse up to homotopy in this relative situation. Consider the commutative diagram below.

$$\begin{array}{ccc} \partial a & \xrightarrow{u} & X \\ \downarrow & & \downarrow{\scriptstyle r} \\ a & \xrightarrow{v} & S \end{array}$$

We want to prove the existence of a map $w : a \to X$ such that $w_{|\partial a} = u$ and $rw = v$. As X is fibrant, there exists a map $k : I\times a \to X$ whose retriction to $I\times\partial a$ is $h(1_I\times u)$, while $k_0 = iv$. Let us put $w = k_1$. Then

$$\partial w = w_{|\partial a} = (h(1_I\times u))_1 = h_1 u = u\,.$$

It is thus sufficient to prove that $v = rw$. But k and $h(1_I\times w)$ coincide on $I\times\partial a\cup\{1\}\times a$ and thus, by virtue of Proposition 2.5, we must have $k_0 \sim (h(1_I\times w))_0$. In other words, we have $iv \sim irw$. As $ri = 1_S$, this implies that $v \sim rw$, and, by minimality of S, that $v = rw$. □

Lemma 2.9. — *Let X be a minimal complex and $f : X \to X$ a map which is I-homotopic to the identity. Then f is an isomorphism.*

Proof. — Let us choose once and for all a map $h : I\times X \to X$ such that $h_0 = 1_X$ and $h_1 = f$. We will prove that the map $f_a : X_a \to X_a$ is bijective by induction on the dimension d of a. If a is of dimension < 0, there is nothing to prove because there is no such a. Assume that the map $f_b : X_b \to X_b$ is bijective for any object b of dimension

$< d$. Consider two sections $x, y : a \to \mathrm{X}$ such that $f(x) = f(y)$. Then, as f is injective in dimension lesser than d, the equations

$$f\,\partial x = \partial f(x) = \partial f(y) = f\,\partial y$$

imply that $\partial x = \partial y$. On the other hand, we can apply Proposition 2.5 to the maps $h(1_{\mathrm{I}} \times x)$ and $h(1_{\mathrm{I}} \times y)$ for $\varepsilon = 0$, and we deduce that $x \sim y$. As X is minimal, this proves that $x = y$. It remains to prove the surjectivity. Let $y : a \to \mathrm{X}$ be a section. For any map $\sigma : b \to a$ in A such that b is of degree lesser than d, there is a unique section $x_\sigma : b \to \mathrm{X}$ such that $f(x_\sigma) = \sigma^*(y) = y\sigma$. This implies that there is a unique map $z : \partial a \to \mathrm{X}$ such that $fz = \partial y$. The map $\mathrm{I} \times \partial a \xrightarrow{1_{\mathrm{I}} \times z} \mathrm{I} \times a \xrightarrow{h} \mathrm{X}$, together with the map $\{1\} \times a = a \xrightarrow{y} \mathrm{X}$, define a map $\varphi = (h(1_{\mathrm{I}} \times z), x)$, and we can choose a filling k in the diagram below.

$$\begin{array}{ccc}
\mathrm{I} \times \partial a \cup \{1\} \times a & \xrightarrow{\varphi} & \mathrm{X} \\
\big\downarrow & \overset{k}{\nearrow} & \\
\mathrm{I} \times a & &
\end{array}$$

Let us put $x = k_0$. Then $\partial x = z$, and thus $\partial f(x) = \partial y$. Applying Proposition 2.5 to the maps k and $h(1_{\mathrm{I}} \times x)$ for $\varepsilon = 1$, we conclude that $f(x) \sim y$. The object X being a minimal complex, this proves that $f(x) = y$. □

Proposition 2.10. — *Let* X *and* Y *be two minimal complexes. Then any weak equivalence* $f : \mathrm{X} \to \mathrm{Y}$ *is an isomorphism of presheaves.*

Proof. — If $f : \mathrm{X} \to \mathrm{Y}$ is a weak equivalence, as both x and Y are cofibrant and fibrant, there exists $g : \mathrm{Y} \to \mathrm{X}$ such that fg and gf are homotopic to the identify of Y and of X, respectively. By virtue of the preceding lemma, the maps gf and fg must be isomorphisms, which imply right away that f is an isomorphism. □

Theorem 2.11. — *Let* X *be a fibrant object of* $\widehat{\mathrm{A}}$. *The following conditions are equivalent.*

(i) *The object* X *is a minimal complex.*
(ii) *Any trivial fibration of the form* $\mathrm{X} \to \mathrm{S}$ *is an isomorphim.*
(iii) *Any trivial cofibration of the form* $\mathrm{S} \to \mathrm{X}$, *with* S *fibrant, is an isomorphism.*
(iv) *Any weak equivalence* $\mathrm{X} \to \mathrm{S}$, *with* S *a minimal complex, is an isomorphism.*
(v) *Any weak equivalence* $\mathrm{S} \to \mathrm{X}$, *with* S *a minimal complex, is an isomorphism.*

Proof. — It follows immediately from Proposition 2.10 that condition (i) is equivalent to condition (iv) as well as to condition (v). Therefore, condition (v) implies condition (iii): if $i : \mathrm{S} \to \mathrm{X}$ is a trivial cofibration with S fibrant and X minimal, then S must be minimal as well, so that i has to be an isomorphism. Let us prove that condition (iii) implies condition (ii): any trivial fibration $p : \mathrm{X} \to \mathrm{S}$ admits a section $i : \mathrm{S} \to \mathrm{X}$ which has to be a trivial cofibration with fibrant domain, and thus an isomorphism. It is now sufficient to prove that condition (ii) implies condition (i). By virtue of Theorem 2.7, there exists a minimal model of X, namely a trivial cofibration $\mathrm{S} \to \mathrm{X}$ with S a minimal complex. This cofibration has a retraction which, by virtue of Proposition 2.8, is a trivial fibration. Condition (ii) implies that S is isomorphic to X, and thus that X is minimal as well. □

Definition 2.12. — A fibration $p : X \to Y$ in $\widehat{A}$ is *minimal* if it is a minimal complex as an object of $\widehat{A}/Y = \widehat{A/Y}$ for the induced model category structure (whose, weak equivalences, fibrations and cofibrations are the maps which have the corresponding property in $\widehat{A}$, by forgetting the base).

Proposition 2.13. — *The class of minimal fibrations is stable by pullback.*

Proof. — Consider a pullback square

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

in which p' is a minimal fibration. Let $x, y : a \to X$ two global sections which are ∂-equivalent over Y (i.e. ∂-equivalent in X, seen as a fibrant object of $\widehat{A/Y}$). Then $u(x)$ and $u(y)$ are ∂-equivalent in X′ over Y′, and thus $u(x) = u(y)$. As $p(x) = p(y)$, this means that $x = y$. In other words, p is a minimal fibration. □

Everything we proved so far about minimal complexes has its counterpart in the language of minimal fibrations. Let us mention the properties that we will use later.

Theorem 2.14. — *For any fibration* $p : X \to Y$, *there exists a trivial fibration* $r : X \to S$ *and a minimal fibration* $q : S \to Y$ *such that* $p = qr$.

Proof. — By virtue of Theorem 2.7 applied to p, seen as a fibrant presheaf over A/Y, there exists a trivial cofibration $i : S \to X$ such that $q = p_{|S} : S \to Y$ is a minimal fibration. As both X and S are fibrant (as presheaves over A/Y), the embedding i is a strong deformation retract, so that, by virtue of Proposition 2.8 (applied again in the context of presheaves over A/Y), there exists a trivial fibration $r : X \to S$ such that $ri = 1_S$, and such that $qr = p$. □

Remark 2.15. — In the factorisation $p = qr$ given by the preceding theorem, q is necessarily a retract of p. Therefore, if p belongs to a class of maps which is stable under retracts, the minimal fibration must have the same property. Similarly, as r is a trivial fibration, if p belongs to a class which is defined up to weak equivalences, then so does q. This means that this theorem can be used to study classes of fibrations which are more general than classes of fibrations of model category structures.

Proposition 2.16. — *For any minimal fibrations* $p : X \to Y$ *and* $p' : X' \to Y$, *any weak equivalence* $f : X \to X'$ *such that* $p'f = p$ *is an isomorphism.*

Proof. — This is a reformulation of Proposition 2.10 in the context of presheaves over A/Y. □

Lemma 2.17. — *For any cofibration $v : Y \to Y'$ and any trivial fibration $p : X \to Y$, there exists a trivial fibration $p' : X' \to Y'$ and a pullback square of the following form.*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

Proof. — We use Joyal's trick. The pullback functor $v^* : \widehat{A}/Y' \to \widehat{A}/Y$ has a left adjoint $v_!$ and a right adjoint v_*. We see right away that $v^* v_!$ is isomorphic to the identity (i.e. that $v_!$ is fully faithful), so that, by transposition, $v^* v_*$ is isomorphic to the identity as well. Moreover, the functor v_* preserves trivial fibrations because its left adjoint v^* preserves monomorphisms. We define the trivial fibration p' as $v_*(p : X \to Y)$. □

Proposition 2.18. — *Consider a commutative diagram of the form*

$$\begin{array}{ccccc} X_0 & \xrightarrow{w} & X_1 & \xrightarrow{i_1} & X'_1 \\ & {\scriptstyle p_0}\searrow & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p'_1} \\ & & Y & \xrightarrow{j} & Y' \end{array}$$

in which p_0, p_1 and p'_1 are fibrations, w is a weak equivalence, j is a cofibration, and the square is cartesian. Then there exists a cartesian square

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & X'_0 \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p'_0} \\ Y & \xrightarrow{j} & Y' \end{array}$$

in which p'_0 is a fibration, as well as a weak equivalence $w' : X'_0 \to X'_1$ such that $p'_1 w = p'_0$ and $i_1 w = w' i_0$.

Proof. — By virtue of Theorem 2.14, we can choose a trivial fibration $r'_1 : X'_1 \to S'$ and a minimal fibration $q' : S \to Y'$ such that $p'_1 = q' r'_1$. Let us write $S = Y \times_{Y'} S'$, and $k : S \to S'$ for the second projection. The canonical map $r_1 : X_1 \to S$ is a trivial fibration (being the pullback of such a thing), and the projection $q : S \to Y$ is a minimal fibration by Proposition 2.13. We have thus a factorisation $p_1 = q r_1$. Moreover, the map $r_0 = r_1 w$ is a trivial fibration. To see this, let us choose a minimal model $u : T \to X_0$ of the fibration p_0. Then the map $r_1 w u$ is a weak equivalence between minimal fibrations and is thus an isomorphism by Proposition 2.16. This means that r_0 is isomorphic to a retraction of the map u, and is therefore a trivial fibration by Proposition 2.8. The diagram we started from now has the following form.

$$\begin{array}{ccccc} X_0 & \xrightarrow{w} & X_1 & \xrightarrow{i_1} & X'_1 \\ & \searrow^{r_0} & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r'_1} \\ & & S & \xrightarrow{k} & S' \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ & & Y & \xrightarrow{j} & Y' \end{array}$$

Moreover, both squares are cartesian. This means that we can replace j by k. In other words, without loss of generality, it is sufficient to prove the proposition in the case where p_0, p_1 and p'_1 are trivial fibrations. Under these additional assumptions, we obtain a cartesian square

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & X'_0 \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p'_0} \\ Y & \xrightarrow{j} & Y' \end{array}$$

in which p'_0 is a trivial fibration by Lemma 2.17. The lifting problem

$$\begin{array}{ccccc} X_0 & \xrightarrow{w} & X_1 & \xrightarrow{i_1} & X'_1 \\ \downarrow{\scriptstyle i_0} & & {\scriptstyle w'} \nearrow & & \downarrow{\scriptstyle p'_1} \\ X'_0 & & \xrightarrow{p'_0} & & Y' \end{array}$$

has a solution because i_0 is a cofibration and p'_1 a trivial fibration. Moreover, any lift w' must be a weak equivalence because both p'_0 and p'_1 are trivial fibrations. □

Remark 2.19. — One can prove the preceding proposition in a much greater generality, without using the theory of minimal fibrations: the proof of [**KLV12**, Theorem 3.4.1] can be carried out in any topos endowed with a model category structure whose cofibrations precisely are the monomorphisms.

Lemma 2.20. — *Assume that the model category structure on $\widehat{A}$ is right proper. For any trivial cofibration $v : Y \to Y'$ and any minimal fibration $p : X \to Y$, there exists a minimal fibration $p' : X' \to Y'$ and a pullback square of the following form.*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

Proof. — Let us factor the map vp as a trivial cofibration $u' : X \to X''$ followed by a fibration $p'' : X'' \to Y'$. By virtue of Theorem 2.14, we can factor p'' into a trivial fibration $q : X'' \to X'$ followed by a minimal fibration $p' : X' \to Y'$. We thus get a commutative square

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

with $u = qu'$. The projection $Y \times_{Y'} X' \to Y$ is a minimal fibration (Proposition 2.13). On the other hand, the model category structure being right proper, the comparison map $X \to Y \times_{Y'} X'$ is a weak equivalence over Y. Therefore, Proposition 2.16 implies that this comparison map is an isomorphism, and thus that this commutative square is cartesian. □

Proposition 2.21. — *Assume that the model category structure on $\widehat{A}$ is right proper. For any trivial cofibration $v : Y \to Y'$ and any fibration $p : X \to Y$, there exists a fibration $p' : X' \to Y'$ and a pullback square of the following form.*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

Proof. — By virtue of Theorem 2.14, there exists a factorisation of p as $p = qr$ with r a trivial fibration and q a minimal fibration. We can extend q and then r, using Lemmata 2.20 and 2.17 successively. □

3. Second construction: extension of Voevodsky's proof

3.1. — Let A be a small category. A class of presheaves $\mathscr{C}$ on A is *saturated by monomorphisms* is it satisfies the following properties.

(a) The empty presheaf is in $\mathscr{C}$.
(b) For any pushout square

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

in which X, X′ and Y are in $\mathscr{C}$ and i is a monomorphism, then Y′ is in $\mathscr{C}$.
(c) For any well ordered set α and any functor $X : \alpha \to \widehat{A}$ such that the natural map $X_i \to X_j$ is a monomorphism for any $i < j$ in α, if X_i is in $\mathscr{C}$ for any $i \in \alpha$, then $\varinjlim_{i\in\alpha} X_i$ is in $\mathscr{C}$.
(d) Any retract of an object in $\mathscr{C}$ is in $\mathscr{C}$.

Proposition 3.2. — *If* A *is an elegant Reedy category, any class $\mathscr{C}$ of presheaves on* A *which is saturated by monomorphisms and contains the representable presheaves contains all the presheaves on* A.

Proof. — As the boundary inclusions $\partial a \to a$ form a generating family for the class of monomorphisms, it is sufficient to prove that the boundaries ∂a belong to $\mathscr{C}$ for any representable presheaf a. We proceed by induction on the dimension d of a. If $d \leq 0$, then the only proper subobject of a is the empty presheaf, which belongs to $\mathscr{C}$ by definition. If $d > 0$, consider the set E of proper subobjects K of a which are in $\mathscr{C}$. It is clear that E is non-empty because the empty subobject of a is an element of E. Note that a subobject K of a is proper if and only if the identity of a is not contained in the set of non-degenerate sections of K. Therefore, proper subobjects are stable by arbitrary unions in a. Since any totally ordered set has a cofinal well ordered subset, by Zorn's lemma, there exists a maximal element K in E. Let us prove that $\mathrm{K} = \partial a$. If not, then let us choose a section $u : b \to \partial a$ of ∂a which does not belong to K. We may choose u such that the dimension of b is minimal with respect to this property. Thus u must be non-degenerate, and the boundary $\partial u : \partial b \to \partial a$ must factor through K. We then have a pushout square of the following form, where $\mathrm{L} \subset \partial a$ denotes the union of K and of the image of u in ∂a.

$$\begin{array}{ccc} \partial b & \xrightarrow{\partial u} & \mathrm{K} \\ \downarrow & & \downarrow \\ b & \xrightarrow{u} & \mathrm{L} \end{array}$$

By induction, ∂b is in $\mathscr{C}$, and as b is representable, it belongs to $\mathscr{C}$. As K is in $\mathscr{C}$ as well, L must be in $\mathscr{C}$, which gives a contradiction. Therefore, the boundary $\partial a = \mathrm{K}$ is in $\mathscr{C}$. □

Proposition 3.3. — *If* A *is an elegant Reedy category, then any* A*-localizer is regular. In other words, for any model category structure on* $\widehat{\mathrm{A}}$ *whose cofibrations precisely are the monomorphisms, and for any presheaf* X *on* A*, the family of sections of* X *exhibit* X *as an homotopy colimit of representable presheaves (see* [**Cis06**, Définition 3.4.13]*).*

Proof. — This follows right away from [**Cis06**, Exemple 3.4.10, Proposition 3.4.22] and from the preceding proposition. □

A consequence of the preceding proposition is that, in the category of presheaves on an elegant Reedy category, the notion of weak equivalence is local in the following sense.

Corollary 3.4. — *Let* A *be an elegant Reedy category, and assume that* $\widehat{\mathrm{A}}$ *is endowed with a model category structure whose cofibrations precisely are the monomorphisms. Consider a commutative triangle of the form*

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathrm{S} & \end{array}$$

in which both p and q are fibrations. Then f is a weak equivalence if and only if, for any representable presheaf a and any section $s : a \to \mathrm{S}$, *the induced morphism* $a \times_{\mathrm{S}} \mathrm{X} \to a \times_{\mathrm{S}} \mathrm{Y}$ *is a weak equivalence.*

Proof. — This obviously is a necessary condition (because, the pullback functor along $a \to \mathrm{S}$ is a right Quillen functor from $\widehat{\mathrm{A}}/\mathrm{S}$ to $\widehat{\mathrm{A}}/a$, and thus, by Ken Brown's lemma, preserves weak equivalences between fibrant objects). The converse follows from the preceding proposition and from [**Cis06**, Corollaire 3.4.47]. One can also give a more elementary proof using directly Proposition 3.2 as follows. Replacing A by A/S, we may assume without loss of generality that S is the terminal object. The class of presheaves Z such that the induced map $\mathrm{Z} \times \mathrm{X} \to \mathrm{Z} \times \mathrm{Y}$ is a weak equivalence is saturated and contains the representable presheaves, so that it contains the terminal object by Proposition 3.2. □

3.5. — Let A be an elegant Reedy category, and assume that $\widehat{\mathrm{A}}$ is endowed with a proper model category structure whose cofibrations precisely are the monomorphisms. Given two fibrations $p : \mathrm{X} \to \mathrm{S}$ and $q : \mathrm{Y} \to \mathrm{S}$, we will write

$$\mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y}) \to \mathrm{S} \tag{3.5.1}$$

for the map corresponding to the internal Hom of $\widehat{\mathrm{A}/\mathrm{S}}$ through the equivalence $\widehat{\mathrm{A}}/\mathrm{S} \simeq \widehat{\mathrm{A}/\mathrm{S}}$. In other words, given a map $\mathrm{T} \to \mathrm{S}$, morphisms from T to $\mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y})$ over S correspond bijectively to morphisms of the form $\mathrm{T} \times_{\mathrm{S}} \mathrm{X} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{Y}$ over T. Given any map $\mathrm{T} \to \mathrm{S}$, we have the canonical pullback square below.

$$\begin{array}{ccc} \mathit{Hom}_{\mathrm{T}}(\mathrm{T} \times_{\mathrm{S}} \mathrm{X}, \mathrm{T} \times_{\mathrm{S}} \mathrm{Y}) & \longrightarrow & \mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathrm{T} & \longrightarrow & \mathrm{S} \end{array} \tag{3.5.2}$$

Remark that, if p and q are fibrations, then the map (3.5.1) is a fibration as well: as our model category structure is right proper, the pullback functor along p is a left Quillen functor from $\widehat{\mathrm{A}}/\mathrm{S}$ to $\widehat{\mathrm{A}}/\mathrm{S}$, so that its right adjoint is a right Quillen functor, hence preserves fibrant objects. In this case, we define a subpresheaf $\mathit{Eq}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y}) \subset \mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y})$ by requiring that, for any map $\mathrm{T} \to \mathrm{S}$, a section of $\mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y})$ over T factors through $\mathit{Eq}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y})$ if and only if the corresponding map $\mathrm{T} \times_{\mathrm{S}} \mathrm{X} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{Y}$ is a weak equivalence. This actually defines a subpresheaf of $\mathit{Hom}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y})$ over S precisely because of Corollary 3.4.

Proposition 3.6. — *Under the assumptions of paragraph 3.5, for any fibrations* $\mathrm{X} \to \mathrm{S}$ *and* $\mathrm{Y} \to \mathrm{S}$, *the structural map* $\mathit{Eq}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y}) \to \mathrm{S}$ *is a fibration.*

Proof. — Consider a commutative square of the following form

$$\begin{array}{ccc} \mathrm{K} & \xrightarrow{k} & \mathit{Eq}_{\mathrm{S}}(\mathrm{X}, \mathrm{Y}) \\ {\scriptstyle j}\downarrow & & \downarrow \\ \mathrm{L} & \longrightarrow & \mathrm{S} \end{array}$$

in which the map j is a trivial cofibration. As the structural map $Hom_S(X,Y) \to S$ is a fibration, one can find a lift l in the solid commutative square below.

$$\begin{array}{ccc} K & \xrightarrow{k} & Hom_S(X,Y) \\ {\scriptstyle j}\downarrow & \overset{l}{\nearrow} & \downarrow \\ L & \longrightarrow & S \end{array}$$

The equation $lj = k$ means that we have a pullback square of the form

$$\begin{array}{ccc} K \times_S X & \xrightarrow{k} & K \times_S Y \\ {\scriptstyle j\times_S 1_X}\downarrow & & \downarrow{\scriptstyle j\times_S 1_X} \\ L \times_S X & \xrightarrow{l} & L \times_S Y \end{array}$$

in which the two vertical maps are weak equivalence (by right properness) as well as k. Therefore, the map $l : L \to Hom_S(X,Y)$ factors through $Eq_S(X,Y)$, which produces a lift in the commutative square we started from. □

Definition 3.7. — Let A be a small category. A *strongly proper* model category structure on $\widehat{A}$ is a proper model category structure whose cofibrations precisely are the monomorphisms, and such that the notion of fibration is local over A in the following sense: any morphism of presheaves $p : X \to Y$, the following conditions are equivalent.

(i) The map $p : X \to Y$ is a fibration.
(ii) For any representable presheaf a and any section $a \to Y$, the first projection $a \times_Y X \to a$ is a fibration.

3.8. — Let A be a small category, and assume that $\widehat{A}$ is endowed with a strongly proper model category structure. Consider an infinite regular cardinal κ which is greater than the cardinal of the set of arrows of A. We will use the construction of Hofmann and Streicher [**Str14**] of a universe of fibrations with κ-small fibers. We denote by Set_κ some full subcategory of the category of sets of cardinal lesser than κ, such that, for any cardinal $\alpha < \kappa$, there exists a set of cardinal α in Set_κ. Let W be the presheaf whose set of sections over an object a of A is the set of functors $(A/a)^{op} \to Set_\kappa$. Given a map $f : a \to b$ in A, the precomposition with the induced functor $A/a \to A/b$ defines the corresponding map $f^* : W_b \to W_a$. Similarly, let $\overline{W}$ be the presheaf whose set of sections consists of couples (X, s), where X is a presheaf on A/a with values in Set_κ (i.e. an element of W_a), and s is a global section of X. Forgetting the sections defines a mophism of presheaves $\rho : \overline{W} \to W$. Note that, since we have canonical equivalences $\widehat{A}/a \simeq \widehat{A/a}$, any element X of W_a determines canonically a morphism $p_X : X \to a$; in fact, one can identify the elements of W_a as the data of a presheaf X on A with values in Set_κ, together with a map $X \to a$, as well as with specified cartesian squares

$$\begin{array}{ccc} f^*(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ b & \xrightarrow{f} & a \end{array}$$

for any morphism $f : b \to a$ in A (with $f^*(X)$ a presheaf with values in Set_κ). We define the presheaf U as the subpresheaf of W whose sections over a representable presheaf a are the elements X such that the corresponding morphism $p_X : X \to a$ is a fibration. We define $\overline{U}$ by the following pullback square.

$$\begin{array}{ccc} \overline{U} & \longrightarrow & \overline{W} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\ U & \longrightarrow & W \end{array}$$

The following lemma is straightforward.

Lemma 3.9. — *Assume that there are two cartesian squares of the following form.*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{x} & \overline{W} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \rho} \\ Y & \xrightarrow{y} & W \end{array}$$

If p' has κ-small fibers and v is a monomorphism, then there exists a map $y' : Y' \to W$ such that $y'v = y$. In particular, the case where Y is empty tells us that a morphism of presheaves over A has κ-small fibers if and only if it can be obtained as a pullback of the morphism $\rho : \overline{W} \to W$.

Proposition 3.10. — *Under the assumptions of paragraph 3.8, let $p : X \to Y$ be map with κ-small fibers, and choose a classifying cartesian square.*

$$\begin{array}{ccc} X & \longrightarrow & \overline{W} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \rho} \\ Y & \xrightarrow{y} & W \end{array}$$

Then p is a fibration if and only if the classifying map y factors through $U \subset W$.

Proof. — This is a reformulation of the last part of the definition of strong properness. □

Corollary 3.11. — *Assume that there are two cartesian squares of the following form.*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{v} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{x} & \overline{U} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{y} & U \end{array}$$

If p' is a fibration with κ-small fibers and if v is a monomorphism, then there exists a map $y' : \mathrm{Y}' \to \mathrm{U}$ such that $y'v = y$.

Theorem 3.12. — *Under the assumptions of paragraph 3.8, if A is an Eilenberg-Zilber category and if κ is an inaccessible cardinal, the map $\pi : \overline{\mathrm{U}} \to \mathrm{U}$ is a univalent fibration between fibrant objects which classifies fibrations with κ-small fibers.*

Proof. — The fact that U is fibrant is a reformulation of proposition 2.21 and of Corollary 3.11. The fact that π is a fibration follows straight away from Proposition 3.10. Let $\pi_0 : \overline{\mathrm{U}}_0 = \overline{\mathrm{U}} \times \mathrm{U} \to \mathrm{U} \times \mathrm{U}$ and $\pi_1 : \overline{\mathrm{U}}_1 = \mathrm{U} \times \overline{\mathrm{U}} \to \mathrm{U} \times \mathrm{U}$ be the pullbacks of the fibration π along the first and second projection of $\mathrm{U} \times \mathrm{U}$ to U, respectively. By virtue of Proposition 3.6, we have a canonical fibration

$$(s,t) : Eq_{\mathrm{U}\times\mathrm{U}}(\overline{\mathrm{U}}_0, \overline{\mathrm{U}}_1) \to \mathrm{U} \times \mathrm{U}\,.$$

As the pullback of the fibration π_i along the diagonal $\mathrm{U} \to \mathrm{U} \times \mathrm{U}$ is canonically isomorphic to π for $i = 0, 1$, the fibration (s,t) has a canonical section over the diagonal $\mathrm{U} \to \mathrm{U} \times \mathrm{U}$, which provides a morphism

$$id : \mathrm{U} \to Eq_{\mathrm{U}\times\mathrm{U}}(\overline{\mathrm{U}}_0, \overline{\mathrm{U}}_1)$$

such that $(s,t)id$ is the diagonal (or equivalently, such that $sid = tid = 1_{\mathrm{U}}$). The property that π is univalent means that this map id is a weak equivalence. It is thus sufficient to prove that the fibration

$$t : Eq_{\mathrm{U}\times\mathrm{U}}(\overline{\mathrm{U}}_0, \overline{\mathrm{U}}_1) \to \mathrm{U}$$

is a trivial fibration. Consider a cofibration $j : \mathrm{Y} \to \mathrm{Y}'$. Then a commutative square

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\ \xi\ } & Eq_{\mathrm{U}\times\mathrm{U}}(\overline{\mathrm{U}}_0, \overline{\mathrm{U}}_1) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle t} \\ \mathrm{Y}' & \xrightarrow{\ \xi'\ } & \mathrm{U} \end{array}$$

consists essentially of a commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{X}_0 & \xrightarrow{\ w\ } & \mathrm{X}_1 & \xrightarrow{\ i_1\ } & \mathrm{X}'_1 \\ & {\scriptstyle p_0}\searrow & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p'_1} \\ & & \mathrm{Y} & \xrightarrow{\ j\ } & \mathrm{Y}' \end{array}$$

in which p_0, p_1 and p'_1 are fibrations (with κ-small fibers), w is a weak equivalence, and the square is cartesian (where the triple (p_0, w, p_1) correponds to ξ, the fibration p'_1 corresponds to ξ', and the cartesian square to the equation $\xi' j = t\xi$). Therefore, Proposition 2.18 together with Corollary 3.11 give a map $\zeta : \mathrm{Y}' \to Eq_{\mathrm{U}\times\mathrm{U}}(\overline{\mathrm{U}}_0, \overline{\mathrm{U}}_1)$ such that $t\zeta = \xi'$ and $\zeta j = \xi$. □

Proposition 3.13. — *Let A and B be a small categories, with B having the structure of an elegant Reedy category. Assume that $\widehat{\mathrm{A}}$ is endowed with a strongly proper model category structure, and consider the associated injective model category structure on the category* *of presheaves on* B *with values in* $\widehat{A}$ *(for which cofibrations and weak equivalences are defined termwise with respect to evaluation at objects of* B*). Then this defines a strongly proper model category structure on* $\widehat{A \times B}$*.*

Proof. — Given a presheaf X on $C = A \times B$ and a presheaf F on B, we obtain a presheaf $\mathrm{Hom}_{\widehat{B}}(F, X)$ on A whose sections over an object a are given by

$$\mathrm{Hom}_{\widehat{B}}(F, X)_a = \mathrm{Hom}_{\widehat{B}}(F, X_a)$$

where X_a is the presheaf on B obtained by evaluating X at a. Given an object b of B, the Yoneda lemma for presheaves over B gives the identification

$$X_b = \mathrm{Hom}_{\widehat{B}}(b, X)\,,$$

and we set

$$M_b X = \mathrm{Hom}_{\widehat{B}}(\partial b, X)\,.$$

The injective model category structure on the category of presheaves on B with values in $\widehat{A}$ coincides with the Reedy model structure. This means that a morphism of presheaves $p : X \to Y$ on C is a fibration if and only if, for any object b of B, the induced map

$$q_b : X_b \to Y_b \times_{M_b Y} M_b X$$

is a fibration of $\widehat{A}$. Consider a map $p : X \to Y$ such that, for any representable presheaf c on C and any section $c \to Y$, the canonical map $c \times_Y X \to c$ is a fibration. Let b be an object of B. We want to prove that q_b is a fibration of $\widehat{A}$. But the model category structure on $\widehat{A}$ being strongly proper, it sufficient to prove the existence of lifts in commutative diagrams of the from

$$\begin{array}{ccccc}
K & & \longrightarrow & & X_b \\
{\scriptstyle j}\big\downarrow & & \nearrow & & \big\downarrow{\scriptstyle q_b} \\
L & \longrightarrow & a & \longrightarrow & Y_b \times_{M_b Y} M_b X
\end{array}$$

in which j is a trivial cofibration and a is a representable presheaf on A. Such a lifting problem is equivalent to a lifting problem of the form

$$\begin{array}{ccccc}
L \boxtimes \partial b \cup K \boxtimes b & & \longrightarrow & & X \\
\big\downarrow & & \nearrow & & \big\downarrow{\scriptstyle p} \\
L \boxtimes b & \longrightarrow & a \boxtimes b & \longrightarrow & Y
\end{array}$$

in which, for any presheaves E and F on A and B respectively, we write $E \boxtimes F$ for the cartesian product of the pullbacks of E and F along the projections $A \times B \to A$ and $A \times B \to B$ respectively. But $c = a \boxtimes b$ is then a representable presheaf on C, and we are reduced to a lifting problem of the following form.

$$\begin{array}{ccc}
L \boxtimes \partial b \cup K \boxtimes b & \longrightarrow & c \times_Y X \\
\big\downarrow & \nearrow & \big\downarrow \\
L \boxtimes b & \longrightarrow & c
\end{array}$$

As the projection $c \times_{Y} X \to c$ is a fibration, this achieves the proof. □

Examples of Eilenberg-Zilber test categories are the simplicial category Δ, the cubical category $\square$ (1.5), the cubical category with connections $\square^c$ (1.6), and Joyal's categories Θ_n for $1 \leq n \leq \omega$ (1.8). Once we are here, we can get a much more explicit proof of Corollary 1.4, at least in the case of Eilenberg-Zilber local test categories: we apply Theorem 3.12 to the Grothendieck model structure on the category of presheaves on an Eilenberg-Zilber local test category (see 1.2), which is meaningful for we have the following result.

Theorem 3.14. — *Let* A *be an elegant local test category. The Grothendieck model category structure on* $\widehat{A}$ *is strongly proper.*

Proof. — We already know that the Grothendieck model category structure is proper (this does not use the property that A is an elegant Reedy category and is true for any local test category). Let $p : X \to Y$ be morphism such that, for any section $a \to Y$, the induced map $a \times_{Y} X \to a$ is a fibration. We want to prove that p is a fibration. Note that A/Y is a again an elegant local test category and that, under the identification $\widehat{A}/Y \simeq \widehat{A/Y}$, the Grothendieck model structure on $\widehat{A/Y}$ coincides with the model category structure on $\widehat{A}/Y$ induced by the Grothendieck model structure on $\widehat{A}$. Therefore, replacing A by A/Y, we may assume that Y is the terminal object. We thus have a presheaf X on A such that $a \times X \to a$ is a fibration for any representable presheaf a, and we want to prove that X is fibrant. We will consider the minimal model structure on $\widehat{A}$ (corresponding to the minimal A-localizer; see [**Cis06**, Théorème 1.4.3]), and will prove first that X is fibrant for the minimal model structure. Let us choose an interval I such that the projection $Z \times I \to Z$ belongs to the minimal A-localizer (e.g. I might be the subobject classifier; see [**Cis06**, 1.3.9]). By virtue of [**Cis06**, Remarque 1.3.15, Proposition 1.3.36], we have to check that the map from X to the terminal presheaf has the right lifting property with respect to the inclusions of the form $I \times \partial a \cup \{\varepsilon\} \times a \to I \times a$ for any representable presheaf a and $\varepsilon = 0, 1$. But lifting problems of shape

$$\begin{array}{ccc} I \times \partial a \cup \{\varepsilon\} \times a & \xrightarrow{u} & X \\ \downarrow & \overset{v}{\nearrow} & \\ I \times a & & \end{array}$$

are in bijection with lifting problems of the form

$$\begin{array}{ccc} I \times \partial a \cup \{\varepsilon\} \times a & \xrightarrow{(p,u)} & a \times X \\ \downarrow & \overset{(pr_2,v)}{\nearrow} & \downarrow \\ I \times a & \xrightarrow{pr_2} & a \end{array}$$

where $p : I \times \partial a \cup \{\varepsilon\} \times a \to a$ is the restriction of the second projection $I \times a \to a$. Hence X is fibrant for the minimal model structure (because the projection being a fibration for the Grothendieck model structure, it is also a fibration for the minimal model structure). By virtue of Proposition 3.3 and [**Cis06**, Proposition 6.4.26], the Grothendieck model category structure is the left Bousfield localization of the minimal model category structure on $\widehat{A}$ by the set of maps between representable presheaves. It is thus sufficient to prove that, for any map between representable presheaves $u : a \to b$, the map

$$\mathrm{Map}(b, X) \to \mathrm{Map}(a, X)$$

is a weak equivalence (where the mapping spaces are constructed from the minimal model structure). The latter is equivalent to the map

$$\mathrm{Map}_b(b, b \times X) \to \mathrm{Map}_b(a, b \times X)$$

where Map_b denotes the mapping space with respect to the model category structure on $\widehat{A}/b$ induced by the minimal model structure on $\widehat{A}$. The projection from $b \times X$ to b being a fibration of the Grothendieck model category structure, we deduce that X is local with respect to the left Bousfield localization by the maps between representable presheaves, and thus that X is fibrant in the Grothendieck model category structure. □

Remark 3.15. — The preceding theorem, together with Proposition 3.13, gives a new proof, in the case of Eilenberg-Zilber categories, of Shulman's result that the injective model structure for simplicial presheaves supports a model of intensional type theory with univalent universes [**Shu13**, Theorem 5.6].

Remark 3.16. — The proof of Theorem 3.14 would have been much easier if we would have exhibited a generating set of trivial cofibrations of the Grothendieck model category structure of the form $K \to a$ with a representable. This happens in practice (e.g. horn inclusions for simplicial sets, open boxes for cubical sets), but I don't know if this is true for a general elegant local test category. In fact, there is a candidate for a counter-example. Let Ω be the category of finite rooted trees considered by Weiss and Moerdijk for their notion of dendroidal sets. In a short note in preparation (in collaboration with D. Ara and I. Moerdijk), it will be shown that Ω is a test category. Although Ω is not an elegant Reedy category for the simple reason that it is not a Reedy category, for any normal dendroidal set X (i.e. such that, for any tree T, the automorphisms of T act freely on the set of sections of X over T), the category Ω/X is an Eilenberg-Zilber Reedy category. As a consequence, given any reasonable model of the operad E_∞ (that is any weakly contractible normal dendroidal set), the category Ω/E_∞ is an Eilenberg-Zilber test category. On the other hand the dendroidal horns are well understood: the right lifting property with respect to dendroidal inner horns define the ∞-operads, which are models for topological (coloured) symmetric operads. But if we consider the right lifting property with respect to all dendroidal horns, Bašić and Nikolaus [**BN12**] have shown that we obtain models of infinite loop spaces. This means that we have an Eilenberg-Zilber test category Ω/E_∞ for which there really is no natural candidate for a generating family of trivial cofibrations with representable codomains.

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