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Dendroidal sets as models for homotopy operads

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Abstract

The homotopy theory of ∞-operads is defined by extending Joyal's homotopy theory of ∞-categories to the category of dendroidal sets. We prove that the category of dendroidal sets is endowed with a model category structure, the fibrant objects of which are the ∞-operads (that is, dendroidal inner Kan complexes). This extends the theory of ∞-categories in the sense that the Joyal model category structure on simplicial sets, the fibrant objects of which are the ∞-categories, is recovered from the model category structure on dendroidal sets by simply slicing over the monoidal unit.

Introduction

The notion of dendroidal set is an extension of that of simplicial set, suitable for defining and studying nerves of (coloured) operads in the same way as nerves of categories feature in the theory of simplicial sets. It was introduced by one of the authors and Weiss in [**12**]. As explained in that paper, the category $d\mathcal{S}et$ of dendroidal sets carries a symmetric monoidal structure, which is closely related to the Boardman–Vogt tensor product for operads [**2**]. There is also a corresponding internal Hom of dendroidal sets. The category of dendroidal sets extends the category $s\mathcal{S}et$ of simplicial sets, in the precise sense that there are adjoint functors (left adjoint on the left)

$$i_! : s\mathcal{S}et \rightleftarrows d\mathcal{S}et : i^*$$

with good properties. In particular, the functor $i_!$ is strong monoidal and fully faithful, and identifies $s\mathcal{S}et$ with the slice category $d\mathcal{S}et/\eta$, where η is the unit of the monoidal structure on $d\mathcal{S}et$. (In fact, this adjunction is an open embedding of toposes.)

Using these adjoint functors $i_!$ and i^*, we can say more precisely how various constructions and results from the theory of simplicial sets extend to that of dendroidal sets. For example, the nerve functor $N : \mathcal{C}at \to s\mathcal{S}et$ and its left adjoint, which we denote by τ, naturally extend to a pair of adjoint functors

$$\tau_d : d\mathcal{S}et \rightleftarrows \mathcal{O}perad : N_d,$$

which plays a central role in our work.

The goal of this paper is to lay the foundations for a homotopy theory of dendroidal sets and "∞-operads" (or "operads-up-to-homotopy", or "quasi-operads"), extending the simplicial theory of ∞-categories (or "quasi-categories"), which has recently been developed by Joyal, Lurie and others. Our main result is the existence of a Quillen closed model structure on the category of dendroidal sets, having the following properties.

1. This Quillen model structure on $d\mathcal{S}et$ is symmetric monoidal (in the sense of [**8**]) and left proper.
2. The fibrant objects of this model structure are precisely the ∞-operads.
3. The induced model structure on the slice category $d\mathcal{S}et/\eta$ is precisely the Joyal model structure on simplicial sets [**10**, **11**].

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The existence of such a model structure was suggested in [**12**]. The ∞-operads referred to in property (2) are the dendroidal analogues of the ∞-categories forming the fibrant objects in the Joyal model structure. They are the dendroidal sets satisfying a lifting condition analogous to the weak Kan condition of Boardman–Vogt, and were introduced in [**12**, **13**] under the name "(dendroidal) inner Kan complexes". The dendroidal nerve of every operad is such an ∞-operad; conversely, intuitively speaking, ∞-operads are operads in which the composition of operations is only defined up to homotopy, in a way that is associative up to homotopy. For example, the homotopy coherent nerve of a symmetric monoidal topological category is an ∞-operad. The theory of ∞-operads contains the theory of ∞-categories, as well as the theory of symmetric monoidal ∞-categories and of operads in them. The theory of ∞-operads is also likely to be of use in studying the notion of ∞-category enriched in a symmetric monoidal ∞-category (for example, the various notions of A_∞-categories, dg categories, weak n-categories, etc.).

The proof of our main theorem is based on three sources: First of all, we use the general methods of constructing model structures on presheaf categories developed in [**3**] (we only use the first chapter and Section 8.1 of that book, which are both elementary). Secondly, we use some fundamental properties of dendroidal inner Kan complexes proved in [**13**]. And finally, we use some important notions and results from Joyal's seminal paper [**9**]: namely, the theory of join operations and the notions of left or right fibration of simplicial sets. Apart from these sources, our proof is entirely self-contained. In particular, we do not use the Joyal model structure in our proof, but instead deduce this model structure as a corollary, as expressed in property (3) above.

It is known that there are several (Quillen) equivalent models for ∞-categories: one is given by a left Bousfield localization of the Reedy model structure on simplicial spaces and has as its fibrant objects Rezk's complete Segal spaces; another is given by a Dwyer–Kan style model structure on topological categories established by Bergner, in which all objects are fibrant. The equivalence of these approaches is extensively discussed in Lurie's book [**11**]; see also [**1**, **10**]. It is natural to ask whether analogous models exist for ∞-operads. In the subsequent paper [**4**] and in a forthcoming paper entitled 'Dendroidal sets and simplicial operads', we will show that this is indeed the case. We will prove there that the model structure on dendroidal sets described above is equivalent to a model structure on topological operads in which all objects are fibrant (the weak equivalences being defined as the fully faithful and essentially surjective maps, as in [**6**, Definition 12.2]), as well as to a model structure on dendroidal spaces, the fibrant objects of which are "dendroidal complete Segal spaces". The models for ∞-categories just mentioned as well as the equivalences between them will again emerge simply by slicing over suitable unit objects of the respective monoidal structures. Together these model categories fit into a row of Quillen equivalences

$$\begin{tikzcd} sOperad \arrow[r, "\sim"] & dSet \arrow[r, "\sim"] & dSpaces \\ sCat \arrow[u] \arrow[r, "\sim"] & sSet \arrow[u] \arrow[r, "\sim"] & sSpaces, \arrow[u] \end{tikzcd}$$

in which the vertical arrows are (homotopy) full embeddings.

This paper is organized as follows. In the first section, we recall the basics about dendroidal sets. In Section 2, we state the main results of this paper: the existence of a model category structure on the category of dendroidal sets, the fibrant objects of which are the ∞-operads, as well as its main properties. In Section 3, we construct this model structure through rather formal arguments. At this stage, it is clear, by construction, that the fibrant objects are ∞-operads, but the converse is not obvious. Sections 4 and 5 provide the tools to prove that any ∞-operad is fibrant, following the arguments known to hold in the case of simplicial sets for the theory of ∞-categories. More precisely, in Section 4, we develop a dendroidal analogue of Joyal's join operations, and prove a generalization of a theorem of Joyal, which ensures a right lifting property for inner Kan fibrations with respect to certain non-inner horns, under an additional hypothesis of weak invertibility of some 1-cells. In Section 5, we construct and examine a subdivision of cylinders of trees in terms of dendroidal horns. Finally, in Section 6, we prove that any ∞-operad is fibrant, and study some of the good properties of fibrations between ∞-operads. This is done by proving an intermediate result, which is important by itself: a morphism of diagrams in an ∞-operad is weakly invertible if and only if it is locally (that is, objectwise) weakly invertible (this is where Sections 4 and 5 have their roles to play).

We also added two appendices, which are independent of the rest of this paper. In Appendix A, we study the join operations on leaves (whereas in Section 4, we studied join operations on roots), and in Appendix B, we study another subdivision of cylinders of trees. In fact, these appendices can be used to provide another proof of our main results: Section 6 might have been written using Appendices A and B instead of Sections 4 and 5, respectively, without any changes (except, sometimes, replacing the evaluation at 1 by the evaluation at 0, whenever necessary). However, these appendices are not formal consequences of the rest of these notes, and it will be useful to have these kinds of results available for further work on the subject.

1. *Dendroidal sets*

1.1. Recall from [**12**] the category of trees Ω. The objects of Ω are non-empty non-planar trees with a designated root, and, given two trees T and T', a map from T to T' is a morphism of the corresponding operads, which, in these notes, we will denote by T and T' again. Hence, by definition, the category of trees is a full subcategory of the category of operads. Recall that the category dSet of *dendroidal sets* is defined as the category of presheaves of sets on the category of trees Ω. Given a tree T, we denote by $\Omega[T]$ the dendroidal set represented by T.

Let 0 be the tree with only one edge, and set $\eta = \Omega[0]$. Then the category $\Omega/0$ identifies canonically with the category Δ of simplices, so that the category dSet/η is canonically equivalent to the category sSet of simplicial sets. The corresponding functor

$$i : \Delta \longrightarrow \Omega, \quad [n] \longmapsto i[n] = n \tag{1.1}$$

is fully faithful and its image is a sieve in Ω. This functor i induces an adjunction

$$i_! : \mathit{sSet} \rightleftarrows \mathit{dSet} : i^* \tag{1.2}$$

(where $i_!$ is the left Kan extension of i). Under the identification $\mathit{sSet} = \mathit{dSet}/\eta$, the functor $i_!$ is simply the forgetful functor from dSet/η to dSet. The functor $i_!$ is fully faithful and makes sSet into an open subtopos of dSet. In other words, if there is a map of dendroidal sets $X \to Y$ with Y a simplicial set, then X has to be a simplicial set as well.

We also recall the pairs of adjoint functors

$$\tau : \mathit{sSet} \rightleftarrows \mathit{Cat} : N \quad \text{and} \quad \tau_d : \mathit{dSet} \rightleftarrows \mathit{Operad} : N_d, \tag{1.3}$$

where N and N_d denote the nerve functors from the category of categories to the category of simplicial sets and from the category of (symmetric coloured) operads to the category of dendroidal sets.

The category of operads is endowed with a closed symmetric monoidal structure: the tensor product is defined as the Boardman–Vogt tensor product; see [**12**, Section 5]. This defines canonically a unique closed symmetric monoidal structure on the category of dendroidal sets such that the functor τ_d is symmetric monoidal, and such that, for two trees T and S, we have

$$\Omega[T] \otimes \Omega[S] = N_d(T \otimes_{\mathrm{BV}} S),$$

where $T \otimes_{\mathrm{BV}} S$ is the Boardman–Vogt tensor product of operads. We will denote internal Hom objects by $\mathcal{H}om(A, X)$ or by X^A.

Note that the functor $i_! : s\mathcal{S}et \to d\mathcal{S}et$ is a symmetric monoidal functor, if we consider $s\mathcal{S}et$ with its closed cartesian monoidal structure. In more concrete terms, for any simplicial sets K and L, we have a canonical isomorphism

$$i_!(K \times L) \simeq i_!(K) \otimes i_!(L).$$

This turns the category of dendroidal sets into a simplicial category; given two dendroidal sets A and X, we will write $\mathcal{h}om(A, X)$ for $i^*(\mathcal{H}om(A, X))$, the simplicial set of maps from A to X.

1.2. We recall here from [**12**] the different kinds of faces of trees in Ω.

Let T be a tree.

If e is an inner edge of T, we will denote by T/e the tree obtained from T by contracting e. We then have a canonical inclusion

$$\partial_e : T/e \longrightarrow T. \tag{1.4}$$

A map of type (1.4) is called an *inner face* of T.

If v is a vertex of T, with the property that all but one of the edges incident to v are outer, then we will denote by T/v the tree obtained from T by removing the vertex v and all the outer edges incident to it. We then have a canonical inclusion

$$\partial_v : T/v \longrightarrow T. \tag{1.5}$$

If $T = C_n$ is the nth corolla, that is, the tree with exactly one vertex v and n leaves, $n \geqslant 0$, then, there are exactly $n + 1$ inclusions of shape

$$\eta = T/v \longrightarrow T, \tag{1.6}$$

with n of these corresponding to each of the leaves of C_n, whereas another one corresponds to the root. A map of type (1.5) or (1.6) is called an *outer face* of T.

A map of type (1.4), (1.5) or (1.6), will be called an *elementary face* of T.

We define $\partial\Omega[T]$ as the union in $d\mathcal{S}et$ of all the images of elementary face maps $\Omega[T/x] \to \Omega[T]$. We thus have, by definition, an inclusion

$$\partial\Omega[T] \longrightarrow \Omega[T]. \tag{1.7}$$

Maps of shape (1.7) are called *boundary inclusions*. The image of a face map ∂_x will sometimes be denoted by $\partial_x(T)$ for short.

We will call *faces* the maps of Ω obtained, up to an isomorphism, as compositions of elementary faces. It can be checked that faces are exactly the monomorphisms in Ω; see [**12**, Lemma 3.1].

1.3. A monomorphism of dendroidal sets $X \to Y$ is *normal* if for any tree T, any non-degenerate dendrex $y \in Y(T)$ that does not belong to the image of $X(T)$ has a trivial stabilizer $\mathrm{Aut}(T)_y \subset \mathrm{Aut}(T)$. A dendroidal set X is *normal* if the map $\emptyset \to X$ is normal.

There is an obvious functor e from Ω to the category of finite sets, which sends a tree T to the set $e(T)$ of its edges. This functor preserves monomorphisms. One deduces from this property that, for any trees T and S, the group $\mathrm{Aut}(S)$ acts freely on $\Omega[T](S) = \mathrm{Hom}_\Omega(S, T)$. In particular, for any tree T, the dendroidal set $\Omega[T]$ is normal.

PROPOSITION 1.4. *The class of normal monomorphisms is stable by pushouts, transfinite compositions and retracts. Furthermore, this is the smallest class of maps in $d\mathcal{S}et$ closed under* *pushouts and transfinite compositions, and containing the boundary inclusions* $\partial\Omega[T] \to \Omega[T]$, $T \in \Omega$.

Proof. The category Ω satisfies the axioms of [**3**, 8.1.1] (see [**14**, Chapter 2]), and the definition of normal dendroidal set given here is equivalent to the one given in [**3**, 8.1.23]. This proposition thus follows from [**3**, Proposition 8.1.35]. □

PROPOSITION 1.5. *A monomorphism of dendroidal sets* $X \to Y$ *is normal if and only if for any tree* T, *the action of* $\mathrm{Aut}(T)$ *on* $Y(T) - X(T)$ *is free.*

Proof. It is easily seen that the class of monomorphisms satisfying the above property is stable by pushouts and transfinite compositions, and contains the boundary inclusions $\partial\Omega[T] \to \Omega[T]$. It thus follows from the preceding proposition that any normal monomorphism has this property. But it is also obvious that any monomorphism with this property is normal. □

COROLLARY 1.6. *A dendroidal set* X *is normal if and only if for any tree* T, *the action of the group* $\mathrm{Aut}(T)$ *on* $X(T)$ *is free.*

COROLLARY 1.7. *Given any map of dendroidal sets* $X \to Y$, *if* Y *is normal, then* X *is normal.*

COROLLARY 1.8. *Any monomorphism* $i : A \to B$ *with* B *normal is a normal monomorphism.*

PROPOSITION 1.9. *Let* $A \to B$ *and* $X \to Y$ *be two normal monomorphisms. The induced map*

$$A \otimes Y \amalg_{A\otimes X} B \otimes X \longrightarrow B \otimes Y$$

is a normal monomorphism.

Proof. As the class of normal monomorphisms is generated by the boundary inclusions, it is sufficient to check this property in this case; see, for example, [**8**, Lemma 4.2.4].

Consider now two trees S and T. We have to show that the map

$$\partial\Omega[S] \otimes \Omega[T] \amalg_{\partial\Omega[S]\otimes\partial\Omega[T]} \Omega[S] \otimes \partial\Omega[T] \longrightarrow \Omega[S] \otimes \Omega[T]$$

is a normal monomorphism. But as $\Omega[S] \otimes \Omega[T]$ is the dendroidal nerve of the Boardman–Vogt tensor product of S and T, which is Σ-free, it is a normal dendroidal set. Hence we are reduced to prove that the above map is a monomorphism. This latter property is equivalent to the fact that the commutative square

$$\begin{tikzcd}
\partial\Omega[S] \otimes \partial\Omega[T] \arrow[r] \arrow[d] & \partial\Omega[S] \otimes \Omega[T] \arrow[d] \\
\Omega[S] \otimes \partial\Omega[T] \arrow[r] & \Omega[S] \otimes \Omega[T]
\end{tikzcd}$$

is a pullback square in which any map is a monomorphism. As the nerve functor preserves pullbacks, this reduces to the following property: for any elementary faces $S/x \to S$ and $T/y \to$

T, the commutative square

$$\begin{tikzcd} S/x \otimes_{\mathrm{BV}} T/y \arrow[r] \arrow[d] & S/x \otimes_{\mathrm{BV}} T \arrow[d] \\ S \otimes_{\mathrm{BV}} T/y \arrow[r] & S \otimes_{\mathrm{BV}} T \end{tikzcd}$$

is a pullback square of monomorphisms in the category of operads. This is an elementary consequence of the definitions involved. □

1.10. Under the assumptions of Proposition 1.9, we shall write $A \otimes Y \cup B \otimes X$ instead of $A \otimes Y \amalg_{A\otimes X} B \otimes X$.

2. *Statement of main results*

In this section, we state the main results of this paper.

2.1. Recall from [**13**, Section 5] the notion of *inner horn*. Given an inner edge e in a tree T, we get an inclusion

$$\Lambda^e[T] \longrightarrow \Omega[T], \tag{2.1}$$

where $\Lambda^e[T]$ is obtained as the union of all the images of elementary face maps, which are distinct from the face $\partial_e : T/e \to T$. The maps of shape (2.1) are called *inner horn inclusions*.

A map of dendroidal sets is called an *inner anodyne extension* if it belongs to the smallest class of maps stable by pushouts, transfinite composition and retracts, and containing the inner horn inclusions.

A map of dendroidal sets is called an *inner Kan fibration* if it has the right lifting property with respect to the class of inner anodyne extensions (or, equivalently, to the set of inner horn inclusions).

A dendroidal set X is an *inner Kan complex* if the map from X to the terminal dendroidal set is an inner Kan fibration. We will also call inner Kan complexes *∞-operads*. For example, for any operad $\mathcal{P}$, the dendroidal set $N_d(\mathcal{P})$ is an ∞-operad; see [**13**, Proposition 5.3]. In particular, for any tree T, the dendroidal set $\Omega[T]$ is an ∞-operad. For a simplicial set K, its image by $i_!$ is an ∞-operad if and only if K is an *∞-category* (that is, K is a quasi-category in the sense of [**9**]).

A map of dendroidal sets will be called a *trivial fibration* if it has the right lifting property with respect to normal monomorphisms.

Note that the small object argument implies that we can factor any map of dendroidal sets into a normal monomorphism followed by a trivial fibration (or, into an inner anodyne extension followed by an inner Kan fibration, respectively).

Remark 2.2. A morphism between normal dendroidal sets is a trivial fibration if and only if it has the right lifting property with respect to monomorphisms: this follows immediately from Corollaries 1.7 and 1.8.

2.3. Recall the naive model structure on the category of operads [**12**, **14**]: the weak equivalences are the equivalences of operads, that is, the maps $f : \mathcal{P} \to \mathcal{Q}$ that are fully faithful and essentially surjective: for any $(n+1)$-tuple of objects $(a_1, \dots, a_n, a)$ in $\mathcal{P}$, f induces a bijection

$$\mathcal{P}(a_1, \ldots, a_n; a) \longrightarrow \mathcal{Q}(f(a_1), \ldots, f(a_n); f(a)),$$

and any object of $\mathcal{Q}$ is isomorphic to the image of some object in $\mathcal{P}$. The fibrations are *operadic fibrations*, that is, the maps $f : \mathcal{P} \to \mathcal{Q}$ such that, given any isomorphism $\beta : b_0 \to b_1$ in $\mathcal{Q}$, and any object a_1 in $\mathcal{P}$ such that $f(a_1) = b_1$, there exists an isomorphism $\alpha : a_0 \to a_1$ in $\mathcal{P}$, such that $f(\alpha) = \beta$.

This model structure is closely related to the naive model structure on $\mathcal{Cat}$ (for which the weak equivalences are the equivalences of categories). In fact, the latter can be recovered from the one on operads by slicing over the unit operad (which is also the terminal category). The fibrations of the naive model structure on $\mathcal{Cat}$ will be called the *categorical fibrations*.

THEOREM 2.4. *The category of dendroidal sets is endowed with a model category structure for which the cofibrations are the normal monomorphisms, the fibrant objects are the ∞-operads, and the fibrations between fibrant objects are the inner Kan fibrations between ∞-operads, the image by τ_d of which is an operadic fibration. The class of weak equivalences is the smallest class of maps of dendroidal sets W that satisfies the following three properties.*

(a) (*'2 out of 3 property'*) *In any commutative triangle, if two maps are in W, then so is the third.*

(b) *Any inner anodyne extension is in W.*

(c) *Any trivial fibration between ∞-operads is in W.*

Proof. See Proposition 3.12 for a formal construction of this model category structure. To go further, the fibrations between fibrant objects remain to be described as well as the weak equivalences. Note that, for an ∞-operad X, the category underlying $\tau_d(X)$ is canonically isomorphic to $\tau i^*(X)$; see Section 4.1. From there, we see that Theorem 6.10 and Corollary 6.11 complete the proof. □

PROPOSITION 2.5. *The adjunction $\tau_d : d\mathcal{Set} \rightleftarrows \mathcal{Operad} : N_d$ is a Quillen pair. Moreover, the two functors τ_d and N_d both preserve weak equivalences. In particular, a morphism of operads is an equivalence of operads if and only if its dendroidal nerve is a weak equivalence.*

Proof. See Section 6.17. □

PROPOSITION 2.6. *The model category structure of Theorem 2.4 has the following additional properties.*

(a) *It is left proper.*

(b) *It is cofibrantly generated* (*it is even combinatorial*).

(c) *It is symmetric monoidal.*

Proof. See Propositions 3.12 and 3.17. □

PROPOSITION 2.7. *For any normal dendroidal set A and any ∞-operad X, the set of maps $\mathrm{Hom}_{\mathrm{Ho}(d\mathcal{Set})}(A, X)$ is canonically identified with the set of isomorphism classes of objects in the category $\tau\mathcal{hom}(A, X)$.*

Proof. See Proposition 6.20. □

Corollary 2.8. *Let $f : X \to Y$ be a morphism of ∞-operads. The following conditions are equivalent.*

(a) *The map $f : X \to Y$ is a weak equivalence.*

(b) *For any normal dendroidal set A, the map*

$$\tau_d \mathcal{H}om(A, X) \longrightarrow \tau_d \mathcal{H}om(A, Y)$$

is an equivalence of operads.

(c) *For any normal dendroidal set A, the map*

$$\tau hom(A, X) \longrightarrow \tau hom(A, Y)$$

is an equivalence of categories.

Proof. Remember that, by definition (and any ∞-operad being fibrant), the map f is a weak equivalence if and only if, for any normal dendroidal set A, the induced map

$$\mathrm{Hom}_{\mathrm{Ho}(dSet)}(A, X) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(dSet)}(A, Y)$$

is bijective. This corollary is thus a direct consequence of Propositions 2.5 and 2.7 and of the fact that the model category structure on $dSet$ is monoidal. □

Corollary 2.9. *Let $u : A \to B$ be a morphism of normal dendroidal sets. The following conditions are equivalent.*

(a) *The map $u : A \to B$ is a weak equivalence.*

(b) *For any ∞-operad X, the map*

$$\tau_d \mathcal{H}om(B, X) \longrightarrow \tau_d \mathcal{H}om(A, X)$$

is an equivalence of operads.

(c) *For any ∞-operad X, the map*

$$\tau hom(B, X) \longrightarrow \tau hom(A, X)$$

is an equivalence of categories.

Proof. The fibrant objects of $dSet$ are exactly the ∞-operads. Hence, the map $u : A \to B$ is a weak equivalence if and only if, for any ∞-operad X, the map

$$\mathrm{Hom}_{\mathrm{Ho}(dSet)}(B, X) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(dSet)}(A, X)$$

is bijective. We conclude the proof using the same arguments as in the proof of Corollary 2.8. □

Corollary 2.10 (Joyal). *The category of simplicial sets is endowed with a left proper, cofibrantly generated, symmetric monoidal model category structure for which the cofibrations are the monomorphisms, the fibrant objects are the ∞-categories and the fibrations between fibrant objects are the inner Kan fibrations between ∞-categories, the image by τ of which is a categorical fibration.*

Proof. The model category structure on $dSet$ induces a model category structure on $dSet/\eta \simeq sSet$; see also Remark 3.14 for $B = \eta$. □

Remark 2.11. Note that the functor $i_! : sSet \to dSet$ is fully faithful and symmetric monoidal. Moreover, for any simplicial sets A and X, we have $hom(i_!(A), i_!(X)) = X^A$. We deduce from this that the induced map

$$\mathrm{Hom}_{\mathrm{Ho}(sSet)}(A, X) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(dSet)}(i_!(A), i_!(X))$$

is bijective (where Ho($s\mathcal{S}et$) denotes the homotopy category of the Joyal model structure, given by Corollary 2.10). As a consequence, we also have formally the simplicial analogues of Corollaries 2.7–2.9.

As the reader can check immediately from the results above, the model category structure on $d\mathcal{S}et$ is a simplicial model category, if we consider the Joyal model category structure on $s\mathcal{S}et$. However, the model category structure on $d\mathcal{S}et$ is *not* a simplicial model category in the usual sense.

3. *Construction of an abstract model category for ∞-operads*

This section is devoted to the construction of a model category structure on $d\mathcal{S}et$. The construction is relatively formal and uses very little of the theory of dendroidal sets. By definition, we will have that any fibrant object of this model category is an ∞-operad. The proof of the converse (any ∞-operad is fibrant) is the raison d'être of the next sections.

PROPOSITION 3.1. *Let $A \to B$ and $X \to Y$ be an inner anodyne extension and a normal monomorphism, respectively. The induced map*

$$A \otimes Y \cup B \otimes X \longrightarrow B \otimes Y$$

is an inner anodyne extension.

Proof. Using [**3**, Corollary 1.1.8], we see that it is sufficient to check this property when $A \to B$ is an inner horn inclusion and when $X \to Y$ is a boundary inclusion. This proposition thus follows from [**13**, Proposition 9.2]. □

3.2. We denote by J the nerve of the contractible groupoid with two objects 0 and 1 (that is, J is the nerve of the fundamental groupoid of $\Delta[1]$). We will write $J_d = i_!(J)$ for the corresponding dendroidal set.

A morphism of dendroidal sets is a *J-anodyne extension* if it belongs to the smallest class of maps that contains the inner anodyne extensions and the maps

$$\partial\Omega[T] \otimes J_d \cup \Omega[T] \otimes \{e\} \longrightarrow \Omega[T] \otimes J_d \quad T \in \Omega, e = 0, 1,$$

and that is closed under pushouts, transfinite compositions and retracts.

A morphism of dendroidal sets will be called a *J-fibration* if it has the right lifting property with respect to J-anodyne extensions.

A dendroidal set X is J-fibrant if the map from X to the terminal dendroidal set is a J-fibration.

PROPOSITION 3.3. *Let $A \to B$ and $X \to Y$ be a J-anodyne extension and a normal monomorphism respectively. The induced map*

$$A \otimes Y \cup B \otimes X \longrightarrow B \otimes Y$$

is a J-anodyne extension.

Proof. Using [**3**, Corollary 1.1.8], this follows formally from the definition and from Proposition 3.1. □

3.4. Let B be a dendroidal set. Denote by An_B the class of maps of $d\mathcal{S}et/B$, the image in $d\mathcal{S}et$ of which is J-anodyne. For each dendroidal set X over B, with structural map $a : X \to B$, we define a cylinder of X over B

$$\begin{tikzcd} X \amalg X \arrow[r, "(\partial^0_X, \partial^1_X)"] \arrow[dr, "(a,a)"'] & J_d \otimes X \arrow[r, "\sigma_X"] \arrow[d, "a'"] & X \arrow[dl, "a"] \\ & B, & \end{tikzcd} \tag{3.1}$$

in which ∂^e_X is the tensor product of $\{e\} \to J_d$ with 1_X, whereas σ_X is the tensor product of $J_d \to \eta$ with 1_X, and a' is the composition of $1_{J_d} \otimes a$ with the map σ_B.

These cylinders over B define the notion *J-homotopy over B* (or *fibrewise J-homotopy*) between maps in $d\mathcal{S}et/B$. Given two dendroidal sets A and X over B, we define $[A, X]_B$ as the quotient of the set $\mathrm{Hom}_{d\mathcal{S}et/B}(A, X)$ by the equivalence relation generated by the relation of J-homotopy over B. A morphism $A \to A'$ of dendroidal sets over B is a *B-equivalence* if, for any dendroidal set X over B such that the structural map $X \to B$ is a J-fibration, the map

$$[A', X]_B \longrightarrow [A, X]_B$$

is bijective.

In the case B is normal, any monomorphism over B is normal; see Corollaries 1.7 and 1.8. We see from Proposition 3.3 and from [**3**, Lemma 1.3.52] that the class An_B is a class of anodyne extensions with respect to the functorial cylinder (3.1) in the sense of [**3**, Definition 1.3.10]. In other words, the functorial cylinder (3.1) and the class An_B form a *homotopical structure* on the category $d\mathcal{S}et/B$ in the sense of [**3**, Definition 1.3.14]. As a consequence, a direct application of [**3**, Theorem 1.3.22, Proposition 1.3.36 and Lemma 1.3.52] leads to the following statement (the results of [**3**] are stated for presheaves categories, so that, strictly speaking, to apply them, we implicitly use the canonical equivalence of categories between $d\mathcal{S}et/B$ and the category of presheaves on Ω/B).

Proposition 3.5. *For any normal dendroidal set B, the category $d\mathcal{S}et/B$ of dendroidal sets over B is endowed with a left proper cofibrantly generated model category structure for which the weak equivalences are the B-equivalences, the cofibrations are the monomorphisms and the fibrant objects are the dendroidal sets X over B such that the structural map is a J-fibration. Moreover, a morphism between fibrant objects is a fibration in $d\mathcal{S}et/B$ if and only if its image in $d\mathcal{S}et$ is a J-fibration.*

Remark 3.6. Any J-anodyne extension over B is a trivial cofibration in the model structure of the preceding proposition; see [**3**, Proposition 1.3.31].

Lemma 3.7. *Let $p : X \to Y$ be a trivial fibration between normal dendroidal sets. Any section $s : Y \to X$ is a J-anodyne extension.*

Proof. This is a particular case of [**3**, Corollary 1.3.35] applied to the homotopical structure defined in Section 3.4 on $d\mathcal{S}et/Y$. □

3.8. We fix once and for all a normalization E_∞ of the terminal dendroidal set: that is, we choose a normal dendroidal set E_∞ such that the map from E_∞ to the terminal dendroidal set is a trivial fibration.

Lemma 3.9. *For any normal dendroidal set X, and any map $a : X \to E_\infty$, the map $(a, 1_X) : X \to E_\infty \times X$ is a J-anodyne extension.*

Proof. This follows immediately from Lemma 3.7 because $(a, 1_X)$ is a section of the projection $X \times E_\infty \to X$, which is a trivial fibration by definition of E_∞. □

LEMMA 3.10. *Let $i : A \to B$ be a morphism of normal dendroidal sets, and $p : X \to Y$ a morphism of dendroidal sets. The map p has the right lifting property with respect to i in $d\mathcal{S}et$ if and only if, for any morphism $B \to E_\infty$, the map $1_{E_\infty} \times p$ has the right lifting property with respect to i in $d\mathcal{S}et/E_\infty$.*

Proof. Suppose that $1_{E_\infty} \times p$ has the right lifting property with respect to i, and consider the lifting problem below.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "\ell"'] & Y \end{tikzcd}$$

As B is normal, there exists a map $\beta : B \to E_\infty$. If we write $\alpha = \beta\, i$, we see immediately that the lifting problem above is now equivalent to the lifting problem

$$\begin{tikzcd} A \arrow[r, "(\alpha,a)"] \arrow[d, "i"'] & E_\infty \times X \arrow[d, "1_{E_\infty} \times p"] \\ B \arrow[r, "(\beta,b)"'] \arrow[ur, dotted, "(\beta,\ell)"] & E_\infty \times Y, \end{tikzcd}$$

and this proves the lemma. □

3.11. Given a normal dendroidal set A and a J-fibrant dendroidal set X, we denote by $[A, X]$ the quotient of $\mathrm{Hom}_{d\mathcal{S}et}(A, X)$ by the equivalence relation generated by the J-homotopy relation (that is, with the notation of Section 3.4, $[A, X] = [A, X]_e$, where e denotes the terminal dendroidal set). Note that the functor $J_d \otimes (-)$ is a functorial cylinder object for normal dendroidal sets, with respect to the following model category structure.

PROPOSITION 3.12. *The category of dendroidal sets is endowed with a left proper cofibrantly generated model category in which the cofibrations are the normal monomorphisms, the fibrant objects are the J-fibrant dendroidal sets, and the fibrations between fibrant objects are the J-fibrations. Furthermore, given a normal dendroidal set A and a J-fibrant dendroidal set X, we have a canonical identification*

$$[A, X] = \mathrm{Hom}_{\mathrm{Ho}(d\mathcal{S}et)}(A, X).$$

Proof. Proposition 3.5 applied to $B = E_\infty$ gives us a model category structure on $d\mathcal{S}et/E_\infty$. Consider the adjunction

$$p_! : d\mathcal{S}et/E_\infty \rightleftarrows d\mathcal{S}et : p^*,$$

where p^* is the functor $X \mapsto E_\infty \times X$. It follows obviously from Lemma 3.9 that the functor $p^* p_!$ is a left Quillen equivalence from the category $d\mathcal{S}et/E_\infty$ to itself. This implies immediately that the adjunction $(p_!, p^*)$ satisfies all the necessary hypotheses to define a model structure on $d\mathcal{S}et$ by transfer; see, for example, [5] or [3, Proposition 1.4.23]. In other words, the category of dendroidal sets is endowed with a cofibrantly generated model category structure for which the weak equivalences or the fibrations are the maps the image by p^* of which is a weak equivalence or a fibration, respectively, in $d\mathcal{S}et/E_\infty$. The description of cofibrations follows from Proposition 1.4. We know that the fibrations between fibrant objects in $d\mathcal{S}et/E_\infty$ are the maps with image in $d\mathcal{S}et$ of a J-fibration; see Proposition 3.5. The description of fibrant objects and of fibrations between fibrant objects in $d\mathcal{S}et$ as J-fibrant objects and J-fibrations is thus a direct consequence of Lemma 3.10. The identification $[A, X] = \mathrm{Hom}_{\mathrm{Ho}(d\mathcal{S}et)}(A, X)$ is obtained from the general description of the set of maps from a cofibrant object to a fibrant object in an abstract model category. It remains to prove left properness: this follows from the left properness of the model category structure of Proposition 3.5 for $B = E_\infty$ (which is obvious, as any object over E_∞ is cofibrant), and from the fact that p^* preserves cofibrations as well as colimits and it preserves and detects weak equivalences. □

3.13. The weak equivalences of the model structure defined in Proposition 3.12 will be called the *weak operadic equivalences*.

Given a dendroidal set A, a *normalization* of A is a trivial fibration $A' \to A$ with A' normal. For instance, the projection $E_\infty \times A \to A$ is a normalization of A (as E_∞ is normal, it follows from Corollary 1.7 that $E_\infty \times A$ is normal). For a morphism of dendroidal sets $f : A \to B$, the following conditions are equivalent.

(a) The map f is a weak operadic equivalence.

(b) For any commutative square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the horizontal maps are normalizations, and for any J-fibrant dendroidal set X, the map $[B', X] \to [A', X]$ is bijective.

(c) There exists a commutative square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the horizontal maps are normalizations such that, for any J-fibrant dendroidal set X, the map $[B', X] \to [A', X]$ is bijective.

Remark 3.14. Given a normal J-fibrant dendroidal set B, the model structure induced on $d\mathcal{S}et/B$ by the model structure of Proposition 3.12 coincides with the model structure of Proposition 3.5 (this follows, for instance, from the fact that these model structures have the same cofibrations and fibrations between fibrant objects).

Remark 3.15. The model category structure of Proposition 3.12 is cofibrantly generated. The generating cofibrations are the inclusions of shape $\partial\Omega[T] \to \Omega[T]$ for any tree T. We do not know of any explicit set of generating trivial cofibrations. However, we know (from the proof of Proposition 3.12) that there exists a generating set of trivial cofibrations $\mathcal{J}$ for the model structure on $d\mathcal{S}et/E_\infty$, such that $p_!(\mathcal{J})$ is a generating set of trivial cofibrations of $d\mathcal{S}et$. In particular, there exists a generating set of trivial cofibrations of $d\mathcal{S}et$, which consists of trivial cofibrations between normal dendroidal sets. Statements about trivial cofibrations will often be reduced to statements about J-anodyne extensions using the following argument.

Proposition 3.16. *The class of trivial cofibrations between normal dendroidal sets is the smallest class $\mathcal{C}$ of monomorphisms between normal dendroidal sets that contains J-anodyne* *extensions, and such that, given any monomorphisms between normal dendroidal sets*

$$A \xrightarrow{i} B \xrightarrow{j} C,$$

if j and ji are in $\mathcal{C}$, so is i.

Proof. Let $i : A \to B$ be a monomorphism between normal dendroidal sets. As B is normal, we can choose a map from B to E_∞. We can then choose a commutative diagram over E_∞

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, "b"'] & B', \end{tikzcd}$$

in which a and b are J-anodyne extensions, A' and B' are fibrant in $d\mathcal{S}et/E_\infty$ and i' is a monomorphism: this follows, for instance, from the fact that any J-fibrant resolution functor constructed with the small object argument applied to the generating set of J-anodyne extensions preserves monomorphisms; see [**3**, Proposition 1.2.35]. Applying [**3**, Corollary 1.3.35] to the model structure of Proposition 3.5 for $B = E_\infty$, we see that i is a trivial cofibration if and only if i' is a J-anodyne extension. This proves the proposition. □

PROPOSITION 3.17. *The model category structure on $d\mathcal{S}et$ is symmetric monoidal.*

Proof. As we already know that normal monomorphisms are well behaved with respect to the tensor product (Proposition 1.9), it just remains to prove that, given a normal monomorphism $i : A \to B$ and a trivial cofibration $j : C \to D$, the induced map

$$A \otimes D \cup B \otimes C \longrightarrow B \otimes D$$

is a trivial cofibration. According to [**8**, Lemma 4.2.4], we can assume that i is a generating cofibration, and j a generating trivial cofibration. In particular, we can assume that i and j are monomorphisms between normal dendroidal sets; see Remark 3.15. It is thus sufficient to prove that, given a normal dendroidal set A, the functor $X \mapsto A \otimes X$ preserves trivial cofibrations between normal dendroidal sets. By Proposition 3.16, it is even sufficient to prove that the tensor product by A preserves J-anodyne extensions, which follows from Proposition 3.3. □

4. *The join operation on trees*

The aim of this section is to study a dendroidal analogue of the join operations on simplicial sets introduced by Joyal in [**9**]. We shall prove a generalization of [**9**, Theorem 2.2]; see Theorem 4.2.

4.1. Let X be a ∞-operad. A 1-simplex of X (that is, a map $\Delta[1] \to i^*(X)$) will be called *weakly invertible* if the corresponding morphism in the category $\tau(i^*(X))$ is an isomorphism.

Note that, for any ∞-operad X, the category $\tau(i^*(X))$ is canonically isomorphic to the category underlying the operad $\tau_d(X)$: this comes from the explicit description of $\tau(i^*(X)$ given by Boardman and Vogt (see [**9**, Proposition 1.2]) and from its dendroidal generalization, which describes $\tau_d(X)$ explicitly; see [**13**, Proposition 6.10]. As a consequence, weakly invertible 1-cells in X can be described as the maps $i_!\Delta[1] = \Omega[1] \to X$, which induce invertible morphisms in the underlying category of the operad $\tau_d(X)$.

In order to state the main result of this section, we still have to introduce some notation.

Let T be a tree with at least two vertices, among which is a unary vertex r at the root. We denote by $\partial_r : T/r \to T$ the corresponding outer face; see (1.5). We define the *outer horn*

$\Lambda^r[T]$ *associated with* r as the subobject of $\partial\Omega[T]$ defined as the union of all elementary faces except ∂_r (we only defined inner horns in Section 2.1). Given a dendroidal set X and a map $f : \Lambda^r[T] \to X$, we write $f(r)$ for the map $i_!\Delta[1] \to X$ obtained by composing f with the map $i_!\Delta[1] \to \Lambda^r[T]$, which corresponds to the vertex r in T.

Theorem 4.2. *Let T be a tree with at least two vertices, among which is a unary vertex r at the root, and let $p : X \to Y$ be an inner Kan fibration between ∞-operads. Then any solid commutative square of the form*

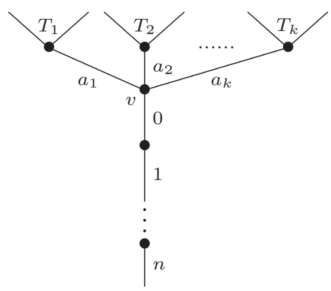
$$\begin{tikzcd} \Lambda^r[T] \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Omega[T] \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y, \end{tikzcd}$$

in which $f(r)$ is weakly invertible in X has a diagonal filling h.

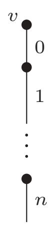
4.3. In order to prove this theorem, we will introduce join operations on forests.

A *forest* is a finite set of trees (that is, of objects of Ω). Given a forest $\mathcal{T} = (T_1, \dots, T_k)$, $k \geqslant 0$, we write $\mathcal{T}/d\mathcal{S}et$ for the category of dendroidal sets under the coproduct $\Omega[\mathcal{T}] = \amalg_{i=1}^k \Omega[T_i]$. The objects of $\mathcal{T}/d\mathcal{S}et$ are thus of shape $(X, x_i) = (X, x_1, \dots, x_k)$, where X is a dendroidal set, and $x_i \in X(T_i)$, for $1 \leqslant i \leqslant k$. Morphisms $(X, x_i) \to (Y, y_i)$ are maps $f : X \to Y$ such that $f(x_i) = y_i$ for all i, $1 \leqslant i \leqslant k$.

Given an integer $n \geqslant 0$, we construct the tree $\mathcal{T} \star n$ by joining the trees $T_1, \dots, T_k$ together over a new vertex v, and then grafting the result onto $i[n]$ (that is, onto $[n]$ viewed as a tree).

$(T_1, \dots, T_k) \star n =$  (4.1)

We remark that the forest $\mathcal{T}$ might be empty: for $k = 0$, we have the following.

$(\,) \star n =$  (4.2)

As each T_i, $1 \leqslant i \leqslant k$, embeds canonically into $\mathcal{T} \star n$, we can view $\Omega[\mathcal{T} \star n]$ as an object of $\mathcal{T}/d\mathcal{S}et$. One checks that there is a unique functor

$$\Delta \longrightarrow \Omega, \quad [n] \longmapsto \mathcal{T} \star n,$$

such that the inclusions $T_i \to \mathcal{T} \star n$ are functorial in T_i and such that the canonical inclusion $i[n] \to \mathcal{T} \star n$ is functorial in $[n]$. This defines a functor

$$\mathcal{T} \star (-) : \Delta \longrightarrow \mathcal{T}/d\mathcal{S}et. \tag{4.3}$$

By Kan extension, we obtain a colimit preserving functor, which extends (4.3):

$$\mathcal{T} \star (-) : sSet \longrightarrow \mathcal{T}/dSet. \tag{4.4}$$

We have $\mathcal{T} \star \Delta[n] = \Omega[\mathcal{T} \star n]$. The functor (4.4) has a right adjoint

$$\mathcal{T}\backslash(-) : \mathcal{T}/dSet \longrightarrow sSet. \tag{4.5}$$

For a one tree forest $\mathcal{T} = (T)$, we will simply write $\mathcal{T} \star K = T \star K$ and $\mathcal{T}\backslash X = T\backslash X$ for any simplicial set K and any dendroidal set X under $\Omega[T]$. Under these conventions, these operations extend the join operations introduced by Joyal in [**9**], in the sense that we have the following formulas:

$$i[n] \star i_!(K) = i_!(\Delta[n] \star K),$$
$$i[n]\backslash i_!(L) = i_!(\Delta[n]\backslash L).$$

Note that the inclusions $\Omega[n] \to \mathcal{T} \star \Delta[n]$ in $dSet$ induce a natural projection map

$$\pi_X : \mathcal{T}\backslash X \longrightarrow i^*(X) \tag{4.6}$$

for any dendroidal set X under $\mathcal{T}$.

Remark 4.4. Note that any tree with at least one vertex T is obtained by joining a forest with an ordinal, that is, as $T = \mathcal{T} \star n$ for some forest $\mathcal{T}$ and some integer $n \geqslant 0$. A tree T has at least two vertices and a unary vertex at the root (as in the statement of Theorem 4.2) if and only if there exists a forest $\mathcal{T}$ such that $T = \mathcal{T} \star 1$.

4.5. In order to prove Theorem 4.2, we will also have to consider some specific maps of forests. For this purpose, we introduce the following terminology.

Let T be a tree. A set A of edges in T is called *admissible* if, for any input edge e of T, and any vertex v in T, if A contains a path (branch) from e to v, then A contains all the edges above v.

If A is an admissible set of edges in T, we will define a forest $\partial_A(T)$, and for each tree S in $\partial_A(T)$, a face map $S \to T$ in the category Ω. Roughly speaking, one deletes from T all edges in A, and defines $\partial_A(T)$ as the resulting connected components. A formal definition is by induction on the cardinality of A.

(i) If A is empty, then $\partial_A(T) = T$.

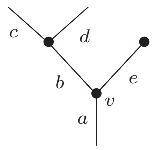
(ii) If A contains the root edge e of T, let $T_1, \ldots, T_k$ be the trees obtained from T by deleting e and the vertex immediately above it, let $A_i = T_i \cap A$, and define $\partial_A(T)$ as the union of the forests $\partial_{A_i}(T_i)$, $1 \leqslant i \leqslant k$.

(iii) If A contains an input edge a of T, it must contain all the edges above the vertex v just below a. Let $T_{(v)}$ be the tree obtained from T by pruning away v and all the edges above it. Let $A_{(v)} = T_{(v)} \cap A$, and define $\partial_A(T) = \partial_{A_{(v)}}(T_{(v)})$.

(iv) If A contains an inner edge a of T, let T/a be the tree obtained by contracting a, and define $\partial_A(T)$ to be $\partial_{A-\{a\}}(T/a)$.

One can check that the steps (i)–(iv) can be performed in any order, so that the forest $\partial_A(T)$ is well defined. Each tree S in this forest $\partial_A(T)$ is a face of T, and hence comes with a canonical map $S \to T$.

Example 4.6. The tree

$T =$ 

has two input edges c and d. The edges b and c form a path from c down to v. So any admissible set A that contains b and c, for example, must contain d and e as well.

4.7. This construction extends to forests in the following way. Let $\mathcal{T} = (T_1, \dots, T_k)$ be a forest. An *admissible subset of edges* A in $\mathcal{T}$ is a k-tuple $A = (A_1, \dots, A_k)$, where A_i is an admissible set of edges of T_i for $1 \leqslant i \leqslant k$. We can then define the forest $\partial_A(T)$ as the union of the forests $\partial_{A_i}(T_i)$. Given any integer $n \geqslant 0$, we have a canonical map

$$\partial_A(\mathcal{T}) \star n \longrightarrow \mathcal{T} \star n, \tag{4.7}$$

which is characterized by the fact that, given any tree S in some $\partial_{A_i}(T_i)$, for $1 \leqslant i \leqslant k$, the diagram

$$\begin{tikzcd} S \star n \arrow[r] \arrow[d] & T_i \star n \arrow[d] \\ \partial_A(\mathcal{T}) \star n \arrow[r] & \mathcal{T} \star n \end{tikzcd} \tag{4.8}$$

commutes. The map (4.7) is a monomorphism of trees in Ω and is natural in $[n]$ (as an object of Δ). More generally, given an inclusion $A \subset B$ between admissible subsets of edges in $\mathcal{T}$, we have canonical monomorphisms of trees

$$\partial_B(\mathcal{T}) \star n \longrightarrow \partial_A(\mathcal{T}) \star n \tag{4.9}$$

(which is just another instance of (4.7) for the forest $\partial_A(\mathcal{T})$ with admissible subset of edges given by the sets $B_i \cap \partial_{A_i}(T_i)$). The maps (4.9) define a contravariant functor from the set of admissible subsets of edges in $\mathcal{T}$ (partially ordered by inclusion) to Ω. Given an inclusion $A \subset B$ of admissible subsets of edges in $\mathcal{T}$, there exists a unique morphism

$$\Omega[\partial_B(\mathcal{T})] \longrightarrow \Omega[\partial_A(\mathcal{T})] \tag{4.10}$$

such that the following diagram commutes for any simplicial set K:

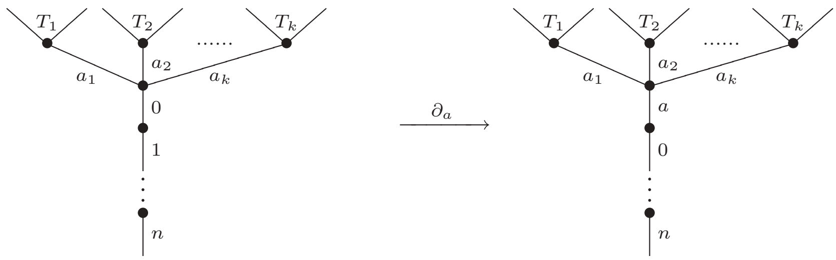
$$\begin{tikzcd} \Omega[\partial_B(\mathcal{T})] \arrow[r] \arrow[d] & \Omega[\partial_A(\mathcal{T})] \arrow[d] \\ \partial_B(\mathcal{T}) \star K \arrow[r] & \partial_A(\mathcal{T}) \star K. \end{tikzcd} \tag{4.11}$$

By adjunction, we also have natural morphisms of simplicial sets

$$\partial_A(\mathcal{T}) \backslash X \longrightarrow \partial_B(\mathcal{T}) \backslash X \tag{4.12}$$

for all dendroidal sets X under $\Omega[\partial_A(\mathcal{T})]$.

Example 4.8. If a is the root of T, and if T is obtained by grafting trees T_i with root edges a_i onto a corolla, then the map of type (4.9) for $A = \emptyset$ and $B = \{a\}$ is the map ∂_a given by contracting a as follows.



4.9. We will now study an elementary combinatorial situation, which we will have to consider twice to prove Theorem 4.2: in the proof of Proposition 4.11 and in the proof of 4.15.2.

Consider a tree T. Assume that $T = \mathcal{T} \star n$ for a forest $\mathcal{T} = (T_1, \ldots, T_k)$ and an ordinal $[n]$, $n > 0$.

Let i, $0 \leqslant i < n$, be an integer, and $\{A_1, \ldots, A_s\}$, $s \geqslant 1$, a finite family of admissible subsets of edges in $\mathcal{T}$. Define

$$C \subset D \subset \Omega[T]$$

by

$$C = \left(\bigcup_{r=1}^{s} \partial_{A_r}(\mathcal{T}) \star \Lambda^i[n] \right) \cup \Omega[n] \quad \text{and} \quad D = \bigcup_{r=1}^{s} \partial_{A_r}(\mathcal{T}) \star \Delta[n],$$

where $\Omega[n]$ is considered as a subcomplex of $\Omega[T]$ through the canonical map.

Lemma 4.10. *Under the assumptions of Section* 4.9, *the map* $C \to D$ *is an inner anodyne extension.*

Proof. For $T = \mathcal{T} \star n$, we will write $\Lambda^i[T]$ for the horn corresponding to the face $\mathcal{T} \star (n-1) \to \mathcal{T} \star n$ induced by the face $\partial_i : [n-1] \to [n]$; see Section 2.1. (As $i < n$, $\Lambda^i[T]$ is always an inner horn in $\Omega[T]$.)

If $\mathcal{T}$ is the empty forest, we must have $s = 1$ and $A_1 = \emptyset$, so that $D = \Omega[T]$, and $C = \Lambda^i[T]$ is an inner horn. From now on, we will assume that $\mathcal{T}$ is non-empty.

Given a forest $\mathcal{T}'$, the number of edges in $\mathcal{T}'$ is simply defined as the sum of the number of edges in each of the trees that occur in $\mathcal{T}'$. For each integer $p \geqslant 0$, write $\mathcal{F}_p$ for the set of faces F that belong to D but not to C, and are of shape $F = \Omega[\partial_A(\mathcal{T}) \star n]$ for an admissible subset of edges A in $\mathcal{T}$, such that $\partial_A(\mathcal{T})$ has exactly p edges.

Define a filtration

$$C = C_0 \subset C_1 \subset \ldots \subset C_p \subset \ldots \subset D$$

by

$$C_p = C_{p-1} \cup \bigcup_{F \in \mathcal{F}_p} F, \quad p \geqslant 1.$$

We have $D = C_p$ for p big enough, and it is sufficient to prove that the inclusions $C_{p-1} \to C_p$ are inner anodyne for $p \geqslant 1$. It is sufficient to prove that, for any subset Φ of $\mathcal{F}_p$, the inclusion $C_{p-1} \to C_{p-1} \cup \bigcup_{F \in \Phi} F$ is an inner anodyne extension. If F and F' are distinct elements in $\mathcal{F}_p$, then $F \cap F'$ is in C_{p-1}. Therefore, for any face F' in $\mathcal{F}_p$ that does not belong to Φ, the map

$$C_{p-1} \cup \bigcup_{F \in \Phi} F \longrightarrow F' \cup C_{p-1} \cup \bigcup_{F \in \Phi} F$$

is a pushout of the map $F' \cap C_{p-1} \to C_{p-1}$. To finish the proof of this lemma, we are thus reduced to prove that, for any face F in $\mathcal{F}_p$, the inclusion $F \cap C_{p-1} \to C_{p-1}$ is an inner anodyne extension. For an admissible subset of edges A such that $F = \Omega[\partial_A(\mathcal{T}) \star n]$ belongs to $\mathcal{F}_p$, we will prove that

$$F \cap C_{p-1} = \Lambda^i[\partial_A(\mathcal{T}) \star n],$$

which will complete our argument. Indeed, the intersection $F \cap C_{p-1}$ clearly contains the inner horn $\Lambda^i[\partial_A(\mathcal{T}) \star n]$ (by definition of $C \subset C_{p-1}$); on the other hand, $F \cap C_{p-1}$ has to be distinct from F (because F is in $\mathcal{F}_p$), and the image of the face $\partial_i : \partial_A(\mathcal{T}) \star (n-1) \to \partial_A(\mathcal{T}) \star n$ cannot be in C_{p-1} (otherwise, it would contain the boundary of F, whence the whole of F, by construction), which forces $F \cap C_{p-1}$ to fit in $\Lambda^i[\partial_A(\mathcal{T}) \star n]$. □

PROPOSITION 4.11. *Let $\mathcal{T} = (T_1, \dots, T_k)$ be a forest, and $n \geqslant 1$, $0 \leqslant i < n$, be integers. The inclusion $(\mathcal{T} \star \Lambda^i[n]) \cup \Omega[n] \to \mathcal{T} \star \Delta[n]$ is an inner anodyne extension.*

Proof. This is a particular case of the preceding lemma. □

4.12. Remember from [**9**] that a morphism of simplicial sets is called a *left* or *right fibration* if it has the right lifting property with respect to inclusions of shape $\Lambda^i[n] \to \Delta[n]$ for $n \geqslant 1$ and $0 \leqslant i < n$ or $0 < i \leqslant n$ respectively.

A morphism between ∞-categories $X \to Y$ is *conservative* if the induced functor $\tau(X) \to \tau(Y)$ is conservative (which can be reformulated by saying that a 1-simplex of X is weakly invertible if and only if its image in Y is weakly invertible). For instance, by virtue of [**9**, Proposition 2.7], any left or right fibration between ∞-categories is conservative.

PROPOSITION 4.13. *Let $p : X \to Y$ be an inner Kan fibration of dendroidal sets under a forest $\mathcal{T}$. The map induced by the projections* (4.6)

$$\mathcal{T}\backslash X \longrightarrow \mathcal{T}\backslash Y \times_{i^*(Y)} i^*(X)$$

is a left fibration.

In particular, for any ∞-operad X under a forest $\mathcal{T}$, the map $\mathcal{T}\backslash X \to i^*(X)$ is a left fibration.

Proof. For any inclusion of simplicial sets $i : K \to L$, lifting problems of shape

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & \mathcal{T}\backslash X \arrow[d] \\ L \arrow[r] \arrow[ur, dotted] & \mathcal{T}\backslash Y \times_{i^*(Y)} i^*(X) \end{tikzcd}$$

are in bijection with lifting problems of shape

$$\begin{tikzcd} (\mathcal{T} \star K) \cup L \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathcal{T} \star L \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

so that we conclude immediately with Proposition 4.11. □

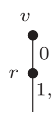
Corollary 4.14. *For any ∞-operad X and any forest $\mathcal{T}$ over X, the simplicial set $\mathcal{T}\backslash X$ is an ∞-category. Similarly, for any inner Kan fibration between ∞-operads $X \to Y$ and any forest $\mathcal{T}$ over X, the simplicial set $\mathcal{T}\backslash Y \times_{i^*(Y)} i^*(X)$ is an ∞-category.*

Proof. If X is an ∞-operad, then $i^*(X)$ is clearly an ∞-category. As the projection $\mathcal{T}\backslash X \to i^*(X)$ is a left fibration, this implies the above corollary. □

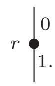
As a warm-up to prove Theorem 4.2, we shall consider a particular case.

Lemma 4.15. *Theorem 4.2 is true if $T = (\ ) \star 1$ (where $(\ )$ denotes the empty forest).*

Proof. In this case, T is a tree of shape



and $\Lambda^r[T]$ is the union of the two faces

 and 

In other words, we get $\Lambda^r[T] = (\ ) \star \Lambda^1[1] \cup \Omega[1]$. Thus, a lifting problem of shape

$$\begin{tikzcd}
\Lambda^r[T] \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\
\Omega[T] \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y
\end{tikzcd}$$

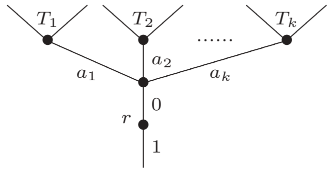
is equivalent to a lifting problem of shape

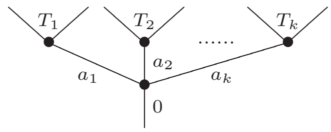
$$\begin{tikzcd}
\{1\} \arrow[r, "\tilde{f}"] \arrow[d] & (\ )\backslash X \arrow[d, "\varphi"] \\
\Delta[1] \arrow[r, "\tilde{g}"] \arrow[ur, dotted, "\tilde{h}"] & (\ )\backslash Y \times_{i^*(Y)} i^*(X).
\end{tikzcd}$$

By virtue of Proposition 4.13, the map φ is a left fibration, and, as left fibrations are stable by pullback and by composition, so is the projection of $(\ )\backslash Y \times_{i^*(Y)} i^*(X)$ to $i^*(X)$. The image of $\tilde{g}$ by the latter is nothing but $f(r)$, and, as we know that left fibrations between ∞-categories are conservative (see [**9**, Proposition 2.7]), the 1-cell $\tilde{g}$ is quasi-invertible in $(\ )\backslash Y \times_{i^*(Y)} i^*(X)$. We conclude the proof using [**9**, Propositions 2.4 and 2.7]. □

Proof of Theorem 4.2. Let T be a tree with at least two vertices and a unary vertex r at the root. There is a forest $\mathcal{T} = (T_1, \dots, T_k)$, $k \geqslant 0$, such that $T = \mathcal{T} \star 1$. By virtue of Lemma 4.15, we may assume that $\mathcal{T}$ is not the empty forest, or, equivalently, that $k \geqslant 1$. We will write

$T' = \mathcal{T} \star 0$. The trees T and T' can be represented as follows.

$T =$ 

$T' =$ 

The dendroidal set $\Lambda^r[T]$ can be written as

$$\Lambda^r[T] = \bigcup_A \partial_A(T') \star \Delta[0],$$

where we identify $\Delta[0] = \{1\}$ and where A runs through the family of *non-empty* admissible subsets of edges A in T' (ordered by inclusion). Therefore, given a dendroidal set X, a map $\Lambda^r[T] \to X$ corresponds to a compatible family of maps of dendroidal sets

$$\partial_A(T') \star \Delta[0] \longrightarrow X,$$

which, by adjunction, corresponds to a compatible family of maps of simplicial sets

$$\{1\} = \Delta[0] \longrightarrow \partial_A(T')\backslash X.$$

This family thus corresponds to a map

$$\Delta[0] \longrightarrow \varprojlim_A \partial_A(T')\backslash X.$$

By separating the case $A = \{0\}$ (the root edge of T') from the others, the map $\Lambda^r[T] \to X$ corresponds in fact to a commutative square of shape

$$\begin{tikzcd}
\Delta[0] \arrow[r] \arrow[d, "\partial_0"'] & \mathcal{T}\backslash X \arrow[d] \\
\Delta[1] \arrow[r] & \varprojlim \partial_{\bar{B}}(\mathcal{T})\backslash X,
\end{tikzcd}$$

in which the limit $\varprojlim \partial_{\bar{B}}(\mathcal{T})\backslash X$ is over the non-empty admissible subsets of edges B in T' with $0 \notin B$, and $\bar{B} = (B \cap T_1, \ldots, B \cap T_k)$.

Consider from now on an inner Kan fibration between ∞-operads $p : X \to Y$. Lifting problems of shape

$$\begin{tikzcd}
\Lambda^r[T] \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\
\Omega[T] \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y
\end{tikzcd}$$

correspond to lifting problems

$$\begin{tikzcd}
\Delta[0] \arrow[r, "\tilde{f}"] \arrow[d, "\partial_0"'] & P \arrow[d, "\varphi"] \\
\Delta[1] \arrow[r, "\tilde{g}"] \arrow[ur, dotted, "\tilde{h}"] & Q
\end{tikzcd}$$

where $P = \mathcal{T}\backslash X$ and $Q = U \times_W V$, with

$$U = \varprojlim \partial_{\bar{B}}(\mathcal{T})\backslash X, \quad V = \mathcal{T}\backslash Y, \quad W = \varprojlim \partial_{\bar{B}}(\mathcal{T})\backslash Y.$$

Exactly as in the proof of Proposition 4.15, it now suffices to prove the following three statements.

(i) The map $\varphi : P \to Q$ is a left fibration.
(ii) The simplicial set Q is an ∞-category.
(iii) If $f(r)$ is weakly invertible in X, then the 1-cell $\tilde{g}$ is weakly invertible in $i^*(Q)$.

Note that, as left fibrations are conservative and are stable by pullback and composition, statements (ii) and (iii) will follow from the following two assertions.

(iv) The map $V \to W$ is a left fibration.
(v) The map $U \to i^*(X)$ is a left fibration.

But (iv) is a particular case of (i): just replace p by the map from Y to the terminal dendroidal set. It thus remains to prove (i) and (v).

4.15.1. *Proof of* (i).

For $0 \leqslant i < n$, a lifting problem of the form

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & P \arrow[d, "\varphi"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Q \end{tikzcd}$$

corresponds to a lifting problem of the form

$$\begin{tikzcd} \Lambda^i[\mathcal{T} \star n] \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Omega[\mathcal{T} \star n] \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

As $\Lambda^i[\mathcal{T} \star n]$ is an inner horn, (i) thus follows from the fact that p is an inner Kan fibration.

4.15.2. *Proof of (v).*

For $0 \leqslant i < n$, a lifting problem of the form

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & U \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & i^*(X) \end{tikzcd}$$

corresponds to a lifting problem of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ D, \arrow[ur, dotted] & \end{tikzcd}$$

where the inclusion $C \to D$ can be described as follows. The dendroidal set D is the union of all the faces $\partial_x(\mathcal{T} \star n)$ given by contracting an inner edge or a root edge in one of the trees T_i, or by deleting a top vertex in the tree $\mathcal{T} \star n$. The dendroidal set C is the union of the image of $\Omega[n] \to \Omega[\mathcal{T} \star n]$ and all the 'codimension 2' faces of $\Omega[\mathcal{T} \star n]$ of shape $\partial_j \partial_x(\mathcal{T} \star n)$, where ∂_x is as above, and $0 \leqslant j \leqslant n$ is distinct from i. It is now sufficient to check that the inclusion $C \to D$ is inner anodyne extension, which follows from a straightforward application of Lemma 4.10. $\square$

5. *Subdivision of cylinders*

5.1. Let S be a tree with at least one vertex, and consider the tensor product $\Omega[S] \otimes \Delta[1]$. It has a subobject

$$A_0 = \partial\Omega[S] \otimes \Delta[1] \cup \Omega[S] \otimes \{1\},$$

where $\{1\} \to \Delta[1]$ is $\partial_0 : \Delta[0] \to \Delta[1]$. In this section, we will prove the following result.

THEOREM 5.2. *There exists a filtration of* $\Omega[S] \otimes \Delta[1]$ *of the form*

$$A_0 \subset A_1 \subset \ldots \subset A_{N-1} \subset A_N = \Omega[S] \otimes \Delta[1],$$

where:

(i) *the inclusion* $A_i \to A_{i+1}$ *is inner anodyne for* $0 \leqslant i < N-1$;

(ii) *the inclusion* $A_{N-1} \to A_N$ *fits into a pushout of the form*

$$\begin{tikzcd} \Lambda^r[T] \arrow[r] \arrow[d] & A_{N-1} \arrow[d] \\ \Omega[T] \arrow[r] & A_N \end{tikzcd}$$

for a tree T *with at least two vertices and a unary vertex* r *at the root*;

(iii) *the map*

$$\Delta[1] \longrightarrow \Lambda^r[T] \longrightarrow A_{N-1} \subset \Omega[S] \otimes \Delta[1]$$

corresponding to the vertex r *in* (ii) *coincides with the inclusion*

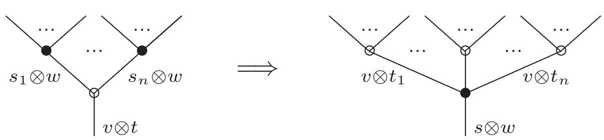
$$\{e_S\} \otimes \Delta[1] \longrightarrow \Omega[S] \otimes \Delta[1]$$

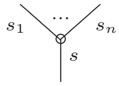
where e_S *is the edge at the root of the tree* S.

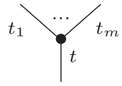
5.3. The proof of Theorem 5.2 is in fact very similar to that of [**13**, Proposition 9.2], stated here as Proposition 3.1. We recall from [**13**, Proposition 9.2] that, for any two trees S and T, one can write

$$\Omega[S] \otimes \Omega[T] = \bigcup_{i=1}^{N} \Omega[T_i],$$

where $\Omega[T_i] \to \Omega[S] \otimes \Omega[T]$ are 'percolation schemes'. Drawing vertices of S as white, and those of T as black, these percolation schemes are partially ordered in a natural way, starting with the tree obtained by stacking a copy of the black tree T on top of each input edge of the white tree S, and ending with the tree obtained by stacking copies of S on top of T. The intermediate trees are obtained by letting the black vertices of T percolate through the white tree S, by successive 'moves' of the form



with 

in S, and 

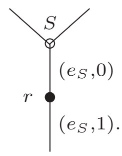
in T.

In the special case where $T = [1]$, the filtration referred to in Theorem 5.2 is given by

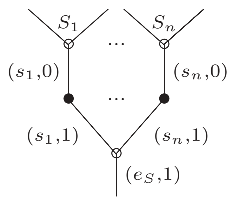
$$A_i = A_0 \cup \Omega[T_1] \cup \ldots \cup \Omega[T_i],$$

where $T_1, \ldots, T_N$ is any linear order on the percolation schemes extending the natural partial order.

Remark 5.4. For any tree S with at least one vertex, and a root edge named e_S (e for ‘exit’), the last tree T_N in the partial order of percolation schemes for $\Omega[S] \otimes \Delta[1]$ is of shape

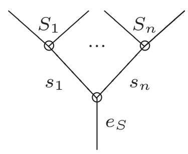
$T_N =$ 

It always has a unique predecessor T_{N-1} of the form

$T_{N-1} =$ 

,

where S is of the form $(S_1, \ldots, S_n) \star [0]$.

$S =$ 

This observation already enables us to get the following.

Proof of parts (ii) *and* (iii) *of Theorem* 5.2. Consider all the faces of T_N. For such a face $F \to T_N$, there are three possibilities:

(a) it misses an S-colour entirely (that is, there is an edge s in S so that neither $(s, 0)$ nor $(s, 1)$ are in F, so that $\Omega[F]$ factors through $\partial\Omega[S] \otimes \Delta[1]$;
(b) it is given by contracting the edge $(e_S, 0)$, in which case $\Omega[F]$ factors through $\Omega[T_{N-1}]$ (as the face F then coincides with the face of T_{N-1} obtained by contracting $(s_1, 1), \ldots, (s_n, 1)$);
(c) it is given by chopping off the edge $(e_S, 1)$ and the black vertex above it, that is, $\Omega[F] = \Omega[S] \otimes \{0\}$. This face cannot factor through A_0, nor through any of the earlier percolation schemes as none of these has an edge coloured $(e_S, 0)$.

Thus, $\Omega[T_N] \cap A_{N-1} = \Lambda^r[T_N]$, where r denotes the black vertex as pictured above. This shows that

$$\begin{tikzcd} \Lambda^r[T] \arrow[r] \arrow[d] & A_{N-1} \arrow[d] \\ \Omega[T] \arrow[r] & A_N \end{tikzcd}$$

is a pushout, exactly as stated in part (ii) of Theorem 5.2. Moreover, the statement of part (iii) of Theorem 5.2 is obvious from the construction. □

5.5. The proof of part (i) of Theorem 5.2 is more involved, but it is completely analogous to the proof of [**13**, Proposition 9.2]. The difference with the situation in [**13**, Proposition 9.2] is that, now, we are dealing with an inclusion of the form

$$\partial\Omega[S] \otimes \Omega[T] \cup \Omega[S] \otimes \Lambda^e[T] \longrightarrow \Omega[S] \otimes \Omega[T],$$

where e is an outer edge of $T = i[1]$, whereas in [**13**, Proposition 9.2], we dealt with

$$\Omega[S] \otimes \partial\Omega[T] \cup \Lambda^e[S] \otimes \Omega[T] \longrightarrow \Omega[S] \otimes \Omega[T],$$

where e is an inner edge of S. This forces us to look at different 'spines' and 'characteristic edges' compared with that in [**13**, Proposition 9.2] (notice also in this connection that, although the tensor product is symmetric, the partial order on the percolation schemes is reversed).

The following lemma was also used (implicitly) in [**13**].

LEMMA 5.6. *Let T_i and T_j be two distinct percolation schemes for $\Omega[S] \otimes \Delta[1]$. Then*

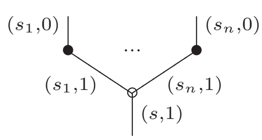
$$\Omega[T_i] \cap \Omega[T_j] \subset \cup_k \Omega[T_k]$$

as subobjects of $\Omega[S] \otimes \Delta[1]$, where the union ranges over all the percolation schemes T_k that precede both T_i and T_j in the partial order.

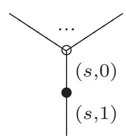
Proof. Let F be a common face of $\Omega[T_i]$ and $\Omega[T_j]$. If $T_j \leqslant T_i$ in the partial order, there is nothing to prove. Otherwise, we will give an algorithm for replacing T_j by successively earlier percolation schemes,

$$T_j = T_{j_0} \geqslant T_{j_1} \geqslant T_{j_2} \geqslant \dots$$

each having F as a face, and eventually preceding T_i in the partial order. As a first step, T_j is obtained from an earlier percolation scheme $T_{j'}$ by changing



into



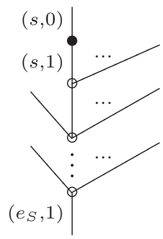
in $T_{j'}$ in T_j.

If F is also a face of $T_{j'}$, we 'push up the black vertices' by replacing T_j by $T_{j_1} = T_{j'}$. If not, then the colour $(s, 0)$ must occur in F, hence in T_j as well as in T_i. So the occurrence of $(s, 0)$ in T_j is not the reason that $T_j \nleqslant T_i$, and we put $T_{j_1} = T_j$. Treating all black vertices in this way, we can push them up if they occur below black vertices in T_i, until we eventually reach a percolation scheme $T_{j_n} \leqslant T_j$, still having F as a face, for which $T_{j_n} \leqslant T_i$. □

5.7. We return to the proof of Theorem 5.2. Consider the inclusion

$$A_k \longrightarrow A_{k+1} = A_k \cup \Omega[T_{k+1}], \tag{5.1}$$

for $k+1 < N$. The percolation scheme T_{k+1} will have at least one black vertex. Consider all the black vertices in T_{k+1}, and the corresponding faces of T_{k+1} formed by paths from these black vertices to the root of T_{k+1} as follows.

$\beta =$ 

(5.2)

The face β is the minimal external face that contains the given black vertex as well as the root edge. We call a face β of T_{k+1} of this form a *spine* in T_{k+1}. Note that the vertex just above $(e_S, 1)$ is indeed white, as in the picture, because $k+1 < N$. Note also that the outer face of β given by chopping off this vertex misses the colour e_S, hence belongs to $\partial\Omega[S] \otimes \Delta[1] \subset A_0$. Furthermore, the outer face of β given by chopping off its black top vertex belongs to $\Omega[S] \otimes \{1\} \subset A_0$. Finally, all the inner faces of β miss an S-colour, hence factor through $\partial\Omega[S] \otimes \Delta[1]$, except possibly that given by contracting the edge $(s,1)$ near the top. However, if this last face $\partial_{(s,1)}(\beta)$ of β belongs to A_k, then some earlier T_i, $1 \leqslant i \leqslant k$, contains the edge $(s,0)$, hence all of β. Thus, either $\Omega[\beta]$ is contained in A_k, or we can adjoin it by an inner anodyne extension

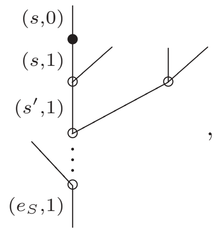
$$\begin{tikzcd}
\Lambda^{(s,1)}[\beta] \arrow[r] \arrow[d] & A_k \arrow[d] \\
\Omega[\beta] \arrow[r] & A_k \cup \Omega[\beta].
\end{tikzcd} \tag{5.3}$$

Such a spine β is an example of an *initial segment* of T_{k+1}. Recall from [**13**] that a face $R \to T_{k+1}$ is called an *initial segment* if it is obtained by successively chopping off top vertices. Our strategy will be to adjoin more initial segments of T_{k+1} to A_k, starting with the spines. To this end, we need the following definition and lemma from [**13**], in which we use the notation $m(R) \subset \Omega[T_{k+1}]$ for the image of the map $\Omega[R] \to \Omega[T_{k+1}]$ given by an initial segment R.

DEFINITION 5.8 ([**13**]). Let $R, Q_1, \ldots, Q_p$ be initial segments of T_{k+1}, and let $B = m(Q_1) \cup \ldots \cup m(Q_p)$. Suppose that, for every top face F of R, we have $m(F) \subset A_k \cup B$. In this situation, an inner edge ξ of R is called *characteristic* with respect to $Q_1, \ldots, Q_p$, if for any inner face F of R, if $m(F/\xi)$ is contained in $A_k \cup B$, then so is $m(F)$ (where $F/\xi \to F$ is the face obtained by contracting ξ).

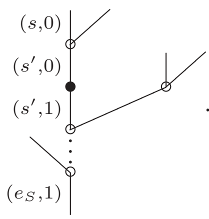
EXAMPLE 5.9. In any spine β as in picture (5.2), the edge $\xi = (s,1)$ is characteristic with respect to any family of initial segments.

Example 5.10. More generally, suppose R is an initial segment of T_{k+1} given by a spine β expanded by one (or more) white vertices, say

$R =$ 

,

then $\xi = (s,1)$ is again characteristic with respect to any family $Q_1, \dots, Q_p$. Indeed, if R/ξ is a face of an initial segment Q_i, then so is R itself; see [**13**, Remark 9.6 (iv)]. And if R/ξ is a face of T_j for a percolation scheme T_j, then T_j either contains R, or looks like



.

But, by Lemma 5.6, we can assume that T_j comes before T_{k+1} in the partial order, so this is impossible. Finally, if $\Omega[R/\xi] \to \Omega[S] \otimes \Delta[1]$ factors through A_0, then R/ξ misses an S-colour, and hence so does R.

5.11. The proof of Theorem 5.2 (i) is based on a repeated use of arguments like the preceding one in Example 5.10. We quote the following lemma on characteristic edges from [**13**].

Lemma 5.12 ([**13**, Lemma 9.7]). *Let R, $Q_1, \dots, Q_p$ be initial segments of T_{k+1}. Let $B = m(Q_1) \cup \dots \cup m(Q_p)$, and suppose each top face of R has the property that $m(F)$ is contained in $A_k \cup B$. If R possesses a characteristic edge with respect to $Q_1, \dots, Q_p$, then the inclusion*

$$A_k \cup B \longrightarrow A_k \cup B \cup m(R)$$

of subobjects of $\Omega[S] \otimes \Delta[1]$ is an inner anodyne.

Lemma 5.13. *Let R, $Q_1, \dots, Q_p$ be initial segments of T_{k+1}, satisfying condition* (i) *in Definition* 5.8*, and let β be a spine in R. Then the edge $\xi = (s,1)$ immediately below the black vertex on the spine is a characteristic edge for R.*

Hint for a proof. This is proved exactly as Example 5.10; cf. also [**13**, Lemma 9.8]. □

5.14. Using the characteristic edges from Lemma 5.13, one can now copy the proof of [**13**, Lemma 9.9], repeated below as Lemma 5.15, verbatim. This proof is by induction on l, and describes a precise strategy for adjoining more and/or larger initial segments of T_{k+1} to A_k.

Lemma 5.15. *Fix $l \geqslant 0$, and let $Q_1, \ldots, Q_p$ be a family of initial segments in T_{k+1}, each containing at least one spine, and at most l spines (so, necessarily, $p = 0$ if $l = 0$). Let $R_1, \ldots, R_q$ be initial segments, which each contain exactly $l+1$ spines. Then the inclusion $A_k \to A_k \cup B \cup C$ is inner anodyne, where $B = m(Q_1) \cup \ldots \cup m(Q_p)$ and $C = m(R_1) \cup \ldots \cup m(R_q)$.*

5.16. This strategy terminates when one arrives at the number l of *all* spines in T_{k+1}. Indeed, for this l and $p = 0$, $q = 1$, Lemma 5.15 states for $R_1 = T_{k+1}$ that $A_k \to A_{k+1}$ is inner anodyne, as asserted in Theorem 5.2 (i). This completes the proof of Theorem 5.2.

6. ∞-*Operads as fibrant objects*

6.1. The aim of this section is to characterize ∞-operads as the fibrant objects of the model category structure on the category of dendroidal sets given by Proposition 3.12. This characterization is stated in Theorem 6.10.

Given an ∞-category X, we denote by $k(X)$ the maximal Kan complex contained in X; see [**9**, Corollary 1.5].

Recall that, given two dendroidal sets A and X, we write

$$\mathcal{hom}(A, X) = i^* \mathcal{H}om(A, X).$$

Note that, by virtue of Proposition 3.1, if X is an ∞-operad, and if A is normal, then $\mathcal{H}om(A, X)$ is an ∞-operad, so that $\mathcal{hom}(A, X)$ is an ∞-category.

For an ∞-operad X and a simplicial set K, we will write $X^{(K)}$ for the subcomplex of $\mathcal{H}om(i_!(K), X)$, which consists of dendrices

$$a : \Omega[T] \times i_!(K) \longrightarrow X$$

such that, for any 0-cell u in T, the induced map

$$a_u : K \longrightarrow i^*(X)$$

factors through $k(i^*(X))$ (that is, all the 1-cells in the image of a_u are weakly invertible in $i^*(X)$).

For an ∞-operad X and a normal dendroidal set A, we will write $k(A, X)$ for the subcomplex of $\mathcal{hom}(A, X)$, which consists of maps

$$u : A \otimes i_!(\Delta[n]) \longrightarrow X$$

such that, for all vertices a of A (that is, maps $a : \eta \to A$), the induced map

$$u_a : \Delta[n] \longrightarrow i^*(X)$$

factors through $k(i^*(X))$. So, by definition, for any normal dendroidal set A, any simplicial set K, and any ∞-operad X, there is a natural bijection:

$$\mathrm{Hom}_{s\mathcal{S}et}(K, k(A, X)) \simeq \mathrm{Hom}_{d\mathcal{S}et}(A, X^{(K)}). \tag{6.1}$$

Remark 6.2. The simplicial set $k(A, X)$ is by definition the ∞-category of objectwise weakly invertible 1-cells in $\mathcal{hom}(A, X)$. We can reformulate the definition of $k(A, X)$ as follows (still with A normal and X an ∞-operad). Define

$$\mathrm{Ob}\, A = \coprod_{A_0} \eta. \tag{6.2}$$

We have a unique monomorphism $i : \mathrm{Ob}\, A \to A$, which is the identity on 0-cells. As A is normal, i is a normal monomorphism. We also have

$$k(\mathrm{Ob}\, A, X) = k(\mathcal{hom}(\mathrm{Ob}\, A, X)) = \prod_{A_0} k(i^* X), \tag{6.3}$$

and $k(A, X)$ fits by definition in the following pullback square:

$$\begin{tikzcd} k(A,X) \arrow[r] \arrow[d] & \mathcal{hom}(A,X) \arrow[d] \\ k(\mathcal{hom}(\mathrm{Ob}\, A, X)) \arrow[r] & \mathcal{hom}(\mathrm{Ob}\, A, X). \end{tikzcd} \tag{6.4}$$

In particular, the projection of $k(A, X)$ on $k(\mathrm{Ob}\, A, X)$ is an inner Kan fibration, and as the latter is a Kan complex, this shows that $k(A, X)$ is an ∞-category. One of the key results of this section asserts that $k(A, X)$ is a Kan complex as well, which can be reformulated by saying that the inclusion $k(\mathcal{hom}(A, X)) \subset k(A, X)$ is in fact an equality. In other words, a map in the ∞-category $\mathcal{hom}(A, X)$ is weakly invertible if and only it is objectwise weakly invertible; see Corollary 6.8.

6.3. Before stating the next theorem, we recall that, for a morphism between ∞-categories $f : X \to Y$, the induced map $\tau(f) : \tau(X) \to \tau(Y)$ is a categorical fibration if and only if the map

$$ev_1 : X^{(\Delta[1])} \longrightarrow Y^{(\Delta[1])} \times_Y X$$

induced by evaluating at 1 (that is, by the inclusion $\{1\} \to \Delta[1]$) has the right lifting property with respect to the map $\emptyset = \partial\Delta[0] \to \Delta[0]$ (that is, is surjective on 0-cells); see [**9**, Proposition 2.4].

THEOREM 6.4. *Let $p : X \to Y$ be an inner Kan fibration between ∞-operads. The map $ev_1 : X^{(\Delta[1])} \to Y^{(\Delta[1])} \times_Y X$ has the right lifting property with respect to inclusions $\partial\Omega[S] \to \Omega[S]$ for any tree S with at least one vertex. Consequently, the functor $\tau i^*(p)$ is a categorical fibration if and only if the evaluation at 1 map $X^{(\Delta[1])} \to Y^{(\Delta[1])} \times_Y X$ is a trivial fibration of dendroidal sets.*

Proof. Consider a tree S with at least one vertex and a solid commutative square

$$\begin{tikzcd} \partial\Omega[S] \arrow[r, "f"] \arrow[d] & X^{(\Delta[1])} \arrow[d] \\ \Omega[S] \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y^{(\Delta[1])} \times_Y X. \end{tikzcd}$$

We want to prove the existence of a diagonal filling h. This corresponds by adjunction to a filling $\tilde{h}$ in the following commutative square:

$$\begin{tikzcd} \partial\Omega[S] \otimes \Delta[1] \cup \Omega[S] \otimes \{1\} \arrow[r, "\tilde{f}"] \arrow[d] & X \arrow[d] \\ \Omega[S] \otimes \Delta[1] \arrow[r, "\tilde{g}"] \arrow[ur, dotted, "\tilde{h}"] & Y \end{tikzcd}$$

(as the inclusion of $\partial\Omega[S]$ in $\Omega[S]$ is bijective on objects, and as the restriction of $\tilde{h}$ to $\partial\Omega[S] \otimes \Delta[1] \cup \Omega[S] \otimes \{1\}$ coincides with $\tilde{f}$, the map $\Omega[S] \to X^{\Delta[1]}$ corresponding to a filling $\tilde{h}$ will automatically factor through $X^{(\Delta[1])}$).

Consider the filtration

$$\partial\Omega[S] \otimes \Delta[1] \cup \Omega[S] \otimes \{1\} = A_0 \subset A_1 \subset \ldots \subset A_{N-1} \subset A_N = \Omega[S] \otimes \Delta[1]$$

given by Theorem 5.2. As the map $X \to Y$ is an inner Kan fibration, using Theorem 5.2 (i), it is sufficient to find a filling in a solid commutative diagram of shape

$$\begin{tikzcd} A_{N-1} \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ \Omega[S] \otimes \Delta[1] \arrow[r, "\tilde{g}"] \arrow[ur, dotted, "\tilde{h}"] & Y, \end{tikzcd}$$

in which the restriction of f' to $\partial\Omega[S] \otimes \Delta[1] \cup \Omega[S] \otimes \{1\}$ coincides with $\tilde{f}$. By virtue of Theorem 5.2 (ii), it is even sufficient to find a filling k in a solid commutative diagram of shape

$$\begin{tikzcd} \Lambda^r[T] \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \Omega[T] \arrow[r, "b"] \arrow[ur, dotted, "k"] & Y, \end{tikzcd}$$

in which T is a tree with unary vertex r at the root, and a is the restriction of f' to $\Lambda^r[T] \subset A_{N-1}$. Furthermore, by Theorem 5.2(iii), we may assume that $a(r)$ is weakly invertible in $i^*(X)$. Thus, the existence of the filling k is ensured by Theorem 4.2.

The last assertion of the theorem follows from 6.3. □

LEMMA 6.5. *Any left fibration between Kan complexes is a Kan fibration.*

Proof. This follows from [**9**, Theorem 2.2 and Proposition 2.7]. □

LEMMA 6.6. *A morphism of simplicial sets $X \to Y$ is a right or left fibration if and only if it has the right lifting property with respect to maps of shape*

$$\partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{e\} \longrightarrow \Delta[n] \times \Delta[1]$$

for $e = 1$ (or $e = 0$, respectively) and $n \geqslant 0$.

Proof. The map $\partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{0\} \to \Delta[n] \times \Delta[1]$ is obtained as a finite composition of pushouts of horns of shape $\Lambda^k[n+1] \to \Delta[n+1]$ with $0 \leqslant k < n+1$; see (the dual version of) [**7**, Chapter IV, 2.1.1].

Conversely, the inclusion map $\Lambda^k[n] \to \Delta[n]$, $0 \leqslant k < n$, is a retract of the map $\Lambda^k[n] \times \Delta[1] \cup \Delta[n] \times \{0\} \to \Delta[n] \times \Delta[1]$; see [**7**, Chapter IV, 2.1.3].

We deduce easily from this that a morphism of simplicial sets $X \to Y$ is a left fibration if and only if the evaluation at 0 map $X^{\Delta[1]} \to Y^{\Delta[1]} \times_Y X$ is a trivial fibration (that is, has the right lifting property with respect to monomorphisms). The case of right fibrations follows by duality. □

PROPOSITION 6.7. *Let $p : X \to Y$ be an inner Kan fibration between ∞-operads. If $\tau i^*(p)$ is a categorical fibration, then, for any monomorphism between normal dendroidal sets $A \to B$, the map*

$$k(B, X) \longrightarrow k(B, Y) \times_{k(A,Y)} k(A, X)$$

is a Kan fibration between Kan complexes.

Proof. The functor $i_! : s\mathcal{S}et \to d\mathcal{S}et$ being symmetric monoidal and preserving inner anodyne extensions, Proposition 3.1 implies that the map

$$\mathcal{H}om(B,X) \longrightarrow \mathcal{H}om(B,Y) \times_{\mathcal{H}om(A,Y)} \mathcal{H}om(A,X)$$

is an inner Kan fibration between ∞-categories. This implies that the map

$$k(B,X) \longrightarrow k(B,Y) \times_{k(A,Y)} k(A,X)$$

is an inner Kan fibration between ∞-categories.

We claim that this map has the right lifting property with respect to the inclusion $\{1\} \to \Delta[1]$. Using the identification (6.1), we see that lifting problems of shape

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & k(B,X) \arrow[d] \\ \Delta[1] \arrow[r] \arrow[ur, dotted] & k(B,Y) \times_{k(A,Y)} k(A,X) \end{tikzcd} \tag{6.5}$$

correspond to lifting problems of shape

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^{(\Delta[1])} \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & Y^{(\Delta[1])} \times_Y X \end{tikzcd} \tag{6.6}$$

so that our claim follows from Theorem 6.4.

More generally, the map $k(B,X) \to k(B,Y) \times_{k(A,Y)} k(A,X)$ has the right lifting property with respect to maps of shape

$$\partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{1\} \longrightarrow \Delta[n] \times \Delta[1], \quad n \geqslant 0. \tag{6.7}$$

We have just checked it above in the case where $n = 0$, so that it remains to prove the case where $n > 0$. Consider a lifting problem of shape

$$\begin{tikzcd} \partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{1\} \arrow[r, "u"] \arrow[d] & k(B,X) \arrow[d] \\ \Delta[n] \times \Delta[1] \arrow[r, "v"'] \arrow[ur, dotted, "g"] & k(B,Y) \times_{k(A,Y)} k(A,X)\,. \end{tikzcd} \tag{6.8}$$

This lifting problem gives rise to a lifting problem of shape

$$\begin{tikzcd} \partial\Omega[n] \otimes B \cup \Omega[n] \otimes A \arrow[r] \arrow[d] & X^{(\Delta[1])} \arrow[d] \\ \Omega[n] \otimes B \arrow[r] \arrow[ur, dotted, "h"] & Y^{(\Delta[1])} \times_Y X. \end{tikzcd} \tag{6.9}$$

The existence of the lifting h is provided again by Theorem 6.4. The map h defines a map

$$l : i_!(\Delta[n] \times \Delta[1]) \otimes B \longrightarrow X.$$

As a consequence, it is sufficient to check that, for every non-degenerate m-simplex $\delta : \Delta[m] \to \Delta[n] \times \Delta[1]$, $m \geqslant 1$, and for any object $b : \eta \to B$, the map $(li_!(\delta))_b : \Delta[m] \to i^*(X)$ factors through $k(i^*(X))$. Using the 'two out of three property' for weakly invertible 1-cells in $i^*(X)$, we can assume that $m = 1$. But then, as $n > 0$, using again the 'two out of three property' for weakly invertible 1-cells, we may assume that δ factors through $\partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{1\}$, which implies then that $(l \circ i_!(\delta)) \otimes 1_B$ factors through the subcomplex $i_!(\partial\Delta[n] \times \Delta[1] \cup \Delta[n] \times \{1\}) \otimes B$: the required property thus follows from the fact that the restriction of the transpose of h to the object $i_!(\Delta[n] \times \Delta[1] \cup \partial\Delta[n] \times \{1\}) \otimes B$ corresponds to the map u in (6.8).

By virtue of Lemma 6.6, the map $k(B, X) \to k(B, Y) \times_{k(A,Y)} k(A, X)$ is a left fibration, hence, by [**9**, Proposition 2.7], is conservative. By applying [**9**, Corollary 1.4], we deduce, from the case where $A = \emptyset$ and Y is the terminal dendroidal set, that $k(B, X)$ is a Kan complex for any normal dendroidal set B and any ∞-operad X. As any left fibration between Kan complexes is a Kan fibration (Lemma 6.5), the maps $k(B, X) \to k(A, X)$ are thus Kan fibrations between Kan complexes for any monomorphism between normal dendroidal sets $A \to B$ and any ∞-operad X. As a consequence, Kan fibrations being stable by pullback, we see that the fibre product $k(B, Y) \times_{k(A,Y)} k(A, X)$ is a Kan complex. Again, using Lemma 6.5, we conclude that $k(B, X) \to k(B, Y) \times_{k(A,Y)} k(A, X)$ is a Kan fibration between Kan complexes. □

Corollary 6.8. *For any normal dendroidal set A and any ∞-operad X, we have*

$$k(\mathcal{hom}(A, X)) = k(A, X).$$

For any inner Kan fibration between ∞-operads $p : X \to Y$ such that $\tau i^*(p)$ is a categorical fibration, and for any monomorphism between normal dendroidal sets $A \to B$, we have

$$k(\mathcal{hom}(B, Y) \times_{\mathcal{hom}(A,Y)} \mathcal{hom}(A, X)) = k(B, Y) \times_{k(A,Y)} k(A, X).$$

Proof. If A is normal, then, for any ∞-operad X, $k(A, X)$ is a Kan complex that contains $k(\mathcal{hom}(A, X))$. As $k(\mathcal{hom}(A, X))$ is the maximal sub Kan complex contained in the ∞-category $\mathcal{hom}(A, X)$, this proves the first assertion. The second assertion is proved similarly. □

Corollary 6.9. *Let $p : X \to Y$ be an inner Kan fibration between ∞-operads. If $\tau i^*(p)$ is a categorical fibration, then, for any anodyne extension of simplicial sets $K \to L$, the map*

$$X^{(L)} \longrightarrow Y^{(L)} \times_{Y^{(K)}} X^{(K)}$$

is a trivial fibration of dendroidal sets.

Proof. This follows from Proposition 6.7 and from the natural identification (6.1). □

Theorem 6.10. *A dendroidal set is J-fibrant if and only if it is an ∞-operad. An inner Kan fibration between ∞-operads $p : X \to Y$ is a J-fibration (that is, a fibration for the model category structure of Proposition* 3.12) *if and only if $\tau i^*(p)$ is a categorical fibration.*

Proof. Let $p : X \to Y$ be an inner Kan fibration between ∞-operads. We have to prove that, for $e = 0, 1$, the anodyne extension $\{e\} \to J$ induces a trivial fibration of dendroidal sets

$$X^{J_d} \longrightarrow Y^{J_d} \times_Y X$$

if and only if $\tau i^*(p)$ is a categorical fibration. But, for any ∞-operad Z, we clearly have $Z^{J_d} = Z^{(J)}$ and $Z = Z^{(\{e\})}$. Hence, by virtue of Corollary 6.9, if $\tau i^*(p)$ is a categorical fibration, then p is a J-fibration. The converse is a direct consequence of [**9**, Corollary 1.6]. □

Corollary 6.11. *The class of weak operadic equivalences is the smallest class of maps of dendroidal sets W that satisfies the following three properties.*

(a) (*'Two out of three property'*) *In any commutative triangle, if two maps are in W, then so is the third.*

(b) *Any inner anodyne extension is in W.*

(c) *Any trivial fibration between ∞-operads is in W.*

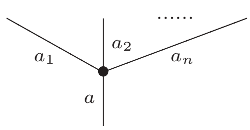
Proof. Consider a class of maps W satisfying conditions (a), (b) and (c) above. We want to prove that any weak operadic equivalence is in W.

Let $f : A \to B$ be a morphism of dendroidal sets. Using the small object argument applied to the set of inner horns, we can see that there exists a commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"] & Y, \end{tikzcd}$$

in which the maps a and b are inner anodyne extensions, and X and Y are ∞-operads. It is clear that f is a weak operadic equivalence (respectively, is in W) if and only if p has the same property. Hence it is sufficient to prove that any weak operadic equivalence between ∞-operads is in W. As any trivial fibration between ∞-operads is in W by assumption, and as ∞-operads are the fibrant objects of a model category, this corollary follows from Ken Brown's Lemma [8, Lemma 1.1.12]. □

6.12. We will write C_n for the corolla with $n+1$ edges as follows.

$C_n =$ 

Let X be an ∞-operad. Given an $(n+1)$-tuple of 0-cells $(x_1, \dots, x_n, x)$ in X, the space of maps $X(x_1, \dots, x_n; x)$ is obtained by the pullback below, in which the map p is the map induced by the inclusion $\eta \amalg \dots \amalg \eta \to \Omega[C_n]$ (with $n+1$ copies of η, corresponding to the $n+1$ objects $(a_1, \dots, a_n, a)$ of C_n).

$$\begin{tikzcd} X(x_1, \dots, x_n; x) \arrow[r] \arrow[d] & \mathcal{H}om(\Omega[C_n], X) \arrow[d, "p"] \\ \eta \arrow[r, "{(x_1, \dots, x_n, x)}"'] & X^{n+1} \end{tikzcd}$$

Using the identification $s\mathcal{S}et = d\mathcal{S}et/\eta$, we shall consider $X(x_1, \dots, x_n; x)$ as a simplicial set.

Proposition 6.13. *The simplicial set $X(x_1, \dots, x_n; x)$ is a Kan complex.*

Proof. The first assertion of Corollary 6.8 for $A = \Omega[C_n]$ can be reinterpreted by saying we have the pullback square below (see Remark 6.2).

$$\begin{tikzcd} k(\mathcal{h}om(\Omega[C_n], X)) \arrow[r] \arrow[d] & \mathcal{H}om(\Omega[C_n], X) \arrow[d] \\ k(i^* X)^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

As the terminal simplicial set η is certainly a Kan complex, it thus follows from the construction of $X(x_1, \ldots, x_n; x)$ that we have a pullback square

$$\begin{tikzcd} X(x_1,\ldots,x_n;x) \arrow[r] \arrow[d] & k(\mathcal{hom}(\Omega[C_n],X)) \arrow[d] \\ \eta \arrow[r, "(x_1,\ldots,x_n,x)"'] & k(i^*X)^{n+1}, \end{tikzcd}$$

in which the right vertical map in this diagram is a Kan fibration (by Proposition 6.7, applied for $A = \eta \amalg \ldots \amalg \eta$ and $B = \Omega[C_n]$). The stability of Kan fibrations by pullback achieves the proof. □

Proposition 6.14. *There is a canonical bijection*

$$\pi_0(X(x_1,\ldots,x_n;x)) \simeq \tau_d(X)(x_1,\ldots,x_n;x).$$

Proof. We will use the explicit description of $\tau_d(X)$ given by [**13**, Lemma 6.4 and Proposition 6.6]. The unit map $X \to N_d\tau_d(X)$ induces a map

$$X(x_1,\ldots,x_n;x) \longrightarrow (N_d\tau_d(X))(x_1,\ldots,x_n;x).$$

It is easily seen that $(N_d\tau_d(X))(x_1,\ldots,x_n;x)$ is the discrete simplicial set associated to $\tau_d(X)(x_1,\ldots,x_n;x)$, so that we get a surjective map

$$\pi_0(X(x_1,\ldots,x_n;x)) \longrightarrow \tau_d(X)(x_1,\ldots,x_n;x).$$

Using the explicit description of $\tau_d(X)$ given by [**13**, Lemma 6.4 and Proposition 6.6], it is now sufficient to prove that, if f and g are two 0-simplices of $X(x_1,\ldots,x_n;x)$ homotopic along the edge 0 in the sense of [**13**, Definition 6.2], then they belong to the same connected component. But then, f and g define two objects of $\tau(\mathcal{hom}(\Omega[C_n],X))$ that are isomorphic. This can be expressed by the existence of a map

$$h : \Delta[1] \longrightarrow k(\mathcal{hom}(\Omega[C_n],X)),$$

which connects f and g. Using the fact that $k(\mathcal{hom}(\Omega[C_n],X)) \to i^*(X)^{n+1}$ is a Kan fibration between Kan complexes, we can see by a path lifting argument that such a map h is homotopic under $\partial\Delta[1]$ to a map $\Delta[1] \to X(x_1,\ldots,x_n;x)$ that connects f and g. □

Lemma 6.15. *Let $X \to Y$ be a trivial fibration between ∞-operads. Then, for any $(n+1)$-tuple of 0-cells $(x_1,\ldots,x_n,x)$ in X, the induced map*

$$X(x_1,\ldots,x_n;x) \longrightarrow Y(f(x_1),\ldots,f(x_n);f(x))$$

is a trivial fibration of simplicial sets.

Proof. We know that the map

$$\mathcal{H}om(\Omega[C_n],X) \longrightarrow \mathcal{H}om(\Omega[C_n],Y) \times_{Y^{n+1}} X^{n+1}$$

is a trivial fibration (this follows from Proposition 1.9 by adjunction). As we have a pullback of shape

$$\begin{tikzcd} X(x_1,\dots,x_n;x) \arrow[r] \arrow[d] & \mathcal{H}om(\Omega[C_n],X) \arrow[d] \\ Y(f(x_1),\dots,f(x_n);f(x)) \arrow[r] & \mathcal{H}om(\Omega[C_n],Y)\times_{Y^{n+1}} X^{n+1}, \end{tikzcd}$$

this proves the lemma. □

Proposition 6.16. *The functor $\tau_d : d\mathcal{S}et \to \mathcal{O}perad$ sends weak operadic equivalences to equivalences of operads.*

Proof. We know that τ_d sends inner horn inclusions to isomorphisms of operads (this follows from [**13**, Theorem 6.1] by the Yoneda Lemma). As τ_d preserves colimits, we deduce that τ_d sends inner anodyne extensions to isomorphisms of operads. By virtue of Corollary 6.11, it is thus sufficient to prove that τ_d sends trivial fibrations between ∞-operads to equivalences of operads. Let $f : X \to Y$ be a trivial fibration between ∞-operads. By virtue of Proposition 6.14 and Lemma 6.15, we see that $\tau_d(f)$ is fully faithful. As f is obviously surjective on 0-cells, $\tau_d(f)$ has to be an equivalence of operads. □

Corollary 6.17. *The adjunction $\tau_d : d\mathcal{S}et \rightleftarrows \mathcal{O}perad : N_d$ is a Quillen pair. Moreover, the two functors τ_d and N_d both preserve weak equivalences. In particular, a morphism of operads is an equivalence of operads if and only if its dendroidal nerve is a weak operadic equivalence.*

Proof. The functor τ_d preserves cofibrations, so that this is a direct consequence of Proposition 6.16. Note that any operad is fibrant, so that the dendroidal nerve functor N_d preserves weak equivalences. Hence the last assertion comes from the fact that N_d is fully faithful and τ_d preserves weak equivalences. □

Remark 6.18. Theorem 6.10 also asserts that the functor τ_d preserves fibrations between ∞-operads.

6.19. If A is a normal dendroidal set, and if X is an ∞-operad, then we have

$$\mathrm{Hom}_{\mathrm{Ho}(d\mathcal{S}et)}(A,X) = [A,X] \simeq \pi_0(k(\mathit{hom}(A,X))). \tag{6.10}$$

Indeed, $J \otimes A$ is a cylinder of A, and morphisms

$$J \otimes A \longrightarrow X$$

correspond to morphisms

$$J \longrightarrow k(\mathit{hom}(A,X)),$$

so that this formula follows from the fact that X is J-fibrant. The next statement is a reformulation of (6.10).

Proposition 6.20. *Let A be a normal dendroidal set, and X an ∞-operad. The set $[A,X] = \mathrm{Hom}_{\mathrm{Ho}(d\mathcal{S}et)}(A,X)$ can be canonically identified with the set of isomorphism classes of objects in the category $\tau\mathit{hom}(A,X)$ (which is also the category underlying $\tau_d(\mathcal{H}om(A,X))$).*

Proof. This proposition is a direct application of the explicit description of the operad $\tau_d(\mathcal{H}om(A, X))$ given by [**13**, Proposition 6.6] and of (6.10). □

Appendix A. *Grafting orders onto trees*

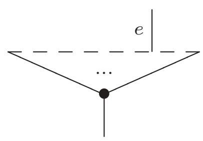
The main goal of the technical Sections 4 and 5 was to deduce Theorem 6.4, and from it, Corollary 6.9, which is the main step towards the proof of Theorem 6.10. There is an asymmetry in this approach, in that Theorem 6.4 was only proved for evaluation at one of the end points, and the symmetry was established in Corollary 6.9 by using the theory of left fibrations between simplicial sets.

In these two appendices, we will prove the analogues of Theorems 4.2 and 5.2, from which one can deduce directly the symmetric version of Theorem 6.4 (for evaluation at 0). These two appendices can also be used as an alternative approach to the results in Section 6. Moreover, they are of interest by themselves, as they form the basis of a theory of right fibrations of dendroidal sets.

However, as the left–right duality for simplicial sets does not extend to dendroidal sets, the results of these appendices cannot be deduced from their analogues proved earlier.

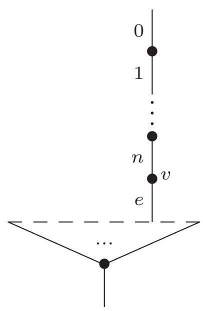
We begin by studying the analogue of Theorem 4.2 (see Theorem A.7).

A.1. Let T be a tree endowed with an input edge (leaf) e.

$T =$ 

(A.1)

Given an integer $n \geqslant 0$, we define the tree $n \star_e T$ as the tree obtained by joining the n-simplex to the edge e by a new vertex v.

$n \star_e T =$ 

(A.2)

This defines a unique functor

$$\Delta \longrightarrow \Omega, \quad [n] \longmapsto n \star_e T \tag{A.3}$$

such that the obvious inclusions $i[n] \to n \star_e T$ are functorial. We thus get a functor

$$(-) \star_e T : \Delta \longrightarrow T/d\mathcal{S}et \tag{A.4}$$

(where $T/d\mathcal{S}et$ denotes the category of dendroidal sets under $\Omega[T]$). By Kan extension, we obtain a colimit preserving functor

$$(-) \star_e T : s\mathcal{S}et \longrightarrow T/d\mathcal{S}et. \tag{A.5}$$

We have $\Delta[n] \star_e T = \Omega[n \star_e T]$. The functor (A.5) has a right adjoint

$$(-)/_e T : T/d\mathcal{S}et \longrightarrow s\mathcal{S}et. \tag{A.6}$$

Remark A.2 (Functoriality in T). We shall say that a face map $R \to T$ is *e-admissible* if it does not factor through the external face map that chops off e. For such a face $R \to T$, e is also a leaf of R, and there are natural maps

$$n \star_e R \longrightarrow n \star_e T. \tag{A.7}$$

Thus, we obtain, for each simplicial set K, and each dendroidal set X under T (that is, under $\Omega[T]$), natural maps

$$K \star_e R \longrightarrow K \star_e T \tag{A.8}$$

and

$$X/_e T \longrightarrow X/_e R. \tag{A.9}$$

Similarly, the inclusions $\Omega[n] \to \Delta[n] \star_e T$ induce a projection

$$X/_e T \longrightarrow i^*(X) \tag{A.10}$$

for any dendroidal set X under T.

A.3. Let $0 < i \leqslant n$ be integers. Let $\{R_1, \dots, R_t\}$, $t \geqslant 1$, be a finite family of e-admissible faces of T, and define

$$C \subset D \subset \Omega[n \star_e T]$$

by

$$C = \left(\bigcup_{s=1}^{t} \Lambda^i[n] \star_e R_s \right) \cup \Omega[n] \quad \text{and} \quad D = \bigcup_{s=1}^{t} \Delta[n] \star_e R_s,$$

where $\Omega[n]$ is seen as a subcomplex of $\Omega[n \star_e T]$ through the canonical embedding.

Lemma A.4. *Under the assumptions of* A.3*, the map $C \to D$ is an inner anodyne extension.*

Proof. For $p \geqslant 1$, write $\mathcal{F}_p$ for the set of faces F of $\Omega[n \star_e T]$ that belong to D but not to C, and are of the form $F = \Omega[n \star_e R]$ for an e-admissible face R of T with exactly p edges. Define a filtration

$$C = C_0 \subset C_1 \subset \ldots \subset C_p \subset \ldots \subset D$$

by

$$C_p = C_{p-1} \cup \bigcup_{F \in \mathcal{F}_p} F, \quad p \geqslant 1.$$

We have $D = C_p$ for p big enough, and it is sufficient to prove that the inclusions $C_{p-1} \to C_p$ are inner anodyne for $p \geqslant 1$. If F and F' are distinct in $\mathcal{F}_p$, then $F \cap F'$ is in C_{p-1}. Moreover, if $F = \Omega[n \star_e R]$ for an e-admissible face R of T, then we have

$$F \cap C_{p-1} = \Lambda^i[n \star_e R],$$

which is an inner horn. Hence we can describe the inclusion $C_{p-1} \to C_p$ as a finite composition of pushouts by inner horn inclusions of shape $F \cap C_{p-1} \to F$ for $F \in \mathcal{F}_p$. □

Proposition A.5. *Let $0 < i \leqslant n$ be integers. The inclusion*

$$(\Lambda^i[n] \star_e T) \cup \Omega[n] \longrightarrow \Omega[n \star_e T]$$

is an inner anodyne extension.

Proof. Apply Lemma A.4. □

PROPOSITION A.6. *For any inner Kan fibration $p : X \to Y$ under T, the morphism $X/_e T \to Y/_e \times_{i^*(Y)} i^*(X)$ is a right fibration of simplicial sets.*
In particular, for any ∞-operad X under T, the map $X/_e T \to i^*(X)$ is a right fibration between ∞-categories.

Proof. This follows from Proposition A.5 by a standard adjunction argument. □

THEOREM A.7. *Let S be a tree with at least two vertices, let v be a unary top vertex in S, and let $p : X \to Y$ be an inner Kan fibration between ∞-operads. Then any solid commutative square of the form*

$$\begin{tikzcd} \Lambda^v[S] \arrow[r, "\varphi"] \arrow[d] & X \arrow[d, "p"] \\ \Omega[S] \arrow[r, "\psi"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

in which $\varphi(v)$ is weakly invertible in X, has a diagonal filling h.

Proof. The tree S has to be of shape $S = 1 \star_e T$ for a tree T with a given leaf e. Under this identification, we have $\Lambda^v[S] = \Lambda^0[1 \star_e T]$. A lifting h in the solid commutative square

$$\begin{tikzcd} \Lambda^0[1 \star_e T] \arrow[r, "\varphi"] \arrow[d] & X \arrow[d, "p"] \\ \Omega[1 \star_e T] \arrow[r, "\psi"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

is thus equivalent to a lifting k in the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "\tilde{\varphi}"] \arrow[d] & P \arrow[d] \\ \Delta[1] \arrow[r, "\tilde{\psi}"] \arrow[ur, dotted, "k"] & Q, \end{tikzcd}$$

in which $P = X/_e T$ and $Q = U \times_W V$, with

$$U = \varprojlim_R X/_e R\ , \quad V = Y/_e T\ , \quad W = \varprojlim_R Y/_e R,$$

where R ranges over all the proper e-admissible faces of T. As in the proof of Theorem 4.2, it is now sufficient to prove the three following properties.

(i) The map $P \to Q$ is a right fibration.
(ii) The simplicial set Q is an ∞-category.
(iii) If $\varphi(x)$ is weakly invertible in X, then so is the 1-cell $\tilde{\psi}$ in Q.

Properties (ii) and (iii) will follow from the two assertions below.

(iv) The map $V \to W$ is a right fibration.
(v) The map $U \to i^*(X)$ is a right fibration.

As (iv) is a particular case of (i), we are thus reduced to prove (i) and (v).

A.7.1. *Proof of* (i).

A lifting problem of shape

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & P \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Q \end{tikzcd} \qquad 0 < i \leqslant n$$

is equivalent to a lifting problem of shape

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Omega[n \star_e T] \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

where C is the union of $\Lambda^i[n] \star_e T$ with the union of the faces of $\Omega[n \star_e T]$ that are of the form $n \star_e S \to n \star_e T$, where S ranges over the e-admissible elementary faces of T. In other words, $C = \Lambda^i[n \star_e T]$ is an inner horn, so that the required lifting exists, because p is assumed to be an inner Kan fibration.

A.7.2. *Proof of* (v).

A lifting problem of shape

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & U \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & i^*(X) \end{tikzcd} \qquad 0 < i \leqslant n$$

is equivalent to a lifting problem of shape

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ D \arrow[ur, dotted] & \end{tikzcd}$$

in which the inclusion $C \to D$ can be described as follows:

$$C = \Omega[n] \cup \bigcup_R \Lambda^i[n] \star_e R \subset D = \bigcup_R \Omega[n \star_e R] \subset \Omega[n \star_e T],$$

where R ranges over the e-admissible elementary faces of T. It is easily seen that the inclusion $C \to D$ is an inner anodyne extension by Lemma A.4. $\square$

Appendix B. *Another subdivision of cylinders*

B.1. We will refer to the horn inclusions of shape $\Lambda^x[S] \to \Omega[S]$, where S is a tree with a unary top vertex x, as *end extensions*. A composition of pushouts of end extensions will be called an *end anodyne* map.

The goal of this section is to prove a dual version of Theorem 5.2, namely:

Theorem B.2. *Let T be a tree with at least one vertex, and consider the subobject*

$$B_0 = \{0\} \otimes \Omega[T] \cup \Delta[1] \otimes \partial\Omega[T] \subset \Delta[1] \otimes \Omega[T].$$

There exists a filtration of $\Delta[1] \otimes \Omega[T]$ of the form

$$B_0 \subset B_1 \subset \ldots \subset B_{N-1} \subset B_N = \Delta[1] \otimes \Omega[T]$$

where, for each i, $0 \leqslant i < N$, the map $B_i \to B_{i+1}$ is either inner anodyne or end anodyne.

Moreover, the end anodyne maps are all pushouts of the form

$$\begin{tikzcd} \Lambda^v[S] \arrow[r] \arrow[d] & B_i \arrow[d] \\ \Omega[S] \arrow[r] & B_{i+1} \end{tikzcd}$$

with the following properties.

(i) *The tree S has at least two vertices, and v is a unary top vertex;*

(ii) *The map*

$$\Delta[1] \longrightarrow \Lambda^v[S] \longrightarrow B_i \subset \Delta[1] \otimes \Omega[T],$$

corresponding to the vertex v in S, coincides with an inclusion of shape

$$\Delta[1] \otimes \{t\} \longrightarrow \Delta[1] \otimes \Omega[T]$$

for some edge t in T.

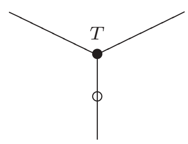
B.3. As in the proof of Theorem 5.2, we will follow the convention of [**13**], and write

$$\Omega[S] \otimes \Omega[T] = \bigcup_{i=1}^{m} \Omega[T_i],$$

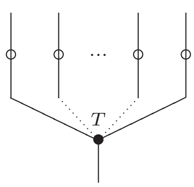
where the union ranges over the partially ordered set of percolation schemes, starting with a number of copies of T grafted on top of S, and ending with the reverse grafting. For

$S = [1] =$  ,

the first tree is of shape

$T_1 =$  , (B.1)

and the last one is

$T_m =$  . (B.2)

Let us fix a linear order on the percolation schemes for $\Delta[1] \otimes \Omega[T]$ that extends the natural partial order. Such a linear ordering induces a filtration on the tensor product $\Delta[1] \otimes \Omega[T]$,

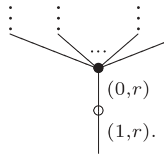
$$C_0 \subset C_1 \subset \ldots \subset C_{m-1} \subset C_m = \Delta[1] \otimes \Omega[T] \tag{B.3}$$

by setting

$$C_0 = B_0 = \{0\} \otimes \Omega[T] \cup \Delta[1] \otimes \partial\Omega[T] \text{ and } C_i = B_0 \cup \Omega[T_1] \cup \cdots \cup \Omega[T_i], \tag{B.4}$$

the filtration of Theorem B.2 will be a refinement of this one.

Let us start by considering T_1. If the root edge of T is called r, then T_1 looks like



(B.5)

With the exception of the faces $\partial_{(0,r)}(T_1)$ (which contracts $(0,r)$) and $\partial_{(1,r)}(T_1)$ (which chops off $(1,r)$ as well as the white vertex), any face F of T_1 misses a colour of T (by this, we mean there is an edge a in T such that no edge in F is named (i,a)). Hence, $\Omega[F] \subset \Delta[1] \otimes \partial\Omega[T]$ for these F. Moreover, $\partial_{(1,r)}(T_1) = \{0\} \otimes T_1$. So $\Omega[T_1] \cap B_0 = \Lambda^{(0,r)}[T_1]$, and

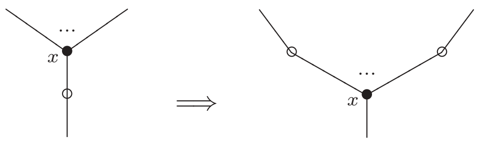
$$\begin{tikzcd} \Lambda^{(0,r)}[T_1] \arrow[r] \arrow[d] & B_0 \arrow[d] \\ \Omega[T_1] \arrow[r] & B_0 \cup \Omega[T_1] \end{tikzcd} \tag{B.6}$$

is a pushout. So, if we let $B_1 = C_1$, then $B_0 \to B_1$ is obviously an inner anodyne.

Suppose we have defined a filtration up to some B_l

$$B_0 \subset B_1 \subset \ldots \subset B_l \quad l \geqslant 1, \tag{B.7}$$

so that $B_l = C_k$ for some k, $1 \leqslant k \leqslant m$. We will extend this filtration as $B_l \subset B_{l+1} \subset \ldots \subset B_{l'}$, so that $B_{l'} = C_{k+1}$. The percolation scheme T_{k+1} is obtained from an earlier one T_j by pushing a white vertex in T_j one step up through a black vertex x, as in

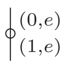


(B.8)

in T_j in T_{k+1}

(we have denoted by x the black vertex in both trees, although it would be more accurate to write x for the relevant vertex of T, and write $0 \otimes x$ and $1 \otimes x$ for the corresponding vertices in T_j and T_{k+1} respectively). The Boardman–Vogt relation states that, as subobjects of $\Delta[1] \otimes \Omega[T]$, the face of T_{k+1} obtained by contracting all input edges of x coincides with the face of T_j obtained by contracting the output edge of x in T_j. In particular, note that if x has no input edges at all (that is, if x is a 'nullary operation' in T), then T_{k+1} is a face of T_j, so $C_{k+1} = C_k$, and we let $B_{l'} = B_l$, and there is nothing proven. Therefore, from now on, we will assume that the set of input edges of x, denoted *input*(x), is non-empty, and we proceed as follows.

Let E be the set of all colours (edges) e in T for which



(B.9)

occurs in T_{k+1}. For $U \subset E$, let

$$T_{k+1}^{(U)} \subset T_{k+1} \tag{B.10}$$

be the face given by contracting all the edges $(1,e)$ for $e \in E$ but e *not* in U. Note that if $U \cap$ *input*$(x) = \emptyset$, then $\Omega[T_{k+1}^{(U)}] \subset B_l$ by the Boardman–Vogt relation just mentioned. Therefore, we will only consider U with $U \cap$ *input*$(x) \neq \emptyset$. We will successively adjoin $\Omega[T_{k+1}^{(U)}]$ to B_l for larger and larger such U, until we reach the case where $U = E$ and $T_{k+1}^{(U)} = T_{k+1}$.

If $U = \{e\}$ is a singleton (with e an input edge of x), then the face $\partial_{(1,e)}\Omega[T^{(\{e\})}_{k+1}]$ is contained in B_l as said, whereas the face $\partial_{(0,e)}\Omega[T^{(\{e\})}_{k+1}]$ is not (it cannot belong to an earlier T_i, and is obviously not contained in $B_0 = C_0$). Any other face of $T^{(\{e\})}_{k+1}$ misses a colour of T and hence is contained in B_0. Thus,

$$\Omega[T^{(\{e\})}_{k+1}] \cap B_l \subset \Omega[T^{(\{e\})}_{k+1}] \tag{B.11}$$

is either an inner horn (if $(0, e)$ is an inner edge of T_{k+1}) or an end extension (if $(1, e)$ is an input edge of T_{k+1}). In either case, we can adjoin $\Omega[T^{(\{e\})}_{k+1}]$ by forming the pushout below.

$$\begin{tikzcd} \Omega[T^{(\{e\})}_{k+1}] \cap B_l \arrow[r] \arrow[d] & B_l \arrow[d] \\ \Omega[T^{(\{e\})}_{k+1}] \arrow[r] & \Omega[T^{(\{e\})}_{k+1}] \cup B_l \end{tikzcd} \tag{B.12}$$

We successively adjoin $\Omega[T^{(\{e\})}_{k+1}]$ to B_l in this way for all e in E that are input edges of x in T: if these are $e_1, \ldots, e_p$, let

$$B_{l+r} = B_l \cup \Omega[T^{(\{e_1\})}_{k+1}] \cup \ldots \cup \Omega[T^{(\{e_r\})}_{k+1}]. \tag{B.13}$$

Then, for each $r < p$, the map $B_{l+1} \to B_{l+r+1}$ is inner anodyne or end anodyne.

In general, we proceed by induction on U. Choose $U \subset E$ with $U \cap \mathit{input}(x) \neq \emptyset$, and assume we have adjoined $\Omega[T^{(U')}_{k+1}]$ already, for all U' of smaller cardinality than U. We will write $B_{l''}$ for the last object in the filtration constructed up to that point. Fix an order on the set of elements of U, and write it as

$$U = \{\alpha_1, \ldots, \alpha_s\}. \tag{B.14}$$

Consider $\Omega[T^{(U)}_{k+1}]$. The tree $T^{(U)}_{k+1}$ has edges $(0, c)$ or $(1, c)$ for c not in E, and the corresponding face misses the colour c altogether, hence $\partial_{(i,c)}\Omega[T^{(U)}_{k+1}]$ is contained in B_0 for these c. Next, the tree $T^{(U)}_{k+1}$ has edges coloured $(1, \alpha_i)$ for $1 \leqslant i \leqslant s$, and contracting any of these gives a face

$$\partial_{(1,\alpha_i)}\Omega[T^{(U)}_{k+1}] = \Omega[T^{(U-\{\alpha_i\})}_{k+1}], \tag{B.15}$$

which is contained in $B_{l''}$ by the inductive assumption on U. None of the faces given by contracting (if it is inner) or by chopping off (if it is outer) an edge $(0, \alpha_i)$ in $T^{(U)}_{k+1}$ can be contained in $B_{l''}$, however.

Let $A_1, \ldots, A_t$ be all the subsets of the set of these edges $\{(0, \alpha_1), \ldots, (0, \alpha_s)\}$ of $T^{(U)}_{k+1}$ that contain $(0, \alpha_1)$, and order them by some linear order extending the inclusion order. So there are $t = 2^{s-1}$ such A_i, and we could fix the order to be

$$\begin{aligned} A_1 &= \{(0, \alpha_1)\} \\ A_2 &= \{(0, \alpha_1), (0, \alpha_2)\} \\ &\vdots \qquad \vdots \\ A_s &= \{(0, \alpha_1), (0, \alpha_s)\} \\ A_{s+1} &= \{(0, \alpha_1), (0, \alpha_2), (0, \alpha_3)\} \\ &\vdots \qquad \vdots \\ A_t &= \{(0, \alpha_1), \ldots, (0, \alpha_s)\}\,. \end{aligned}$$

For $q = 1, \ldots, t$, let $T_{k+1}^{(U,q)}$ be the tree obtained from $T_{k+1}^{(U)}$ by contracting or chopping off all the edges $(0, \alpha_i)$ not in A_q. So

$$T_{k+1}^{(U,1)} = \partial_{(0,\alpha_s)} \partial_{(0,\alpha_{s-1})} \ldots \partial_{(0,\alpha_2)} T_{k+1}^{(U)} \tag{B.16}$$

and

$$T_{k+1}^{(U,t)} = T_{k+1}^{(U)}. \tag{B.17}$$

We will successively adjoin these $\Omega[T_{k+1}^{(U,q)}]$ to the filtration, to form the part

$$B_{l''} \subset B_{l''+1} \subset \ldots \subset B_{l''+t} = B_{l''} \cup \Omega[T_{k+1}^{(U)}] \tag{B.18}$$

of the filtration, as

$$B_{l''+q} = B_{l''} \cup \Omega[T_{k+1}^{(U,1)}] \cup \ldots \Omega[T_{k+1}^{(U,q)}]. \tag{B.19}$$

We start with $T_{k+1}^{(U,1)}$. The only face of $\Omega[T_{k+1}^{(U,1)}]$ not contained in $B_{l''}$ is the one given by the edge $(0, \alpha_1)$. Thus

$$\Omega[T_{k+1}^{(U,1)}] \cap B_{l''} \subset \Omega[T_{k+1}^{(U,1)}] \tag{B.20}$$

is either an inner horn (if $(0, \alpha_1)$ is an inner edge) or an end extension (if $(0, \alpha_1)$ is an input edge of $T_{k+1}^{(U)}$, that is, α_1 is an input edge of T). So the pushout $B_{l''} \to B_{l''+1}$ is either inner anodyne or end anodyne.

Suppose we have adjoined $\Omega[T_{k+1}^{(U,q')}]$ for all $1 \leqslant q' < q$, so have arrived at the stage $B_{l''+q-1}$ of the filtration. Consider now A_q and the corresponding dendroidal set $\Omega[T_{k+1}^{(U,q)}]$. As before, its faces given by edges coloured by (i, c) for $i = 0, 1$ with c not in E are contained in B_0, and its faces given by edges coloured $(1, e)$ with $e \in U$ are contained in $\Omega[T_{k+1}^{(U')}]$ for a smaller $U' = U - \{e\}$, hence are contained in $B_{l''}$. Let us consider the remaining faces given by the edges $(0, \alpha_1), \ldots, (0, \alpha_r)$ in A_q. If $i \neq 1$, the face of $\Omega[T_{k+1}^{(U,q)}]$ given by $(0, \alpha_i) \in A_q$ is contained in $\Omega[T_{k+1}^{(U,q')}]$ for some $q' < q$ with $A_{q'} = A_q - \{(0, \alpha_i)\}$. So the only face that is missing is the one given by $(0, \alpha_1)$, that is,

$$\Omega[T_{k+1}^{(U,q)}] \cap B_{l''+q-1} = \Lambda^{(0,\alpha_1)}[T_{k+1}^{(U,q)}]\,. \tag{B.21}$$

Therefore, the induced pushout $B_{l''+q-1} \to B_{l''+q}$ is either inner anodyne or end anodyne (depending on whether α_1 is an external edge of T or not). At the end, when $q = t$, we have adjoined all of $\Omega[T_{k+1}^{(U)}]$.

This completes the construction of the segment of the filtration for the subset $U \subset E$. As said, we continue this construction until we reach the stage $U = E$, when $T_{k+1}^{(U)} = T_{k+1}$, which completes the construction of the segment of the filtration from B_l until $B_{l'}$, interpolating between C_k and C_{k+1}. This completes the description of the filtration. From it, the last part of the theorem is clear.

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ERRATUM TO THE ARTICLE 'DENDROIDAL SETS AS MODELS FOR HOMOTOPY OPERADS'

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Proposition 1.9 of our paper [1] is wrong as stated, and should be modified as we will explain below. This modification does not affect any of the main results of this paper and its two sequels [2, 3]: the existence of the model structure on dendroidal sets in [1], the equivalent model structures for dendroidal complete Segal spaces and for Segal operads in [2], and the Quillen equivalence to the model category of simplicial operads in [3]. However, the error does affect all the statements concerning the monoidality of the model structures.

Recall from [1] the inclusion $i\colon \Delta \to \Omega$, and the induced left Quillen functor $i_!\colon s\mathcal{S}et \to d\mathcal{S}et$. Let us call a tree (an object of Ω) *linear* if it lies in the image of i, and a dendroidal set *simplicial* if it lies in the image of $i_!$. Also, let us call a tree *open* if it has no vertices of valence zero (i.e. no vertices without input edges). If $S \to T$ is a morphism in Ω and T is open, then S is necessarily open as well. Therefore, the open trees define a subobject U of the terminal object in $d\mathcal{S}et$. Let us call a dendroidal set *open* if the unique map to the terminal object factors through U. These open dendroidal sets form a full subcategory $d\mathcal{S}et/U$ of $d\mathcal{S}et$. More generally, we call a normal monomorphism $X \to Y$ *linear* if it is obtained by attaching linear trees (i.e. $X \to Y$ lies in the saturation of $\partial\Omega[T] \to \Omega[T]$ for T linear) and *open* if it is obtained by attaching open trees.

The modified version of Proposition 1.9 in [1] should be:

Proposition 1.9. *Let $A \to B$ and $X \to Y$ be normal monomorphisms. If one of them is linear or both are open, then the induced map*

$$A \otimes Y \cup_{A\otimes X} B \otimes X \to B \otimes Y$$

is again a normal monomorphism (and is open as well in the second case).

By the usual induction, this follows from the following lemma (whose proof is an elementary but tedious combinatorial argument, and we refer the reader to [4] for the details), which should be added at the very end of the first section of [1]:

Lemma 1.11. *Let S and T be trees. If one of them is linear or both are open, then the pushout-product map*

$$\partial\Omega[S] \otimes \Omega[T] \cup_{\partial\Omega[S]\otimes\partial\Omega[T]} \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$$

is a normal monomorphism (and is open as well in the second case).

The main result of [5] (namely Proposition 9.2) is wrong as stated, but its proof says the following: given two trees S and T, as well as an inner edge e of S, if W denotes the the image of the map

$$\Lambda^e[S] \otimes \Omega[T] \cup_{\Lambda^e[S]\otimes\partial\Omega[T]} \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]\,,$$

then the inclusion

$$W \subset \Omega[S] \otimes \Omega[T]$$

is inner anodyne. This means that [1, Proposition 3.1] should be replaced by the following statement.

Proposition 3.1. *Let $A \to B$ and $X \to Y$ be normal monomorphisms. If one of them is linear or both are open, and if one of them is inner anodyne, then the induced map*

$$A \otimes Y \cup_{A \otimes X} B \otimes X \to B \otimes Y$$

is again an inner anodyne extension.

Proposition 3.3 of [1] should be modified accordingly (replacing, in Proposition 3.1 above, the expression "inner anodyne" by "J-anodyne").

Let us call a model category $\mathcal{M}$ *Joyal simplicial* if it satisfies the axioms for a simplicial model category, but with respect to the Joyal model structure on simplical sets instead of the classical Kan-Quillen structure. The model structure on $d\mathcal{S}et$ established in [1] is not monoidal. Instead, Proposition 3.17 of [1] should be replaced by the following result, which follows immediately from the propositions above and from the arguments explained in the original 'proof' of [1, Prop. 3.17]:

Proposition 3.17. *The Boardman-Vogt tensor product turns the model structure on $d\mathcal{S}et$ into a Joyal simplicial model structure, and induces a symmetric monoidal model structure on the category $d\mathcal{S}et/U$ of open dendroidal sets.*

In the proof of the existence of the model structure, it is only the "linear half" of Propositions 1.9 and 3.1 which are used, and this proof is unaffected. Indeed, [1, Propositions 3.1 and 3.3] are used only in the case where one of the maps is in the image of $i_!$.

We close this erratum with a list of places where the reference to (consequences of) monoidality should be reformulated in accordance with the previous two propositions: In paper [1], these are point 1 in the introduction, Proposition 2.6(c), Corollary 2.8(b), Corollary 2.9(b); In the proof of Lemma 6.15, the first sentence should begin as 'We know that the image by the functor i^* of the map...'; Also the statement between brackets at the very end of Proposition 6.20 should be skipped (and the proof of Prop. 6.20 should refer to Boardman and Vogt's explicit description of the category associated to a quasi-category, instead of its dendroidal analogue). In [2, 3], no use of the alleged monoidality of the model structure is made, and the necessary changes all concern inessential references to the monoidality in [1]: in paper [2], the places where we recall the monoidality from paper [1], are in the abstract and the introduction, in Theorem 1.1, Remarks 6.14 and 8.16, and finally in the proof of Propposition 6.11, where it is Proposition 3.1 above which should be used rather than the monoidality; and in paper [3], the mentioning of monoidality is in the introduction as well as in Proposition 2.8 and Theorem 5.7.

We repeat that this error only affects the monoidality of the model structures, and none of the main results about the existence of the model structure on $d\mathcal{S}et$ [1] and the Quillen equivalent model categories presented in [2, 3].

References

[1] D.-C. Cisinski and I. Moerdijk, *Dendroidal sets as models for ∞-operads*, J. Topol. **4** (2011), no. 2, 257–299.

[2] ———, *Dendroidal Segal spaces and ∞-operads*, J. Topol. **6** (2013), no. 3, 675–704.

[3] ———, *Dendroidal sets and simplicial operads*, J. Topol. **6** (2013), no. 3, 705–756.

[4] ———, *Note on the tensor product of dendroidal sets*.

[5] I. Moerdijk and I. Weiss, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. **221** (2009), 343–389.