BATANIN HIGHER GROUPOIDS
AND HOMOTOPY TYPES

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To Ross Street for his 60th birthday

ABSTRACT. We prove that any homotopy type can be recovered canonically from its associated weak ω-groupoid. This implies that the homotopy category of CW-complexes can be embedded in the homotopy category of Batanin's weak higher groupoids.

INTRODUCTION

The idea that the homotopy category of CW-complexes should be equivalent to some reasonnable homotopy category of ∞-groupoids comes from Grothendieck's letter to Quillen in *Pursuing stacks* [Gro83]. But we know that such a result is false as far as we consider only strict higher groupoids; see e.g. [Ler94, Ber99]. However, we still hope to prove this equivalence by considering some weaker notion of higher groupoid. M. Batanin [Bat98] did give a more precise formulation of this: he defined a reasonnable notion of weak ω-groupoid and a nice functor from spaces to the category of weak ω-groupoids, and made the hypothesis that this should give an answer to this problem; see [Bat98, Hypothesis page 98] for a precise formulation. Batanin's definition of weak ω-groupoids is based on the notion of ω-operad. These are operads for which the operations are not indexed by integers but by (finite planar level) trees. The underlying algebra has been intensively studied by R. Street and M. Batanin; see [Bat98, Str00, BS00]. C. Berger [Ber02] started to study how one can define reasonnable homotopy theories from this kind of object. In particular, thanks to Berger's insights, a good notion of weak equivalence between weak ω-groupoids is now available, so that our problem is more precise. In these notes, we prove that Batanin's functor (or a very slightly modified version of it)

$$\Pi_\infty : \mathsf{Ho}(\mathscr{T}op) \longrightarrow \mathsf{Ho}(\omega\text{-}\mathscr{G}roupoid)$$

is faithful and conservative. This comes from the fact that one can reconstruct canonically the homotopy type of a space X from its associated weak ω-groupoid $\Pi_\infty(X)$ (see 4.6 and its proof). This is proved by relating Batanin's construction of the functor Π_∞ with Berger's results (this is essentially the contents of Theorem 3.15 and its corollary).

1. Omega operads

1.1. We first recall the notion of ω-operad (see [Bat98, Str00, BS00]). Let $\mathbb{G}$ be the globe category. The objects of $\mathbb{G}$ are the $\overline{n}$'s for each integer $n \geq 0$. The globular operators (i.e. the maps in $\mathbb{G}$) are generated by the cosource and cotarget maps $s, t : \overline{n} \longrightarrow \overline{n+1}$ subject to the relations $ss = ts$ and $tt = st$. Recall that an ω-*graph* is a presheaf (of sets) on the category $\mathbb{G}$. The category of ω-graphs is denoted by ω-$\mathscr{G}raph$. The obvious forgetful functor

$$U : \omega\text{-}\mathscr{C}at \longrightarrow \omega\text{-}\mathscr{G}raph$$

from the category of *strict* ω-categories to the category of ω-graphs has a left adjoint

$$F : \omega\text{-}\mathscr{G}raph \longrightarrow \omega\text{-}\mathscr{C}at \ .$$

The functor U is *monadic*. This means that if ε denotes the counit of the adjunction (F, U), the functor

$$Y \longmapsto (UY, U\varepsilon_Y)$$

induces an equivalence of categories ω-$\mathscr{C}at \simeq \mathbf{Alg}_{UF}$. In other words, if we define $\underline{\omega} = UF$, the category ω-$\mathscr{C}at$ is canonically equivalent to the category of algebras on the monad $\underline{\omega}$ (see for example [Ber02, Theorem 1.12]).

An ω-*collection* A is a pair $(\underline{A}, p_A)$ where A is an endofunctor of the category ω-$\mathscr{G}raph$, and p_A is a *cartesian* natural transformation

$$p_A : \underline{A} \longrightarrow \underline{\omega} \ .$$

This means that for any map $X \longrightarrow Y$ of ω-graphs, the commutative square

$$
\begin{array}{ccc}
\underline{A}(X) & \longrightarrow & \underline{\omega}(X) \\
\downarrow & & \downarrow \\
\underline{A}(Y) & \longrightarrow & \underline{\omega}(Y)
\end{array}
$$

is a pullback square in ω-$\mathscr{G}raph$. A map of ω-collections $A \longrightarrow B$ is a natural transformation $q : \underline{A} \longrightarrow \underline{B}$ such that $p_B q = p_A$. The category of ω-collections is endowed with a monoidal structure induced by the composition of endofunctors: $A \otimes B = (\underline{A} \circ \underline{B}, p_{A \otimes B})$, where $p_{A \otimes B}$ is the composition below.

$$\underline{A} \circ \underline{B} \xrightarrow{A \star p_B} \underline{A} \circ \underline{\omega} \xrightarrow{p_A \star \underline{\omega}} \underline{\omega} \circ \underline{\omega} \xrightarrow{\mu} \underline{\omega}$$

This tensor product is well defined as the natural transformation μ above is cartesian (see [Str00]). An ω-*operad* is a monoid object in the category of ω-collections. In other words, an ω-operad is an ω-collection $A = (\underline{A}, p_A)$ endowed with a structure of a monad on $\underline{A}$ over $\underline{\omega}$. For example, $\omega = (\underline{\omega}, 1_{\underline{\omega}})$ is an ω-operad. This is clearly the *terminal ω-operad*, that is the terminal object in the category of ω-operads.

1.2. Let X be an ω-graph. For $n \geq 0$, we denote by X_n the set $X(\overline{n})$ of n-cells in X. We define $\leq_X$ to be the preorder relation on the set $\mathrm{II}_{n \geq 0} X_n$ generated by

$$s^*(x) \leq_X x \quad \text{and} \quad x \leq_X t^*(x) \quad \text{for any } x \text{ in } X_n, \, n \geq 1,$$

where $s^*, t^* : X_n \longrightarrow X_{n-1}$ are the maps induced by the operators $s, t : \overline{n-1} \longrightarrow \overline{n}$. We define a (finite planar level) *tree* as a non empty ω-graph T such that the set $\mathrm{II}_{n \geq 0} T_n$ is finite and $\leq_T$ is a total order[1].

For example, for any $n \geq 0$, the ω-graph $\overline{n}$ (that is the ω-graph represented by the corresponding object in $\mathbb{G}$) is a tree. We will say that a tree T is *linear* if $T \simeq \overline{n}$ for some integer $n \geq 0$. A tree is *non linear* if it is not linear.

A tree T is of height $\leq n$ if for any integer $k > n$, $T^k = \varnothing$. We define the height of T as the integer

$$\mathrm{ht}(T) = \min\{n \,|\, T \text{ is of height} \leq n\} \, .$$

For an ω-graph X and a tree T, one defines X^T as the set of maps

$$X^T = \mathrm{Hom}_{\omega\text{-}\mathscr{G}raph}(T, X) \, .$$

A morphism of trees is just a morphism of ω-graphs. Any morphism of trees is a monomorphisms (see for example [Ber02, Lemma 1.3]). We denote by Θ_0 the full subcategory of ω-$\mathscr{G}raph$ whose objects are trees. We will consider Θ_0 as a site with the Grothendieck topology defined by epimorphic families of maps (see [Ber02, Definition 1.5]). One checks easily that a family of maps $\{T_i \longrightarrow T\}_{i \in I}$ in Θ_0 is a covering if and only if the induced map $\mathrm{II}_{i \in I} T_i \longrightarrow T$ is an epimorphism in the category ω-$\mathscr{G}raph$.

1.3. Let A be an ω-operad. We denote by $\mathbf{Alg}_A$ the category of algebras over the monad $\underline{A}$. We then have a free $\underline{A}$-algebra functor

$$\omega\text{-}\mathscr{G}raph \longrightarrow \mathbf{Alg}_A \quad , \qquad X \longmapsto \underline{A}(X)$$

whose right adjoint is the forgetful functor

$$\underline{A} \longrightarrow \omega\text{-}\mathscr{G}raph \quad , \qquad C \longmapsto C \, .$$

We define the category Θ_A as the full subcategory of $\mathbf{Alg}_A$ spanned by objects of the form $\underline{A}(T)$ for any tree T (this is the full subcategory of the Kleisli category of the monad $\underline{A}$ spanned by trees). One has by definition a functor

$$i : \Theta_0 \longrightarrow \Theta_A \quad , \qquad T \longmapsto \underline{A}(T) \, .$$

An *A-cellular set* is a presheaf of sets on the category Θ_A.

The inclusion functor of Θ_A in $\mathbf{Alg}_A$ induces a nerve functor from $\mathbf{Alg}_A$

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[1]This definition of trees is not the usual one (see for example [Bat98, Ber02]), but is equivalent to it: finite planar level trees in the usual sense correspond bijectively to ω-graphs satisfying the properties listed above (see [Str00]). The definition we have chosen is related with the canonical embedding of trees in ω-graphs (see also [Bat98, p. 61 and p. 80]).

to the category $\widehat{\Theta}_A = \mathscr{S}et^{\Theta_A^{op}}$ of presheaves of sets on the category Θ_A

$$\mathscr{N}_A : \mathbf{Alg}_{\underline{A}} \longrightarrow \widehat{\Theta}_A \ , \quad C \longmapsto \big(\underline{A}(T) \longmapsto C^T = \mathrm{Hom}_{\mathbf{Alg}_{\underline{A}}}(\underline{A}(T), C)\big) \ .$$

The nerve functor $\mathscr{N}_A$ is a right adjoint to the left Kan extension of the inclusion functor of Θ_A in $\mathbf{Alg}_{\underline{A}}$

$$cat_A : \widehat{\Theta}_A \longrightarrow \mathbf{Alg}_{\underline{A}} \ .$$

We have finally an inverse image functor

$$i^* : \widehat{\Theta}_A \longrightarrow \widehat{\Theta}_0 \ .$$

Following [Ber02], we say that a presheaf X on Θ_A is a Θ_A-*model* if its restriction $i^*(X)$ on Θ_0 is a sheaf for the Grothendieck topology defined in 1.2.

Theorem 1.4 (C. Berger)**.** *The nerve functor* $\mathscr{N}_A : \mathbf{Alg}_{\underline{A}} \longrightarrow \widehat{\Theta}_A$ *is fully faithful. Its essential image consists of the* Θ_A-*models.*

Proof. See [Ber02, Theorem 1.17]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 1.5. If $\varnothing$ denotes the initial ω-operad (that is the corresponding monad is the identity of ω-$\mathscr{G}raph$), then $\Theta_\varnothing = \Theta_0$ is the already defined category of trees. By definition, the corresponding category of algebras is the category of ω-graphs. Hence Theorem 1.4 says that the category ω-$\mathscr{G}raph$ is canonically equivalent to the category of sheaves on Θ_0 for the Grothendieck topology defined in 1.2.

Example 1.6. If ω is the terminal ω-operad, the category Θ_ω is the opposite of Joyal's category of ω-disks (see [Ber02, Proposition 2.2]). By definition, the corresponding category of algebras is the category of strict ω-categories.

Lemma 1.7. *A presheaf X on Θ_0 is a sheaf if and only if for any tree T, the map*

$$\mathrm{Hom}_{\widehat{\Theta}_0}(T, X) \longrightarrow \varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\widehat{\Theta}_0}(\overline{n}, X)$$

is a bijection.

Proof. This comes from the fact that for any tree T, the map

$$\varinjlim_{\overline{n} \longrightarrow T} \overline{n} \longrightarrow T$$

is an isomorphism in the category of ω-graphs. $\qquad\qquad\qquad\qquad\qquad\square$

1.8. For a tree T, we denote by $\Theta_A[T]$ the presheaf on Θ_A represented by $\underline{A}(T)$. Hence for any tree S, we have

$$\Theta_A[T](S) \simeq \mathrm{Hom}_{\Theta_A}(\underline{A}(S), \underline{A}(T)) \ .$$

We also have the identification $\mathscr{N}_A(\underline{A}(T)) = \Theta_A[T]$.

Proposition 1.9. *Let A be an ω-operad. Then an A-cellular set X is (isomorphic to) the nerve of an $\underline{A}$-algebra if and only if for any tree T, the map*

$$\mathrm{Hom}_{\widehat{\Theta}_A}(\Theta_A[T], X) \longrightarrow \varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\widehat{\Theta}_A}(\Theta_A[\overline{n}], X)$$

is a bijection.

Proof. If $i_! : \widehat{\Theta}_0 \longrightarrow \widehat{\Theta}_A$ denotes the left Kan extension of the canonical functor i from Θ_0 to Θ_A, one has $i_! T \simeq \underline{A}(T)$ for any tree T. This implies that the given condition is equivalent to saying that for any tree T, the map

$$\mathrm{Hom}_{\widehat{\Theta}_0}(T, i^*(X)) \longrightarrow \varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\widehat{\Theta}_0}(\overline{n}, i^*(X))$$

is a bijection. Moreover, we know that X is the nerve of an $\underline{A}$-algebra if and only if $i^*(X)$ is a sheaf on Θ_0. The result follows by Lemma 1.7. $\qquad\square$

1.10. For a tree T we define its *boundary* to be the presheaf on Θ_0

$$\partial T = \bigcup_{S \subset T, S \neq T} S \ ,$$

where the S's run over the proper subtrees of T. For an ω-operad A, we define $\partial \Theta_A[T] = i_!(\partial T)$. We then have a canonical inclusion of A-cellular sets

$$\partial \Theta_A[T] \longrightarrow \Theta_A[T] \ .$$

Proposition 1.11. *Let A be an ω-operad. Then for any Θ_A-model X, and any non linear tree T, the canonical map*

$$\mathrm{Hom}_{\widehat{\Theta}_A}(\Theta_A[T], X) \longrightarrow \mathrm{Hom}_{\widehat{\Theta}_A}(\partial \Theta_A[T], X)$$

is an isomorphism.

Proof. It is sufficient to prove that the canonical map

$$\varinjlim_{S \subset T, S \neq T} \underline{A}(S) \simeq \underline{A}\big(\varinjlim_{S \subset T, S \neq T} S \big) \longrightarrow \underline{A}(T)$$

is an isomorphism in the category of $\underline{A}$-algebras. As the functor $\underline{A}$ from ω-graphs to $\underline{A}$-algebras preserves colimits, we can suppose for this that A is the initial ω-operad. But the inclusions $S \subset T$, $S \neq T$, form an epimorphic family in the category of presheaves over Θ_0, which implies that ∂T is a covering sieve of T. This proves that the sheaf associated to the presheaf ∂T is canonically isomorphic to T. Hence the result. $\qquad\square$

2. CONTRACTIBLE B-OPERADS

2.1. Let $\mathscr{T}op$ be the category of (compactly generated) topological spaces. We define a functor

(2.1.1) $$b : \mathbb{G} \longrightarrow \mathscr{T}op$$

by $b(\overline{n}) = B_{top}^n$, where B_{top}^n denotes the euclidian n-dimensional ball. The operator s (resp. t) from $\overline{n-1}$ to $\overline{n}$ is sent to the continuous map s_{top} (resp.

t_{top}) which sends B_{top}^{n-1} to the north hemisphere of B_{top}^n (resp. to the south hemisphere of B_{top}^n); see [Bat98, Example 2, p. 58]. This defines a functor

$$(2.1.2) \quad b^* : \mathscr{T}op \longrightarrow \omega\text{-}\mathscr{G}raph \quad , \quad X \longmapsto (\overline{n} \longmapsto \mathrm{Hom}_{\mathscr{T}op}(B_{top}^n, X)) \ .$$

The aim of this section is to explain how we can produce some ω-operads A such that for any topological space X, the ω-graph $b^*(X)$ is endowed with a structure of an $\underline{A}$-algebra[2]. Taking the left Kan extension of b leads to a left adjoint to b^*, that is the unique cocontinuous functor

$$b_! : \omega\text{-}\mathscr{G}raph \longrightarrow \mathscr{T}op$$

whose restriction to $\mathbb{G}$ is b. By restriction to Θ_0, this defines a functor

$$R_0 : \Theta_0 \longrightarrow \mathscr{T}op \quad , \quad T \longmapsto B_{top}^T = b_!(T) \ .$$

More explicitely, one has a canonical isomorphism

$$\varinjlim_{\overline{n} \longrightarrow T} B_{top}^n \simeq B_{top}^T \ .$$

2.2. According to [Bat98, Proposition 7.2], the coglobular topological space b defines an ω-operad $B = E^{op}(b)$. We will call B the *topological ω-operad* (see also [Bat98, section 9, p. 97 sq]). The corresponding monad $\underline{B}$ on $\omega\text{-}\mathscr{G}raph$ is defined as follows. For an ω-graph X and an integer n, the ω-graph $\underline{B}(X)$ is given by the formula

$$(2.2.1) \qquad \underline{B}(X)_n = \coprod_{\mathrm{ht}(T) \leq n} \mathrm{Hom}(B_{top}^n, B_{top}^T) \times X^T$$

where $\mathrm{Hom}(B_{top}^n, B_{top}^T)$ denotes the set of maps between the corresponding coglobular spaces[3]. The natural transformation

$$p_B : \underline{B} \longrightarrow \underline{\omega}$$

is defined by the projections

$$\mathrm{Hom}(B_{top}^n, B_{top}^T) \times X^T \longrightarrow X^T$$

once we remember the canonical identification (see [Ber02, Definition 1.8 and Theorem 1.12])

$$(2.2.2) \qquad\qquad \underline{\omega}(X)_n \simeq \coprod_{\mathrm{ht}(T) \leq n} X^T \ .$$

Proposition 2.3. *There is a canonical functor*

$$R_B : \Theta_B \longrightarrow \mathscr{T}op$$

---

[2]This construction is due to Batanin [Bat98].

[3]Any tree has a canonical structure of a coglobular ω-graph (see e.g. [Ber02, Definition 1.8]). Hence, by functoriality, for any tree T, B_{top}^T is canonically endowed with a structure of a coglobular space.

such that the following diagram commutes.

$$
\begin{array}{ccc}
\Theta_0 & \xrightarrow{\ R_0\ } & \mathscr{T}op \\
\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle R_B} & \\
\Theta_B & &
\end{array}
$$

Proof. As the functor i is bijective on objects, we only have to define the functor R_B on morphisms. Let $n \geq 0$ be an integer, and let T be a tree. We have

$$
\mathrm{Hom}_{\Theta_B}(\underline{B}(\overline{n}), \underline{B}(T)) \simeq \underline{B}(T)_n
$$

$$
\simeq \coprod_{\mathrm{ht}(U) \leq n} \mathrm{Hom}(B_{top}^n, B_{top}^U) \times \mathrm{Hom}_{\Theta_0}(U, T)
$$

hence the maps

$$
\mathrm{Hom}(B_{top}^n, B_{top}^U) \times \mathrm{Hom}_{\Theta_0}(U, T) \longrightarrow \mathrm{Hom}_{\mathscr{T}op}(B_{top}^n, B_{top}^T)
$$

defined by $(g, f) \longmapsto R_0(f) \circ g$ induce a map

$$
\mathrm{Hom}_{\Theta_B}(\underline{B}(\overline{n}), \underline{B}(T)) \longrightarrow \mathrm{Hom}_{\mathscr{T}op}(B_{top}^n, B_{top}^T) \ .
$$

For two trees S and T, we obtain a map

$$
\varprojlim_{\overline{n} \longrightarrow S} \mathrm{Hom}_{\Theta_B}(\underline{B}(\overline{n}), \underline{B}(T)) \longrightarrow \varprojlim_{\overline{n} \longrightarrow S} \mathrm{Hom}_{\mathscr{T}op}(B_{top}^n, B_{top}^T)
$$

that is a map

$$
\mathrm{Hom}_{\Theta_B}(\underline{B}(S), \underline{B}(T)) \longrightarrow \mathrm{Hom}_{\mathscr{T}op}(B_{top}^S, B_{top}^T) \ .
$$

This defines the functor R_B and ends the proof. $\qquad\qquad\square$

2.4. We define the functor

$$
|-|_B : \widehat{\Theta}_B \longrightarrow \mathscr{T}op \quad , \qquad X \longmapsto |X|_B
$$

as the left Kan extension of the functor R_B given by Proposition 2.3. It has a right adjoint

$$
\Pi_\infty^B : \mathscr{T}op \longrightarrow \widehat{\Theta}_B \quad , \qquad X \longmapsto \left(\underline{B}(T) \longmapsto \mathrm{Hom}_{\mathscr{T}op}(B_{top}^T, X) \right) \ .
$$

Proposition 2.5. *For any (compactly generated) topological space X, the B-cellular set $\Pi_\infty^B(X)$ is a $\underline{B}$-algebra.*

Proof. According to Proposition 1.9, it is sufficient to check that for any tree T, the map

$$
\mathrm{Hom}_{\widehat{\Theta}_B}(\Theta_B[T], \Pi_\infty^B(X)) \longrightarrow \varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\widehat{\Theta}_B}(\Theta_B[\overline{n}], \Pi_\infty^B(X))
$$

is a bijection. The canonical identifications

$$\varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\widehat{\Theta}_B}(\Theta_B[\overline{n}], \Pi_\infty^B(X)) \simeq \varprojlim_{\overline{n} \longrightarrow T} \mathrm{Hom}_{\mathscr{T}op}(B_{top}^n, X)$$

$$\simeq \mathrm{Hom}_{\mathscr{T}op}(B_{top}^T, X)$$

$$\simeq \mathrm{Hom}_{\widehat{\Theta}_B}(\Theta_B[T], \Pi_\infty^B(X))$$

end this proof. $\square$

Corollary 2.6. *For any non linear tree T, the continuous map*

$$|\partial \Theta_B[T]|_B \longrightarrow |\Theta_B[T]|_B = B_{top}^T$$

is an homeomorphism.

Proof. By Theorem 1.4 and Proposition 1.9, this is reformulation of Propositions 1.11 and 2.5 and of the Yoneda Lemma applied to $\mathscr{T}op$. $\square$

Definition 2.7. A *B-operad* is an ω-operad A endowed with a morphism of ω-operads $\varphi_A : A \longrightarrow B$ from A to the topological ω-operad B.

Remark 2.8. For a B-operad A, the morphism φ_A defines a functor

$$\varphi_A : \Theta_A \longrightarrow \Theta_B \quad , \qquad \underline{A}(T) \longmapsto \underline{B}(T) .$$

We define the functor

$$(2.8.1) \qquad\qquad\qquad R_A : \Theta_A \longrightarrow \mathscr{T}op$$

as the composition of φ_A and of the functor R_B (see 2.4). In other words, for any tree T, we have $R_A(\underline{A}(T)) = R_B(\underline{B}(T)) = B_{top}^T$. We define the functor

$$(2.8.2) \qquad\qquad\qquad | - |_A : \widehat{\Theta}_A \longrightarrow \mathscr{T}op$$

as the left Kan extension of R_A. It has a right adjoint

$$(2.8.3) \qquad\qquad\qquad \Pi_\infty^A : \mathscr{T}op \longrightarrow \widehat{\Theta}_A$$

defined by $\Pi_\infty^A = \varphi_A^* \Pi_\infty^B$, where $\varphi_A^* : \widehat{\Theta}_B \longrightarrow \widehat{\Theta}_A$ is the inverse image functor of the functor φ_A above. As the functor φ_A^* obviously sends Θ_B-models on Θ_A-models, it follows from Theorem 1.4 and Proposition 2.5 that Π_∞^A can be defined as a functor

$$(2.8.4) \qquad\qquad\qquad \Pi_\infty^A : \mathscr{T}op \longrightarrow \mathbf{Alg}_{\underline{A}} .$$

We remark that for any topological space X, the underlying ω-graph of $\Pi_\infty^A(X)$ is the ω-graph $b^*(X)$ introduced in 2.1.

2.9. Following [Ber02, Definition 1.20][4], we say that an ω-operad A is *contractible* if for any ω-graph X, the canonical map $\underline{A}(X) \longrightarrow \underline{\omega}(X)$ has the

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[4]The definition of contractible ω-operads in terms of lifting properties in fact appeared first in Lemma 1.1 from [Bat02].

right lifting property with respect to inclusions $\partial\overline{n} \longrightarrow \overline{n}$, $n \geq 0$. In other words, we ask that any solid commutative square

(2.9.1)
$$
\begin{array}{ccc}
\partial\overline{n} & \longrightarrow & \underline{A}(X) \\
\downarrow & \nearrow^{f} & \downarrow \\
\overline{n} & \longrightarrow & \underline{\omega}(X)
\end{array}
$$

admits a diagonal filler f such that the diagram still commutes. As the natural transformation $\underline{A} \longrightarrow \underline{\omega}$ is cartesian, it is sufficient to check the lifting property above in the case where $X = e$ is the terminal ω-graph (see [Ber02, Definition 1.20]). It is then easy to check that this definition is somehow equivalent to the notion of ω-*operad with contractions* as defined in [Lei04a, Lei04b][5]. More precisely, an ω-operad with contractions as defined by Leinster corresponds to a contractible ω-operad A with a given lift f for all the commutative squares of type (2.9.1) where X is the terminal ω-graph.

Proposition 2.10. *The topological ω-operad B is contractible.*

Proof. Let X be an ω-graph. For any integer $n \geq 0$, we have by construction
$$
\mathrm{Hom}_{\omega\text{-}\mathscr{G}raph}(\overline{n}, \underline{B}(X)) \simeq \underline{B}(X)_n = \coprod_{\mathrm{ht}(T) \leq n} \mathrm{Hom}(B^n_{top}, B^T_{top}) \times X^T \ .
$$

We deduce from this that
$$
\mathrm{Hom}_{\omega\text{-}\mathscr{G}raph}(\partial\overline{n}, \underline{B}(X)) \simeq \coprod_{\mathrm{ht}(T) \leq n} \mathrm{Hom}(\partial B^n_{top}, B^T_{top}) \times X^T
$$

where $\partial B^n_{top} = S^{n-1}$ is the $n-1$-dimensional sphere. Using the formula (2.2.2), one concludes that it is sufficient to check that any map from S^{n-1} to B^T_{top} can be extended to B^n_{top}. This follows from the fact that B^T_{top} is contractible. $\qquad\square$

Theorem 2.11 (T. Leinster). *There is a universal[6] contractible ω-operad K.*

Proof. By [Lei04a, Proposition 9.2.2 and Appendix G], the category of ω-operads with contractions as defined in [Lei04a, Definition 9.2.1] has an initial object K. The assertion now comes from the fact that any contractible ω-operad A can be endowed with the structure of an ω-operad with contractions (just chose the lift f for all the commutative squares of type (2.9.1) where X is the terminal ω-graph), so that K is also a weak initial object in the category of contractible ω-operad. $\qquad\square$

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[5]However it is not obvious at all that this definition is equivalent to Batanin's definition of contractible ω-operad with system of compositions [Bat98, Definitions 8.2 and 8.4]. Any contractible ω-operad as defined in 2.9 is a contractible ω-operad with system of compositions in Batanin's sense, but the converse is still a conjecture.

[6]This is in the sense defined in [Bat98, Definition 8.5]: universal means weakly initial.

Remark 2.12. This theorem has to be compared with Batanin's result [Bat98, Theorem 8.1] saying that there is a universal contractible ω-operad with system of compositions.

2.13. It is then reasonnable to define a *weak ω-category* as a $\underline{K}$-algebra (and any such weak ω-category is a weak ω-category in Batanin's sense [Bat98], but we don't know if the converse is true).

As the topological ω-operad B is contractible, one has a map of ω-operads $\varphi_K : K \longrightarrow B$. In other words, K is a B-operad (see 2.7). It is obvious that K is also the initial contractible B-operad. As explained in Remark 2.8, the map φ_K induces a canonical functor

$$(2.13.1) \qquad\qquad \varphi_K : \Theta_K \longrightarrow \Theta_B .$$

as well as a functor

$$(2.13.2) \qquad\qquad \Pi_\infty^K : \mathscr{T}op \longrightarrow \mathbf{Alg}_{\underline{K}} .$$

3. Topological realizations of cellular spaces

3.1. Let Δ be the category of simplices. The objects are the totally ordered sets $[n] = \{0, \ldots, n\}$ for any integer $n \geq 0$, and the morphisms are the order-preserving maps. The category $\widehat{\Delta}$ of simplicial sets is the category of presheaves of sets on Δ. The simplicial set represented by $[n]$ is denoted by $\Delta[n]$, and the boundary of $\Delta[n]$ is denoted by $\partial\Delta[n]$. For any integers $0 \leq k \leq n$, $n \geq 1$, $\Lambda^k[n]$ is the k-horn of $\Delta[n]$ (see [GZ67, GJ99]).

3.2. Let A be an ω-operad. We denote by $\mathbf{s}\widehat{\Theta}_A$ the category of simplicial A-cellular sets. We first recall here how one can produce a reasonable homotopy theory of simplicial A-cellular sets following C. Berger's construction [Ber02]. We define three sets of maps of simplicial A-cellular sets as follows. The set I_A consists of the inclusions

$$\partial\Theta_A[T] \otimes \Delta[n] \cup \Theta_A[T] \otimes \partial\Delta[n] \longrightarrow \Theta_A[T] \otimes \Delta[n]$$

for any tree T and any integer $n \geq 0$. The set J'_A consists of the inclusions

$$\partial\Theta_A[T] \otimes \Delta[n] \cup \Theta_A[T] \otimes \Lambda^k[n] \longrightarrow \Theta_A[T] \otimes \Delta[n]$$

for any tree T and any integers $0 \leq k \leq n$, $n \geq 1$. Finally, the set J''_A is consists of the inclusions

$$\Theta_A[S] \otimes \Delta[n] \cup \Theta_A[T] \otimes \partial\Delta[n] \longrightarrow \Theta_A[T] \otimes \Delta[n]$$

for any map of trees $S \longrightarrow T$ and any integer $n \geq 0$.

Theorem 3.3 (C. Berger). *Let A be an ω-operad. The category of simplicial A-cellular sets is endowed with a left proper cofibrantly generated model category structure. The cofibrations are generated by the set I_A, and the trivial cofibrations are generated by the set of maps $J'_A \cup J''_A$.*

Proof. See [Ber02, Proposition 4.11]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.4. We did not say how the weak equivalences are defined: this is not necessary as the cofibrations and the trivial cofibrations of this model structure are defined so that all the structure is already determined; see [Ber02]. However, a way to define this model structure is to prove that the category $\mathbf{s}\widehat{\Theta}_A$ is endowed with a proper cofibrantly generated model category structure with the termwise simplicial weak equivalences as weak equivalences. For this model structure, the cofibrations are generated by I_A, and the trivial cofibrations are generated by J'_A. The model structure of Theorem 3.3 can be defined as the left Bousfield localization of the latter by the maps $S \to T$ for any trees S ant T (this is the precise formulation given in [Ber02, Proposition 4.11]).

Definition 3.5. Let A be an ω-operad. A *canonical realization of A* is a functor
$$R : \Theta_A \longrightarrow \mathscr{T}op$$
satisfying the following properties.

R1 For any tree T, the space $R(\underline{A}(T))$ is (weakly) contractible.

R2 If $R_! = | - |_R : \widehat{\Theta}_A \longrightarrow \mathscr{T}op$ denotes the left Kan extension of R, then for any tree T, the canonical continuous map
$$|\partial\Theta_A[T]|_R \longrightarrow |\Theta_A[T]|_R = R(\underline{A}(T))$$
is a cofibration of topological spaces (for the usual model category structure on $\mathscr{T}op$).

Example 3.6. For any B-operad A, the functor R_A (2.8.1) is a canonical realization of A: the spaces B^T_{top} are contractible CW-complexes, and for any tree T, the map
$$|\partial\Theta_A[T]|_A \longrightarrow |\Theta_A[T]|_A = B^T_{top}$$
is a cofibration. To see this, as any homeomorphism is a cofibration of topological spaces, it is sufficient to check this property when T is linear (Corollary 2.6). But if $T = \overline{n}$, then $|\partial\Theta_A[T]|_A$ is the $n-1$-dimensionnal sphere S^{n-1}, and the canonical inclusion of S^{n-1} in the n-dimensionnal ball is obviously a cofibration.

Example 3.7. As the category Θ_ω is the opposite category of Joyal's category of ω-disks, the canonical ω-disk structure on the coglobular space b (2.1.1) defines a functor
$$D_\omega : \Theta_\omega \longrightarrow \mathscr{T}op$$
that happends to be a canonical realization of the terminal ω-operad (see [Ber02, Propositions 2.2 and 2.6]).

If A is an ω-operad, we have a canonical functor
$$p : \Theta_A \longrightarrow \Theta_\omega \quad , \qquad \underline{A}(T) \longrightarrow \underline{\omega}(T) \ .$$
We can define a functor
$$D_A : \Theta_A \longrightarrow \mathscr{T}op$$

by the formula $D_A = D_\omega \circ p$. One checks immediately that D_A is a canonical realization of A. We will call D_A the ω-*disks realization* of A.

Example 3.8. Let A be an ω-operad. We define the *categorical realization* H of A as follows. For this, we recall the following general construction. Given a (small) category C and a presheaf of sets X on C, we define the *category of elements of* X as the category C/X whose objects are pairs (c, s), where c is an object of C, and s a section of X over c. A map in C/X from (c, s) to (c', s') is a map $u : c \longrightarrow c'$ in C such that $u^*(s') = s$. We have a canonical forgetful functor

$$C/X \longrightarrow C \quad , \qquad (c, s) \longmapsto c .$$

For a cellular space X, we define the category Θ_A/X as the category of elements of X. This defines a functor from the category of cellular sets to the category of small categories. Taking the topological realization of the nerve of the Θ_A/X's thus defines a functor

$$| - |_H : \widehat{\Theta}_A \longrightarrow \mathscr{T}op .$$

One can show that this functor is cocontinuous and sends monomorphisms of A-cellular sets to cofibrations of topological spaces: it follows from [Cis02, Corollaire A.1.12] that the functor

$$X \longmapsto \text{simplicial nerve of } \Theta_A/X$$

is cocontinuous and preserves monomorphisms, so that this is a consequence of the fact that the topological realization functor of simplicial sets is cocontinuous and sends monomorphisms to cofibrations of topological spaces. We conclude that the functor

$$H : \Theta_A \longrightarrow \mathscr{T}op \quad , \qquad \underline{A}(T) \longmapsto |\Theta_A[T]|_H$$

is a canonical realization of A: we have already verified R2, and for R1, it is obvious that $|\Theta_A[T]|_H$ is contractible as it is the realization of a category with a terminal object.

3.9. Given a canonical realization R of an ω-operad A, we get a functor

$$\Theta_A \times \Delta \longrightarrow \mathscr{T}op \quad , \qquad (\underline{A}(T), [n]) \longmapsto R(\underline{A}(T)) \times \Delta^n_{top}$$

where Δ^n_{top} is the topological n-simplex. The left Kan extension of the latter is a cocontinuous functor

$$| - |^{\mathbf{s}}_R : \mathbf{s}\widehat{\Theta}_A \longrightarrow \mathscr{T}op$$

whose right adjoint

$$\text{Sing}_R : \mathscr{T}op \longrightarrow \mathbf{s}\widehat{\Theta}_A$$

is defined by

$$X \longmapsto (\underline{A}(T), [n]) \longmapsto \text{Hom}_{\mathscr{T}op}(R(\underline{A}(T)) \times \Delta^n_{top}, X)) .$$

Proposition 3.10. *For any canonical realization R of an ω-operad A, the functor*

$$| - |_R^{\mathbf{s}} : \mathbf{s}\widehat{\Theta}_A \longrightarrow \mathscr{T}op$$

is a left Quillen functor.

Proof. It is sufficient to check that the functor $| - |_R$ sends the elements of I_A (resp. of $J_A' \cup J_A''$) to cofibrations (resp. to trivial cofibrations) of topological spaces. As the model category of topological spaces is monoidal with respect to the cartesian product, it is sufficient to check the following facts.

(a) For any integer $n \geq 0$, the topological realization of the inclusion $\partial\Delta[n] \longrightarrow \Delta[n]$ is a cofibration of topological spaces.
(b) For any tree T, the map $|\partial\Theta_A[T]|_R \longrightarrow |\Theta_A[T]|_R$ is a cofibration of topological spaces.
(c) For any integers $0 \leq k \leq n$, $n \geq 1$, the topological realization of $\Lambda^k[n]$ is contractible.
(d) For any map of trees $S \to T$, the map $|\Theta_A[S]|_R \longrightarrow |\Theta_A[T]|_R$ is a weak equivalence.

Properties (a) and (c) are well known. Property (b) (resp. (d)) follows from condition R2 (resp. R1) of Definition 3.5. $\square$

3.11. We denote by

$$| - |_R^{\mathbf{L}} : \mathsf{Ho}(\mathbf{s}\widehat{\Theta}_A) \longrightarrow \mathsf{Ho}(\mathscr{T}op)$$

the total left derived functor of the functor $| - |_R^{\mathbf{s}}$.

3.12. Let $\mathscr{M}$ be a model category. For a given small category I, let $\mathscr{M}^I$ be the category of functors from I to $\mathscr{M}$. We recall the following facts about homotopy colimits in model categories (see e.g. [CS02, Cis03, DHKS04]). The colimit functor

$$\varinjlim_I : \mathscr{M}^I \longrightarrow \mathscr{M}$$

has a total left derived functor

$$\mathbf{L}\varinjlim_I : \mathsf{Ho}(\mathscr{M}^I) \longrightarrow \mathsf{Ho}(\mathscr{M})$$

where $\mathsf{Ho}(\mathscr{M}^I)$ denotes the localization of $\mathscr{M}^I$ by the class of termwise weak equivalences. The functor $\mathbf{L}\varinjlim_I$ is a left adjoint to the functor

$$\mathsf{Ho}(\mathscr{M}) \longrightarrow \mathsf{Ho}(\mathscr{M}^I) \quad , \qquad X \longmapsto X_I$$

(where X_I is the constant diagram with value X). Moreover, any left Quillen functor

$$F : \mathscr{M} \longrightarrow \mathscr{M}' ,$$

induces a functor

$$F : \mathscr{M}^I \longrightarrow \mathscr{M}'^I$$

that has a total left derived functor (see [Cis03, Proposition 6.12])

$$\mathbf{L}F : \mathsf{Ho}(\mathscr{M}^I) \longrightarrow \mathsf{Ho}(\mathscr{M'}^I) \ .$$

By virtue of [Cis03, Proposition 6.12], we also know that if F is cocontinuous, the functors $\mathbf{L}F$ preserve homotopy colimits in the following sense. For any functor Φ from I to $\mathscr{M}$, the canonical map

$$\mathbf{L}\varinjlim_I \mathbf{L}F(\Phi) \longrightarrow \mathbf{L}F(\mathbf{L}\varinjlim_I \Phi)$$

is an isomorphism in $\mathsf{Ho}(\mathscr{M'})$.

3.13. Let A be an ω-operad. Given a functor

$$U : \Theta_A \longrightarrow \mathscr{T}op \ ,$$

one defines a functor

$$U^{\mathbf{L}} : \mathbf{s}\widehat{\Theta}_A \longrightarrow \mathsf{Ho}(\mathscr{T}op)$$

as follows. First of all, the functor U induces a functor

$$U^{\dagger} : \Theta_A \times \Delta \longrightarrow \mathscr{T}op$$

defined by $U^{\dagger}(\underline{A}(T), [n]) = U(\underline{A}(T)) \times \Delta_{top}^n$. Given a simplicial A-cellular set X, one has its category of elements as a presheaf over $\Theta_A \times \Delta$

$$E(X) = (\Theta_A \times \Delta)/X$$

(see example 3.8). We have a canonical functor φ_X from $E(X)$ to $E(e) = \Theta_A \times \Delta$ (where e denotes the terminal simplicial A-cellular set). We define the functor

$$U_X^{\dagger} : E(X) \longrightarrow \mathscr{T}op$$

as the composition of φ_X with $U^{\dagger}$. The functor $U^{\mathbf{L}}$ is at last defined by the formula

$$U^{\mathbf{L}}(X) = \mathbf{L}\varinjlim_{E(X)} U_X^{\dagger} \ .$$

It is obvious that if V is another functor from Θ_A to $\mathscr{T}op$, any termwise weak equivalence $U \longrightarrow V$ induces an isomorphism of functors $U^{\mathbf{L}} \simeq V^{\mathbf{L}}$.

We define the functor

(3.13.1) $$|| - || : \mathbf{s}\widehat{\Theta}_A \longrightarrow \mathsf{Ho}(\mathscr{T}op)$$

by the formula $||X|| = P^{\mathbf{L}}(X)$, where $P : \Theta_A \longrightarrow \mathscr{T}op$ is the functor which sends every object of Θ_A to the terminal topological space.

Let R be a canonical realization of A. One interesting fact is that one has a canonical isomorphism

(3.13.2) $$R^{\mathbf{L}} \simeq || - ||$$

(thanks to the condition R1 of Definition 3.5). Moreover, we also have the following.

Proposition 3.14. *For any simplicial A-cellular set X, one has a natural isomorphism*

$$R^{\mathbf{L}}(X) \simeq |X|_R^{\mathbf{L}} \ .$$

Proof. Recall from 3.13 that $E(X)$ is the category of elements of X. By composing the canonical functor from $E(X)$ to $\Theta_A \times \Delta$ with the Yoneda embedding into $\mathbf{s}\widehat{\Theta}_A$, we get a functor

$$h_X : E(X) \longrightarrow \mathbf{s}\widehat{\Theta}_A \ .$$

We have a canonical map

$$z_X : \mathbf{L}\varinjlim_{E(X)} h_X \longrightarrow \varinjlim_{E(X)} h_X \simeq X$$

in $\mathsf{Ho}(\mathbf{s}\widehat{\Theta}_A)$. We claim that this map is an isomorphism. This is sufficient to prove our result: once we know that z_X is an isomorphism, we have

$$R^{\mathbf{L}}(X) = \mathbf{L}\varinjlim_{E(X)} |h_X|_R^{\mathbf{L}}$$
$$\simeq |\mathbf{L}\varinjlim_{E(X)} h_X|_R^{\mathbf{L}} \quad \text{(by the end of 3.12)}$$
$$\simeq |X|_R^{\mathbf{L}} \ .$$

Hence we just have to prove our claim that the map z_X above is an isomorphism. For this, we need some more abstract results. By [Ber02, Proposition 4.11], the category of simplicial A-cellular sets is endowed with a (left) proper cofibrantly generated model category category structure whose weak equivalences are the termwise simplicial weak equivalences. The cofibrations (resp. the trivial cofibrations) are generated by the set I_A (resp. by the set J'_A); see 3.2. It follows that the identity of $\mathbf{s}\widehat{\Theta}_A$ is a left Quillen functor from the latter model structure to the model structure of Theorem 3.3. Hence it is sufficent to check that z_X is an isomorphism in the localization of $\mathbf{s}\widehat{\Theta}_A$ by the termwise simplicial weak equivalences. But as the definition of the homotopy category of a model category and of the homotopy colimit functors $\mathbf{L}\varinjlim$ only depend on the class of weak equivalences, it is sufficient to prove our claim by considering our favorite model category structure on $\mathbf{s}\widehat{\Theta}_A$ for which the class of weak equivalences consists of the termwise simplicial weak equivalences. If we take the model structure where the cofibrations are the monomorphisms, then the fact that z_X is an isomorphism comes from [Cis02, Proposition 4.4.24]. (But if the reader prefers the model structure where the fibrations are the termwise Kan fibrations, we can also refer to [Dug01b, Proposition 2.9].) $\qquad\square$

Theorem 3.15. *Let A be an ω-operad. If R_1 and R_2 are two canonical realizations of A, then the corresponding total left derived functors $| - |_{R_1}^{\mathbf{L}}$ and $| - |_{R_2}^{\mathbf{L}}$ are isomorphic.*

Proof. For any simplicial A-cellular set X, we have

$$|X|_{R_1}^{\mathbf{L}} \simeq ||X|| \simeq |X|_{R_2}^{\mathbf{L}}$$

by Proposition 3.14. $\qquad\square$

Corollary 3.16. *Let A be a contractible ω-operad. Then for any canonical realization R of A, the functor*

$$| - |_R^{\mathbf{s}} : \mathbf{s}\widehat{\Theta}_A \longrightarrow \mathscr{T}\!op$$

is a left Quillen equivalence.

Proof. Given a canonical realization R of A, the left Quillen functor $| - |_{\mathbf{R}}^{\mathbf{s}}$ is a Quillen equivalence if and only if its total left derived functor is an equivalence of categories. Hence, by Theorem 3.15, it is sufficient to prove that there exists a canonical realization R of A such that $| - |_R^{\mathbf{s}}$ is a Quillen equivalence. But we know this is the case when $R = D_A$ is the ω-disks realization of A (3.7) by virtue of [Ber02, Theorems 3.9 and 4.14]. $\qquad\square$

4. The weak ω-groupoid functor

Theorem 4.1 (C. Berger). *Let A be an ω-operad. Then the category $\mathbf{sAlg}_{\underline{A}}$ of simplicial $\underline{A}$-algebras is a left proper cofibrantly generated model category with the following definition: a map of simplicial $\underline{A}$-algebras is a weak equivalence (resp. a fibration) if its nerve (1.3) is a weak equivalence (resp. a fibration) of simplicial A-cellular sets. Moreover, the nerve functor*

$$\mathscr{N}_{\underline{A}} : \mathbf{sAlg}_{\underline{A}} \longrightarrow \mathbf{s}\widehat{\Theta}_A$$

is a right Quillen equivalence.

Proof. See [Ber02, Theorem 4.13]. $\qquad\square$

4.2. Let A be an ω-operad. We say that a map of $\underline{A}$-algebras is a *weak equivalence* if it is a weak equivalence of simplicial $\underline{A}$-algebras. We denote by $\mathsf{Ho}(\mathbf{Alg}_{\underline{A}})$ the localization of the category $\mathbf{Alg}_{\underline{A}}$ by the weak equivalences. The canonical functor from $\mathbf{Alg}_{\underline{A}}$ to $\mathbf{sAlg}_{\underline{A}}$ induces a functor

$$(4.2.1) \qquad\qquad \mathsf{Ho}(\mathbf{Alg}_{\underline{A}}) \longrightarrow \mathsf{Ho}(\mathbf{sAlg}_{\underline{A}}) \ .$$

It follows from the following result that this latter functor is an equivalence of categories.

Proposition 4.3 (C. Berger). *Let A be an ω-operad. Then for any fibrant simplicial A-cellular set for the model category structure of Theorem 3.3, the canonical map $X_0 \longrightarrow X$ is a weak equivalence (where X_0 denotes the A-cellular set obtained from the evaluation of X at $\Delta[0]$, seen as a discrete simplicial A-cellular set in the canonical way).*

Proof. See [Ber02, Proposition 4.17]. $\qquad\square$

Theorem 4.4. *Let A be a contractible B-operad. Then the functor*

$$\Pi_\infty^A : \mathscr{T}\!op \longrightarrow \mathbf{Alg}_{\underline{A}}$$

preserves weak equivalences, and the induced functor

$$\Pi_\infty^A : \mathsf{Ho}(\mathscr{T}\!op) \longrightarrow \mathsf{Ho}(\mathbf{Alg}_{\underline{A}})$$

is an equivalence of categories.

Proof. For two compactly generated topological spaces X and Y, we denote by $\mathscr{H}om(X, Y)$ the space of continuous maps from X to Y endowed with the compact-open topology. If X is an object of $\mathscr{T}op$, we get a simplicial space $S(X)$ defined by

$$S(X)_n = \mathscr{H}om(\Delta_{top}^n, X) \ .$$

Applying the functor Π_∞^A termwise to this simplicial space defines a simplicial $\underline{A}$-algebra $\Pi_\infty^A S(X)$. One can see easily that the functor $X \longmapsto \mathscr{N}_{\underline{A}} \Pi_\infty^A S(X)$ is a right adjoint to the left Quillen functor $| \ - \ |_R^{\mathsf{s}}$ where $R = R_A$ is the canonical realization of A of Example 3.6. Hence the functor $\mathscr{N}_{\underline{A}} \Pi_\infty^A S$ is a right Quillen functor. As any topological space is fibrant, this implies that the functor $\mathscr{N}_{\underline{A}} \Pi_\infty^A S$ preserves weak equivalences. Our result then follows trivially from Corollary 3.16, Theorem 4.1, Proposition 4.3, and from the equivalence of categories (4.2.1). $\qquad\square$

4.5. Let K be the universal contractible ω-operad (see Theorem 2.11). We defined in 2.13 the weak ω-categories to be the $\underline{K}$-algebras. The ω-operad K is also a contractible ω-operad with system of compositions as defined in [Bat98, Definitions 8.2 and 8.4]. Following M. Batanin, a *weak ω-groupoid* is a weak ω-category in which every cell is weakly invertible in the precise sense given by [Bat98, 9.5]. For example, for any topological space X, $\Pi_\infty^K(X)$ is a weak ω-groupoid; see [Bat98, Theorem 9.1]. We denote by ω-$\mathscr{G}roupoid$ the full subcategory of $\mathbf{Alg}_{\underline{K}}$ whose objects are the weak ω-groupoids. A morphism of weak ω-groupoids is a *weak equivalence* if it is a weak equivalence of $\underline{K}$-algebras. We will write $\mathsf{Ho}(\omega$-$\mathscr{G}roupoid)$ for the localization of the category of weak ω-groupoids by the weak equivalences. The functor Π_∞^K of (2.13.2) thus induces by Theorem 4.4 a functor

$$\Pi_\infty : \mathsf{Ho}(\mathscr{T}op) \longrightarrow \mathsf{Ho}(\omega\text{-}\mathscr{G}roupoid) \ .$$

Corollary 4.6. *The functor* $\Pi_\infty : \mathsf{Ho}(\mathscr{T}op) \longrightarrow \mathsf{Ho}(\omega$-$\mathscr{G}roupoid)$ *is faithful and conservative.*

Proof. The inclusion functor from the category of weak ω-groupoids to the category of weak ω-categories induces a functor

$$i : \mathsf{Ho}(\omega\text{-}\mathscr{G}roupoid) \longrightarrow \mathsf{Ho}(\mathbf{Alg}_{\underline{K}}) \ .$$

But it follows from Theorem 4.4 that the functor $i\Pi_\infty = \Pi_\infty^K$ is an equivalence of categories. This implies obviously our result. $\qquad\square$

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