

Global descent obstructions for varieties

Jean-Marc Couveignes* and Emmanuel Hallouin†

September 11, 2009

Abstract

We show how to transport descent obstructions from the category of covers to the category of varieties. We deduce examples of curves having \mathbf{Q} as field of moduli, that admit models over every completion of \mathbf{Q} , but have no model over \mathbf{Q} .

AMS classification 11R34, 12G05, 11G35, 14D22.

Contents

1	Introduction	2
1.1	Some categories with a Galois action	2
1.2	Weil descent	3
1.3	Strategy	4
2	Cancellation of the automorphism group of the base curve	4
3	Quasi-projective surfaces	8
3.1	Construction of the surface	8
3.2	Field of moduli and fields of definition	8
4	Projective normal surfaces	9
4.1	Construction of the surface \mathcal{S} and first properties	9
4.2	The group of automorphisms of \mathcal{S}	11
4.3	Field of moduli and fields of definition of \mathcal{S}	12
5	Curves	13
5.1	Two stable curves	13
5.2	Deformations	17
5.3	Fields of moduli and fields of definition of fibers	18
6	Seven lemmas about curves and surfaces	19
6.1	About curves and products of two curves	19
6.2	Deformation of an automorphism of a nodal curve	20
6.3	Automorphisms of curves in a family	21

*This work is supported by the French Agence Nationale pour la Recherche through project ALGOL (ANR-07-BLAN-0248).

†Institut de Mathématiques de Toulouse, Université de Toulouse et CNRS, Université de Toulouse 2 le Mirail, 5 allées Antonio Machado 31058 Toulouse cédex 9

1 Introduction

In this article, we construct descent obstructions in the category of varieties. For example, we show the following theorem:

Theorem 1.1 *There exists a projective, integral and smooth $\overline{\mathbf{Q}}$ -curve, having \mathbf{Q} as field of moduli, which is defined over all the completions of \mathbf{Q} but not over \mathbf{Q} itself.*

The main idea is to start from a descent obstruction in the category of covers of curves, and to transport it into various other categories: the category of quasi-projective surfaces, the category of projective surfaces, and finally the category of smooth curves. This process is summarized by the following theorem:

Theorem 1.2 *Let K be a field with characteristic zero and let K^s be an algebraic closure of K . Let \mathcal{B} be a projective, geometrically integral, and smooth K -curve and let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ be a connected, possibly ramified, K^s -cover having K as field of moduli. There exists a projective, integral, and smooth K^s -curve having K as field of moduli and having exactly the same fields of definition as the initial cover φ .*

Examples of obstructions to descent have been mostly constructed in the categories of G -covers and covers [CH85, DF94, CG94] and in the category of dynamical systems [Sil95]. A key technical point is that, in many cases, one can measure these obstructions in terms of the Galois cohomology of a finite abelian group. As far as we know, no example of global obstructions was known for varieties. Mestre gave some examples of local descent obstructions for hyperelliptic curves in [Mes91]. Dèbes and Emsalem [DE99] give a criterion for a curve to be defined over its field of moduli. This criterion involves a particular model for the quotient of the curve by its automorphism group. Dèbes and Emsalem prove that the local-global principle applies to the descent problem for a curve *together with its automorphisms*. However they leave open the question of the local-global principle for a curve (and a variety in general).

Global descent obstructions for covers have been constructed by Ros and Couveignes:

Theorem 1.3 (cf.[CR04], Corollaire 2) *There exists a connected ramified $\overline{\mathbf{Q}}$ -cover of $\mathbf{P}_{\mathbf{Q}}^1$ having \mathbf{Q} as field of moduli, which is defined over all the completions of \mathbf{Q} but which does not admit any model over \mathbf{Q} .*

If we apply theorem 1.2 to these obstructions, we prove theorem 1.1.

1.1 Some categories with a Galois action

Let K be a field of characteristic zero. Let K^s be an algebraic closure of K . Call $\Gamma_K = \text{Gal}(K^s/K)$ the absolute Galois group. By definition a K -variety \mathcal{V} is a separated and geometrically integral scheme of finite type over K . If $f \in K(\mathcal{V})$ is a function on \mathcal{V} and $x \in K$ a scalar, we denote by $(f)_x$ the positive part of the divisor of $f - x$. We denote by $(f)_\infty$ the negative part of the divisor of f . We consider various categories with a functorial action of Γ_K , beginning with the category of K^s -varieties. We also consider the full subcategories of smooth projective curves over K^s , of smooth quasi-projective surfaces over K^s , of projective normal surfaces over K^s . Let \mathcal{B} be a smooth K -curve. By a *cover* of \mathcal{B} we mean a finite K -morphism $\varphi : \mathcal{C} \rightarrow \mathcal{B}$ where \mathcal{C} is a projective, geometrically integral and smooth K -curve. In particular, we assume \mathcal{C} to be geometrically connected, but we do

not assume ϕ to be étale unless explicitly stated. We shall be interested in the category of K^s -covers of \mathcal{B} , that is the category of covers of $\mathcal{B} \otimes_K K^s$. We shall be interested also in the full subcategory of étale covers.

Let $\mathcal{O} \rightarrow \mathrm{Spec}(K^s)$ be an object in one of the preceding categories and take σ in Γ_K . The object ${}^\sigma\mathcal{O} \rightarrow \mathrm{Spec}(K^s)$ is defined by the composition $\mathcal{O} \rightarrow \mathrm{Spec}(K^s) \xrightarrow{\mathrm{Spec}(\sigma^{-1})} \mathrm{Spec}(K^s)$. This gives a covariant functor also denoted by σ . The stabilizer in Γ_K of the K^s -isomorphism class of an object is a finite index subgroup. The subfield of K^s which is fixed by this subgroup is called the *field of moduli* of the object. Let L be a field such that $K \subset L \subset K^s$. We say that \mathcal{O} is defined over L if it is K^s -isomorphic to the pullback of an L -object $\mathcal{O}' \rightarrow \mathrm{Spec}(L)$ by $\mathrm{Spec}(K^s) \rightarrow \mathrm{Spec}(L)$. We also say that L is a *field of definition* of \mathcal{O} or that $\mathcal{O}' \rightarrow \mathrm{Spec}(L)$ is a *model over L* of $\mathcal{O} \rightarrow \mathrm{Spec}(K^s)$. The field of moduli is contained in all the fields of definition.

The notions of field of moduli and of field definition are relative to the base field K . All the above definitions make sense for an arbitrary field K . In this work, we always assume that K has characteristic zero. There are two main reasons for that: firstly we often use the fact that K is infinite, and secondly the proof of lemma 5.2 relies on the fact that the order of a certain automorphisms group is prime to the characteristic of K .

1.2 Weil descent

A detailed treatment of all topics presented in this section can be found in [Mil08, Chapter 16].

Let $\mathcal{O} \rightarrow \mathrm{Spec}(K^s)$ be an object in one of the categories of paragraph 1.1. We denote by $\mathbb{G}(\mathcal{O}, K)$ the full subcategory whose objects are the ${}^\sigma\mathcal{O}$ for $\sigma \in \Gamma_K$. This is a groupoid called the *groupoid of conjugates* of the object \mathcal{O} . It is connected if and only if K is the field of moduli of \mathcal{O} . Objects in $\mathbb{G}(\mathcal{O}, K)$ are parametrized by elements in Γ_K .

A K^s/K system of descent on \mathcal{O} is a family $(I_\sigma)_{\sigma \in \Gamma_K}$ of isomorphisms $I_\sigma : {}^\sigma\mathcal{O} \rightarrow \mathcal{O}$ satisfying the cocycle condition:

$$\forall \sigma, \tau \in \Gamma_K, \quad I_\sigma \circ {}^\sigma I_\tau = I_{\sigma\tau}.$$

Let L be a subfield of K^s containing K . The system $(I_\sigma)_\sigma$ splits over L if there exists a model $\mathcal{O}_L \rightarrow \mathrm{Spec}(L)$ of \mathcal{O} and a morphism $I : \mathcal{O} \rightarrow \mathcal{O}_L \otimes_K K^s$ such that $I_\sigma = I^{-1} \circ {}^\sigma I$ for every $\sigma \in \Gamma_K$ fixing L .

A descent system is said to be *continuous* if it splits over a finite extension of K ; such a system is called a *descent data*. A descent data is called *effective* if it splits over K itself. Descent data form a category denoted by $\mathbb{D}(\mathcal{O}, K)$. A *morphism* between two descent data $(I_\sigma)_\sigma$ and $(J_\sigma)_\sigma$ is a K^s -automorphism ψ of \mathcal{O} such that:

$$\forall \sigma \in \Gamma_K, \quad \psi \circ I_\sigma = J_\sigma \circ {}^\sigma \psi.$$

Since every morphism is an isomorphism, $\mathbb{D}(\mathcal{O}, K)$ is a groupoid. For every K -extension L contained in K^s , we define in the same way the category $\mathbb{D}(\mathcal{O}, L)$ of descent data to L .

In all the categories we are interested in, descent data are always effective. This is a result of Weil often called *Weil descent* [Wei56, Mil08]. So the category $\mathbb{D}(\mathcal{O}, L)$ is equivalent to the category $\mathbb{M}(\mathcal{O}, L)$ of L -models of \mathcal{O} , whose objects are the models over L of \mathcal{O} and whose morphisms are L -isomorphisms.

Let \mathcal{O}_1 and \mathcal{O}_2 be two K^s -objects and let us consider the categories $\mathbb{G}(\mathcal{O}_1, K)$, $\mathbb{D}(\mathcal{O}_1, K)$, $\mathbb{M}(\mathcal{O}_1, K)$ and $\mathbb{G}(\mathcal{O}_2, K)$, $\mathbb{D}(\mathcal{O}_2, K)$, $\mathbb{M}(\mathcal{O}_2, K)$. Suppose that there exists a Galois equivariant functor \mathbb{F} from $\mathbb{G}(\mathcal{O}_1, K)$ to $\mathbb{G}(\mathcal{O}_2, K)$. Then if $\mathbb{G}(\mathcal{O}_1, K)$ is connected, so is $\mathbb{G}(\mathcal{O}_2, K)$ and K is

the field of moduli of both objects. Furthermore, \mathbb{F} maps every *descent system* on \mathcal{O}_1 to a *descent system* on \mathcal{O}_2 . Suppose that there exists moreover a finite K -extension L and a *descent data* of \mathcal{O}_1 to L which is mapped by \mathbb{F} onto a *descent data* of \mathcal{O}_2 to L . Then \mathbb{F} maps every *descent data* of \mathcal{O}_1 to a finite extension of K , to a *descent data* of \mathcal{O}_2 to the same extension. In that situation we say that the functor \mathbb{F} is *continuous*.

Under this hypothesis, \mathbb{F} induces a functor from $\mathbb{D}(\mathcal{O}_1, K)$ to $\mathbb{D}(\mathcal{O}_2, K)$ and also from $\mathbb{M}(\mathcal{O}_1, K)$ to $\mathbb{M}(\mathcal{O}_1, K)$ since all descent data are effective. We obtain, the same way, a functor between $\mathbb{M}(\mathcal{O}_1, L)$ and $\mathbb{M}(\mathcal{O}_2, L)$ for every K -extension L contained in K^s . In particular, this implies that every field of definition of \mathcal{O}_1 is a field of definition of \mathcal{O}_2 .

We stress that the objects \mathcal{O}_1 and \mathcal{O}_2 need not belong to the same category. They only have to belong each to a fibered category over the étale site of K where descent data are effective.

1.3 Strategy

All the descent obstructions contained in this work follow from the obstructions construct by Ros and Couveignes in the category of covers of curves (see theorem 1.3 above).

In order to transport these descent obstructions for covers into various other categories, we present several constructions that build an object in a target category out of an object in a source category. Of course these constructions must preserve fields of moduli and fields of definition. To this end, we use the method described in paragraph 1.2: we construct continuous functors between the groupoids of conjugates of two objects belonging to different categories.

In section 2, we start with a K^s -cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ where \mathcal{B} is a K -curve and we construct a K -curve \mathcal{B}' without any non-trivial K^s -automorphism and a K^s -cover $\mathcal{C}' \rightarrow \mathcal{B}' \otimes_K K^s$ having the same field of moduli and the same fields of definition as φ .

In section 3, we start with a K^s -cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ where \mathcal{B} is a K -curve and we construct a quasi-projective K^s -surface having the same field of moduli and the same fields of definition as φ . If the base curve \mathcal{B} of φ has no non-trivial K^s -automorphism, this surface is the complementary open subset to the graph of the cover. Thanks to the result of section 2, one can reduce to this case.

In section 4, we start with a K^s -cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ where \mathcal{B} is a K -curve. We assume that the field of moduli of this cover is K and we construct a projective normal K^s -surface having K as field of moduli and the same fields of definition as φ . This projective surface is constructed as a cover of $\mathcal{C} \times (\mathcal{B} \otimes_K K^s)$ which is strongly ramified along the graph of φ .

Finally, in section 5, we construct a projective K^s -curve, having K as field of moduli, and having the same fields of definition as the initial cover φ . This curve is a curve on the previous surface. It is obtained by deformation of a stable curve chosen to have the same automorphism group as the surface.

So our construction goes in several steps and the key point is that, at every step, we must carefully control the automorphism groups of all the objects involved.

Acknowledgements: We thank Qing Liu for communicating to us a simple proof of lemma 6.7 below.

2 Cancellation of the automorphism group of the base curve

In this section, K is a field of characteristic zero, K^s an algebraic closure of it, $L \subset K^s$ an algebraic extension of K , and \mathcal{B} a smooth, projective, geometrically integral K -curve. We assume we are given a geometrically connected L -cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ having K as field of moduli. We want to construct other covers having K as field of moduli and having the same fields of definition as φ , and

satisfying additional properties. In particular, we want to show that one can assume that the base curve has no non-trivial K^s -automorphism.

We first prove that the degree of the cover can be multiplied by any prime integer not dividing the initial degree.

Lemma 2.1 *Let \mathcal{B} be a smooth projective geometrically integral K -curve and let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ be an L -cover of degree d . The curve \mathcal{C} is assumed to be geometrically integral. For every prime p prime to d , there exists a geometrically integral L -curve \mathcal{C}' and a L -cover $\varphi' : \mathcal{C}' \rightarrow \mathcal{B} \otimes_K L$ of degree pd , having the same field of moduli and the same fields of definition as φ .*

Proof — Let $f \in K(\mathcal{B})$ be a non-constant function whose divisor is simple and does not meet the ramification locus of φ . The equation $Y^p = f$ defines a degree p extension of $K(\mathcal{B})$. We denote by \mathcal{B}' the smooth projective curve corresponding to this function field. This is a geometrically integral K -curve which covers \mathcal{B} by a degree p cover ν ; and $\nu \otimes_K K^s$ is Galois. The function field extensions $L(\mathcal{C})$ and $L(\mathcal{B}')$ are linearly disjoint over $L(\mathcal{B})$. Let \mathcal{C}' be the smooth projective curve corresponding to the compositum of $L(\mathcal{C})$ and $L(\mathcal{B}')$. By construction, this is a geometrically integral L -curve which covers $\mathcal{B} \otimes_K L$ by a cover ψ of degree pd :

$$\begin{array}{ccc}
 & \mathcal{C}' & \\
 \swarrow & & \searrow \\
 \mathcal{B}' \otimes_K L & & \mathcal{C} \\
 \searrow & \downarrow \psi & \swarrow \varphi \\
 & \mathcal{B} \otimes_K L &
 \end{array}$$

Since the function f has been chosen in $K(\mathcal{B})$, the above construction is Galois equivariant. Thus we obtain a Γ_K -equivariant functor:

$$\mathbb{F} : \mathbb{G}(\mathcal{C} \otimes_L K^s \xrightarrow{\varphi} \mathcal{B} \otimes_K K^s, K) \longrightarrow \mathbb{G}(\mathcal{C}' \otimes_L K^s \xrightarrow{\psi} \mathcal{B} \otimes_K K^s, K)$$

which is continuous because the starting L -model (more precisely the descent data corresponding to it) maps to an L -model (more precisely the descent data corresponding to it). Then the field of moduli of $\mathcal{C}' \xrightarrow{\psi} \mathcal{B} \otimes_K L$ is necessarily included in the one of $\mathcal{C} \xrightarrow{\varphi} \mathcal{B} \otimes_K L$ and each field of definition of φ is a field of definition of ψ .

To prove the converse, we construct a functor the other way around. Indeed, \mathbb{F} defines a one-to-one correspondence between the objects of the two groupoids of conjugates (because, in both cases, the objects are parameterized by Γ_K). We denote by \mathbb{F}^{-1} the inverse map on objects. This map can be extended to a functor by defining the image of a morphism. Take $\sigma \in \Gamma_K$ and consider the object $\sigma \mathcal{C}' \xrightarrow{\sigma \psi} \mathcal{B} \otimes_K K^s$ of the groupoid $\mathbb{G}(\mathcal{C}' \otimes_L K^s \xrightarrow{\psi} \mathcal{B} \otimes_K K^s, K)$. Then the maximal subcover which is unramified at the support of f is nothing but the corresponding object $\sigma \mathcal{C} \xrightarrow{\sigma \varphi} \mathcal{B} \otimes_K K^s$ in the first groupoid $\mathbb{G}(\mathcal{C} \otimes_L K^s \xrightarrow{\varphi} \mathcal{B} \otimes_K K^s, K)$. Therefore, if $\sigma, \tau \in \Gamma_K$, each K^s -isomorphism I' between $\tau \psi$ and $\sigma \psi$, induces a *unique* K^s -isomorphism $I : \tau \mathcal{C} \rightarrow \sigma \mathcal{C}$ between the characteristic subcovers $\tau \varphi$ and $\sigma \varphi$. In other words, the diagram:

$$\begin{array}{ccc}
 \tau \mathcal{C}' & \xrightarrow{I'} & \sigma \mathcal{C}' \\
 \downarrow & & \downarrow \\
 \tau \mathcal{C} & \xrightarrow{I} & \sigma \mathcal{C} \\
 \downarrow & & \downarrow \\
 \mathcal{B} & \xrightarrow{\text{Id}} & \mathcal{B}
 \end{array}$$

commutes. Sending I' to I turns \mathbb{F}^{-1} into a functor. This functor is continuous since it maps ψ/L to φ/L . Thus the field of moduli of $\mathcal{C}' \xrightarrow{\psi} \mathcal{B} \otimes_K L$ contains the one of $\mathcal{C} \xrightarrow{\varphi} \mathcal{B} \otimes_K L$ and every field of definition of ψ is also a field of definition of φ . \square

Next, we show that the base curve can be assumed to have genus greater than 2.

Lemma 2.2 *Let \mathcal{B} be a smooth projective geometrically integral K -curve and let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ be a L -cover of degree d . The curve \mathcal{C} is assumed to be geometrically integral. Then there exists a smooth projective geometrically integral K -curve \mathcal{B}' of genus greater than 2 and a L -cover $\varphi' : \mathcal{C}' \rightarrow \mathcal{B}' \otimes_K L$ having the same field of moduli and the same fields of definition as φ .*

Proof — We keep the notation of the preceding construction. We suppose that the chosen function f has degree at least 3. We look at the cover $\varphi' : \mathcal{C}' \rightarrow \mathcal{B}' \otimes_K L$. Since the degree of f is greater than 3, by Hurwitz genus formula, the curve \mathcal{B}' has a genus greater than or equal to 2. With the same kind of arguments as in the previous proof, one shows that φ and φ' share the same field of moduli and the same fields of definition. \square

Last, we prove that one can assume the base curve to have no non-trivial K^s -automorphism.

Lemma 2.3 *Let \mathcal{B} be a smooth projective geometrically integral K -curve and let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ be a L -cover of degree d . The curve \mathcal{C} is assumed to be geometrically integral. Then there exists a smooth projective geometrically integral K -curve \mathcal{B}' of genus greater than 2 and such that $\mathcal{B}' \otimes_K K^s$ does not have any non-trivial automorphism and there exists a L -cover $\varphi' : \mathcal{C}' \rightarrow \mathcal{B}' \otimes_K L$ having the same field of moduli and the same fields of definition as φ .*

Proof — Thanks to lemma 2.2, one can assume that the genus of \mathcal{B} is greater than 2. Consequently, the group A of K^s -automorphisms of \mathcal{B} is finite.

Let $p \geq 3$ be a prime integer. To begin with, we show that there exists a non-constant function $f \in K(\mathcal{B})$ which is non-singular above 2, -2 and ∞ , of degree greater than $2 + 4p(g(\mathcal{B}) - 1) + 2p^2$, such that the set $f^{-1}(\{-2, 2\})$ is not invariant by any non-trivial automorphism of $\mathcal{B} \otimes_K K^s$, and such that the set of singular values of φ does not meet the set $f^{-1}(\{2, -2, \infty\})$.

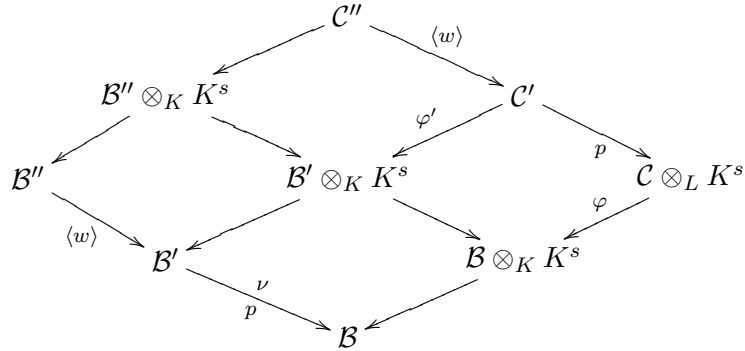
The kernels of the linear forms $\alpha \pm \text{Id}$ for $\alpha \in A \setminus \{\text{Id}\}$ are finitely many strict K^s -subspaces of the K^s -vector space $K^s(\mathcal{B})$. So there exists non-constant function $f \in K^s(\mathcal{B})$ such that $\alpha(f) \neq \pm f$ for all $\alpha \in A \setminus \{\text{Id}\}$. We now prove that this function can be chosen in order to be non singular above infinity and such that its singular values do not meet the singular values of φ . We first need to point out two easy facts concerning the linear space $\mathcal{L}(\Delta)$ associated to any *effective* and *simple* divisor Δ on \mathcal{B} : firstly this linear space is not contained in the kernel of $\alpha + \text{Id}$ because $1 \notin \text{Ker}(\alpha + \text{Id})$; and secondly if the degree of Δ is large enough then the functions of $\mathcal{L}(\Delta)$ generate the K^s -extension $K^s(\mathcal{B})$ and thus $\mathcal{L}(\Delta)$ is not contained in the kernel $\alpha - \text{Id}$. We choose such an effective simple divisor with large enough degree Δ . We can further assume that its support avoids the singular values of φ . Then all the intersections of $\mathcal{L}(\Delta)$ with the kernels of the linear functions $\alpha \pm \text{Id}$ are strict subvector spaces of $\mathcal{L}(\Delta)$: therefore there exists a non-constant function $f \in \mathcal{L}(\Delta)$ avoiding all these kernels. If the degree of Δ is large enough, the functions of $\mathcal{L}(\Delta)$ of degree less than the degree of Δ are also contained in a finite union of strict subvector spaces. So one can assume that $\deg(f) = \deg(\Delta)$. One can also assume that $f \in K(\mathcal{B})$. We finally can assume that the degree of the divisor Δ is greater than $2 + 4p(g(\mathcal{B}) - 1) + 2p^2$. Then so is the degree of f .

By construction, the function f^2 does not have any non-trivial automorphism (in short one has $\text{Aut}_{K^s(f^2)}(K^s(\mathcal{B})) = \{\text{Id}\}$). By lemma 6.3, we deduce that almost all the fibers of f^2 are non

singular and not fixed by any non trivial automorphism of A . In particular, there exists $\lambda \in K^*$ such that the fiber of f^2 above λ^2 is non singular, not fixed by any non-trivial automorphism in A and does not meet the singular values of φ . The function $2f/\lambda$ satisfies all the properties we want. Let us denote it by f .

Now the equation $X^p + X^{-p} - f = 0$ defines a regular extension of $K(\mathcal{B})$, that becomes Galois with Galois group equal to the dihedral group D_p when extending the scalars to K^s . Let \mathcal{B}'' be the smooth projective curve associated to this function field and $\mu : \mathcal{B}'' \rightarrow \mathcal{B}$ be the corresponding K -cover of degree $2p$. The K^s -automorphism group of μ is equal to D_p . The set of singular values of μ is $f^{-1}(\{2, -2, \infty\})$. We denote by w the automorphism of \mathcal{B}'' given by $w(X) = X^{-1}$ and by \mathcal{B}' the quotient $\mathcal{B}''/\langle w \rangle$. This is a smooth projective K -curve that covers \mathcal{B} by a K -cover $\nu : \mathcal{B}' \rightarrow \mathcal{B}$ of degree p . This cover does not have any non-trivial automorphism since $\langle w \rangle$ is self-normalized in D_p .

Because the ramification loci do not meet, the function fields $K^s(\mathcal{B}'')$ and $K^s(\mathcal{C})$ are necessarily linear disjoint over $K^s(\mathcal{B})$. Let \mathcal{C}' et \mathcal{C}'' be the smooth projective curves corresponding to the compose extension of $K^s(\mathcal{C})$ with $K^s(\mathcal{B}')$ and $K^s(\mathcal{B}'')$. We have the following diagram:



The cover $\mathcal{C}'' \rightarrow \mathcal{C} \otimes_L K^s$ is again a Galois cover with Galois group equal to D_p and the cover $\mathcal{C}'' \rightarrow \mathcal{C}'$ is of degree 2.

We show that the cover $\varphi' : \mathcal{C}' \rightarrow \mathcal{B}' \otimes_K K^s$ satisfies the conditions of the conclusion of lemma. First of all, it has the same field of moduli and the same fields of definition as φ . The proof goes as in lemmas 2.2 and 2.3: the subcover $\varphi : \mathcal{C} \otimes_L K^s \rightarrow \mathcal{B} \otimes_K K^s$ of $\nu \circ \varphi' : \mathcal{C}' \rightarrow \mathcal{B} \otimes_K K^s$ is characterized as the maximal subcover unramified at $f^{-1}(\{2, -2, \infty\})$.

Last we have to prove that the curve $\mathcal{B}' \otimes_K K^s$ does not have any non-trivial automorphism. Let \mathfrak{b}' be an automorphism of $\mathcal{B}' \otimes_K K^s$. The image \mathcal{I} of $\nu \times (\nu \circ \mathfrak{b}') : \mathcal{B}' \otimes_K K^s \rightarrow (\mathcal{B} \times \mathcal{B}) \otimes_K K^s$ stands between $\mathcal{B}' \otimes_K K^s$ and $\mathcal{B} \otimes_K K^s$. The bi-degree of \mathcal{I} is $\leq (p, p)$; so, by lemma 6.1, its virtual arithmetic genus is less than $1 + 2p(g(\mathcal{B}) - 1) + p^2$. If \mathcal{I} is not isomorphic to $\mathcal{B} \otimes_K K^s$ then \mathcal{I} is birationally equivalent to $\mathcal{B}' \otimes_K K^s$ and thus its geometric genus is $> \frac{1}{4} \deg(f)p \geq 1 + 2p(g(\mathcal{B}) - 1) + p^2$ by Hurwitz genus formula. Contradiction. Therefore \mathcal{I} is a correspondence of bi-degree $(1, 1)$ which defines an automorphism \mathfrak{b} of $\mathcal{B} \otimes_K K^s$ such that $\mathfrak{b} \circ \nu = \nu \circ \mathfrak{b}'$. Such an automorphism preserves the ramification data of ν , the one of its galois closure μ and also the one of the unique subcover of degree 2 of μ . Since this last cover is exactly ramified above $f^{-1}(\{-2, 2\})$, we deduce that $\mathfrak{b} = \text{Id}$ and then that \mathfrak{b}' is a K^s -automorphism of the cover ν . Since ν does not have any non-trivial automorphism, necessarily $\mathfrak{b}' = \text{Id}$. \square

3 Quasi-projective surfaces

Let K be a field of characteristic zero. In this section, we give a general process which associates to each K^s -cover of curves $\varphi : \mathcal{C} \rightarrow \mathcal{B}$, a smooth quasi-projective integral K^s -surface with the same field of moduli and fields of definition.

Theorem 3.1 *Let K be a field of characteristic zero, \mathcal{B} a smooth projective and geometrically integral K -curve, and $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ a K^s -cover (\mathcal{C} is assumed to be integral). There exists a smooth quasi-projective integral K^s -surface having the same field of moduli and the same fields of definition as φ .*

3.1 Construction of the surface

First of all, by lemmas 2.2 and 2.3, one can assume that the base curve $\mathcal{B} \otimes_K K^s$ has genus greater than 2 and has no non-trivial K^s -automorphism.

We consider $\mathcal{P} = (\mathcal{B} \otimes_K K^s) \times \mathcal{C}$ the product of the two curves and \mathcal{G} the graph of φ inside \mathcal{P} . Let \mathcal{U} be the open complementary set of \mathcal{G} in \mathcal{P} . We denote by $p_{\mathcal{C}} : \mathcal{P} \rightarrow \mathcal{C}$ and $p_{\mathcal{B}} : \mathcal{P} \rightarrow \mathcal{B}$ the projections onto each components.

The surface we are looking for is nothing else than the open set \mathcal{U} . We call it the *mark* of the cover (\mathcal{C}, φ) and we now prove that it has the same field of moduli and the same fields of definition as φ .

3.2 Field of moduli and fields of definition

We need two lemmas.

Lemma 3.2 *Let \mathcal{U} and \mathcal{V} be the marks of two non-trivial connected K^s -covers $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ and $\psi : \mathcal{D} \rightarrow \mathcal{B} \otimes_K K^s$, where \mathcal{B} is a smooth projective geometrically integral K -curve of genus greater than 2 and having no non-trivial K^s -automorphism. Then every surjective K^s -morphism from \mathcal{U} to \mathcal{V} is equal to $\text{Id} \times \gamma$ where $\gamma : \mathcal{C} \rightarrow \mathcal{D}$ is a K^s -morphism between the covers $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ and $\psi : \mathcal{D} \rightarrow \mathcal{B} \otimes_K K^s$.*

Proof — A K^s -morphism between the covers (\mathcal{C}, φ) and (\mathcal{D}, ψ) is a K^s -morphism of curves $\gamma : \mathcal{C} \rightarrow \mathcal{D}$ such that $\psi\gamma = \varphi$. The product isomorphism $\text{Id} \times \gamma$ from $(\mathcal{B} \otimes_K K^s) \times \mathcal{C}$ to $(\mathcal{B} \otimes_K K^s) \times \mathcal{D}$ maps the graph of φ to the graph of ψ and also the mark \mathcal{U} to the mark \mathcal{V} .

Conversely, let v be a surjective K^s -morphism from \mathcal{U} to \mathcal{V} . Let c be a closed K^s -point of \mathcal{C} . The restriction of $p_{\mathcal{D}} \circ v$ to $((\mathcal{B} \otimes_K K^s) \times \{c\}) \cap \mathcal{U}$ is a constant function because the genus of \mathcal{B} is less than the one of \mathcal{D} . We denote by $\gamma(c)$ this constant; this defines a morphism $\gamma : \mathcal{C} \rightarrow \mathcal{D}$ which cannot be constant since v is surjective. The restriction of $p_{\mathcal{B}} \circ v$ to $((\mathcal{B} \otimes_K K^s) \times \{c\}) \cap \mathcal{U}$ is a morphism β_c with values in \mathcal{B} . Let $F \subset \mathcal{C}$ the set of closed K^s -points of \mathcal{C} such that the morphism β_c is constant. This is a closed set; and a finite one because v is surjective. For a closed K^s -point $c \notin F$ the morphism β_c induces an automorphism of \mathcal{B} , which is trivial since \mathcal{B} does not have any non-trivial automorphism. Thus we have $v(b, c) = (b, \gamma(c))$ for every closed K^s -point b on \mathcal{B} and c on \mathcal{C} with $c \notin F$ and $(b, c) \in \mathcal{U}$. Let b be a closed K^s -point of $\mathcal{B} \otimes_K K^s$. The restriction of $p_{\mathcal{B}} \circ v$ to $(\{b\} \times \mathcal{C}) \cap \mathcal{U}$ is constant and equal to b on the non-empty open set $(\{b\} \times (\mathcal{C} - F)) \cap \mathcal{U}$. So it is a constant function. So F is empty and v is the restriction of $\text{Id} \times \gamma$ to \mathcal{U} . Thus $\text{Id} \times \gamma$ maps \mathcal{U} to \mathcal{V} and therefore $\psi\gamma = \varphi$. \square

Lemma 3.3 *Let \mathcal{U} be the mark of a connected K^s -cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ of degree greater than 2, where \mathcal{B} is a smooth projective geometrically integral K -curve of genus greater than 2 and having no non-trivial K^s -automorphism. Then:*

1. *the group of K^s -automorphisms of \mathcal{U} is equal to the group of K^s -automorphisms of the cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$;*
2. *the field of moduli of \mathcal{U} (in the category of quasi-projective varieties) is the field of moduli of the cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$;*
3. *an algebraic extension of K is a field of definition of \mathcal{U} if and only if it is the field of definition of the cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$.*

Proof — Using lemma 3.2 with $\mathcal{V} = {}^\sigma\mathcal{U}$ and $\psi = {}^\sigma\varphi$, we show that there exists an isomorphism of groupoids from $\mathbb{F} : \mathbb{G}(\mathcal{C} \xrightarrow{\varphi} \mathcal{B} \otimes_K K^s, K) \rightarrow \mathbb{G}(\mathcal{U}, K)$. This functor and its inverse are moreover Galois equivariant. The first two conclusions of lemma 3.3 result directly from the existence of this isomorphism. To prove the third one, we must verify that \mathbb{F} and \mathbb{F}^{-1} are continuous functors.

Let $L \subset K^s$ be a finite extension of K and let $(I_\sigma)_{\sigma \in \text{Gal}(K^s/L)}$ be a descent datum for φ to L . Since descent is effective, this datum splits over L itself: there exists an L -model $\varphi_L : \mathcal{C}_L \rightarrow \mathcal{B} \otimes_K L$ of φ and $I : \mathcal{C} \rightarrow \mathcal{C}_L \otimes_L K^s$ a K^s -isomorphism between the covers φ and $\varphi_L \otimes_L K^s$ such that $I_\sigma = I^{-1} \circ {}^\sigma I$. We denote by \mathcal{U}_L the complementary in $(\mathcal{B} \otimes_K L) \times \mathcal{C}_L$ of the graph of φ_L . This is clearly an L -model of \mathcal{U} . One checks that this L -model splits the descent system $(\mathbb{F}(I_\sigma))_{\sigma \in \text{Gal}(K^s/L)}$ for \mathcal{U} to L . So the image $(\mathbb{F}(I_\sigma))_{\sigma \in \text{Gal}(K^s/L)}$ is a descent data for \mathcal{U} to L . This proves that \mathbb{F} is continuous. To conclude, let us notice that, by definition, the continuity of \mathbb{F}^{-1} is equivalent to the continuity of \mathbb{F} . \square

4 Projective normal surfaces

In this section K is a field of characteristic zero. We start from a cover of curves, having K as field of moduli, and we construct out of it, a projective normal integral surface over K^s , having the same field of moduli and the same fields of definition as the original cover.

Theorem 4.1 *Let K be a field of characteristic zero and let \mathcal{B} be a smooth, projective and geometrically integral curve over K . Let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K K^s$ be a K^s -cover, having K as field of moduli. We assume that \mathcal{C} is integral. Then, there exists a projective, normal and integral surface \mathcal{S} over K^s , having K as field of moduli, and having the same fields of definition as φ .*

The proof of this theorem is given in the next two paragraphs.

4.1 Construction of the surface \mathcal{S} and first properties

We start from a cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ having K as field of moduli and defined over an algebraic extension $L \subset K^s$ of K . We may assume that the degree d of φ is at least 2 (otherwise the theorem is trivial). According to lemma 2.3, we may assume that the genus of \mathcal{B} is at least 2 and that \mathcal{B} has no non-trivial automorphism over K^s . We shall construct a cover $\psi : \mathcal{D} \rightarrow \mathcal{B} \otimes_K L$ and a cover of $(\mathcal{B} \otimes_K L) \times \mathcal{D}$ strongly ramified along the graph of ψ .

To this end, we first choose a finite generating system $(h_j)_{1 \leq j \leq J}$ of $K(\mathcal{B})$ over K . We assume that none of the h_j is a power in $K^s(\mathcal{B})$. We set $I = 2J$ and let $\Pi = \prod_{i=1}^I p_i$ be the product of the

first I prime integers greater than the degree d of φ . We choose two distinct prime integers a and b , both bigger than $1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$. For every $1 \leq j \leq J$ we set :

$$f_j = h_j^a, \quad \text{and} \quad f_{j+J} = h_j^b.$$

We can choose a and b in such a way that none of the functions $f_i - \lambda$ is a p_i -th power in $K^s(\mathcal{B})$ for $\lambda \in K^s$ and $1 \leq i \leq I$: this is evident for $\lambda = 0$. If $\lambda \neq 0$ and if $h_i^a - \lambda = \prod_{0 \leq k \leq a-1} (h_i - \zeta_a^k \lambda^{\frac{1}{a}})$ is a power, then h_i has at least a distinct singular values. This is impossible if we choose an a bigger than the number of singular values of h_i .

We note also that the $(f_i)_{1 \leq i \leq I}$ generate $K(\mathcal{B})$ over K . And they all have a degree greater than $1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$. Let M (resp. m) be the maximum (resp. minimum) among the degrees of the f_i . Then :

$$\forall 1 \leq i \leq I, \quad 1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2 < m \leq \deg(f_i) \leq M.$$

Let p be a prime integer bigger than $(g(\mathcal{B}) + IM)\Pi$. We call \mathcal{D} the L -curve and $\psi : \mathcal{D} \rightarrow \mathcal{B} \otimes_K L$ the degree pd cover given by lemma 2.1. The genus of \mathcal{D} is bigger than $dp > (g(\mathcal{B}) + IM)\Pi$ and the covers φ and ψ have the same field of moduli and the same fields of definition. We also check that the function $f_i \circ \psi - \lambda$ is not a p_i -th power in $K^s(\mathcal{D})$, for any $\lambda \in K^s$ and $1 \leq i \leq I$ (because this is already true for $f_i - \lambda$ and the degree pd of ψ is prime to p_i).

We now define the function g_i on $(\mathcal{B} \otimes_K L) \times \mathcal{D}$ as :

$$g_i(P, Q) = f_i(\psi(Q)) - f_i(P).$$

The negative part $(g_i)_\infty$ of the divisor of g_i is :

$$(g_i)_\infty = (f_i)_\infty \times \mathcal{D} + \mathcal{B} \times (f_i \circ \psi)_\infty.$$

The positive part $(g_i)_0$ is denoted Δ_i . We have :

$$\text{pgcd}_i(\Delta_i) = \mathcal{G},$$

where \mathcal{G} is the graph of ψ .

Let $L((\mathcal{B} \otimes_K L) \times \mathcal{D})$ be the field of functions of $(\mathcal{B} \otimes_K L) \times \mathcal{D}$. We define a regular radicial extension of $L((\mathcal{B} \otimes_K L) \times \mathcal{D})$ by setting

$$y_i^{p_i} = g_i.$$

We denote by \mathcal{S} the normalization of $(\mathcal{B} \otimes_K L) \times \mathcal{D}$ in the later radicial extension.

The surface \mathcal{S} is normal by construction. We denote by χ the ramified cover :

$$\chi : \mathcal{S} \rightarrow (\mathcal{B} \otimes_K L) \times \mathcal{D}.$$

If we base change χ to K^s we obtain a Galois cover $\chi \otimes_K K^s$ with Galois group $\prod_{i=1}^I \mathbf{Z}/p_i \mathbf{Z}$. We introduce a family of curves on \mathcal{S} that will help studying the group of K^s -automorphisms of \mathcal{S} . For any point Q on \mathcal{D} , we call \mathcal{E}_Q the inverse image of $\mathcal{B} \times Q$ by χ . We denote by $\chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B} \times Q$ the restriction of χ to \mathcal{E}_Q . If Q is the generic point on $\mathcal{D} \otimes_L K^s$, then \mathcal{E}_Q is a geometrically integral curve and χ_Q is a degree Π , geometrically connected cover. The degree of the ramification divisor of this cover is bounded above by the product $2IM$ (where I is the number of functions in the family

$(f_i)_i$ and M is the maximum of the degrees of these functions). Therefore, the geometric genus of \mathcal{E}_Q can be upper-bounded :

$$g(\mathcal{E}_Q) \leq (g(\mathcal{B}) + IM)\Pi < g(\mathcal{D}). \quad (1)$$

We also have a lower bound for the genus of any non-trivial subcover of χ_Q . Indeed, such a cover has degree at least $p_1 \geq 3$ and its ramification divisor has degree at least m (where m is the minimum among the degrees of the functions f_i). So :

$$1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2 < m \leq g(\text{non-trivial subcover of } \chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B}). \quad (2)$$

Inequalities (1) and (2) will help computing the automorphism group of \mathcal{S} .

4.2 The group of automorphisms of \mathcal{S}

We denote by A the group of K^s -automorphisms of ψ . An element in A induces a K^s -automorphism of the surface $(\mathcal{B} \otimes_K L) \times \mathcal{D}$, and this later automorphism can be lifted uniquely to an automorphism of $K^s(\mathcal{S})/K^s$ that fixes all y_i and stabilizes $K^s(\mathcal{B} \otimes_K L \times \mathcal{D})$. In the sequel we shall use the same notation for an automorphism of ψ , the induced automorphism of $(\mathcal{B} \otimes_K L) \times \mathcal{D}$ and its lift to \mathcal{S} . In other words, A can be identified with a subgroup of $\text{Aut}_{K^s}(\mathcal{S})$, the group of K^s -automorphisms of \mathcal{S} .

We know another subgroup of $\text{Aut}_{K^s}(\mathcal{S})$, namely the Galois group $B = \prod_i \mathbf{Z}/p_i \mathbf{Z}$ of the extension $K^s(\mathcal{S})/K^s((\mathcal{B} \otimes_K L) \times \mathcal{D})$.

To summarize, an element α of A , seen as a K^s -automorphism of \mathcal{S} , makes the following diagram commute :

$$\begin{array}{ccc} \mathcal{S} \otimes_L K^s & \xrightarrow{\alpha} & \mathcal{S} \otimes_L K^s \\ \chi \downarrow & & \downarrow \chi \\ ((\mathcal{B} \otimes_K L) \times \mathcal{D}) \otimes_L K^s & \xrightarrow{\alpha} & ((\mathcal{B} \otimes_K L) \times \mathcal{D}) \otimes_L K^s \end{array}$$

while an element β of B makes the following diagram commute :

$$\begin{array}{ccc} \mathcal{S} \otimes_L K^s & \xrightarrow{\beta} & \mathcal{S} \otimes_L K^s \\ & \searrow \chi & \swarrow \chi \\ & ((\mathcal{B} \otimes_K L) \times \mathcal{D}) \otimes_L K^s & \end{array}$$

It is clear that $A \times B \subset \text{Aut}_{K^s}(\mathcal{S})$. We now prove that this inclusion is an equality.

Lemma 4.2 *The group of K^s -automorphisms of \mathcal{S} is equal to $A \times B$.*

Proof — Let \mathfrak{a} be a K^s -automorphism of \mathcal{S} .

First of all, let Q be the generic point of $\mathcal{D} \otimes_L K^s$. We know from inequality (1) of section 4.1 that $g(\mathcal{E}_Q) < g(\mathcal{D})$. We deduce that $\mathfrak{a}(\mathcal{E}_Q) = \mathcal{E}_{\alpha(Q)}$ where α is a K^s -automorphism of \mathcal{D} .

We now prove that the isomorphism between \mathcal{E}_Q and $\mathcal{E}_{\alpha(Q)}$ induced by \mathfrak{a} makes the following diagram commute :

$$\begin{array}{ccc} \mathcal{E}_Q & \xrightarrow{\mathfrak{a}} & \mathcal{E}_{\alpha(Q)} \\ \chi_Q \downarrow & & \downarrow \chi_{\alpha(Q)} \\ \mathcal{B} \times Q & \xrightarrow{\text{Id} \times \alpha} & \mathcal{B} \times \alpha(Q) \end{array}$$

Indeed, the cartesian product of the maps χ_Q and $\chi_{\alpha(Q)} \circ \mathfrak{a}$ defines a morphism :

$$\mathcal{E}_Q \xrightarrow{\chi_Q \times (\chi_{\alpha(Q)} \circ \mathfrak{a})} \mathcal{B} \times \mathcal{B},$$

whose image \mathcal{N} is a divisor with bidegree $\leq (\Pi, \Pi)$. Using lemma 6.1 we deduce that the arithmetic genus of \mathcal{N} is smaller than or equal to $1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$. On the other hand, using the projection on the first factor $\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$, we find that \mathcal{N} is sandwiched between \mathcal{E}_Q and $\mathcal{B} \otimes_K K^s$. But we know from inequality (2) of section 4.1 that such a subcover of $\chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B} \otimes_K K^s$, if non-trivial, has geometric genus greater than or equal to $m > 1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$. We deduce that the image of $\chi_Q \times (\chi_{\alpha(Q)} \circ \mathfrak{a})$ is K^s -isomorphic to \mathcal{B} and is therefore a correspondence of bidegree $(1, 1)$. Since \mathcal{B} has no non-trivial K^s -automorphism we deduce that the above diagram can be completed with the identity downstairs. So $\chi \circ \mathfrak{a} = (\text{Id} \times \alpha) \circ \chi$.

We now prove that $\alpha \in A$. To this end, we first notice that if Q is the generic point on $\mathcal{D} \otimes_K K^s$, we just showed that \mathfrak{a} induces an isomorphism between the covers $\chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B}$ and $\chi_{\alpha(Q)} : \mathcal{E}_{\alpha(Q)} \rightarrow \mathcal{B}$. Therefore these two covers have the same ramification data: for every $1 \leq i \leq I$ the points P such that $f_i(P) = f_i(\psi(Q))$ and those such that $f_i(P) = f_i(\psi(\alpha(Q)))$ are the same. Thus :

$$\forall i, \quad f_i(\psi(Q)) = f_i(\psi(\alpha(Q)))$$

therefore $\psi(Q) = \psi(\alpha(Q))$, because the f_i generate $K(\mathcal{B})$ over K . So $\psi = \psi \circ \alpha$, and $\alpha \in A$. Therefore $\chi_{\alpha(Q)} \circ \alpha = (\text{Id} \times \alpha) \circ \chi_Q = \chi_{\alpha(Q)} \circ \mathfrak{a}$. We set $\beta = \mathfrak{a} \circ \alpha^{-1}$ and we check that $\chi_{\alpha(Q)} \circ \beta = \chi_{\alpha(Q)}$. Since Q is generic and α surjective we deduce that $\chi \circ \beta = \chi$ therefore $\beta \in B$. We conclude that $\mathfrak{a} = \beta\alpha \in A \times B$ as was to be shown. \square

Remark 1 *We have just proven something slightly stronger than lemma 4.2: the group of birational K^s -automorphisms of \mathcal{S} is $A \times B$. We shall not need this stronger result.*

4.3 Field of moduli and fields of definition of \mathcal{S}

From the definition of \mathcal{S} we deduce the existence of a Galois equivariant \mathbb{F} between the groupoids $\mathbb{G}(\psi \otimes_L K^s, K)$ and $\mathbb{G}(\mathcal{S} \otimes_L K^s, K)$. This already implies that K is also the field of moduli of $\mathcal{S} \otimes_L K^s$. The functor \mathbb{F} is continuous because the initial L -model (the associated descent datum) is mapped onto an L -model (the associated descent datum). Therefore, every field of definition of ψ is a field of definition of \mathcal{S} .

In order to prove the converse, we construct a continuous functor $\tilde{\mathbb{F}}$ from $\mathbb{G}(\mathcal{S} \otimes_L K^s, K)$ to $\mathbb{G}(\psi \otimes_L K^s, K)$ such that $\tilde{\mathbb{F}} \circ \mathbb{F} = \text{Id}^1$. We observe that \mathbb{F} induces a bijection between the objects of either groupoids: these objects are parameterized by Γ_K in either cases. We denote by $\tilde{\mathbb{F}}$ the reciprocal bijection. In order to turn $\tilde{\mathbb{F}}$ into a functor, there remains to define the image of a morphism.

So take σ and τ in Γ_K and consider ${}^\sigma\mathcal{S}$ and ${}^\tau\mathcal{S}$ the associated objects in $\mathbb{G}(\mathcal{S} \otimes_L K^s, K)$. Let ι be a morphism from ${}^\sigma\mathcal{S}$ to ${}^\tau\mathcal{S}$. Since the group of K^s -automorphisms of the surface \mathcal{S} has a unique subgroup of order Π , the same holds true for ${}^\sigma\mathcal{S}$ et ${}^\tau\mathcal{S}$. And these two subgroups are mapped onto each other by ι . Quotienting out by these two subgroups we obtain a K^s -isomorphism ι' from ${}^\sigma(\mathcal{B} \otimes_K L \times \mathcal{D})$ to ${}^\tau(\mathcal{B} \otimes_K L \times \mathcal{D})$. This isomorphism ι' maps the graph of ${}^\sigma\psi$ into the graph of ${}^\tau\psi$ (because these graphs can be characterized using the ramification data of the quotient covers). Therefore ι' sends the mark of ${}^\sigma\psi$ into the mark of ${}^\tau\psi$. We denote by $\tilde{\mathbb{F}}(\iota) : {}^\sigma\psi \rightarrow {}^\tau\psi$ the isomorphism of covers associated to this isomorphism of marks by lemma 3.2.

¹This is the best we can hope, because \mathbb{F} is not fully faithful.

We thus construct a functor $\widetilde{\mathbb{F}}$ which is continuous because \mathbb{F} is continuous. It is clear that $\widetilde{\mathbb{F}} \circ \mathbb{F}$ is the identity functor. We deduce that any field of definition of \mathcal{S} is a field of definition of ψ also.

According to lemma 2.1, there exists an equivariant isomorphism between the groupoid $\mathbb{G}(\psi \otimes_L K^s, K)$ and the groupoid $\mathbb{G}(\varphi \otimes_L K^s, K)$ associated to the initial cover. We deduce that φ and \mathcal{S} have the same field of modules and the same fields of definition. This finishes the proof of theorem 4.1.

5 Curves

In this section K is a field of characteristic zero. We start from a cover of curves, having K as field of moduli, and we construct out of it, a projective normal integral curve over K^s , having the same field of moduli and the same fields of definition as the original cover. This will prove theorem 1.2.

We shall make use of the surface \mathcal{S} constructed in section 4. So we keep the notation of section 4. We know that \mathcal{S} has field of moduli K and the same fields of definition as the initial cover $\varphi : \mathcal{C}/L \rightarrow \mathcal{B} \otimes_K L$ (or equivalently $\psi : \mathcal{D}/L \rightarrow \mathcal{B} \otimes_K L$).

The main idea is to draw on \mathcal{S} a singular (but stable) curve inheriting the field of moduli and fields of definition of \mathcal{S} ; then to deform it to obtain a smooth projective curve.

5.1 Two stable curves

We go on with the notation of section 4.1. In particular, we have constructed a cover $\chi : \mathcal{S} \rightarrow (\mathcal{B} \otimes_K L) \times \mathcal{D}$ strongly ramified along the graph of ψ . For any point P on \mathcal{B} , we call \mathcal{F}_P the inverse image of $P \times \mathcal{D}$ by χ and $\chi_P : \mathcal{F}_P \rightarrow P \times \mathcal{D}$ the correstriiction of χ to $P \times \mathcal{D}$. We call Γ the union of the supports of all divisors of the functions g_i . It contains the ramification locus of the cover χ .

Lemma 5.1 *There exist two non-constant functions $f, g \in K(\mathcal{B})$ such that :*

- *the divisor $((f)_0 + (f)_\infty) \times \mathcal{D}$ crosses transversally Γ ;*
- *the divisor $(\mathcal{B} \otimes_K L) \times ((g \circ \psi)_0 + (g \circ \psi)_\infty)$ crosses transversally $\Gamma \cup [((f)_0 + (f)_\infty) \times \mathcal{D}]$;*
- *any K^s -automorphism of \mathcal{D} that stabilizes the fiber $(g \circ \psi)_0$ is an automorphism of the cover ψ (note that the preceding condition implies that this fiber is simple);*
- *for any zero P of f , the cover $\kappa_P := g \circ \psi \circ \chi_P : \mathcal{F}_P \rightarrow \mathbf{P}^1$ has no automorphism other than the elements of $A \times B$:*

$$\text{Aut}_{K^s}(g \circ \psi \circ \chi_P) = \text{Aut}_{K^s}(\psi \circ \chi_P) = A \times B.$$

Proof — Let $f \in K(\mathcal{B})$ be a non-constant function. We apply lemma 6.2 to $L, \mathcal{B}, \mathcal{D}, \Gamma$ et f . We deduce that there exist two distinct scalars x and y in L such that $(f)_x \times \mathcal{D}$ and $(f)_y \times \mathcal{D}$ cross transversally Γ . We even can choose x and y in K and such that for every point P in $f^{-1}(x)$ or $f^{-1}(y)$, the fiber of every function $f_i \in K(\mathcal{B})$ above $f_i(P)$ does not meet the singular values of ψ , that is :

$$\forall P \in f^{-1}(x) \cup f^{-1}(y), \quad f_i^{-1}(f_i(P)) \cap \{\text{singular values of } \psi\} = \emptyset. \quad (3)$$

We replace f by $(f - x)/(f - y)$ and the first condition is fulfilled.

Now, for every zero P of f , we see that \mathcal{F}_P is smooth and geometrically integral, because $(f)_0 \times \mathcal{D}$ crosses transversally the ramification locus Γ of χ . We now prove that

$$\text{Aut}_{K^s}(\psi \circ \chi_P) = A \times B.$$

Indeed the function field $K^s(\mathcal{F}_P)$ is the compositum :

$$\begin{array}{ccc}
 & K^s(\mathcal{F}_P) & \\
 & / \quad \backslash & \\
 K^s(\mathcal{D}) & & K^s(\mathcal{B}') \stackrel{\text{d\'ef.}}{=} K^s((f_i - f_i(P))^{\frac{1}{p_i}}, 1 \leq i \leq I) \\
 & \backslash \quad / & \\
 & K^s(\mathcal{B}) & \\
 \text{pd} & & \prod_i p_i
 \end{array}$$

where $\mathcal{B}' \rightarrow \mathcal{B}$ is an abelian cover with Galois group $B = \prod_{i=1}^I \mathbf{Z}/p_i\mathbf{Z}$. The $K^s(\mathcal{B})$ -extensions $K^s(\mathcal{D})$ and $K^s(\mathcal{B}')$ are linearly disjoint (their degrees are coprime and one of them is Galois) and condition (3) implies that the extension $K^s(\mathcal{D})/K^s(\mathcal{B})$ is not ramified above the zeros of the functions $f_i - f_i(P)$.

Now, any subcover of $\mathcal{B}' \rightarrow \mathcal{B}$ is ramified above the zeros of at least one of the functions $f_i - f_i(P)$. The same is true for any subcover of $\mathcal{F}_P \rightarrow \mathcal{D}$. We deduce that $\mathcal{D} \rightarrow \mathcal{B}$ is the maximal subcover of $\mathcal{F}_P \rightarrow \mathcal{B}$ that is not ramified above the zeros of the functions $f_i - f_i(P)$. Therefore any $K^s(\mathcal{B})$ -automorphism of $K^s(\mathcal{F}_P)$ stabilizes $K^s(\mathcal{D})$. Thus :

$$\text{Aut}_{K^s(\mathcal{B})}(K^s(\mathcal{F}_P)) = \text{Aut}_{K^s(\mathcal{B})}(K^s(\mathcal{D})) \times \text{Aut}_{K^s(\mathcal{B})}(K^s(\mathcal{B}')),$$

as was to be shown.

Next we look for a function g in $K(\mathcal{B})$ such that $g \circ \psi$ has no K^s -automorphism but elements of A and, for every zero P of f , the cover $\kappa_P = g \circ \psi \circ \chi_P$ has no K^s -automorphism but elements of $\text{Aut}_{K^s}(\psi \circ \chi_P) = A \times B$. According to lemma 6.4, the functions in $K(\mathcal{B})$ that do not fulfill all these conditions lay in a finite union of strict sub K -algebras. Therefore there exists such a function g .

According to lemma 6.2, the scalars x in K such that $(g \circ \psi)_x$ does not cross $\Gamma \cup [((f)_0 + (f)_\infty) \times \mathcal{D}]$ transversally, are finitely many.

According to lemma 6.3, the x in K such that $(g \circ \psi)_x$ has a K^s -automorphism not in A , are finitely many.

Therefore there exist two distinct scalars x and y in K such that $(g \circ \psi)_x$ and $(g \circ \psi)_y$ cross $\Gamma \cup [((f)_0 + (f)_\infty) \times \mathcal{D}]$ transversally and $(g \circ \psi)_x$ has no automorphism but those in A . We replace g by $(g - x)/(g - y)$ and the last three conditions are satisfied. \square

Now let \mathcal{J}_0 be the curve on $(\mathcal{B} \otimes_K L) \times \mathcal{D}$ with equation :

$$f(P) \times g \circ \psi(Q) = 0.$$

We can control the automorphism group of \mathcal{J}_0 .

Lemma 5.2 *The curve \mathcal{J}_0 is stable and $\text{Aut}_{K^s}(\mathcal{J}_0) \simeq A$.*

Proof — The curve \mathcal{J}_0 is geometrically reduced because the zeros of f and $g \circ \psi$ are simple. The singular points on \mathcal{J}_0 are the couples (P, Q) on $(\mathcal{B} \otimes_K K^s) \times \mathcal{D}$ such that $f(P) = g \circ \psi(Q) = 0$. These are ordinary double points. Therefore \mathcal{J}_0 is semistable. It is geometrically connected also. Its irreducible components (over K^s) are isomorphic to \mathcal{B} or \mathcal{D} . So they all have genus ≥ 2 . Therefore \mathcal{J}_0 is a stable curve.

We now prove that $\text{Aut}_{K^s}(\mathcal{J}_0) \simeq A$: the group of K^s -automorphisms of \mathcal{J}_0 is the group A of K^s -automorphisms of ψ . It is clear that A is included in $\text{Aut}_{K^s}(\mathcal{J}_0)$. Conversely, let a be a K^s -automorphism of \mathcal{J}_0 . Then a permutes the irreducible components of $\mathcal{J}_0 \otimes_K K^s$. Some of

these components are isomorphic to $\mathcal{B} \otimes_K K^s$, and the others are isomorphic to $\mathcal{D} \otimes_L K^s$. Since \mathcal{B} and \mathcal{D} are not K^s -isomorphic, \mathfrak{a} stabilizes the two subsets of components. If we restrict \mathfrak{a} to a component isomorphic to \mathcal{B} then compose with the projection on \mathcal{B} , we obtain a non-constant K^s -morphism from \mathcal{B} to itself. This morphism must be the identity because \mathcal{B} has no non-trivial K^s -automorphism. Therefore \mathfrak{a} stabilizes each component isomorphic to $\mathcal{D} \otimes_L K^s$. The singular points on such a component are the zeros of $g \circ \psi$. The set of these zeroes is stabilized by no K^s -automorphism of \mathcal{D} but those of ψ . So the restriction of \mathfrak{a} to any component isomorphic to $\mathcal{D} \otimes_L K^s$ is an element in A . If we compose \mathfrak{a} with some well chosen element in A , we may assume that \mathfrak{a} is trivial on one component isomorphic to $\mathcal{D} \otimes_L K^s$. Therefore \mathfrak{a} stabilizes every component isomorphic to $\mathcal{B} \otimes_K K^s$. Since these components have no non-trivial automorphism, \mathfrak{a} acts trivially on them. Now let $P \times (\mathcal{D} \otimes_L K^s)$ be a component of \mathcal{J}_0 isomorphic to $\mathcal{D} \otimes_L K^s$. The restriction of \mathfrak{a} to it is an automorphism that fixes the singular points. These points are the zeros of $g \circ \psi$. So \mathfrak{a} restricted to $P \times (\mathcal{D} \otimes_L K^s)$ is an element of A . But A acts faithfully on the set of zeros of $g \circ \psi$. We deduce that \mathfrak{a} acts trivially on every component isomorphic to $\mathcal{D} \otimes_L K^s$. \square

Let \mathcal{K}_0 be the inverse image of \mathcal{J}_0 by χ . This is singular curve over L . The next lemma 5.3 is concerned with the stability and the automorphism group of this curve. Controlling the full group of K^s -automorphisms seems difficult to us. So we shall be interested in the subgroup consisting of *deformable* automorphisms. This subgroup is denoted $\text{Aut}_{K^s}^{\text{def}}(\mathcal{K}_0)$. We now explain what we mean by a deformable automorphism.

We first notice that the components of $\mathcal{K}_0 \otimes_L K^s$ are of two different kinds. Some of them are covers of some $(\mathcal{B} \otimes_K K^s) \times Q$ where Q is a zero of $g \circ \psi$. We denote such a component by \mathcal{E}_Q . The other components are covers of some $P \times (\mathcal{D} \otimes_L K^s)$ where P is a K^s -zero of f . Such a component is denoted by \mathcal{F}_P . We call $\chi_P : \mathcal{F}_P \rightarrow P \times (\mathcal{D} \otimes_L K^s)$ and $\chi_Q : \mathcal{E}_Q \rightarrow (\mathcal{B} \otimes_K K^s) \times Q$ the restrictions of χ to components of \mathcal{K}_0 . Now let T be a singular point on \mathcal{K}_0 such that $\chi(T) = (P, Q)$. So T lays in the intersection of \mathcal{E}_Q and \mathcal{F}_P . The point on \mathcal{E}_Q corresponding to T is denoted R . The point on \mathcal{F}_P corresponding to T is denoted S . So $\chi_Q(R) = P$ and $\chi_P(S) = Q$. And $f \circ \chi_Q$ is a uniformizing parameter for \mathcal{E}_Q at R , while $g \circ \psi \circ \chi_P$ is a uniformizing parameter for \mathcal{F}_P at S . Let \mathfrak{a} be an automorphism of $\mathcal{K}_0 \otimes_L K^s$ and let $T' = (R', S')$ be the image of $T = (R, S)$ by \mathfrak{a} . We write $\chi(T') = (P', Q')$. We observe that $f \circ \chi_{Q'} \circ \mathfrak{a}$ is a uniformizing parameter for \mathcal{E}_Q at R and $g \circ \psi \circ \chi_{P'} \circ \mathfrak{a}$ is a uniformizing parameter for \mathcal{F}_P at S .

We say that \mathfrak{a} is a *deformable automorphism* of \mathcal{K}_0 if for every singular point T of \mathcal{K}_0 we have

$$\frac{f \circ \chi_{Q'} \circ \mathfrak{a}}{f \circ \chi_Q}(R) \times \frac{g \circ \psi \circ \chi_{P'} \circ \mathfrak{a}}{g \circ \psi \circ \chi_P}(S) = 1 \quad (4)$$

where $\chi(T) = (P, Q)$ and $\chi(\mathfrak{a}(T)) = (P', Q')$.

The justification for this definition is given at paragraph 5.2. Deformable automorphisms form a subgroup of the group of K^s -automorphisms of \mathcal{K}_0 .

Lemma 5.3 *The curve \mathcal{K}_0 is stable and $\text{Aut}_{K^s}^{\text{def}}(\mathcal{K}_0) \simeq A \times B$.*

Proof — First, it is clear that $A \times B$ acts faithfully on $\mathcal{K}_0 \otimes_L K^s$, and the corresponding automorphisms are deformable.

The curve \mathcal{K}_0 is drougt on \mathcal{S} . The conditions imposed on functions f and g imply that \mathcal{K}_0 is a stable curve. The intersection of (f) and (g) with the ramification locus Γ of χ , is simple. Therefore every singular point on \mathcal{J}_0 gives rise to $\deg(\chi)$ singular points on \mathcal{K}_0 ; and all these singular points are ordinary double points. To prove that $\mathcal{K}_0 \otimes_L K^s$ is connected, we observe that the function g_i is not a p_i -th power because none of the functions $f_i - \lambda$, $\lambda \in K^s$ is a p_i -th power (and the $f_i \circ \psi - \lambda$ are

not either) as was shown in section 4.1. Also the irreducible components of $\mathcal{K}_0 \otimes_L K^s$ correspond bijectively to those of $\mathcal{J}_0 \otimes_L K^s$.

Now let us prove that $\text{Aut}_{K^s}^{\text{def.}}(\mathcal{K}_0) \simeq A \times B$. The components \mathcal{F}_P and \mathcal{E}_Q have different genera. Therefore no \mathcal{F}_P is K^s -isomorphic to some \mathcal{E}_Q . Thus any K^s -automorphism \mathfrak{a} of \mathcal{K}_0 stabilizes the set of all components \mathcal{F}_P (and also the set of all \mathcal{E}_Q).

Let Q and Q' be two K^s -zeros of $g \circ \psi$ such that $\mathfrak{a}(\mathcal{E}_Q) = \mathcal{E}_{Q'}$. As in the proof of lemma 4.2, we notice that the image of \mathcal{E}_Q in the product $\mathcal{B} \times \mathcal{B}$, by the morphism $\chi_Q \times \chi_{Q'} \circ \mathfrak{a}$, has an arithmetic genus smaller than or equal to $1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$. Again, this implies that this image is K^s -isomorphic to \mathcal{B} (otherwise, this image would have geometric genus bigger than $1 + 2(g(\mathcal{B}) - 1)\Pi + \Pi^2$ by Hurwitz formula). Since \mathcal{B} has no K^s -automorphism, we deduce as before that \mathfrak{a} induces an isomorphism of covers between the restrictions $\chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B}$ and $\chi_{Q'} : \mathcal{E}_{Q'} \rightarrow \mathcal{B}$ de χ . Thus

$$\chi_Q = \chi_{Q'} \circ \mathfrak{a}. \quad (5)$$

This implies that \mathfrak{a} stabilizes every component \mathcal{F}_P where P is any K^s -zero of f . Indeed, let us start from a singular point $T = (R, S) \in \mathcal{E}_Q \cap \mathcal{F}_P$, where P is a K^s -zero of f and Q is a K^s -zero of $g \circ \psi$. We thus have $\chi(T) = (P, Q) \in \mathcal{B} \times \mathcal{D}$. We know there exists $P' \in \mathcal{B}(K^s)$ and $Q' \in \mathcal{D}(K^s)$ such that $\mathfrak{a}(T) \in \mathcal{F}_{P'} \cap \mathcal{E}_{Q'}$. So $\mathfrak{a}(T) \in \mathcal{E}_{Q'} \cap \mathfrak{a}(\mathcal{E}_Q)$ and d'où $\mathfrak{a}(\mathcal{E}_Q) = \mathcal{E}_{Q'}$. We find ourselves in the situation of the last paragraph before the statement of lemma 5.3. Further :

$$P' = \chi_{Q'} \circ \mathfrak{a}(T) = \chi_Q(T) = P.$$

We conclude that $P = P'$ and $\mathfrak{a}(\mathcal{F}_P) = \mathcal{F}_P$.

Now, we deduce from formulae (4) and (5) that :

$$\frac{g \circ \psi \circ \chi_P \circ \mathfrak{a}}{g \circ \psi \circ \chi_P}(S) = 1. \quad (6)$$

Call \mathfrak{a}_P the restriction of \mathfrak{a} to \mathcal{F}_P . This is an automorphism of \mathcal{F}_P . We prove that \mathfrak{a}_P is the restriction to \mathcal{F}_P of an element of $A \times B$. To this end, we introduce the function $\kappa_P = g \circ \psi \circ \chi_P \in K^s(\mathcal{F}_P)$. The degree of κ_P is $\deg(g) \times pd \times \Pi$ and its zeros are all simple. These zeros are the intersection points between \mathcal{F}_P and the other components of \mathcal{K}_0 . Since \mathfrak{a}_P permutes these zeros, the functions $\kappa_P \circ \mathfrak{a}_P$ and κ_P have the same divisor of zeros. Therefore the only possible poles of the function $\frac{\kappa_P}{\kappa_P \circ \mathfrak{a}_P} - 1$ are the poles of κ_P . Thus the degree of $\frac{\kappa_P}{\kappa_P \circ \mathfrak{a}_P} - 1$ is smaller than or equal to the degree of κ_P . But according to (6), the zeros of κ_P are also zeros of $\frac{\kappa_P}{\kappa_P \circ \mathfrak{a}_P} - 1$. So we just proved that if the function $\frac{\kappa_P}{\kappa_P \circ \mathfrak{a}_P} - 1$ is non-zero, it has the same divisor as κ_P . Therefore there exists a constant $c \in K^s$ such that :

$$\frac{\kappa_P}{\kappa_P \circ \mathfrak{a}_P} - 1 = c\kappa_P \quad \text{or equivalently :} \quad \frac{1}{\kappa_P \circ \mathfrak{a}_P} = \frac{1}{\kappa_P} + c.$$

Since \mathfrak{a}_P has finite order e and K has characteristic zero, we deduce that $ce = 0$, then $c = 0$, then $\kappa_P \circ \mathfrak{a}_P = \kappa_P$. In other words, \mathfrak{a}_P is an automorphism of the cover $\kappa_P = g \circ \psi \circ \chi_P : \mathcal{F}_P \rightarrow \mathbf{P}^1$. Because of the hypothesis on the function g , we deduce that \mathfrak{a}_P is the restriction to \mathcal{F}_P of an element in $A \times B$. We replace \mathfrak{a} by \mathfrak{a} composed with the inverse of this element. So we now assume that \mathfrak{a} acts trivially on \mathcal{F}_P . In particular \mathfrak{a} fixes every singular point on \mathcal{F}_P . So \mathfrak{a} stabilizes every component \mathcal{E}_Q for Q any zero of $g \circ \psi$. The restriction \mathfrak{a}_Q of \mathfrak{a} to \mathcal{E}_Q is an automorphism of χ_Q according to (5). Further \mathfrak{a}_Q fixes one point (and even every point) in the unramified fiber above P of the Galois cover $\chi_Q : \mathcal{E}_Q \rightarrow \mathcal{B}$. Therefore \mathfrak{a}_Q is the identity. We have proved that \mathfrak{a} is trivial on every component \mathcal{E}_Q .

To finish with, we now prove that \mathfrak{a} is also trivial on the components \mathcal{F}_P for any zero P of f . Remind we have already assumed this to be true for one of these zeros. We call \mathfrak{a}_P the restriction of \mathfrak{a} to \mathcal{F}_P . We already proved that \mathfrak{a}_P is the restriction of an element in $A \times B$. Further \mathfrak{a}_P fixes all the singular points of \mathcal{K}_0 laying on \mathcal{F}_P . These points are the zeros of $g \circ \psi \circ \chi_P$. So we just need to prove that the action of $A \times B$ on the set of zeros of $g \circ \psi \circ \chi_P$ is free. This is certainly the case for elements in B because the zeros of $g \circ \psi$ are, by hypothesis, unramified in the Galois cover $\chi_P : \mathcal{F}_P \rightarrow \mathcal{D}$. This is true also for elements in $A \times B$ because the action of A on the set of zeros of $g \circ \psi$ is free. \square

5.2 Deformations

In this paragraph we deform the two stable curves \mathcal{J}_0 and \mathcal{K}_0 . If $x \in K^s$ is a scalar, it is natural to consider the curve \mathcal{J}_x drought on the surface $\mathcal{I} = (\mathcal{B} \otimes_K L) \times \mathcal{D}$ and defined by the equation $f(P) \times g(\psi(Q)) = x$. We call \mathcal{K}_x the inverse image of \mathcal{J}_x by χ . In this paragraph and in the next one, we shall prove that for almost all scalars x in K , the curve \mathcal{K}_x is smooth, geometrically integral, with K^s -automorphism group equal to $A \times B$, and with the same field of moduli and the same fields of definition as the original cover φ . To this end, we would like to consider the families $(\mathcal{J}_x)_x$ and $(\mathcal{K}_x)_x$ as fibrations above \mathbf{P}^1 . We should be careful however : the family $(\mathcal{J}_x)_x$ has base points. So we first have to blow up $\mathcal{I} = (\mathcal{B} \otimes_K L) \times \mathcal{D}$ along

$$c = ((f)_\infty \times (g \circ \psi)_0) \cup ((f)_0 \times (g \circ \psi)_\infty).$$

Note that c is the union of $2 \times \deg(f) \times \deg(g \circ \psi)$ simple geometric points. We denote by $\mathcal{I}_{\infty, \infty} \subset \mathcal{I} = (\mathcal{B} \otimes_K L) \times \mathcal{D}$ the complementary open set of $((f)_\infty \times \mathcal{D}) \cup (\mathcal{B} \times (g \circ \psi)_\infty)$ in $(\mathcal{B} \otimes_K L) \times \mathcal{D}$. We similarly define $\mathcal{I}_{0,0}, \mathcal{I}_{0,\infty}, \mathcal{I}_{\infty,0}$. These four open sets cover $(\mathcal{B} \otimes_K L) \times \mathcal{D}$.

Let $\mathbf{P}_L^1 = \text{Proj}(L[X_0, X_1])$ be the projective line over L . We set $F = 1/f$ and $G = 1/g$. Let $\mathcal{J}_{\infty,0} \subset \mathcal{I}_{\infty,0} \times \mathbf{P}_L^1$ be the set of $(P, Q, [X_0 : X_1])$ such that $f(P)X_0 = G(\psi(Q))X_1$. Let $\mathcal{J}_{0,\infty} \subset \mathcal{I}_{0,\infty} \times \mathbf{P}_L^1$ be the set of $(P, Q, [X_0 : X_1])$ such that $g(\psi(Q))X_0 = F(P)X_1$. Let $\mathcal{J}_{\infty,\infty} \subset \mathcal{I}_{\infty,\infty} \times \mathbf{P}_L^1$ be the set of $(P, Q, [X_0 : X_1])$ such that $f(P)g(\psi(Q))X_0 = X_1$. Let $\mathcal{J}_{0,0} \subset \mathcal{I}_{0,0} \times \mathbf{P}_L^1$ be the set of $(P, Q, [X_0 : X_1])$ such that $X_0 = F(P)G(\psi(Q))X_1$. We glue together these four algebraic varieties and obtain a variety $\mathcal{J} \subset \mathcal{I} \times \mathbf{P}_L^1$. Let $b_{\mathcal{I}} : \mathcal{J} \rightarrow \mathcal{I}$ be the projection on the first factor and let $j : \mathcal{J} \rightarrow \mathbf{P}_L^1$ be the projection on \mathbf{P}_L^1 . This is a flat, projective, surjective morphism.

Let $\mathcal{K} \subset \mathcal{S} \times \mathbf{P}_L^1$ be the inverse image of \mathcal{J} by $\chi \times \text{Id}$ where $\text{Id} : \mathbf{P}_L^1 \rightarrow \mathbf{P}_L^1$ is the identity. This is the blow up of \mathcal{S} along $\chi^{-1}(c)$. Note that $\chi^{-1}(c)$ is the union of $\deg(\chi) \times \deg(f) \times \deg(g \circ \psi)$ simple geometrical points because χ is unramified above c . Actually, \mathcal{K} is the normalization of \mathcal{J} in $L(\mathbf{P}^1)(\mathcal{S})$. We denote by $\chi : \mathcal{K} \rightarrow \mathcal{J}$ the corresponding morphism. We call $b_{\mathcal{S}} : \mathcal{K} \rightarrow \mathcal{S}$ the projection on the first factor. We call $k : \mathcal{K} \rightarrow \mathbf{P}_L^1$ the projection on the second factor. This is the composed morphism $k = j \circ \chi$. This is a flat, proper and surjective morphism.

Let $\mathbf{A}_L^1 \subset \mathbf{P}_L^1$ be the spectrum of $L[X]$ where $X = X_1/X_0$. Using the fonction X we identify $\mathbf{P}_L^1(K^s)$ and $K^s \cup \{\infty\}$. If x is a point on $\mathbf{P}^1(K^s)$ we denote by \mathcal{J}_x the fiber of j above x and \mathcal{K}_x the fiber of k above x . The restriction of $b_{\mathcal{I}}$ to \mathcal{J}_x is a closed immersion. So we can see \mathcal{J}_x as a curve drought on $\mathcal{I} = \mathcal{B} \times \mathcal{D}$. Similarly, the restriction of $b_{\mathcal{S}}$ to \mathcal{K}_x is a closed immersion. So we can see \mathcal{K}_x as a curve drought on \mathcal{S} . In particular, the fiber of j at 0 is isomorphic by $b_{\mathcal{I}}$ to the stable curve \mathcal{J}_0 introduced in paragraph 5.1. Similarly, the fiber of k at 0 is isomorphic by $b_{\mathcal{S}}$ to the stable curve \mathcal{K}_0 introduced in paragraph 5.1.

We call \mathcal{J}_η the generic fiber of j and \mathcal{K}_η the generic fiber of k .

We now show that $\mathcal{J}_\eta/L(X)$ is geometrically connected and for almost every $x \in \mathbf{P}^1(K^s)$ the curve \mathcal{J}_x is geometrically connected over $L(x)$. According to Stein's factorization theorem [Liu02,

Chapter 5, Exercise 3.11], we can factor $j : \mathcal{J}/L \rightarrow \mathbf{P}_L^1$ as $j_f \circ j_c$ where j_c has geometrically connected fibers and j_f is finite and dominant. The fiber of j_f above 0 is trivial because \mathcal{J}_0 is connected and reduced. Therefore the degree of j_f is 1 according to [Liu02, Chapter 5, Exercise 1.25]. Therefore j_f is an isomorphism above a non-empty open set of \mathbf{P}_L^1 . The generic fiber \mathcal{J}_η is geometrically connected over $L(X)$ and for almost all $x \in \mathbf{P}^1(K^s)$ the curve \mathcal{J}_x is geometrically connected over $L(x)$.

We now show that \mathcal{J}_η is smooth (and therefore geometrically integral). Indeed, it is smooth outside the points $(P, Q) \in \mathcal{J}_\eta \subset \mathcal{B} \times \mathcal{D}$ where $df(P) = 0$ and $d(g \circ \psi)(Q) = 0$. Such points are defined over K^s . Therefore the function $f(P) \times g(\psi(Q))$ cannot take the transcendental value X at these points.

The ramification locus $\Gamma \subset \mathcal{I}$ of χ cuts the fiber \mathcal{J}_0 transversally. Therefore it cuts the generic fiber \mathcal{J}_η transversally. So \mathcal{K}_η is smooth and geometrically integral. Thus for almost every $x \in K^s$ the fibers \mathcal{J}_x and \mathcal{K}_x are smooth and integral.

Finally, our knowledge of $\text{Aut}_{K^s}^{\text{def.}}(\mathcal{K}_0)$ enables us to show that $\text{Aut}_{K(X)^s}(\mathcal{K}_\eta) \simeq A \times B$. Indeed, set $R = K^s[[X]]$ the completed local ring at $X = 0$ of \mathbf{P}^1 . The curve $\hat{\mathcal{K}} = \mathcal{K} \otimes_{\mathbf{P}_L^1} \text{Spec}(R)$ is stable over the spectrum of R . According to [Liu02, Chapter 10, Proposition 3.38, Remarque 3.39] the functor "automorphism group" $t \mapsto \text{Aut}_t(\hat{\mathcal{K}}_t)$ is representable by finite unramified scheme over $\text{Spec } R$ and the specialization morphism $\text{Aut}_{K^s((X))}(\hat{\mathcal{K}}_\eta) \rightarrow \text{Aut}_{K^s}(\mathcal{K}_0)$ is injective. According to lemma 6.6, the image of this morphism is included in the subgroup of deformable K^s -automorphisms of \mathcal{K}_0 . Since $\text{Spec } R$ has no unramified cover, we deduce the following estimate of the automorphism group of the generic fiber

$$A \times B \subset \text{Aut}_{K(X)^s}(\mathcal{K}_\eta) \subset \text{Aut}_{K^s((X))}(\mathcal{K}_\eta) \subset \text{Aut}_{K^s}^{\text{def.}}(\mathcal{K}_0).$$

We have already seen that the rightmost group is equal to $A \times B$. So que $\text{Aut}_{K(X)^s}(\mathcal{K}_\eta) = A \times B$ as was to be proved.

5.3 Fields of moduli and fields of definition of fibers

We now prove the existence of fibers \mathcal{K}_x where $x \in \mathbf{A}^1(K)$, that are smooth, geometrically integral, and have the same field of moduli (i.e. K) and the same fields of definition as \mathcal{S} . Remind \mathcal{S} has field of moduli K and the same fields of definitions as the initial cover $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$.

Indeed, we proved in paragraph 5.2 that for almost all $x \in \mathbf{A}^1(K)$, the fiber \mathcal{K}_x is smooth and geometrically integral. Using lemma 6.7 on the specialization of the automorphism group in a family of curves we deduce that for almost all $x \in \mathbf{A}^1(K)$, the group of K^s -automorphisms of the fiber \mathcal{K}_x is isomorphic to the group of $K(X)^s$ -automorphisms of the generic fiber. Since the later group is isomorphic to the automorphism group $A \times B$ of the surface \mathcal{S} , we deduce that, for almost all x , the restriction map above is an isomorphism :

$$\text{Aut}_{K^s}(\mathcal{S}) \xrightarrow{\simeq} \text{Aut}_{K^s}(\mathcal{K}_x). \quad (7)$$

Now let $x \in K$ be such that \mathcal{K}_x is smooth and geometrically integral and $\text{Aut}_{K^s}(\mathcal{K}_x) = A \times B$. We prove that \mathcal{K}_x has all the expected properties.

The construction presented in paragraph 5.2 defines a functor \mathbb{F}_x between the groupoids $\mathbb{G}(\mathcal{S}, K)$ and $\mathbb{G}(\mathcal{K}_x, K)$. As far as (iso)morphisms are concerned, this functor is defined by restriction. The functor \mathbb{F}_x is continuous because it maps the initial L -model of \mathcal{S} onto the L -curve \mathcal{K}_x . Therefore the field of moduli of \mathcal{K}_x is K and any field of definition of \mathcal{S} is a field of definition of \mathcal{K}_x .

Further \mathbb{F}_x defines a bijection between the objects in either groupoids (these objects are parametrized by Γ_K on either sides). The identity (7) implies that the functor \mathbb{F}_x is fully faithful. Therefore \mathbb{F}_x is a continuous isomorphism of groupoids. Thus any field of definition of \mathcal{K}_x is a field of définition of \mathcal{S} .

This finishes the proof of theorem 1.2. Theorem 1.1 is an immediate consequence of theorems 1.2 and 1.3.

6 Seven lemmas about curves and surfaces

In this section we state and prove seven lemmas that are needed in the proof of theorem 1.2.

6.1 About curves and products of two curves

Lemma 6.1 *Let K be an algebraically closed field. Let \mathcal{B} and \mathcal{C} be two projective, smooth and integral curves over K . Let β be the genus of \mathcal{B} and let γ be the genus of \mathcal{C} . We fix a geometric point P on \mathcal{B} . We fix a geometric point Q on \mathcal{C} . We identify \mathcal{B} and $\mathcal{B} \times Q$. We identify \mathcal{C} and $P \times \mathcal{C}$. Let Γ be a divisor on $\mathcal{B} \times \mathcal{C}$. Let (b, c) be the bidegree of Γ , i.e. $b = \mathcal{B} \cdot \Gamma$ and $c = \mathcal{C} \cdot \Gamma$. The virtual arithmetic genus π of Γ is at most $1 + bc + c(\beta - 1) + b(\gamma - 1)$. When $b = c$ this bound reads $1 + 2b(\beta - 1) + b^2$.*

Proof — We follow the lines of Weil's proof of the Riemann hypothesis for curves (cf. [Har77, Exercice V-1.10]).

The algebraic equivalence class of the canonical divisor on $\mathcal{B} \times \mathcal{C}$ is $K = 2(\beta - 1)\mathcal{C} + 2(\gamma - 1)\mathcal{B}$. We deduce from adjunction formula that $\pi = \frac{D \cdot (D + K)}{2} + 1$ (cf. [Har77, Exercice V-1.3]). Thus :

$$\pi = \frac{D \cdot (D + 2(\beta - 1)\mathcal{C} + 2(\gamma - 1)\mathcal{B})}{2} + 1 = \frac{D \cdot D + 2c(\beta - 1) + 2b(\gamma - 1)}{2} + 1,$$

and we just need to bound the self intersection $D \cdot D$. We deduce from Castelnuovo's and Severi's inequality (cf. [Har77, Exercice V-1.9]) that $D \cdot D \leq 2bc$. This finishes the proof of the lemma. \square

Lemma 6.2 *Let K be a field. Let \mathcal{B} and \mathcal{C} be two projective, smooth, geometrically integral curves over K . Let Γ be an effective divisor without multiplicity on the surface $\mathcal{B} \times \mathcal{C}$. Let $f \in K(\mathcal{B})$ be a non-constant function. For all but finitely many scalars x in K^s , the divisor $(f)_x \times \mathcal{C}$ crosses transversally Γ , where $(f)_x$ is the positive part of the divisor of $f - x$.*

Proof — We call $p_{\mathcal{B}} : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{B}$ the projection on the first factor. Let E be the set of points in $\mathcal{B}(K^s)$ such that at least one of the following condition holds: $p_{\mathcal{B}}^{-1}(P)$ contains a singular point on Γ , or $p_{\mathcal{B}}^{-1}(P)$ contains a ramified point of the morphism $p_{\mathcal{B}} : \Gamma \rightarrow \mathcal{B}$, or the fiber $p_{\mathcal{B}}^{-1}(P)$ is contained in Γ . The set E is finite. For all $x \in K^s$ but finitely many, the fiber $f^{-1}(x)$ avoids E and it is simple. \square

Lemma 6.3 *Let K be a field. Let \mathcal{B} be a projective, smooth, geometrically integral curve over K . Assume the genus of \mathcal{B} is at least 2. Let $f \in K(\mathcal{B})$ be a non-constant function. We note G the group of K^s -automorphisms of f . This is the set of all K^s -automorphisms α of \mathcal{B} such that $f \circ \alpha = f$. For any $x \in \mathbf{P}^1(K^s)$, we note $(f)_x = f^{-1}(x)$ the fiber above x and G_x the group of K^s -automorphisms of \mathcal{B} that stabilizes the set of K^s -points of $(f)_x$.*

For all x in $\mathbf{P}^1(K^s)$ but finitely many we have $G_x = G$.

Proof — The group $H = \text{Aut}_{K^s}(\mathcal{B})$ of K^s -automorphisms of \mathcal{B} is finite because the genus of \mathcal{B} is at least two. Let α be an automorphism in $H \setminus G$ and let $x \in \mathbf{P}^1(K^s)$. Assume that the K^s -points in $(f)_x$ are permuted by α . Let P be one of them. Then $f \circ \alpha(P) = f(P) = x$. So P is a zero of the non-zero function $f \circ \alpha - f$. For each α there are finitely many such zeros. And the α are finitely many. So the images by f of such P 's are finitely many also. \square

Lemma 6.4 *Let K be a field. Let \mathcal{B} be a projective, smooth, geometrically integral curve over K . Let $L \subset K^s$ be an algebraic extension of K . Let \mathcal{C} be a projective, smooth, geometrically integral curve over L . We assume \mathcal{B} has genus at least 2. Let $\varphi : \mathcal{C} \rightarrow \mathcal{B} \otimes_K L$ be a non-constant L -morphism. Call G the group of K^s -automorphisms of φ . This is the set of K^s -automorphisms α of \mathcal{C} such that $\varphi \circ \alpha = \varphi$. If f is any function in $K^s(\mathcal{B})$ we denote by $G_f = \text{Aut}_{K^s}(f \circ \varphi)$ the groupe of K^s -automorphisms of $f \circ \varphi$. We have $G \subset G_f$. Let $V \subset K(\mathcal{B})$ be the set of functions f such that $G_f \neq G$. The set V is contained in a finite union of strict K -subalgebras of $K(\mathcal{B})$.*

Proof — The statement to be proven concerns the three function fields $K^s(f) \subset K^s(\mathcal{B}) \subset K^s(\mathcal{C})$ and the groups involved are the following ones :

$$\begin{cases} G = \text{Aut}_{K^s}(\varphi) = \text{Aut}_{K^s(\mathcal{B})}(K^s(\mathcal{C})), \\ G_f = \text{Aut}_{K^s}(f \circ \varphi) = \text{Aut}_{K^s(f)}(K^s(\mathcal{C})), \\ A = \text{Aut}_{K^s}(\mathcal{C}) = \text{Aut}_{K^s}(K^s(\mathcal{C})), \end{cases} \quad \Rightarrow \quad G \subset G_f \subset A.$$

Now, the set V can be described as follows :

$$V = \left(\bigcup_{\alpha \in A \setminus G} K^s(\mathcal{C})^\alpha \cap K^s(\mathcal{B}) \right) \cap K(\mathcal{B}) = \bigcup_{\alpha \in A \setminus G} K^s(\mathcal{C})^\alpha \cap K(\mathcal{B}).$$

This is a union of sets indexed by elements in the finite set $A \setminus G$ (remind A is finite because the genus of \mathcal{C} is at least 2). Since $\alpha \notin G$, each $K^s(\mathcal{C})^\alpha \cap K^s(\mathcal{B})$ is a strict subfield of $K^s(\mathcal{B})$ containing K^s . Therefore $K^s(\mathcal{C})^\alpha \cap K(\mathcal{B}) \subsetneq K(\mathcal{B})$. \square

6.2 Deformation of an automorphism of a nodal curve

In this paragraph we give a *necessary* condition for extending an automorphism of a nodal curve to a given deformation of this curve.

Let R be a complete discrete valuation ring. Let π be a uniformizing parameter and let k be the residue field. We assume k is algebraically closed. Let \mathcal{K} be a semi-stable curve over $\text{Spec}(R)$. We note \mathcal{K}_η the generic fiber and \mathcal{K}_0 the special fiber. We assume \mathcal{K}_η is smooth over the fraction field of R . Let T be a singular point of \mathcal{K}_0 . According to [Liu02, Chap 10, Corollaire 3.22], the completion of the local ring of \mathcal{K} at T takes the form :

$$\hat{\mathcal{O}}_{\mathcal{K},T} = R[[f, g]] / \langle fg - \pi^e \rangle$$

where e is a positive integer. This integer is called the *thickness* of \mathcal{K} at T . We also say that f and g form a coordinate system for \mathcal{K} at T . If we reduce modulo π , we obtain the completion of the local ring of \mathcal{K}_0 at T :

$$\hat{\mathcal{O}}_{\mathcal{K}_0,T} = \hat{\mathcal{O}}_{\mathcal{K},T} / \langle \pi \rangle = k[[\bar{f}, \bar{g}]] / \langle \bar{f}\bar{g} \rangle,$$

where $\bar{f} = f \bmod \pi$ and $\bar{g} = g \bmod \pi$. Because T is an ordinary double point, \mathcal{K}_0 has two branches \mathcal{F} and \mathcal{G} at T . These correspond to the two irreducible components of the completion at T . Be careful that these two branches may lay on the same irreducible component of \mathcal{K}_0 . Anyway, the functions \bar{f} and \bar{g} are the uniformizing parameters of either branches. We call P and Q the points of \mathcal{F} and \mathcal{G} above T .

Now let T' be another singular point of \mathcal{K}_0 , and let f', g', e', \mathcal{F}' , and \mathcal{G}' the corresponding data.

Let α be an automorphism of \mathcal{K} over R such that $\alpha(T) = T'$ and $\alpha(\mathcal{F}) = \mathcal{F}'$, $\alpha(\mathcal{G}) = \mathcal{G}'$. One easily checks that the functions $f' \circ \alpha$ and $g' \circ \alpha$ form a coordinate system for \mathcal{K} at T . We deduce that $e' = e$ and that both $f' \circ \alpha/f$ and $g' \circ \alpha/g$ are units in $\hat{\mathcal{O}}_{\mathcal{K},T}$ (indeed, in either fraction, the numerator and denominator have the same Weil divisor). Since $f \times g = \pi^e = f' \circ \alpha \times g' \circ \alpha$, we have $\frac{f' \circ \alpha}{f}(T) \times \frac{g' \circ \alpha}{g}(T) = 1$. We reduce this identity modulo π and obtain the following identity where the first factor is a function on \mathcal{F} evaluated at P while the second factor is a function on \mathcal{G} evaluated at Q :

$$\frac{\bar{f}' \circ \bar{\alpha}}{\bar{f}}(P) \times \frac{\bar{g}' \circ \bar{\alpha}}{\bar{g}}(Q) = 1. \quad (8)$$

This leads us to the following definition.

Definition 6.5 *Let R be a complete discrete valuation ring. Assume that the residue field k is algebraically closed. Let \mathcal{K} be a semi-stable curve over $\text{Spec}(R)$. The generic fiber of \mathcal{K} is assumed to be smooth. Assume we are given a coordinate system at each singular point of the special fiber \mathcal{K}_0 . Let $\bar{\alpha}$ be an automorphism of the special fiber \mathcal{K}_0 . We say that $\bar{\alpha}$ is deformable in $\mathcal{K}/\text{Spec}(R)$ if for every singular point T of \mathcal{K}_0 , the image $\bar{\alpha}(T)$ has the same thickness as T in \mathcal{K} , and if equality (8) holds true.*

We have just proved the following lemma.

Lemma 6.6 *With the notation of definition 6.5, the set of automorphisms of \mathcal{K}_0 that are deformable in $\mathcal{K}/\text{Spec}(R)$ form a subgroup of $\text{Aut}_k(\mathcal{K}_0)$. If α is an automorphism of \mathcal{K} over $\text{Spec}(R)$, its reduction $\bar{\alpha} = \alpha \bmod \pi$ is an automorphism of \mathcal{K}_0 and is deformable in $\mathcal{K}/\text{Spec}(R)$.*

One may compare this statement with [Wew99, Theorem 3.1.1] where the deformation of morphisms between two distinct curves is studied.

It must be pointed out that the converse of lemma 6.6 is not true. For example, consider the elliptic curve E with modular invariant $j = 0$ (or 1728). Every automorphism of E is deformable because there are no singular points on the curve (the condition in definition 6.5 is empty). However, the only automorphisms that can be extended to the generic elliptic curve are the identity and the involution.

6.3 Automorphisms of curves in a family

In this section we state and prove a lemma of specialization of the automorphism group of curves in a family.

Lemma 6.7 *Let K be a field and let \mathcal{U} be a smooth affine curve geometrically integral over K . Let X be a quasi-projective, smooth, geometrically integral surface over K . Let $\pi : X \rightarrow \mathcal{U}$ be a surjective, projective, smooth morphism of relative dimension 1. Assume that for any point x of \mathcal{U} , the fiber X_x at x is geometrically integral. We call η the generic point of \mathcal{U} and $\bar{X}_\eta = X_\eta \otimes_{K(\mathcal{U})} K(\mathcal{U})^s$ the*

generic fiber, seen as a curve over the algebraic closure of the function field of the basis \mathcal{U} . We assume the genus of X_η is at least 2.

There exists a non-empty open subset \mathcal{V} of \mathcal{U} over K such that for any geometric point $\epsilon \in \mathcal{V}(K^s)$ the group of K^s -automorphisms of the fiber at ϵ is equal to the group $\text{Aut}_{K(\mathcal{U})^s}(\bar{X}_\eta)$ of automorphisms of \bar{X}_η .

The following proof was communicated to us by Qing Liu.

Proof — This is a consequence of a general result by Deligne-Mumford. Let $X \rightarrow S$ be a flat projective morphism over a noetherian scheme S . The functor $T \rightarrow \text{Aut}_T(X_T)$ from the category of S -schemes to the category of groups is representable by a group scheme $\text{Aut}_{X/S}$ over S . See [Kol96, Exercise 1.10.2] for example.

When $X \rightarrow S$ is a stable curve with genus at least 2, Deligne and Mumford [DM69, Thm 1.11] prove that the scheme $\text{Aut}_{X/S}$ is finite and unramified over S .

In our lemma, S is a smooth affine curve \mathcal{U} , geometrically integral over a field K . Replacing S by a non-empty open subset, we may assume that $\text{Aut}_{X/S}$ is finite étale over S . At the expense of a finite surjective base change $T \rightarrow S$, we may assume that the generic fiber of $\text{Aut}_{X/S} \rightarrow S$ consists of rational points. So $\text{Aut}_{X/S} \rightarrow S$ is now a disjoint union of étale sections and the fibers have constant degree. In particular, the fibers are constant and the specialization maps $\text{Aut}_S(X) = \text{Aut}_{X/S}(S) \rightarrow \text{Aut}_s(X_s) = \text{Aut}_{X/S}(k(s))$ are isomorphisms. \square

References

- [CG94] Jean-Marc Couveignes and Louis Granboulan. Dessins from a geometric point of view. In Leila Schneps, editor, *The Grothendieck theory of dessins d'enfants*. Cambridge University Press, 1994.
- [CH85] Kevin Coombes and David Harbater. Hurwitz families and arithmetic galois groups. *Duke Mathematical Journal*, 52(4):821–839, 1985.
- [CR04] Jean-Marc Couveignes and Nicolas Ros. Des obstructions globales à la descente des revêtements. *Acta Arithmetica*, 114(4):331–348, 2004.
- [DE99] Pierre Dèbes and Michel Emsalem. On fields of moduli of curves. *Journal of Algebra*, 211:42–56, 1999.
- [DF94] Pierre Dèbes and Michael Fried. Non-rigid constructions in galois theory. *Pacific Journal of Mathematics*, 163(1):81–122, 1994.
- [DM69] Pierre Deligne and David Mumford. The irreducibility of the spaces of curves of given genus. *Publ. Math. I.H.E.S.*, 36:75–109, 1969.
- [Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.
- [Kol96] Janos Kollar. *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1996.
- [Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford, 2002.

- [Mes91] Jean-François Mestre. Construction de courbes de genre 2 à partir de leurs modules. In Teo Mora and Carlo Traverso, editors, *Effective Methods in Algebraic Geometry*, volume 94 of *Progress in Mathematics*, pages 313–334. Birkäuser, 1991.
- [Mil08] James S. Milne. *Algebraic Geometry*. available online: www.jmilne.org/math/CourseNotes/AG510.pdf, 2008.
- [Sil95] Joseph H. Silverman. The field of definition for dynamical systems on \mathbf{p}^1 . *Compositio Math.*, 98(3):269–304, 1995.
- [Wei56] André Weil. The field of definition of a variety. *Amer. J. of Math.*, 78:509–524, 1956.
- [Wew99] Stefan Wewers. Deformation of tame admissible covers of curves. In Helmut VÖLKEIN, David HARBATER, Peter MÜLLER, and J.G. THOMPSON, editors, *Aspects of Galois theory*, volume 256 of *London Mathematical Society Lecture Note Series*, pages 239–282. Cambridge, 1999.