Elliptic periods and primality proving[*]

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Abstract

We define the ring of elliptic periods modulo an integer n and give an elliptic version of the AKS primality criterion.

Contents

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1 Introduction

Agrawal, Kayal and Saxena have proven [1] that primality of an integer n can be tested in deterministic polynomial time $(\log n)^{12+o(1)}$. Their test, often called the AKS test, relies on explicit computation in the multiplicative group of a well chosen free commutative A-algebra B of finite rank, where $A = \mathbb{Z}/n\mathbb{Z}$. More precisely, they take for B the cyclic algebra $A[x]/(x^r - 1)$ where r is a well chosen, and rather large, integer. Lenstra and Pomerance generalized this algorithm and obtained the better deterministic complexity $(\log n)^{6+o(1)}$. The main improvement in Lenstra and Pomerance's approach consists in using a more general construction for the free commutative algebra B. As a consequence, the dimension B is much smaller for a given n, and this results in a faster algorithm. A nice survey [14] has been written by Schoof. Berrizbeitia first [5] and then Cheng [6] have proven that there exists a probabilistic variant of these algorithms that works in time $(\log n)^{4+o(1)}$ provided $n - 1$ has a divisor d bigger than $(\log_2(n))^2$ and smaller than a constant times $(\log_2(n))^2$. Avanzi and Mihăilescu [2] and independently Bernstein [4] explain how to treat a general integer n using a divisor d of $n^f - 1$ instead, where f is a small integer. The initial idea, due to Berrizbeitia and generalized by Cheng, consists in using A-automorphisms of B to accelerate the necessary calculations in B. In these variants, the free commutative A-algebra B has to be constructed in such a way that a non-trivial A-automorphism $\sigma : B \to B$ is effectively given, and can be efficiently applied to any element in B.

All the aforementioned algorithms construct B as a residue ring modulo n of a cyclotomic or Kummer extension of the ring $\mathbb{Z}$ of integers. In this work we propose an AKS-like primality criterion that relies on Kummer theory of elliptic curves in the spirit of [7]. Our primality certificates can be constructed in heuristic and probabilistic time $(\log n)^{4+o(1)}$. They can be verified in rigorous deterministic time $(\log n)^{4+o(1)}$. We stress that in order to *prove* the primality of n one both has to *construct* and *check* a certificate. The space complexity is $(\log n)^{2+o(1)}$. So we obtain the first variant of the AKS test that has both time and space complexity similar to the ECPP method.

In the next section 2 we describe a rather general variant of AKS primality certificates: they consist of a free A-algebra B of rank d together with an A-automorphism $\sigma : B \to B$ of order d. We recall how such certificates can be constructed from multiplicative Kummer theory as in [5]. We also explain how such a big algebra B can be constructed step by step as a direct product of smaller ones. The next four sections are devoted to the construction of such certificates using Kummer theory of elliptic curves. Section 3 is concerned with the explicit description of isogenies between elliptic curves over fields, in the spirit of [17], [16] and [7]. In section 4 the formulae of section 3 are extended to the case of elliptic curves over more general rings. In section 5 we construct rings of elliptic periods modulo an integer n. These are residue rings of fibers of isogenies. We can provide a quite explicit and efficient expression for the multiplication tensor in these rings. We use these rings of elliptic periods to state a primality criterion in the next section 6. The construction of elliptic certificates is detailed in section 7. A refined primality criterion is obtained in section 8.

Notation: If $\vec{\alpha} = (\alpha_i)_{0 \leqslant i \leqslant d-1}$ and $\vec{\beta} = (\beta_i)_{0 \leqslant i \leqslant d-1}$ are two vectors of length d we denote by $\vec{\alpha} \star_j \vec{\beta} = \sum_i \alpha_i \beta_{j-i}$ the j-th component of the convolution product. We denote by $\sigma(\vec{\alpha}) = (\alpha_{i-1})_i$ the cyclic shift of $\vec{\alpha}$. We denote by $\vec{\alpha} \diamond \vec{\beta} = (\alpha_i \beta_i)_i$ the component-wise product and by $\vec{\alpha} \star \vec{\beta} = (\vec{\alpha} \star_i \vec{\beta})_i$ the convolution product.

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2 Ring extensions and primality proving

In this section, we state a general primality criterion in terms of the existence of some commutative free A-algebra B of finite rank fulfilling simple conditions. We rephrase the AKS primality criterion in this slightly different and more general context. We also give a general recipe to construct the ring B as a direct product of smaller ones.

Let $n \geq 2$ be an integer and set $A = \mathbb{Z}/n\mathbb{Z}$. Let B be a finite free commutative A-algebra of rank $d \geq 1$. Then A can be identified to a subring of B. Let $\sigma : B \rightarrow B$ be an A-automorphism of B. Assume σ has finite order d. Let p be a positive prime divisor of n and let $\mathfrak{j}$ be an ideal of B such that $\mathfrak{j} \cap A = pA$. Set $R = B/\mathfrak{j}$. This is a free commutative algebra of finite dimension over the field $\mathbb{F}_p$. We assume that there is a cyclic subgroup H inside R^* and an integer e such that every element in H can be written $f \bmod \mathfrak{j}$ where f is an element of B satisfying the two following identities:

$$\begin{aligned} f^n &= \sigma(f) \bmod \mathfrak{j}, & (1) \\ f^p &= \sigma^e(f) \bmod \mathfrak{j}. & (2) \end{aligned}$$

Definition 1 *Let n, A, B, d, σ, p, $\mathfrak{j}$, R, H, and e be as above. Assume $\#H$ is a positive multiple of some integer S. Then we say that n satisfies condition $\mathcal{C}(p, d, S)$.*

The following lemma is a key ingredient in all variants of the AKS primality test:

Lemma 1 (AKS criterion) *Let $n \geq 2$ be an integer. Let d and S be positive integers such that $S \geq n^{\lfloor \sqrt{d} \rfloor}$. Let p be a positive prime divisor of n. If n satisfies condition $\mathcal{C}(p, d, S)$, then n is a power of p.*

Indeed, set $q = n/p$. There exist four integers i, i', j and j' in $\{0, 1, \ldots, \lfloor\sqrt{d}\rfloor\}$ such that $i(1 - e_p) + je_p$ is congruent to $i'(1 - e_p) + j'e_p$ modulo d. We deduce from equations (2) and (1) that exponentiation by $q^i p^j$ and $q^{i'} p^{j'}$ act similarly on the cyclic group H. We deduce

$$q^i p^j = q^{i'} p^{j'} \bmod \#H \tag{3}$$

We observe that both integers $q^i p^j$ and $q^{i'} p^{j'}$ are bounded above by $n^{\lfloor\sqrt{d}\rfloor} \leq S \leq \#H$. So congruence (3) is an equality and we deduce that n is a power of the prime p. □

Berrizbeitia constructs B as $A[x]/(x^d - a)$ where d divides $n - 1$ and a is a unit in A. We set $n - 1 = dm$ and $\zeta = a^m$. We assume ζ has exact order d in A^*. This means $\zeta^k - 1$ is a unit for every $1 \leq k < d$. We define an A automorphism $\sigma : B \to B$ by setting $\sigma(x) = \zeta x$. Let p be any prime divisor of n and j a maximal ideal of B containing p. Then $\mathrm{j} \cap A = pA$. The main computational step in Berrizbeitia test is to check, by explicit calculation, that the following congruence holds true in B:

$$(x + 1)^n = \zeta x + 1 \bmod (n, x^d - a).$$

Letting σ repeatedly act on the identity above we deduce that

$$(x + \zeta^k)^n = x + \zeta^k \bmod (n, x^d - a)$$

for any $0 \leq k \leq d - 1$. We take for $H \subset (B/\mathrm{j})^*$ the subgroup generated by the $x + \zeta^k \bmod \mathrm{j}$ for $0 \leq k \leq d - 1$. It is clear that condition (1) is satisfied. Condition (2) is satisfied also because $\zeta \bmod p$ has multiplicative order d. Degree considerations similar to those in the original paper [1] show that the order of $H \subset R^*$ is at least 2^d. This lower bound can be improved by several means. See Voloch's work [18] for example. In section 6 we shall adapt this construction to the more general context of Kummer theory of elliptic curves. This way we shall get rid of the condition that d divides $n - 1$.

The following lemma helps collecting information coming from different rings.

Lemma 2 (Glueing lemma) *Let $n \geq 2$ be an integer. Let p be a prime divisor of n. Let $K \geq 2$ be an integer and let d_1, d_2, …, d_K be pairwise coprime integers. For every integer k in $\{1, 2, \ldots, K\}$ we assume $d_k \geq 2$ and n satisfies condition $\mathcal{C}(p, d_k, S_k)$ for some integer $S_k \geq 2$. Assume the S_k are pairwise coprime. Then n satisfies condition $\mathcal{C}(p, d_1 + d_2 + \cdots + d_K, S_1 S_2 \ldots S_K)$.*

Indeed for every k we have a free algebra B_k of dimension d_k over $A = \mathbb{Z}/n\mathbb{Z}$, an A-automorphism $\sigma_k : B_k \to B_k$ of order d_k, an ideal j_k of B_k containing p. We set

$$\begin{aligned} B &= \prod_{1 \leq k \leq K} B_k, \\ d &= \sum_{1 \leq k \leq K} d_k, \end{aligned}$$

$$\begin{aligned}
\sigma &= \prod_{1\leq k\leq K} \sigma_k,\\
\mathfrak{j} &= \prod_{1\leq k\leq K} \mathfrak{j}_k,\\
R &= \prod_{1\leq k\leq K} R_k,\\
H &= \prod_{1\leq k\leq K} H_k.
\end{aligned}$$

Clearly B is a free A-algebra of dimension d, and H is a cyclic subgroup of R^*. The order of H is a multiple of $\prod_k S_k$. □

3 Trace computations

In this section $\mathbf{K}$ is a field with characteristic p and $E/\mathbf{K}$ is an elliptic curve given by a Weierstrass equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\,.$$

Following Vélu [17, 16] and Couveignes and Lercier [7] we state a few identities related to a degree d separable isogeny $E \rightarrow E'$. We exhibit a normal basis for the field extension $\mathbf{K}(E)/\mathbf{K}(E')$ and we study the matrix of the trace form in this basis.

3.1 Some simple elliptic functions

If A is a point in $E(\bar{\mathbf{K}})$ we denote by $\tau_A : E \rightarrow E$ the translation by A. Following [7, Section 2] we set $x_A = x \circ \tau_{-A}$ and $y_A = y \circ \tau_{-A}$. If A and B are two distinct points in $E(\bar{\mathbf{K}})$ we define the function $u_{A,B}$ as in [7, Section 2]. It has two simple poles: one at A and one at B. If A, B and C are pairwise distinct points we set $\Gamma(A,B,C) = u_{A,B}(C)$. The following identities are proven in [7, Section 2].

$$\begin{aligned}
\Gamma(A,B,C) &= (y(C-A) - y(A-B))/(x(C-A) - x(A-B)), && (4)\\
\Gamma(A,B,C) &= \Gamma(B,C,A) = -\Gamma(B,A,C) - a_1 = -\Gamma(-A,-B,-C) - a_1, && (5)\\
u_{A,B} + u_{B,C} + u_{C,A} &= \Gamma(A,B,C) - a_1, && (6)\\
u_{A,B}u_{A,C} &= x_A + \Gamma(A,B,C)u_{A,C} + \Gamma(A,C,B)u_{A,B}\\
&\quad + a_2 + x_A(B) + x_A(C), && (7)\\
u_{A,B}^2 &= x_A + x_B - a_1u_{A,B} + x_A(B) + a_2\,. && (8)
\end{aligned}$$

We further can prove in the same way

$$
\begin{aligned}
x_C u_{A,B} &= \Gamma(A,B,C)x_C + x_B(C)u_{C,B} - x_A(C)u_{C,A} + y_A(C) - y_B(C), && (9)\\
x_A u_{A,B} &= y_A + x_B(A)u_{A,B} - y_B(A), && (10)\\
x_B u_{A,B} &= -y_B - a_1 x_B - a_3 + x_B(A)u_{A,B} - y_B(A). && (11)
\end{aligned}
$$

3.2 Vélu's formulae

Let $d \geq 2$ be an odd integer and let $t \in E(\mathbf{K})$ be a point of order d. For k an integer we set $x_k = x_{kt}$, $y_k = y_{kt}$, $U_k = u_{kt,(k+1)t}$ and $u_k = \mathfrak{a} u_{kt,(k+1)t} + \mathfrak{b}$ where $\mathfrak{a} \neq 0$ and $\mathfrak{b}$ are scalars in $\mathbf{K}$ such that $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} u_k = 1$. Such scalars always exist by [7, Lemma 4]. For k and l distinct and non-zero in $\mathbb{Z}/l\mathbb{Z}$ we set

$$\Gamma_{k,l} = \Gamma(O, kt, lt). \tag{12}$$

Following Vélu we set

$$x' = x + \sum_{1 \leq k \leq d-1} [x_k - x(kt)] \text{ and } y' = y + \sum_{1 \leq k \leq d-1} [y_k - y(kt)]\,. \tag{13}$$

We also set

$$
\begin{aligned}
b_2 &= a_1^2 + 4a_2 && (14)\\
b_4 &= a_1 a_3 + 2a_4 && (15)\\
b_6 &= a_3^2 + 4a_6 && (16)\\
t &= \sum_{1 \leq k \leq (d-1)/2} 6x(kt)^2 + b_2 x(kt) + b_4 && (17)\\
w &= \sum_{1 \leq k \leq (d-1)/2} 10x(kt)^3 + 2b_2 x(kt)^2 + 3b_4 x(kt) + b_6 && (18)\\
a_4' &= a_4 - 5t && (19)\\
a_6' &= a_6 - b_2 t - 7w. && (20)
\end{aligned}
$$

We also set

$$a_1' = a_1 \text{ et } a_2' = a_2 \text{ et } a_3' = a_3. \tag{21}$$

Vélu proves the identity

$$(y')^2 + a_1' x' y' + a_3' y' = (x')^3 + a_2' (x')^2 + a_4' x' + a_6'. \tag{22}$$

So the map $(x, y) \mapsto (x', y')$ defines a degree d isogeny $I : E \to E'$ where E' is the elliptic curve given by the above Weierstrass equation. The system $(u_k)_{k \in \mathbb{Z}/d\mathbb{Z}}$ is a basis of $\mathbf{K}(E)$ over $\mathbf{K}(E')$. For f a function on E we denote by $\mathrm{Tr}(f)$ the sum $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} f \circ \tau_{kt}$. It can be seen as a function on E'. Our goal in this section is to compute $\mathrm{Tr}(u_{O,kt})$, $\mathrm{Tr}(u_k u_l)$ and $\mathrm{Tr}(u_k x)$ as linear combinations of 1, x' and y'.

3.3 Traces of the $u_{O,kt}$

For $1 \leq k \leq d-1$ we set

$$\mathfrak{c}_k = \mathrm{Tr}(u_{O,kt}). \tag{23}$$

It is proven in [7, Section 4.2] that

$$\mathfrak{c}_1 = \mathrm{Tr}(u_{O,t}) = \sum_{1 \leq l \leq d-2} \Gamma_{l,l+1} - a_1\,.$$

Assume k and l and $k+l$ are non-zero in $\mathbb{Z}/d\mathbb{Z}$. Then

$$\mathrm{Tr}(u_{O,(k+l)t}) = \mathrm{Tr}(u_{O,kt}) + \mathrm{Tr}(u_{O,lt}) - d\Gamma_{k,k+l}.$$

If k, l and $k+l$ are non-zero in $\mathbb{Z}/d\mathbb{Z}$ we thus have

$$\mathfrak{c}_{k+l} = \mathfrak{c}_k + \mathfrak{c}_l - d\Gamma_{k,k+l}. \tag{24}$$

3.4 Traces of $u_k u_l$

Assume now that $k \notin \{-1, 0, 1\}$ so O, t, kt and $(k+1)t$ are pairwise distinct. Then

$$\begin{aligned}
U_0 U_k &= u_{O,t}(u_{O,(k+1)t} - u_{O,kt} + \Gamma_{k,k+1}),\\
&= x + \Gamma_{1,k+1} u_{O,(k+1)t} - \Gamma_{1,k+1} u_{O,t} + x(t) + x((k+1)t),\\
&\quad -x - \Gamma_{1,k} u_{O,kt} + \Gamma_{1,k} u_{O,t} - x(t) - x(kt) + \Gamma_{k,k+1} u_{O,t},\\
&= \Gamma_{1,k+1}(u_{O,(k+1)t} - u_{O,t}) - \Gamma_{1,k}(u_{O,kt} - u_{O,t}) + x((k+1)t) - x(kt) + \Gamma_{k,k+1} u_{O,t}.
\end{aligned}$$

So

$$\mathrm{Tr}(U_0 U_k) = \Gamma_{1,k+1}(\mathfrak{c}_{k+1} - \mathfrak{c}_1) - \Gamma_{1,k}(\mathfrak{c}_k - \mathfrak{c}_1) + d(x((k+1)t) - x(kt)) + \Gamma_{k,k+1}\mathfrak{c}_1\,. \tag{25}$$

For $k = 0$, we have $U_0^2 = x + x_t - a_1 u_{0,t} + x(t) + a_2$. And thus,

$$\mathrm{Tr}(U_0^2) = 2x' + d(x(t) + a_2) - a_1\mathfrak{c}_1 + 2\sum_{1 \leq l \leq d-1} x(lt). \tag{26}$$

For $k = -1$, we have

$$\begin{aligned}
U_0 U_{-1} &= u_{O,t} u_{-t,O} = -u_{O,t} u_{O,-t} - a_1 u_{O,t},\\
&= -(x + \Gamma_{1,-1} u_{O,-t} - \Gamma_{1,-1} u_{O,t} + a_2 + x(t) + x(-t)),\\
&= -x + \Gamma_{1,-1}(u_{-t,O} + a_1) + \Gamma_{1,-1} u_{O,t} - a_2 - 2x(t).
\end{aligned}$$

And thus

$$\mathrm{Tr}(U_0U_{-1}) = -x' + 2\Gamma_{1,-1}\mathfrak{c}_1 + d(a_1\Gamma_{1,-1} - a_2) - 2dx(t) - \sum_{1\leq l\leq d-1} x(lt). \tag{27}$$

For $k = 1$, we have

$$\mathrm{Tr}(U_0U_1) = \mathrm{Tr}(U_{-1}U_0) = \mathrm{Tr}(U_0U_{-1}). \tag{28}$$

3.5 Traces of xu_k

For $k \notin \{-1, 0\}$, we have

$$\begin{aligned} xU_k &= x_O u_{kt,(k+1)t} \\ &= \Gamma_{k,k+1}x + x((k+1)t)u_{O,(k+1)t} - x(kt)u_{O,kt} + \\ &\quad y((k+1)t) - y(kt) + a_1(x((k+1)t) - x(kt)). \end{aligned}$$

And thus,

$$\begin{aligned} \mathrm{Tr}(xU_k) &= \Gamma_{k,k+1}(x' + \sum_{1\leq l\leq d-1} x(lt)) + x((k+1)t)\mathfrak{c}_{k+1} - x(kt)\mathfrak{c}_k + \\ &\quad d(y((k+1)t) - y(kt) + a_1(x((k+1)t) - x(kt))). \end{aligned} \tag{29}$$

For $k = 0$, we have

$$xU_0 = x_O u_{O,t} = y + x(t)u_{O,t} + y(t) + a_1x(t) + a_3.$$

And thus,

$$\mathrm{Tr}(xU_0) = y' + x(t)\mathfrak{c}_1 + d(y(t) + a_1x(t) + a_3) + \sum_{1\leq l\leq d-1} y(lt). \tag{30}$$

For $k = -1$, we have

$$xU_{-1} = x_O u_{-t,O} = -y + -a_1x + x(t)u_{-t,O} + y(t) + a_1x(t).$$

And thus,

$$\mathrm{Tr}(xU_{-1}) = -y' - a_1x' + x(t)\mathfrak{c}_1 + d(y(t) + a_1x(t)) - \sum_{1\leq l\leq d-1} (y(lt) + a_1x(lt)). \tag{31}$$

3.6 The trace form

For any k and l we have

$$\mathrm{Tr}(u_k u_l) = \mathfrak{a}^2\,\mathrm{Tr}(U_k U_l) + \mathfrak{b}^2 d + 2\mathfrak{a}\mathfrak{b}\mathfrak{c}_1. \tag{32}$$

We set

$$\mathfrak{e}_k = \mathrm{Tr}(u_0 u_k). \tag{33}$$

The matrix $(\mathrm{Tr}(u_k u_l))_{k,l} = (\mathfrak{e}_{l-k})_{k,l}$ is circulant and its determinant is

$$D = |\mathrm{Tr}(u_k u_l)|_{k,l} = \prod_{0\le k\le d-1}\ \sum_{0\le l\le d-1} \zeta^{kl}\mathfrak{e}_l \tag{34}$$

where ζ is a primitive d-th root of unity.

We compute

$$\sum_{0\le l\le d-1}\mathfrak{e}_l = \sum_{0\le l\le d-1}\mathrm{Tr}(u_0 u_l) = \mathrm{Tr}(u_0 \sum_{0\le l\le d-1} u_l) = \mathrm{Tr}(u_0) = 1.$$

Using equations 25, 26, 27, and 28 we deduce that D is a degree $d-1$ polynomial in x' with leading coefficient

$$\prod_{1\le k\le d-1}(2-\zeta^k-\zeta^{-k}) = d^2.$$

The roots of D are the abscissae of points in the kernel of the dual isogeny $I' : E' \to E$ and they all have multiplicity two. We deduce

$$\psi_I^{2d}(x)D(x') = \mathfrak{a}^{2d-2}\psi_d^2(x), \tag{35}$$

where

$$\psi_I(x) = \prod_{1\le l\le (d-1)/2}(x - x(lt)). \tag{36}$$

is the factor of $\psi_d(x)$ corresponding to points in the kernel of I.

4 Universal Weierstrass curves

All identities stated in section 3 still make sense and hold true for an elliptic curve over a commutative ring under some mild restrictions. Some (but not all) of these identities are proven in this general context in Vélu's thesis [16] and Katz and Mazur's book [9, Chapter 2]. In this section we give an elementary proof for all the required identities. We construct a sort of universal ring for Weierstrass curves with torsion. This ring beeing an integral domain, the identities hold true in its fraction field. There only remains to check the integrality of all quantities involved. By inverting the determinant of formula (34) we define a localization of the universal ring where the system $(u_k)_{k\in\mathbb{Z}/d\mathbb{Z}}$ remains a basis for the function ring extension associated to the isogeny.

Let A_4 and A_6 be indeterminates and set $\Delta = -16(4A_4^3 - 27A_6^2)$. Set

$$\mathcal{A}_0 = \mathbb{Z}[A_4, A_6, \frac{1}{6}, \frac{1}{\Delta}].$$

Let x and y be two more indeterminates. Set

$$\Lambda(A_4, A_6, x, y) = y^2 - x^3 - A_4x - A_6 \in \mathcal{A}_0[x, y].$$

Let E_{aff} be the affine smooth plane curve over $\mathcal{A}_0$ with equation $\Lambda(A_4, A_6, x, y) = 0$. Let E be the projective scheme over $\mathcal{A}_0$ with equation $Y^2Z - X^3 - A_4XZ^2 - A_6Z^3$. We denote by O the section $[0, 1, 0]$. We have $E_{\text{aff}} = E - O$ and E is an elliptic curve over (the spectrum of) $\mathcal{A}_0$ in the sense of [9].

For every integer $k \geq 1$ we denote by $\psi_k(A_4, A_6, x, y)$ the functions in $\mathcal{A}_0[x, y]/\Lambda$ defined recursively as in [15, Exercise 3.7] or [8, Proposition 3.53]. These are in $\mathcal{A}_0[x, y]/\Lambda$ but we can see them as polynomials in $\mathcal{A}_0[x, y]$ with degree 0 or 1 in y. We set

$$\begin{aligned} \phi_k(A_4, A_6, x, y) &= x\psi_k^2 - \psi_{k+1}\psi_{k-1} &\quad (37)\\ \omega_k(A_4, A_6, x, y) &= \frac{\psi_{k+2}\psi_{k-1}^2 - \psi_{k-2}\psi_{k+1}^2}{4y}. &\quad (38)\end{aligned}$$

These are in $\mathcal{A}_0[x, y]/\Lambda$ but we can see them as polynomials in $\mathcal{A}_0[x, y]$ with degree 0 or 1 in y. The ring $\mathcal{A}_0[x, y]/\Lambda$ is an integral domain. We define the following elements of its field of fractions:

$$\begin{aligned} g_k &= x - \frac{\psi_{k+1}\psi_{k-1}}{\psi_k^2} = \frac{\phi_k}{\psi_k^2}\\ h_k &= \frac{\psi_{k+2}\psi_{k-1}^2 - \psi_{k-2}\psi_{k+1}^2}{4y\psi_k^3} = \frac{\omega_k}{\psi_k^3}\end{aligned}$$

The following important relations [8, Propositions 3.52, 3.55] holds true in this field:

$$g_k - g_l = -\frac{\psi_{k+l}\psi_{k-l}}{\psi_k^2\psi_l^2}. \tag{39}$$

We recall that multiplication by k on $E - E[k]$ is given by $(x, y) \mapsto (g_k, h_k)$.

For k odd, ψ_k lies in $\mathcal{A}_0[x]$, and as a polynomial in x, we have $\psi_k = kx^{\frac{k^2-1}{2}} + O(x^{\frac{k^2-3}{2}})$. Let d be an odd integer and let "$x(t)$" and "$y(t)$" be two more indeterminates. Let S be the multiplicative subset in $\mathcal{A}_0[x(t), y(t)]$ generated by all $\psi_k(x(t), y(t))$ for $1 \leq k \leq d-1$. Let

$$\mathcal{A}_0[x(t), y(t), \frac{1}{S}]/\Lambda(A_4, A_6, x(t), y(t))$$

be the ring of fractions of $\mathcal{A}_0[x(t), y(t)]/\Lambda(A_4, A_6, x(t), y(t))$ with respect to S and let $\mathcal{A}_d$ be the quotient ring

$$\mathcal{A}_d = \mathcal{A}_0[x(t), y(t), \frac{1}{S}]/(\psi_d(x(t)), \Lambda(A_4, A_6, x(t), y(t))).$$

This is an integral domain. We denote by $\mathcal{K}_d$ its field of fractions. The point $t = (x(t), y(t))$ defines a section of E_{aff} over $\mathcal{A}_d$. The curve E base changed to $\mathcal{A}_d$ may be seen as the universal reduced Weierstrass elliptic curve with a point of exact order d. For every integer k such that $1 \leq k \leq d-1$ the point kt defines a section of E over $\mathcal{A}_d$. We call $x(kt)$ and $y(kt)$ its coordinates and we have $x(kt) = \frac{\phi_k(A_4,A_6,x(t),y(t))}{\psi_k^2(A_4,A_6,x(t),y(t))}$ and $y(kt) = \frac{\omega_k(A_4,A_6,x(t),y(t))}{\psi_k^3(A_4,A_6,x(t),y(t))}$. We note that due to equation (39) the difference $x(lt) - x(kt)$ is a *unit* in $\mathcal{A}_d$ for any k and l in $\mathbb{Z}/d\mathbb{Z}$ such that k, l, $k+l$ and $k-l$ are not zero. If we base change E to $\mathcal{K}_d$ we obtain an elliptic curve over a field and we can introduce all the scalars and functions of section 3 : the $\Gamma_{k,l}$, the x_k, y_k, U_k, x', y', σ_i, t, w, $\mathfrak{c}_k$, $x(kt)$, $y(kt)$ …The only divisions when defining these quantities occur in the definition of the $\Gamma_{k,l}$. But the denominators are units in $\mathcal{A}_d$. So all these scalars (resp. functions) are in $\mathcal{A}_d$ (resp. $\mathcal{A}_d[x, y]/\Lambda(A_4, A_6, x, y)$). There remains to choose $\mathfrak{a}$ and $\mathfrak{b}$. We just take $\mathfrak{a} = 1$ and $\mathfrak{b} = \frac{1-\mathfrak{c}_1}{d}$. Then the functions $u_k = \mathfrak{a}U_k + \mathfrak{b}$ are in $\mathcal{A}_d[x, y, \frac{1}{d}]/\Lambda(A_4, A_6, x, y)$. All equations from (12) to (36) still hold true because they are true in $\mathcal{K}_d(E)$ and $\mathcal{A}_d[x, y, \frac{1}{d}]/\Lambda(A_4, A_6, x, y)$ embeds in the later field. We recall that D is the determinant

$$D(x') = |\mathrm{Tr}(u_k u_l)|_{k,l}\,. \tag{40}$$

It is a degree $d-1$ polynomial in $\mathcal{A}_d[\frac{1}{d}, x']$ with leading coefficient d^2. The ring

$$\mathcal{A}_d[\frac{1}{d}, \frac{1}{\psi_d(x)}, x, y]/\Lambda(A_4, A_6, x, y)$$

is a free module of rank d over

$$\mathcal{A}_d[\frac{1}{d}, \frac{1}{D(x')}, x', y']/\Lambda(A'_4, A'_6, x', y')$$

and $(u_l)_{1 \leq l \leq d-1}$ is a basis for this module. We deduce the following theorem by specialization.

Theorem 1 *Let n be a positive and prime to 6 integer and let $A = \mathbb{Z}/n\mathbb{Z}$. Let d be an odd integer. Let a_4, a_6, $\mathfrak{x}(t)$ and $\mathfrak{y}(t)$ be elements in A such that*

1. *d is a unit in A,*
2. *$\Delta(a_4, a_6)$ is a unit in A,*
3. *$\psi_d(a_4, a_6, \mathfrak{x}(t), \mathfrak{y}(t)) = 0$,*
4. *$\psi_k(a_4, a_6, \mathfrak{x}(t), \mathfrak{y}(t))$ is a unit in A for any $1 \leq k \leq d-1$.*

Then $t = (\mathfrak{x}(t), \mathfrak{y}(t))$ is a point of exact order d on the Weierstrass elliptic curve given by the equation $y^2 = x^3 + a_4x + a_6$ over A. Set $\mathfrak{a} = 1$ and $\mathfrak{b} = \frac{1-\mathfrak{c}_1}{d}$ and $u_k = \mathfrak{a}U_k + \mathfrak{b}$. Then all equations from (12) to (36) still make sense and hold true in $A[x, y]/\Lambda(a_4, a_6, x, y)$. The ring

$$A[\frac{1}{\psi_d(x)}, x, y]/\Lambda(a_4, a_6, x, y)$$

is a free module of rank d over

$$A[\frac{1}{D(x')}, x', y']/\Lambda(a'_4, a'_6, x', y')$$

and $(u_l)_{1\leq l\leq d-1}$ is a basis for this module.

5 The ring of elliptic periods modulo n

In this section we give a recipe for constructing a ring extension of $\mathbb{Z}/n\mathbb{Z}$ using an elliptic curve over $\mathbb{Z}/n\mathbb{Z}$ and a degree d isogeny $E \to E'$. The resulting ring will be called a ring of elliptic periods. It will be a free algebra of rank d over $\mathbb{Z}/n\mathbb{Z}$. We just adapt the construction of [7, Section 4] to the case where the base ring is no longer a field. So in this section, n is a positive and prime to 6 integer and $A = \mathbb{Z}/n\mathbb{Z}$. We assume we are in the conditions of theorem 1. We have an elliptic curve $E \to \mathrm{Spec}(A)$ over A given by the reduced Weierstrass equation $y^2 = x^3 + a_4x + a_6$. We have a section $t = (x(t), y(t)) \in E(A)$ of exact order d on E where d is a prime to $2n$ integer and $x(t)$, $y(t)$ are in A. We denote by E' the elliptic curve over A given by equation (22). We call $I : E \to E'$ the isogeny given by equations (13). Let $D(x') = |\mathfrak{e}_{l-k}|_{k,l}$ be the polynomial in $A[x']$ defined by equations (34) and (35). Call $\mathcal{L}$ the A-module generated by the u_k for $0 \leq k \leq d-1$.

We further assume we are given a section $a = (x'(a), y'(a)) \in E'(A)$ of $E'_{\text{aff}} \to \mathrm{Spec}(A)$. We assume that $D(x'(a))$ is a unit in A. Geometrically, this means that the section a does not intersect the kernel of the dual isogeny $I' : E' \to I$. This is equivalent to the circulant matrix $(\mathfrak{e}_{l-k}(x'(a)))_{k,l}$ beeing invertible. For every k in $\mathbb{Z}/d\mathbb{Z}$ we write $e_k = \mathfrak{e}_k(x'(a))$. This is an element of A. Saying that the circulant matrix is $(e_{l-k})_{k,l}$ is invertible means that the vector $\vec{e} = (e_k)_{k\in\mathbb{Z}/d\mathbb{Z}}$ is invertible for the convolution product $\star$ on A^d. We denote by $\overrightarrow{e}^{(-1)}$ the inverse of $\vec{e}$ for the convolution product. The ideal $(x' - x'(a), y' - y'(a))$ of $A[x, y, \frac{1}{\psi_d(x)}]/\Lambda(a_4, a_6, x, y)$ is denoted $\mathfrak{F}_a$. We call $B = A[x, y, \frac{1}{\psi_d(x)}]/(\Lambda(a_4, a_6, x, y), \mathfrak{F}_a)$ the residue ring of the fiber $I^{-1}(a)$. We say that B is a ring of elliptic periods. Then B is a free A-module with basis $\Theta = (\theta_k)_{0\leq k\leq d-1}$ where $\theta_k = u_k \bmod \mathfrak{F}_a$. We need an explicit description of the multiplication tensor in this basis. We note that reduction modulo $\mathfrak{F}_a$ defines an isomorphism of A-modulesn :

$$\begin{tikzcd} \mathcal{L} \arrow[r] & B \\ f \arrow[r, mapsto] & f \bmod \mathfrak{F}_a \end{tikzcd}$$

We assume we are given a section $R = (x(R), y(R))$ of $E_{\text{aff}} \to \operatorname{Spec}(A)$ such that the image $S = I(R)$ of R by I is a section $(x'(S), y'(S))$ of $E'_{\text{aff}} \to \operatorname{Spec}(A)$ and $D(x'(S))$ is a unit in A. So the residue ring at $I^{-1}(S)$ is a free A-module of rank d and the evaluation map

$$\begin{tikzcd}
\mathcal{L} \arrow[r] & A^d \\
f \arrow[r, mapsto] & (f(R+kt))_{0\le k\le d-1}
\end{tikzcd}$$

is a bijection. Also the vector

$$\overrightarrow{u_R} = (u_0(R+kt))_{k\in\mathbb{Z}/d\mathbb{Z}} \tag{41}$$

is invertible for the convolution product in A^d. We call $\overrightarrow{u_R}^{(-1)}$ its inverse. We denote by $\vec{x}_R$ the vector

$$\vec{x}_R = (x(R+kt))_{k\in\mathbb{Z}/d\mathbb{Z}} \tag{42}$$

We note $\xi_k = x_k \mod \mathfrak{F}_a$ for every $k \in \mathbb{Z}/d\mathbb{Z}$. Since B is free over A and Θ is a basis for it, there exist constants $(\iota_k)_k$ in A such that $\xi_0 = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} \iota_k\theta_k$. So $\overrightarrow{\iota} = (\iota_k)_k$ is the coordinate vector of ξ_0 in the basis Θ. We similarly call $\overrightarrow{\hat{\iota}} = (\hat{\iota}_k)_k$ the coordinates vector of ξ_0 in the dual basis $\hat{\Theta}$ of Θ for the trace form. We need to compute these constants.

If k is neither 0 nor -1 in $\mathbb{Z}/d\mathbb{Z}$ then we deduce from formula (29) that

$$\begin{aligned}\hat{\iota}_k &= \mathfrak{a}\left(\Gamma_{k,k+1}(x'(a) + \sum_{1\le l\le d-1} x(lt)) + x((k+1)t)\mathfrak{c}_{k+1} - x(kt)\mathfrak{c}_k + d(y((k+1)t) - y(t))\right) \\ &\quad +\mathfrak{b}(x'(a) + \sum_{1\le l\le d-1} x(lt)).\end{aligned} \tag{43}$$

For $k = 0$ we deduce from formula (30) that

$$\hat{\iota}_0 = \mathfrak{a}\left(y'(a) + x(t)\mathfrak{c}_1 + dy(t) + \sum_{1\le l\le d-1} y(lt)\right) + \mathfrak{b}(x'(a) + \sum_{1\le l\le d-1} x(lt)). \tag{44}$$

For $k = -1$ we deduce from formula (31) that

$$\hat{\iota}_{-1} = \mathfrak{a}\left(-y'(a) + x(t)\mathfrak{c}_1 + dy(t) - \sum_{1\le l\le d-1} y(lt)\right) + \mathfrak{b}(x'(a) + \sum_{1\le l\le d-1} x(lt)). \tag{45}$$

The coordinate vectors in the basis Θ and $\hat{\Theta}$ are related by the relation

$$\overrightarrow{\hat{\iota}} = \vec{e} \star \overrightarrow{\iota}. \tag{46}$$

We can now give the multiplication tensor for B in the basis Θ. Let α, β and γ be three elements in B such that $\gamma = \alpha\beta$. Let $\overrightarrow{\alpha} = (\alpha_k)_{k\in\mathbb{Z}/d\mathbb{Z}}$ be the coordinate vector of α in the basis Θ. Define $\overrightarrow{\beta}$ and $\overrightarrow{\gamma}$ in a similar way. Using an argument similar to the one of [7, Section 4.3] we obtain the following theorem.

Theorem 2 *With the above notation the multiplication tensor in the A-basis Θ of the free A-algebra B is given by*

$$\begin{aligned}\overrightarrow{\gamma} = (\mathfrak{a}^2\overrightarrow{\iota}) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta}))\Big) + \\ \overrightarrow{u_R}^{(-1)} \star \Big((\overrightarrow{u_R} \star \vec{\alpha}) \diamond (\overrightarrow{u_R} \star \vec{\beta}) - (\mathfrak{a}^2\vec{x}_R) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta}))\Big)\Big)\end{aligned}$$

6 A primality criterion

In this section we state and prove a primality criterion involving elliptic periods. Assume we are given a prime to 6 positive integer n. We set $A = \mathbb{Z}/n\mathbb{Z}$ and let E be a Weierstrass elliptic curve over A as in section 5. We keep the same notation. Again $d \geq 2$ is a prime to $2n$ integer and $t \in E(A)$ is a section of exact order d. The quotient by $\langle t\rangle$ isogeny $I : E \to E'$ is given by Vélu's formulae. We are given a section $a \in E'_{\text{aff}}(A)$ and we call

$$\mathfrak{F}_a = (x' - x'(a), y' - y'(a))$$

the ideal of $I^{-1}(a)$ in $A[x, y, \frac{1}{\psi_d(x)}]/\Lambda(a_4, a_6, x, y)$. We assume that $D(x'(a))$ is a unit in A.

Let

$$B = A[x, y, \frac{1}{\psi_d(x)}]/(x' - x'(a), y' - y'(a))$$

be the residue ring of $A[x, y, \frac{1}{\psi_d(x)}]/\Lambda(a_4, a_6, x, y)$ at $I^{-1}(a)$.

Let p be a prime factor of n. Set $C = B/pB$. Let $\sigma : B \to B$ be the automorphism induced on B by the translation τ_{-t} :

$$\begin{tikzcd} \sigma : & B \arrow[r] & B \\ & f \bmod \mathfrak{F}_a \arrow[r, mapsto] & f \circ \tau_{-t} \bmod \mathfrak{F}_a. \end{tikzcd}$$

We also denote by $\sigma : C \to C$ the induced map on the quotient $C = B/p$. Let $b \in E(\bar{\mathbb{F}}_p)$ be a geometric point on $E \bmod p$ such that $I(b) = a \bmod p$. Let $\mathfrak{m} \supset p$ be the corresponding maximal ideal of B:

$$\mathfrak{m} = \{f \bmod \mathfrak{F}_a | f(b) = 0\}$$

Let $\mathbf{K} = B/\mathfrak{m} = \mathbb{F}_p(b)$ be the residue field at b. Call $\phi : E \bmod p \to E \bmod p$ the Frobenius endomorphism of $E \bmod p$. There exists $e_p \in \mathbb{Z}/d\mathbb{Z}$ such that $\phi(b) = b - e_p \times (t \bmod p)$. The residual degree $\deg(\mathbf{K}/\mathbb{F}_p)$ is the order of e_p in the additive group $\mathbb{Z}/d\mathbb{Z}$. So for every $f \in A[x, y, \frac{1}{\psi_d(x)}]/\Lambda(a_4, a_6, x, y)$ we have

$$f^p \bmod \mathfrak{m} = f(b)^p = f(\phi(b)) = f(b - e_p t) = \sigma^{e_p}(f \bmod \mathfrak{F}_a) \bmod \mathfrak{m}. \tag{47}$$

Assume now the following equality holds true in the ring B:

$$(\theta_0)^n = (u_0 \bmod \mathfrak{F}_a)^n = u_1 \bmod \mathfrak{F}_a = \theta_1.$$

Since $\sigma(\theta_k) = \theta_{k+1}$ for every $k \in \mathbb{Z}/d\mathbb{Z}$ we deduce that

$$(\theta_k)^n = \sigma(\theta_k) = \theta_{k+1}$$

for every $k \in \mathbb{Z}/d\mathbb{Z}$. Let $G \subset C^*$ consist of all units u in C such that $u^n = \sigma(u)$. In other words $G = \mathrm{Ker}(n - \sigma) \subset C^*$. This is a subgroup of C^*. We just proved $\theta_k \bmod p \in G$ for every $k \in \mathbb{Z}/d\mathbb{Z}$.

Let $H \subset \mathbf{K}^*$ be the image of G by the reduction map modulo $\mathfrak{m}$. Let $f \bmod (\mathfrak{F}_a, p)$ be a unit in $G \subset C^*$. Then

$$f^n \bmod \mathfrak{m} = \sigma(f \bmod \mathfrak{F}_a) \bmod \mathfrak{m}. \tag{48}$$

We now bound $\#H$ from below. We first observe that $\#H = \#G$ because the reduction map modulo $\mathfrak{m}$ is a bijection from G to H. Indeed, let $f \bmod (\mathfrak{F}_a, p) \in G \subset C^*$ be such that $f = 1 \bmod \mathfrak{m}$. Then $f^n = 1 \bmod \mathfrak{m}$. From formula (48) we deduce that $\sigma(f \bmod \mathfrak{F}_a) = 1 \bmod \mathfrak{m}$. So

$$f - 1 \in \cap_{0 \le k \le d-1} \sigma^k(\mathfrak{m}) = \{0\} \subset C.$$

There remains to bound the order of $\#G$ from below. To every subset S of $\mathbb{Z}/d\mathbb{Z}$ we associate the function

$$f_S = \prod_{k \in S} u_k$$

We note that $f_S \bmod (\mathfrak{F}_a, p) = \prod_{k \in S}(\theta_k \bmod p)$ belongs to $G \subset C^*$. Let S_1 and S_2 be two subsets of

$$\{0, 2, 4, \dots, d-3\} \subset \mathbb{Z}/d\mathbb{Z}.$$

Let l_1 and l_2 be two integers that are prime to p. Then $l_1 f_{S_1} \neq l_2 f_{S_2} \bmod (\mathfrak{F}_a, p)$ unless $S_1 = S_2$ and $l_1 = l_2 \bmod p$. Indeed, if $l_1 f_{S_1} = l_2 f_{S_2} \bmod (\mathfrak{F}_a, p)$ then $l_1 f_{S_1} - l_2 f_{S_2} \bmod p$ is a function on $E \bmod p$ with divisor $\ge -\sum_{k \in \mathbb{Z}/d\mathbb{Z}}[kt]$ and it cancels on the degree d divisor $I^{-1}(a) \bmod p$. So $l_1 f_{S_1} = l_2 f_{S_2} \bmod p$. Therefore these two functions have the same poles. We deduce that $S_1 = S_2$. Therefore $l_1 = l_2$ also. There are $2^{\frac{d-1}{2}}$ subsets of $\{0, 2, 4, \dots, d-3\}$.

We deduce the following lemma:

Lemma 3 (Elliptic partial certificate) *Let n be a prime to 6 integer and let E be a Weierstrass elliptic curve over $A = \mathbb{Z}/n\mathbb{Z}$. Let $t \in E(A)$ be a section of exact order d where d is a prime to $2n$ integer. Let E' be the quotient $E/\langle t \rangle$ given by Vélu's formulae. Let $a \in E'_{\mathit{aff}}(A)$ be a section such that the vector $\vec{e} = (\mathfrak{e}_k(x'(a)))_k$ defined by equation (33) is invertible for the convolution product $\star$ on A^d. Assume the congruence*

$$(\theta_0)^n = \theta_1$$

holds true in the ring of elliptic periods $B = A[x, y, \frac{1}{\psi_d(x)}]/(x' - x'(a), y' - y'(a))$.

Then for every prime divisor p of n, there exists an integer S depending on p such that

- *S divides $(p^d - 1)/(p - 1) = 1 + p + p^2 + \cdots + p^{d-1}$,*
- *$S \geq 2^{\frac{d-1}{2}}$,*
- *n satisfies condition $\mathcal{C}(p, d, S)$.*

In that case, we say that (d, E, t, a) is a partial elliptic certificate of degree d for n.

We recall that the condition that the vector $\vec{e}$ be invertible means that the section a does not cross the kernel of the dual isogeny $I' : E' \to E$.

The above lemma can be used as a building block for a primality certificate. We can use it in conjunction with lemma 1 and lemma 2. Indeed, assume we successively apply lemma 3 to K different rings of elliptic periods with pairwise coprime dimensions d_1, …, d_K. For every $1 \leq k \leq K$, the integer S_k given by lemma 3 divides $1 + p + \cdots + p^{d_k - 1}$. So the S_1, S_2, …, S_K are pairwise coprime. In order to prove that n is a power of p we just have to check the inequality

$$2^{\frac{d_1 + d_2 + \cdots + d_K - K}{2}} \geq n^{\sqrt{d_1 + d_2 + \cdots + d_K}}$$

and apply the glueing lemma 2 and the AKS criterion 1.

Theorem 3 *Let n be a prime to 6 integer and let K be a positive integer. Assume that for every $1 \leq k \leq K$ we are given an odd positive integer d_k and a degree d_k elliptic partial certificate for n. Assume further that the d_k are pairwise coprime and*

$$2^{\frac{d_1 + d_2 + \cdots + d_K - K}{2}} \geq n^{\sqrt{d_1 + d_2 + \cdots + d_K}}. \tag{49}$$

Then n is a prime power.

7 Construction of elliptic certificates

In this section we explain how to construct elliptic (partial) certificates of primality for a given integer n using theorem 3. We are given a prime to 6 integer n and we want to construct a ring of elliptic periods modulo n with degree d. If d is large enough, a single certificate will suffice to prove that n is prime. But we may equaly construct several partial certificates of pairwise prime degrees and apply theorem 3 in its full generality.

The construction of the certificate is probabilistic and relies on several heuristics. The number n under consideration in this section is very likely to be prime: it already passed many pseudo-primality tests. Also we shall allow ourselves to use algorithms that are only proven to work under the condition that n is prime. This is not an issue since we do not claim to prove anything here. We set $A = \mathbb{Z}/n\mathbb{Z}$. We want to construct an elliptic curve E over A with a section $t \in E(A)$ of exact order d in the sense of [9, Chapter 1, 1.4]. We should not be too much demanding about d. We may fix an interval $[\mathrm{d}_{\min}, \mathrm{d}_{\max}]$ and accept any d that belongs to this interval. We have to collect enough partial certificates to satisfy inequation (49). We may choose a large $\mathrm{d}_{\min}$. Then we need few different degrees. For example, if $\mathrm{d}_{\min} \geq 4(\log_2 n)^2 + 2$ then a single partial certificate will suffice. Checking the certificate will require time $(\log n)^{4+o(1)}$ and memory $(\log n)^{3+o(1)}$. We may on the contrary choose a small $\mathrm{d}_{\max}$. Then we shall have to collect many partial certificates. A reasonable choice would be to take $\mathrm{d}_{\min} = 2 + \lceil \log_2 n \rceil$ and $\mathrm{d}_{\max} = (\mathrm{d}_{\min})^{1+o(1)}$. We shall then need no more than $\lceil 4 \log_2 n \rceil$ partial certificates. Checking all the certificate will require time $(\log n)^{4+o(1)}$ and memory $(\log n)^{2+o(1)}$. We use complex multiplication theory to produce elliptic partial certificates.

The first step of the algorithm selects quadratic imaginary orders.

We look over the maximal quadratic imaginary orders $\mathcal{O}$ for decreasing fundamental discriminants $-\Delta$. We start with $-\Delta = -7$. For each order $\mathcal{O}$ we first look for a square root δ of $-\Delta$ modulo n using the algorithm of Legendre. Since n is expected to be prime, the algorithm will succeed in probabilistic time $(\log n)^{2+o(1)}$. For a given n, such a square root exists for one quadratic order over two. If we fail to find such a square root, we go to the next quadratic order.

Once we know a square root δ of $-\Delta$ modulo n, we call $\mathfrak{n}$ the ideal $(n, \sqrt{-\Delta} - \delta)$ in $\mathcal{O}$ and we look for an element with norm n in $\mathfrak{n}$. We use fast Cornachia's algorithm. It runs in deterministic time $(\log n)^{1+o(1)}$ and finds such an element $\phi \in \mathcal{O}$ when it exists. We then set $\mathfrak{t} = \mathrm{Tr}(\phi) = \phi + \bar{\phi}$ and look for integers d that satisfie the following conditions

1. $d \in [\mathrm{d}_{\min}, \mathrm{d}_{\max}]$,
2. d is prime to $n(n-1)(n+1)$,
3. there exists an $\epsilon \in \{1, -1\}$ such that d divides $(n + 1 - \epsilon\mathfrak{t})$ and is prime to $(n + 1 - \epsilon\mathfrak{t})/d$,
4. d is prime to all the degrees we have selected before.

If there is no such factor, we go to the next fundamental discriminant $-\Delta$. We stop when we have collected enough pairwise degrees to satisfy inequation (49). We note that the search for split discriminants can be accelerated using the same technique as in the J.O. Shallit fast-ECPP algorithm [11, 12].

If we choose $\mathrm{d}_{\min} \geq (\log n)^2 + 2$ and look for a single quadratic order, we expect to find it in time $(\log n)^{2+o(1)}$. We also expect the values of Δ to be $(\log n)^{o(1)}$. If we choose $\mathrm{d}_{\min} = 2 + \lceil \log_2 n \rceil$ and $\mathrm{d}_{\max} = (\mathrm{d}_{\min})^{1+o(1)}$, we expect to find the $\lceil \log_2 n \rceil$ necessary quadratic orders in time $(\log n)^{3+o(1)}$. In that case, we expect the values of Δ to be $(\log n)^{2+o(1)}$. In either cases, the time spent in finding the necessary fundamental discriminants is negligible compared to the overall complexity of the algorithm. This makes a big difference with the ECPP test. The reason is that, while the ECPP algorithm is looking for curves with order having a single large prime divisor, the elliptic AKS test only requires the existence of a small divisor of this order. This is a much milder condition. The proportion of useful curves in the ECPP algorithm is $(\log n)^{-1+o(1)}$. The proportion of useful curves in the elliptic AKS algorithm is $(\log n)^{o(1)}$. Needless to say, these estimates are heuristic.

The second step of the algorithm constructs a partial elliptic certificate for every selected couple $(-\Delta, d)$.

Once we have found enough quadratic orders $\mathcal{O}$, we compute the associated Hilbert class polynomials. Computing $H_{\mathcal{O}}(X)$ requires quasi-linear time in the size of this polynomial. This polynomial has degree $\Delta^{1/2+o(1)}$ and height $\Delta^{1/2+o(1)}$, where $-\Delta$ is the discriminant of $\mathcal{O}$. So $H_{\mathcal{O}}(X)$ can be computed in time $\Delta^{1+o(1)}$. Finding a root j of $H_{\mathcal{O}}(X)$ modulo n is achieved in probabilistic time

$$\Delta^{1/2+o(1)}(\log n)^{2+o(1)}.$$

So the total time for finding these roots will be $(\log n)^{2+o(1)}$ for a single large degree and $(\log n)^{4+o(1)}$ for many small ones.

Once computed a root of the modular polynomial, we construct an elliptic curve E over $\mathbb{Z}/n\mathbb{Z}$ having modular invariant j. We then construct a random A-section P on E. We expect one and only one among $[n+1-\mathfrak{t}]P$ and $[n+1+\mathfrak{t}]$ to be equal to the zero section O. If this is not the case, we pick another point P. Let $\epsilon \in \{-1, 1\}$ be such that d divides $n+1-\epsilon\mathfrak{t}$. If we have found a section P such that $[n+1-\epsilon\mathfrak{t}]P \neq O$ then we replace E by the its quadratic twist. And we start again with this new curve. If we have found a point P such that $[n+1-\epsilon\mathfrak{t}]P = O$ and $[n+1+\epsilon\mathfrak{t}]P \neq O$, then we multiply P by $(n+1-\mathfrak{t})/d$ and obtain a section t that we hope has exact order d. We can test that t has exact order d by checking that $\psi_k(x(t))$ is a unit in A for every strict divisor k of d. If this condition does not hold, we pick another section P on E.

Once we have found a t of exact order d we construct the quotient isogeny $I : E \rightarrow E'$ using Vélus's formulae. We look for a A-section a on E' having exact order d. This finishes the construction of our certificate. We expect to find t and a as above in time $(\log n)^{1+o(1)}(\log n + d)$ for each certificate of degree d and $(\log n)^{3+o(1)}$ for all certificates.

8 A stronger criterion

In this section, we improve on the primality test described above, at the expense of some more geometry and combinatorics. If we come back to the proof of lemma 3 we find ourselves with an elliptic curve E over a field $\mathbf{K} = \mathbb{F}_p$. We are given a point t of odd order $d \geq 3$ and the corresponding automorphism σ of the field of functions

$$\begin{tikzcd}\sigma : & \mathbf{K}(E) \arrow[r] & \mathbf{K}(E) \\ & f \arrow[r, mapsto] & f \circ \tau_{-t}.\end{tikzcd}$$

We also are given a function u_0 on E. We have a divisor $T = [O] + [t] + [2t] + \cdots + [(d-1)t]$ and the associated linear space $\mathcal{L}(T)$ of dimension d inside $\mathbf{K}(E)$. We consider the $\mathbb{Z}[\sigma]$-module $\mathcal{U}$ generated by u_0 inside $\mathbf{K}(E)^*$. The essential point is that the intersection $\mathcal{U} \cap \mathcal{L}(T)$ is large: the quotient $(\mathcal{U} \cap \mathcal{L}(T))/\mathbf{K}^*$ has cardinality at least $2^{\frac{d-1}{2}}$. All the functions in this intersection have degree $\leq d$.

We want to replace u_0 by a slightly different function and obtain an even larger set of functions with small degree in the corresponding monogenous $\mathbb{Z}[\sigma]$-module. The section is organized as follows. Paragraph 8.1 studies the structure of the $\mathbb{Z}$-module $\mathcal{U} = \mathbf{K}[E- < t >]^*$ of units in $\mathbf{K}[E- < t >]$. We show that the quotient module $\mathcal{U}/\mathbf{K}^*$ is monogenous as a $\mathbb{Z}[\sigma]$-module and we exhibit a generator for it. The determinant computation needed in paragraph 8.1 is postponed to paragraph 8.4. Paragraph 8.2 gives a lower bound for the number of functions with degree $\leq (d-1)/2$ in $\mathcal{U}/\mathbf{K}^*$. The resulting strengthened primality criterion (theorem 4) is stated in paragraph 8.3. It is asymptotically twenty five times faster than the test resulting from theorem 3. Paragraph 8.6 proves a combinatorial lemma. Paragraph 8.5 proves a simple lower bound for binomial coefficients that is useful in paragraph 8.6.

8.1 A group of elliptic units

Let $\mathbf{K}$ be a field and E an elliptic curve over $\mathbf{K}$. Let $d \geq 3$ be an odd integer and let t be a point of order d in $E(\mathbf{K})$. Let $\sigma : \mathbf{K}(E) \to \mathbf{K}(E)$ be the automorphism that sends f to $f \circ \tau_{-t}$.

In this paragraph, we are interested in the group $\mathcal{U}$ of functions in $\mathbf{K}(E)$ having no zeros nor poles outside the group $< t >$ generated by t. There is a unique multiple $\hat{t}$ of t such that $t = 2\hat{t}$. For every k in $\mathbb{Z}/d\mathbb{Z}$ we define u_k as in section 3.2. This is a function having two simple poles : one at kt and one at $(k+1)t$. If $l = 2k \bmod d$ we set $\hat{u}_l = u_k - u_0(\hat{t}) = u_k - u_k(kt+\hat{t}) = \hat{u}_0 \circ \tau_{-l\hat{t}}$. Its divisor is

$$(\hat{u}_l) = -[kt] + 2[\hat{t} + kt] - [(k+1)t] = -[l\hat{t}] + 2[(l+1)\hat{t}] - [(l+2)\hat{t}].$$

It is clear that

$$\prod_{k \in \mathbb{Z}/d\mathbb{Z}} \hat{u}_k \in \mathbf{K}^*.$$

We want to prove that the $\hat{u}_k$ generate the lattice $\mathcal{U}/\mathbf{K}^*$, or equivalently that $(\hat{u}_k)_{0 \leq k \leq d-2}$ is a $\mathbb{Z}$-basis for it. Let $\mathcal{V}$ be the sublattice of $\mathbb{Z}^d$ consisting of vectors $(e_k)_k$ such that $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} e_k = 0$. Let $\mathcal{W}$ be the sublattice of $\mathcal{V}$ consisting of vectors $(e_k)_k$ such that $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} e_k = 0$ and $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} k e_k = 0 \bmod d$. The index of $\mathcal{W}$ in $\mathcal{V}$ is d. We construct a bijection

$$V : \mathcal{U}/\mathbf{K}^* \to \mathcal{W} \tag{50}$$

by associating to every unit u the vector $(v_{k\hat{t}}(u))_k$ consisting of its valuations at all points $k\hat{t}$ for $k \in \mathbb{Z}/d\mathbb{Z}$. In order to prove that $(\hat{u}_k)_{0\le k\le d-2}$ is a $\mathbb{Z}$-basis for $\mathcal{U}/\mathbf{K}^*$ we consider the following $(d-1)\times d$ matrix:

$$\begin{pmatrix} -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

We stress that the $d-1$ lines in this matrix are the images $V(\hat{u}_k)$ of the $\hat{u}_k$ by V, for $0 \le k \le d-2$. We want to show that these lines form a basis of $\mathcal{W}$. We call them W_k for $0 \le k \le d-2$. From equation (54) below, we deduce that the determinant of the rightmost minor in the above matrix is d. So the index of the lattice generated by the $(W_k)_{0\le k\le d-2}$ inside $\mathcal{V}$ is a divisor of d. This implies that this lattice is equal to $\mathcal{W}$.

Lemma 4 *Let $\mathcal{U} \subset \mathbf{K}(E)^*$ be the group of functions having no zero nor pole outside the subgroup $< t >$ generated by t. Then $\mathcal{U}/\mathbf{K}^*$ is a free $\mathbb{Z}$-module and $(\hat{u}_k)_{0\le k\le d-2}$ is a basis for it. As a $\mathbb{Z}[\sigma]$-module, $\mathcal{U}/\mathbf{K}^*$ is monogenous and $\hat{u}_0$ is a generator for it.*

8.2 Elliptic units with small degree

In this paragraph we are interested in the subset $\mathcal{T}$ of $\mathcal{U}$ consisting of functions in $\mathcal{U}$ having degree $\le (d-1)/2$. Recall the definitions of $\mathcal{V}$ and $\mathcal{W}$ given in paragraph 8.1. Let $\mathcal{I}$ be the subset of the lattice $\mathcal{V}$ consisting of vectors having L^1-norm $\le d-1$. Let $\mathcal{J}$ be the intersection of $\mathcal{I}$ and $\mathcal{W}$. The set $\mathcal{T}/\mathbf{K}^*$ is mapped bijectively onto $\mathcal{J}$ by the map V defined in formula (50). We want to bound from below the cardinality of $\mathcal{J}$.

For every k and l in $\mathbb{Z}/d\mathbb{Z}$, the map $\kappa_{k,l} : \mathcal{V} \to \mathcal{V}$ is defined to be the map that increments the k-th coordinate and decrements the l-th one. There are $d(d-1)+1$ such maps. We fix an arbitrary total order on the set consisting of these $d(d-1)+1$ maps. For every vector $\overrightarrow{v}$ in $\mathcal{I}$, there is at least one map $\kappa_{k,l}$ such that $\kappa_{k,l}(\overrightarrow{v})$ is in $\mathcal{J}$:

- if $\overrightarrow{v}$ is zero, we apply the identity $\kappa_{0,0}$ to $\overrightarrow{v}$;
- if $\overrightarrow{v} = (v_m)_{m\in\mathbb{Z}/d\mathbb{Z}}$ is non zero, we assume for example that the l-th coordinate is positive. We set $k = l - \sum_{m\in\mathbb{Z}/d\mathbb{Z}} m v_m$ and we check that $\kappa_{k,l}(\overrightarrow{v})$ is in $\mathcal{W}$ and its norm is not bigger than the norm of $\overrightarrow{v}$.

For every vector $\overrightarrow{v}$ in $\mathcal{I}$, we call $\kappa(\overrightarrow{v})$ the image of $\overrightarrow{v}$ by the smallest map $\kappa_{k,l}$ such that $\kappa_{k,l}(\overrightarrow{v})$ is in $\mathcal{J}$. This way, we define a map $\kappa : \mathcal{I} \to \mathcal{J}$. Every element in $\mathcal{J}$ has at most

$d(d-1)+1$ preimages by κ. Therefore the sizes of $\mathcal{I}$ and $\mathcal{J}$ are related by the following inequation

$$\#\mathcal{J} \geq \frac{\#\mathcal{I}}{d^2}. \tag{51}$$

We know from lemma 7 that $\log \#\mathcal{I} \geq 1.74498 \times d$ if $d \geq 2001$. We deduce that $\log \#\mathcal{J} \geq (1.74498 - 0.0076) \times d$ in this case. Hence the following lemma.

Lemma 5 *If $d \geq 2001$ is an odd integer, the set $\mathcal{T}/\mathbf{K}^*$ consisting of elliptic units (modulo constants) having degree $\leq (d-1)/2$ has cardinality*

$$\#(\mathcal{T}/\mathbf{K}^*) \geq \exp(1.73738 \times d). \tag{52}$$

8.3 Strong primality certificates

Assume we are in the situation of section 6. We are given a prime to 6 positive integer n. We set $A = \mathbb{Z}/n\mathbb{Z}$ and let E be a Weierstrass elliptic curve over A as in section 5. We keep the same notation. Again $d \geq 2001$ is a prime to $2n$ integer and $t \in E(A)$ is a section of exact order d. We call $I : E \to E'$ the quotient by $\langle t \rangle$ isogeny as given by Vélu's formulae. We are given a section $a \in E'_{\text{aff}}(A)$ and we call

$$\mathfrak{I}_a = (x' - x'(a), y' - y'(a))$$

the ideal of $I^{-1}(a)$ in $A[x, y, \frac{1}{\psi_d(x)}]/\Lambda(a_4, a_6, x, y)$. We assume that $D(x'(a))$ is a unit in A. We call $B = A[x, y, \frac{1}{\psi_d(x)}]/(x' - x'(a), y' - y'(a))$ the ring of elliptic periods. We define the functions $(u_l)_{l \in \mathbb{Z}/d\mathbb{Z}}$ as in section 5. There is a unique multiple $\hat{t}$ of t such that $t = 2\hat{t}$. We set $\eta = u_0(\hat{t}) \in A$. If $l = 2k \bmod d$ we set $\hat{u}_l = u_k - \eta$. We set $\theta_k = u_k \bmod \mathfrak{I}_a$ and $\hat{\theta}_l = \theta_k - \eta$.

Assume now the following equality holds true in the ring B:

$$(\hat{\theta}_0)^n = \hat{\theta}_1. \tag{53}$$

Let p be a prime divisor of n and let $C = B/pB$. Let $\hat{\sigma} : A[E - <t>] \to A[E - <t>]$ be the automorphism induced on $A[E - <t>]$ by the translation $\tau_{-\hat{t}}$:

$$\begin{tikzcd}
\hat{\sigma} : & A[E - <t>] \arrow[r] & A[E - <t>] \\
& f \arrow[r, mapsto] & f \circ \tau_{-\hat{t}}.
\end{tikzcd}$$

We also denote by $\hat{\sigma} : B \to B$ and $\hat{\sigma} : C \to C$ the induced map on B and C. Let $G = \operatorname{Ker}(n - \hat{\sigma}) \subset C^*$. This is a subgroup of C^*. From equation (53) we deduce that $\hat{\theta}_0 \bmod p \in G$. We deduce that $\hat{\theta}_k \mod p \in G$ for every $k \in \mathbb{Z}/d\mathbb{Z}$.

Let $\overrightarrow{v}$ be a vector in $\mathcal{J} \subset \mathbb{Z}^d$. Let $(w_k)_{0\leq k\leq d-2}$ be the coordinates of $\overrightarrow{v}$ in the basis $(W_k)_{0\leq k\leq d-2}$ of $\mathcal{W}$ defined at the end of paragraph 8.1. Let $f_{\overrightarrow{v}} = \prod_{0\leq k\leq d-2} \hat{u}_k^{w_k}$ be the unique multiplicative combination of the $\hat{u}_k$ such that $V(f_{\overrightarrow{v}} \bmod p) = \overrightarrow{v}$, where V is the valuation map defined in formula (50). We note that $f_{\overrightarrow{v}} \bmod (\mathfrak{F}_a, p) = \prod_{0\leq k\leq d-2}(\hat{\theta}_k \bmod p)^{w_k}$ belongs to $G \subset C^*$. Since $\overrightarrow{v}$ is in $\mathcal{J}$, we know that $f_{\overrightarrow{v}} \bmod p$ has degree $\leq (d-1)/2$. Let $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ be two distinct vectors in $\mathcal{J}$. Let l_1 and l_2 be two integers that are prime to p. Then $l_1 f_{\overrightarrow{v}_1} \neq l_2 f_{\overrightarrow{v}_2} \bmod (\mathfrak{F}_a, p)$ unless $\overrightarrow{v}_1 = \overrightarrow{v}_2$ and $l_1 = l_2 \bmod p$. Indeed, if $l_1 f_{\overrightarrow{v}_1} = l_2 f_{\overrightarrow{v}_2} \bmod (\mathfrak{F}_a, p)$ then $l_1 f_{\overrightarrow{v}_1} - l_2 f_{\overrightarrow{v}_2} \bmod p$ is a function on $E \bmod p$ with degree $\leq d-1$ and it cancels on the degree d divisor $I^{-1}(a) \bmod p$. So $l_1 f_{\overrightarrow{v}_1} = l_2 f_{\overrightarrow{v}_2} \bmod p$. Therefore $f_{\overrightarrow{v}_1}$ and $f_{\overrightarrow{v}_2}$ have the same divisor. We deduce that $\overrightarrow{v}_1 = \overrightarrow{v}_2$. Therefore $l_1 = l_2 \bmod p$ also. Using the lower bound in lemma 5 we deduce the following lemma:

Lemma 6 (Strong elliptic partial certificate) *Let n be a prime to 6 integer and let E be a Weierstrass elliptic curve over $A = \mathbb{Z}/n\mathbb{Z}$. Let $t \in E(A)$ be a section of exact order d where $d \geq 2001$ is a prime to $2n$ integer. Let E' be the quotient $E/\langle t\rangle$ given by Vélu's formulae. Let $a \in E'_{\mathit{aff}}(A)$ be a section such that the vector $\vec{e} = (\mathfrak{e}_k(x'(a)))_k$ defined by equation (33) is invertible for the convolution product $\star$ on A^d. Assume the congruence*

$$(\hat{\theta}_0)^n = \hat{\theta}_1$$

holds true in the ring of elliptic periods $B = A[x, y, \frac{1}{\psi_d(x)}]/(x' - x'(a), y' - y'(a))$.

Then for every prime divisor p of n, there exists an integer S depending on p such that

- *S divides $(p^d - 1)/(p-1) = 1 + p + p^2 + \cdots + p^{d-1}$,*
- *$S \geq \exp(1.73738 \times d)$,*
- *n satisfies condition $\mathcal{C}(p, d, S)$.*

In that case, we say that (d, E, t, a) is a strong partial elliptic certificate of degree d for n.

We deduce the following strengthened primality criterion.

Theorem 4 *Let n be a prime to 6 integer and let K be a positive integer. Assume that for every $1 \leq k \leq K$ we are given an odd positive integer $d_k \geq 2001$ and a degree d_k strong elliptic partial certificate for n. Assume further that the d_k are pairwise coprime and*

$$\exp(1.73738 \times (d_1 + d_2 + \cdots + d_K)) \geq n^{\sqrt{d_1+d_2+\cdots+d_K}}.$$

Then n is a prime power.

8.4 A determinant

In this paragraph we compute a determinant that was useful in paragraph 8.1. For every integer $n \geq 1$ we define D_n to be the determinant

$$D_n = \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix}$$

In particular $D_1 = 2$ and $D_2 = 3$.

We develop the determinant D_n along the first column and find that

$$D_n = 2D_{n-1} - D_{n-2}$$

for any $n \geq 3$.

We deduce

$$D_n = n + 1 \tag{54}$$

for any $n \geq 1$.

8.5 Lower bounds for binomial coefficients

In this paragraph we compute effective lower bounds for binomial coefficients. These estimates will be useful in the next paragraph 8.6. Let $K \geq 2$ be an integer and let $(d_k)_{1 \leq k \leq K}$ be a family of positive integers. We set $d = \sum_{1 \leq k \leq K} d_k$ and $\alpha_k = d_k/d$. We set $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_K)$ and define the corresponding entropy to be

$$H(\overrightarrow{\alpha}) = H(\alpha_1, \ldots, \alpha_K) = -\alpha_1 \log \alpha_1 - \alpha_2 \log \alpha_2 - \cdots - \alpha_K \log \alpha_K.$$

We recall Robbins effective Stirling formula [13]. For every positive integer d

$$\sqrt{2\pi d} \left(\frac{d}{e}\right)^d \exp(\frac{1}{12d+1}) \leq d! \leq \sqrt{2\pi d} \left(\frac{d}{e}\right)^d \exp(\frac{1}{12d}).$$

We deduce

$$\begin{aligned} (2\pi d)^{\frac{1-K}{2}} \exp(d \times H(\alpha_1, \ldots, \alpha_K) + \frac{1}{13} - \frac{K}{12}) &\leq \binom{d}{d_1 d_2 \ldots d_K} \\ &\leq (2\pi d)^{\frac{1-K}{2}} \exp(d \times H(\alpha_1, \ldots, \alpha_K) + \frac{1}{12} - \frac{K}{13}). \end{aligned}$$

We shall need the following definition.

Definition 2 *Let* $\overrightarrow{\beta} = (\beta_k)_{1\leq k\leq K}$ *be a family of reals in* $]0,1[$ *such that* $\sum_{1\leq k\leq K}\beta_k = 1$*. Let* d *be a positive integer. We assume* $\beta_k > \frac{1}{d}$ *for every* $1 \leq k \leq K$*. For every integer* k *such that* $1 \leq k \leq K-1$ *set* $d_k = \lfloor \beta_k d \rfloor$*. We observe that* d_k *is positive. Set* $d_K = d - \sum_{1\leq k\leq K-1} d_k$*. It is positive also. The rounded multinomial coefficient associated to* d *and* $\overrightarrow{\beta}$ *is defined to be*

$$\binom{d}{\overrightarrow{\beta}} = \left(\begin{array}{c} d \\ d_1, d_2, \ldots, d_K \end{array} \right).$$

In order to find a nice lower bound for this coefficient, we set $\alpha_k = d_k/d$ for every $1 \leq k \leq K$. It is clear that

$$\beta_k - \frac{1}{d} \leq \alpha_k \leq \beta_k,$$

for $1 \leq k \leq K-1$, and

$$\beta_K \leq \alpha_K \leq \beta_K + \frac{K}{d}.$$

We set $\mu = \max(-\log(\min_{1\leq k\leq K}(\beta_k - \frac{1}{d})) - 1, 1)$ and we notice that for any $1 \leq k \leq K$ the derivative of $z \mapsto -z\log z$ is bounded by μ in absolute value between α_k and β_k. Since

$$|\beta_k - \alpha_k| \leq \frac{1}{d}$$

if $1 \leq k \leq K-1$ and $|\beta_K - \alpha_K| \leq \frac{K}{d}$, we deduce

$$|H(\alpha_1, \alpha_2, \ldots, \alpha_K) - H(\beta_1, \beta_2, \ldots, \beta_K)| \leq \frac{2K\mu}{d}.$$

So

$$\frac{1}{d}\log\binom{d}{\overrightarrow{\beta}} \geq H(\overrightarrow{\beta}) - \frac{2K\mu}{d} + \frac{(1-K)\log 2\pi d}{2d} + \frac{1}{d}(\frac{1}{13} - \frac{K}{12}). \tag{55}$$

8.6 An enumeration problem

Let $d \geq 3$ be an odd integer. We are interrested in the set $\mathcal{S}_d$ of vectors $\vec{e} = (e_1, e_2, \ldots, e_d)$ in $\mathbb{Z}^d$ such that

1. the sum $\sum_{1\leq k\leq d} e_k$ of all coordinates is zero,
2. the L^1-norm $\sum_{1\leq k\leq d} |e_k|$ of $\vec{e}$ is $d-1$.

We look for a lower bound for the cardinality of $\mathcal{S}_d$. To every vector $\vec{e}$ in $\mathcal{S}_d$ we associate a partition of $\{1, 2, \ldots, d\}$ in three sets E_0, E_+, E_- corresponding to the indices with zero, positive and negative coordinates respectively. The sum of positive coordinates equals $(d-1)/2$. The sum of negative coordinates equals $-(d-1)/2$.

We fix a real number $\beta \in]0, \frac{1}{2}[$ and define the subset $\mathcal{S}_{d,\beta} \subset \mathcal{S}_d$ consisting of vectors in $\mathcal{S}$ having exactly $\lfloor \beta d \rfloor$ positive coordinates and $\lfloor \beta d \rfloor$ negative coordinates. We assume $\beta d \geq 1$.

The number of elements in $\mathcal{S}_{d,\beta}$ is

$$\#\mathcal{S}_{d,\beta} = \binom{d}{\lfloor \beta d \rfloor, \lfloor \beta d \rfloor, d - 2\lfloor \beta d \rfloor} \binom{\frac{d-1}{2} - 1}{\lfloor \beta d \rfloor - 1} \binom{\frac{d-1}{2} - 1}{\lfloor \beta d \rfloor - 1} \tag{56}$$

The first factor in the product above is the number of corresponding partitions $E_0 \cup E_+ \cup E_-$. The second factor is the number of ways one can write $(d-1)/2$ as a sum of $\lfloor \beta d \rfloor$ strictly positive integers. The third factor is the number of ways one can write $-(d-1)/2$ as a sum of $\lfloor \beta d \rfloor$ strictly negative integers.

We want to choose the real β so as to make the product in formula (56) as big as possible. The logarithm of this product divided by n tends to $H(\beta, \beta, 1-2\beta) + H(2\beta, 1-2\beta)$ as n tends to infinity. This expression is maximal for $\beta = \frac{1}{2+\sqrt{2}}$ and its value is then bigger than 1.7627. We set $\beta = \frac{1}{2+\sqrt{2}}$ and we look for an effective lower bound for every factor in formula (56).

We first apply inequality (55) for $K = 3$, $\overrightarrow{\beta} = (\beta, \beta, 1-2\beta)$, $\mu = 1$, $H(\beta, \beta, 1-2\beta) \geq 1.08439$ and $d \geq 2001$. We find that

$$\frac{1}{d} \log \binom{d}{\lfloor \beta d \rfloor, \lfloor \beta d \rfloor, d - 2\lfloor \beta d \rfloor} \geq 1.08439 - 0.00781 = 1.07658 \tag{57}$$

We now notice that $\lfloor \beta d \rfloor - 1 \geq \frac{1}{2}\left(\frac{d-1}{2} - 1\right)$ and $\lfloor \beta'(d-3) \rfloor \geq \lfloor \beta d \rfloor - 1$ provided $\frac{\beta'}{\beta} \geq \frac{d}{d-3}$ which is guaranteed by setting $\beta' = 0.29334$. So

$$\binom{\frac{d-1}{2} - 1}{\lfloor \beta d \rfloor - 1} \geq \binom{\frac{d-3}{2}}{\lfloor 2\beta' \frac{d-3}{2} \rfloor}. \tag{58}$$

We now apply inequality (55) for $K = 2$, $\overrightarrow{\beta} = (2\beta', 1-2\beta')$, $\mu = 1$, $H(2\beta', 1-2\beta') \geq 0.678$, and $d \geq 999$. We find that

$$\frac{1}{d} \log \binom{d}{\lfloor 2\beta' d \rfloor} \geq 0.678 - 0.0085 = 0.6695.$$

If we substitute d by $(d-3)/2$ in the above formula we obtain

$$\frac{1}{d} \log \binom{\frac{d-3}{2}}{\lfloor 2\beta' \frac{d-3}{2} \rfloor} \geq 0.6695 \times \frac{d-3}{2d} \geq 0.3342. \tag{59}$$

for $d \geq 2001$.

Combining inequations (56), (57), (58), and (59) we deduce the following lemma.

Lemma 7 *Let $d \geq 2001$ be an odd integer and let $\mathcal{S}_d \subset \mathbb{Z}^d$ be the set of vectors having L^1-norm equal to $d-1$ and the sum of all coordinates equal to 0. We have the following lower bound for $\#\mathcal{S}_d$:*

$$\frac{1}{d} \log \#\mathcal{S}_d \geq 1.74498$$

9 First computational results

The annoucement below has been posted to the Number Theory list.

Object: A practical variant of the AKS primality test.

We are pleased to announce some progress on the AKS primality test [AKS04]. We recently came to a variant of this test, no more deterministic, but which is able to prove that an integer N is prime with good asymptotic complexities,that is soft O(log^4 N) in time and soft O(log^2 N) in space.

A first improvement consists in replacing the degree O(log^2 N) extensions of the ring of integers used in previous AKS variants [LP05, B07] by a direct product of O(log N) extensions of degree O(log N). A second ingredient is that we can efficiently construct such extensions, with a non trivial Z/NZ automorphism in the spirit of [B02, C03], using Kummer theory of elliptic curves (cf. [CL07]).

In order to see how practical such an algorithm can be, since its complexities are similar to the ones of the fastECPP test [M07], we develop some pieces of software and launch it on integers of various sizes. At the time of writing, our largest computation is a primality proof for a 4096-bit integers, derived from the Pi constant.

We are currently preparing an article about this, but some details can already be found in [CEL08].

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More precisely, we made use of the MAGMA computer algebra system [MAGMA] to implement the first two steps of the algorithm:

_ The so-called CM step that consists in finding O(log N) endomorphism rings of elliptic curves E_i with d_i-torsion points and coprime d_i’s.

_ The so-called HILBERT step that consists in computing Hilbert Class Polynomial and roots of these polynomials in order to get an explicit model for the curves E_i.

Similar calculations are made when implementing the ECPP test, so we can easily make use of techniques developed to optimize the ECPP test. However, the goal here is different: instead of looking for one elliptic curve with a completely factorized order, we look for numerous ones with small torsion subgroups of coprime orders. This is much easier. As a consequence, this step can be made negligible, both assymptotically and practically.

We finally implemented the last phase of the algorithm in a reasonably optimized C program that makes use of the FFTW library [FFTW] and the ZEN library [ZEN]:

_ The so-called Elliptic AKS test that consists in checking that

(T_i,0 + cste) ^ N = T_i,j + cste for some index j

once constructed an elliptic normal basis (T_i,j), j = 0..d_i, for the elliptic degree d_i extension defined from E_i.

We measured CPU times (in (m)i(n)utes, (h)ours and (d)ays) and space requirements (in (M)ega(b)ytes or (G)iga(b)ytes) on a 1.67 GHz 64-bit Itanium CPU for proving the primality of pseudo prime k-bit integers of the form

N := [2^(k-2)*Pi] + n, n is a as small as possible integer.

This yields the following table.

k bits	1024	2048	4096
CPU time (1 CPU)	12h 2mn	7d 21h	164d 21h
Calendar time (32 CPUs)	48 minutes	10 hours	5 days
Largest d_i	4567	9497	19961
Space	200 Mb	750 Mb	3 Gb

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