# QUASISYMMETRIC CONJUGACY OF ANALYTIC CIRCLE HOMEOMORPHISMS TO ROTATIONS

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## Very very preliminary version[1]

**1.** We denote by $\mathcal{D}^{0,\omega}(\mathbb{T}^1)$ the monoïd

$$\left\{f \in \mathcal{D}^0(\mathbb{T}^1),\ f : \mathbb{R} \to \mathbb{R} \text{ is } \mathbb{R}\text{-analytic}\right\}$$

where

$$\mathcal{D}^0(\mathbb{T}^1) = \left\{f \in \mathrm{Homeo}_+(\mathbb{R}),\ f(x+1) = f(x) + 1, \forall x \in \mathbb{R}\right\}.$$

**Theorem 1.** *If $f \in \mathcal{D}^{0,\omega}(\mathbb{T}^1)$, and $\rho(f) = \alpha$ is a bounded type number, then*

$$f = h \circ R_\alpha \circ h^{-1} \quad \textit{where} \quad h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$$

*i.e. $h$ is a quasisymmetric homeomorphism of $\mathcal{D}^0(\mathbb{T}^1)$ and $R_\alpha(x) = x + \alpha$.*

**2.** If $\alpha \in \mathbb{R} - \mathbb{Q}$, we denote by $(p_n/q_n)_{n \geqslant 0}$ the convergents of $\alpha$. We set

$$\begin{aligned} \widehat{f}^{q_n} &= f^{q_n} - p_n, \\ I_n(x) &= [x, \widehat{f}^{q_n}(x)], \\ J_n(x) &= [x, \widehat{f}^{2q_n}(x)]. \end{aligned}$$

We recall that the intervals

$$(1) \quad f^j(I_n(x)) \bmod 1 \text{ for } 0 \leqslant j < q_{n+1}$$

have pairwise disjoint interiors, and

$$(2) \quad f^j(I_n(x)) \bmod 1 \text{ for } 0 \leqslant j < 2q_{n+1}$$

is a cover of $\mathbb{T}^1$ of multiplicity at most 2.

(0) Also, if $p/q \in \mathbb{Q}$ with $(p, q) = 1$ is a convergent of $\alpha$, then $-p/q$ is a convergent of $-\alpha$.

*Date*: 1987 ?

[1] Translation by Arnaud Chéritat, 2005

## 3.

**Proposition 1.** *We assume that* $f \in \mathcal{D}^0(\mathbb{T}^1)$ *satisfies:*

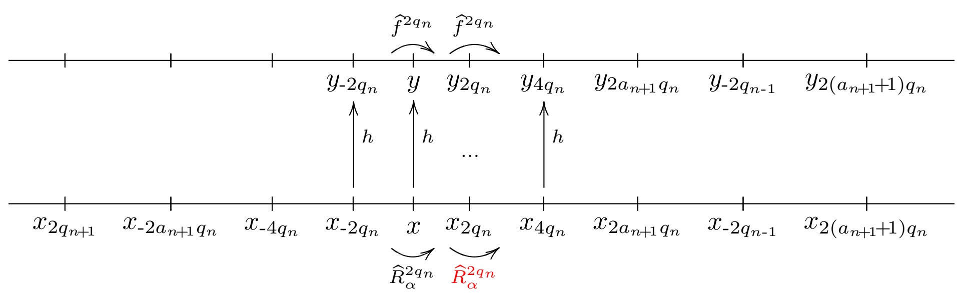
- $\rho(f) = \alpha$ *is a bounded type number;*
- *There exists* $C_1 > 1$ *such that for all* $n \geqslant 0$ *and* $y \in [0,1]$,

$$\frac{1}{C_1} \leqslant \frac{|J_n(y)|}{\left|\widehat{f}^{-2q_n}(J_n(y))\right|} \leqslant C_1; \tag{4}$$

*then* $f = h \circ R_\alpha \circ h^{-1}$ *where* $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ *and* $\widehat{f}^{q_n} = f^{q_n} - p_n$.

The proof is the same as that of [1]. It is not hard to prove that (4) implies $f = h \circ R_\alpha \circ h^{-1}$ with $h \in \mathcal{D}^0(\mathbb{T}^1)$. We can also in theorem 1 use the theorem of J.C. Yoccoz if we prefer.

We have, if $n$ is even (if $n$ is odd we reverse the orientation) the following order of the points[2]:




We argue as in [1] using that (4) and $\sup a_{n+1} < +\infty$ imply that all the intervals $(y_{2kq_n}, y_{2(k+1)q_n})$ in the figure have length ratio bounded from above and from below. Almost all that follows is essentially done by Świątek [2], with the exception of § 8 and 9 (Świątek reasons only about the periodic cycles when $\rho(f) = p/q \in \mathbb{Q}$ and does not look at the case $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$ neither at (4) but it follows very easily from what he does).

## 4.

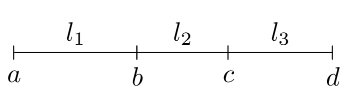
We denote by $\mathcal{L} = \{(a, b, \dots, d) \in \mathbb{R}^4,\ a < b < c < d\}$. If $l \in \mathcal{L}$, we set

$$b(l) = \frac{b-a}{c-a} \Big/ \frac{d-b}{d-c}\,.$$

It is the cross ratio of the 4 points

$$(b, c, a, d)$$

(the cross ratio of $(a, b, c, d)$ is equal to $\frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$).




If $l_1 = b - a$, $l_2 = c - b$, $l_3 = d - c$ we have

$$b(l) = \frac{l_1}{l_1 + l_2}\ \frac{l_3}{l_2 + l_3}$$

where

$$b(l) < 1. \tag{5}$$

[2]where $x \in \mathbb{R}$, $y = h(x)$, $x_k = \widehat{R}_\alpha^k(x)$ and $y_k = \widehat{f}_\alpha^k(y) = h(x_k)$

If $l_2 \leqslant l_1$, $l_2 \leqslant l_3$,

$$(6) \quad b(l) = \frac{1}{1+\frac{l_2}{l_1}} \frac{1}{1+\frac{l_2}{l_3}} \geqslant \frac{1}{4}$$

If $0 < \delta \leqslant b(l)$, we have $b(l) \leqslant l_1/l_2$, $b(l) \leqslant l_3/l_2$, and thus

$$(7) \quad \frac{l_2}{l_1} \leqslant \delta^{-1}$$

$$(8) \quad \frac{l_3}{l_2} \geqslant \delta.$$

**5.** If $l \in \mathcal{L}$ and $h \in \mathcal{D}^0(\mathbb{T}^1)$,

$$D(l,h) = \frac{b\big(h(l)\big)}{b(l)}.$$

where if $l = (a,b,c,d)$ then $h(l) = (h(a), h(b), h(c), h(d))$. We have if $h, g \in \mathcal{D}^0(\mathbb{T}^1)$:

$$\mathrm{D}\left(l, h \circ g\right) = \mathrm{D}\left(g(l), h\right) \mathrm{D}\left(l, g\right)$$

$$(9) \quad \mathrm{D}(l, h^n) = \prod_{j=0}^{n-1} \mathrm{D}\left(h^j(l), h\right).$$

If $h \in \mathcal{D}^1(\mathbb{T}^1)$ there exists $1 \leqslant C(h) < +\infty$ such that for all $l \in \mathcal{L}$ we have

$$C(h)^{-1} \leqslant \mathrm{D}(h,l) \leqslant C(h)$$

where

$$\big(C(h)\big)^{1/4} \leqslant \sup\left(\|Df\|_{C^0}, \|(Df)^{-1}\|_{C^0}\right)$$

suits by the mean value theorem.

**6.**

**Proposition 2.** *If $f \in \mathcal{D}^{0,\omega}(\mathbb{T}^1)$ then*

$$(10) \quad \sup_{l \in \mathcal{L}} \mathrm{D}(l,f) < +\infty.$$

Proof: Let

$$\mathcal{L}_1 = \left\{(a,b,c,+\infty),\ -\infty < a < b < c < +\infty\right\}$$

$$\mathcal{L}_2 = \left\{(-\infty,b,c,d),\ -\infty < b < c < d < +\infty\right\}$$

We set if $l \in \mathcal{L}_1$

$$b(l) = \frac{b-a}{c-a}.$$

It is enough to prove

$$(11) \quad \sup_{l \in \mathcal{L}_1} \mathrm{D}(l,f) < +\infty$$

to get the proposition.

If $l \in \mathcal{L}_1$

$$(12) \quad \mathrm{D}(l,f) = \frac{c-a}{f(c)-f(a)} \frac{f(b)-f(a)}{b-a}.$$

If $\delta > 0$ is fixed, by uniform continuity of $f^{-1}$, we have

$$(13) \quad \sup_{\substack{l \in \mathcal{L}_1 \\ c-a \geqslant \delta}} \mathrm{D}(l,f) < +\infty$$

(we bound $\frac{f(b)-f(a)}{b-a}$ from above by $\|Df\|_{C^0}$).

Let $0 \leqslant \check{c}_1 < \ldots < \check{c}_k < 1$ be the critical points of $f$ on $[0,1[$, $\varepsilon > 0$ and

$$U_{2\varepsilon} = \{x,\ |x-\check{c}_j| < 2\varepsilon,\ j=1,\ldots,k\}.$$

(14) We assume $\varepsilon > 0$ is small enough for $U_{2\varepsilon}$ to be a union of $k$ disjoint intervals and we assume that $\check{c}_{j+1} - \check{c}_j - 4\varepsilon > 2\varepsilon$, $j=1,\ldots,k$ with the convention $\check{c}_{k+1} = \check{c}_1 + 1$.

If $c - a \geqslant \varepsilon$ we bound (12) from above using (13).

If $c - a \leqslant \varepsilon$ and the interval $(a,c)$ is not included in $U_{2\varepsilon}$ we bound (12) from above by

$$\|Df\|_{C^0} \sup_{y \notin U_\varepsilon} \frac{1}{Df(y)}.$$

If $c - a \leqslant \varepsilon$ and the interval $(a,c) \subset U_{2\varepsilon}$, up to assuming $\varepsilon > 0$ small enough, we can pre-compose $f$ by an analytic diffeomorphism $h$ on a neighborhood of $\check{c}_j$ satisfying $h(\check{c}_j) = \check{c}_j$ and boil down to proving (11) for $g_s$ where

$$g_s(x) = x^n + s$$

with $n \in \mathbb{N}^*$, $n$ odd and $s \in \mathbb{R}$. It is enough to prove (11) for $g = x^n$. We set $b = a + l_1$, $c = a + l_1 + l_2$, $l_j > 0$. If $a = 0$ we have

$$\mathrm{D}(l,g) \leqslant 1 \qquad l = (0,b,c,+\infty).$$

If $a \neq 0$. We set

$$\frac{l_1}{a} = x_1, \quad \frac{l_2}{a} = x_2,$$

$x_1 \cdot x_2 > 0$ and $l = (a,b,c,+\infty)$. We have

$$\mathrm{D}(l,g) = \frac{P(x_1+1)}{P(x_1+x_2+1)}$$

where $P(x) = 1 + \cdots + x^{n-1}$. Since $n$ is odd, we have $P(x) > 0$ (if $P(g) = 0$ then $z^n = 1$, $z \neq 1$).

If $x_1 > 0$, since $x_2 > 0$ we have

$$\frac{P(x_1+1)}{P(x_1+x_2+1)} < 1.$$

If $x_1 < -A$ with $A \gg 1$ since $x_2 < 0$, the map $x_2 \mapsto P(x_1+x_2+1)$ is non increasing. We have

$$\frac{P(x_1+1)}{P(x_1+x_2+1)} \leqslant 1.$$

With $-A < x_1 < 0$ we have

$$\frac{P(x_1+1)}{P(x_1+x_2+1)} \leqslant \sup_{|x|<A} P(x+1) / \inf_{x \in \mathbb{R}} P(x) < +\infty.$$

■

**7.** We have the theorem of G. Świątek

**Theorem 2.** *We fix an integer $p \geqslant 2$, $f \in \mathcal{D}^{0,\omega}(\mathbb{T}^1)$, then there exists $C(f,p) > 1$ such that if $(l_i)_{0 \leqslant i \leqslant j-1}$ satisfies: $l_i \in \mathcal{L}$, $l_i = (a_i,b_i,c_i,d_i)$, every $x \in \mathbb{T}^1$ belongs to at most $p$ intervals $(a_i,d_i)$ mod 1; then*

$$\prod_{i=0}^{j-1} \mathrm{D}(l_i,f) \leqslant C(f,p).$$

The important point is that $C(f,p)$ does not depend on $(l_i)_{0\leqslant i\leqslant j-1}$ nor $j$.

Proof: see pages 6–8.

## 8.

**Corollary.** *If $f\in\mathcal{D}^{0,\omega}(\mathbb{T}^1)$, $\rho(f)=\alpha\in\mathbb{R}-\mathbb{Q}$ then there exists $C_1(f)\geqslant 1$ such that for all $x\in\mathbb{R}$, if*

$$l(x)=\begin{cases}\big(\widehat{f}^{-q_n}(x),x,\widehat{f}^{q_n}(x),\widehat{f}^{2q_n}(x)\big) & n \text{ even}\\ \big(\widehat{f}^{2q_n}(x),\widehat{f}^{q_n}(x),x,\widehat{f}^{-q_n}(x)\big) & n\text{ odd}\end{cases}$$

*then for all $0\leqslant j<pq_{n+1}$ and $p\in\mathbb{N}^*$ we have*

$$\text{(15)}\quad \mathrm{D}(l(x),f^j)\leqslant C_1(f)^p.$$

Proof: If $p=1$ this results from (1), (9) and the previous theorem. The case $p=1$ with 5 implies the corollary. ■

## 9. Proof of theorem 1.

It is enough to prove (4). Let $z$ be such that $|\widehat{f}^{q_n}(z)-z|=\min_{x\in\mathbb{R}}|\widehat{f}^{q_n(x)}-x|$. We have

$$b\big(l(z)\big)=\mathrm{D}\left(f^{-j}\big(l(z)\big),f^j\right)\,b\Big(f^{-j}\big(l(z)\big)\Big).$$

By (15) and (6), if $0\leqslant j<pq_{n+1}$, $p\in\mathbb{N}^*$

$$\text{(16)}\quad \frac{1}{4}\leqslant b(l(z))\leqslant C_1(f)^p\;b\big(f^{-j}(l(z))\big).$$

For $j\in\mathbb{N}$, we set

$$z_{-j}=f^{-j}(z)\bmod 1.$$

If $k\in\mathbb{Z}$ and $j$ is fixed we agree that[3]

$$z_{-j+kq_n}=\widehat{f}^{kq_n}(z_{-j}),$$

with the convention $z_0=z$ and obvious abuses of notation.

We fix

$$p=7\text{ and }\delta_0=\frac{1}{4\big(C_1(f)\big)^7}.$$

Up to reversing the orientation we may assume that $n$ is even. The points $z_{-j+iq_n}$ are ordered in $\mathbb{R}$ for $i\geqslant 0$, $i\in\mathbb{N}$, as follows:

$$z_{-j-iq_n}<z_{-j-(i-1)q_n}<\ldots<z_{-j}<z_{-j+q_n}<z_{-j+2q_n}.$$

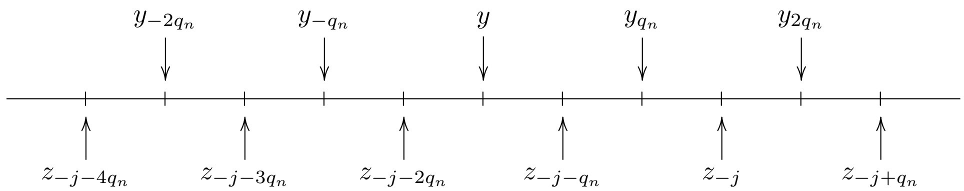
For $0\leqslant j<7q_{n+1}$ we have using (16), (7) and (8)

$$\text{(17)}\quad \frac{-z_{-j+q_n}+z_{-j+2q_n}}{z_{-j+q_n}-z_{-j}}\geqslant\delta_0\,,$$

$$\text{(18)}\quad \frac{-z_{-j}+z_{-j+q_n}}{z_{-j}-z_{-j-q_n}}\leqslant\delta_0^{-1}.$$

[3]In the original, there is a distinction between $Z_{\ldots}$ and $z_{\ldots}$

We consider the points $z_{-j+iq_n}$, $i=-4,\dots,1$ and $y \in (z_{-j-2q_n}, z_{-j-q_n})$.




If $0 < j < 2q_{n+1}$ and $y_{kq_n} = \widehat{f}^k(y)$ then the points are ordered as in the figure above. Let $a_1 = z_{-j-3q_n} - z_{-j-4q_n}$, …, $a_5 = z_{-j+q_n} - z_{-j}$. By (17) and (18) we have

$$\delta_0 \leqslant \frac{a_{i+1}}{a_i} \leqslant \delta_0^{-1}, \quad i=1,\dots,4.$$

Whence

$$\frac{z_{-j} - z_{-j-q_n}}{z_{-j-q_n} - z_{-j-4q_n}} \leqslant \frac{-y + y_{2q_n}}{y - y_{-2q_n}} \leqslant \frac{z_{-j+q_n} - z_{-j-2q_n}}{z_{-j-2q_n} - z_{-j-3q_n}}$$

and thus

$$(19) \quad C(\delta_0)^{-1} \leqslant \frac{-y + y_{2q_n}}{y - y_{-2q_n}} \leqslant C(\delta_0)$$

where $C(\delta_0) > 1$ is a constant which depends only on $\delta_0$.

By (2) and (0), for all $y \in \mathbb{T}^1$, there exists $0 \leqslant j < 2q_{n+1}$ such that

$$y \in (z_{-j-2q_n}, z_{-j-q_n}) \bmod 1.$$

Inequality (19) implies (4) and presupposes only that $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$. Theorem 1 follows from proposition 1 when $\alpha$ is a bounded type number.

Remarks

1. Inequality (4) is true when $f \in \mathcal{D}^{0,\omega}(\mathbb{T}^1)$, $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$. (4) implies the result of J. C. Yoccoz [3] i.e. Denjoy's theorem.

2. Inequality (4) together with the inequality of J. C. Yoccoz when $f \in \mathcal{D}^{0,\omega}(\mathbb{T}^1)$, $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$

$$C(\delta_0) \geqslant \frac{\left|f^{2q_n}(I_n(y))\right|}{\left|I_n(y)\right|} \geqslant C_2(f)\left(Df^{4q_n}(y)\right)^{1/2}$$

where $C_2(f)$ is a positive constant, independant from $n$. This implies that the map $\overline{f} : \mathbb{T}^1 \to \mathbb{T}^1$ induced on $\mathbb{T}^1$ by $f$ is conservative for Haar's measure $m$: if $B$ is $m$-measurable then the sets $\left(\overline{f}^{-j}(B)\right)_{j\in\mathbb{N}}$ are not pairwise disjoint when $m(B) > 0$.

Proof of theorem 2.

We assume $\varepsilon > 0$ small enough satisfying (14) and on $U_{2\varepsilon} - \{c_1,\dots,c_k\}$, $S_f < 0$ ($\Rightarrow \frac{1}{\sqrt{Df}}$ is strictly convex on $U_{2\varepsilon} - \{c_1,\dots,c_k\}$). We set $J = \{0,\dots,j-1\}$. Let

$$J_1 = \left\{i \in J,\ d_i - a_i \geqslant \varepsilon\right\}.$$

We have

$$\# J_1 \leqslant \frac{p}{\varepsilon}$$

and by proposition 2

$$\prod_{i\in J_1} D(l_i, f) \leqslant K_1(f,p)$$

where $K_1$, $K_2$, $K_3$ are constants which depend only on $f$ and $p$. Let

$$J_2 = \left\{i \in J - J_1,\ (a_i, d_i) \bmod 1 \text{ contains a critical point } \check{c}_{k_1} \text{ of } f \text{ on } [0,1[\right\}.$$

We have

$$\# J_2 \leqslant pk$$

where

$$k = \#\{\text{critical points of } f \text{ on } [0,1[\}.$$

Proposition 2 implies

$$\prod_{i \in J_2} D(l_i, f) \leqslant K_2(f,p).$$

Let

$$J_3 = \big\{ i \in J - J_1 - J_2,\ (a_i, d_i) \text{ is not contained in } U_{2\varepsilon} \big\}.$$

We have

$$\log \prod_{i \in J_3} D(l_i, f) \leqslant \sum_{i \in J_3} 2\,\mathrm{var}_{[a_i,d_i]}(\log Df) \leqslant 2p \underset{[0,1]-U_\varepsilon}{\mathrm{var}} \log(Df) < \log\big(K_3(f,p)\big).$$

Let

$$J_4 = J - J_1 - J_2 - J_3.$$

If $i \in J_4$, $(a_i, b_i) \subset U_{2\varepsilon}$. By the next lemma,

$$\prod_{i \in J_4} D(l_i, f) \leqslant 1$$

and we can take $C(f,p) = K_1 K_2 K_3$ where $C(f,p)$ is independant of the $l_i$ and of the integer $j$. ■

**Lemma.** *Let* $f : [a,d] \to \mathbb{R} \quad C^3$, $Df > 0$ *and satisfying*

$$S(f) = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2 < 0$$

*(and thus* $\frac{1}{\sqrt{Df}}$ *is strictly convex). If* $l = a < b < c < d$ *then we have*

$$D(l,f) \leqslant 1.$$

Proof: Composing $f$ on the left and on the right by affine maps we may assume that

$$a = 0 \qquad d = 1$$
$$f(0) = 0 \quad f(1) = 1.$$

Let

$$\phi_\lambda(x) = \frac{x}{\lambda x + 1 - \lambda}, \qquad 1 - \frac{1}{\lambda} \notin [0,1].$$

We have $\phi_\lambda(0) = 0$, $\phi_\lambda(1) = 1$, $\phi_\lambda$ preserves cross ratios and if $0 < x < 1$,

$$\phi_\lambda(x) \longrightarrow 0 \quad \text{if} \quad \lambda \longrightarrow -\infty,$$
$$\phi_\lambda^{-1}(x) \longrightarrow 1 \quad \text{if} \quad \lambda \longrightarrow -\infty.$$

Considering

$$\phi_\lambda^{\pm 1} \circ f = f_\lambda$$

we have

$$D(l, \phi_\lambda^{\pm 1} \circ f) = D(l, f).$$

We may assume that $f = f_\lambda$ satisfies

$$f(0) = 0 < f(b) = b < c < f(1) = 1,$$
$$Sf < 0.$$

Since $\frac{1}{\sqrt{Df}}$ is strictly convex, $f$ has no fixed point apart from 0, $b$ and 1. We want to show that $f(c) > c$. If we had $f(c) < c$ we would have $f(x) < x$ on $]b,1[$ and $f(x) > x$ on $]0,b[$ (if we had $f(x) < x$ on $]0,b[$ by Rolle's theorem, there would exist

$0 < y_1 < b < y_2 < 1$ such that we have $Df(y_1) = Df(b) = Df(y_2)$). We therefore have $Df(b) \leqslant 1$, $Df(0) \geqslant 1$ and $Df(1) \geqslant 1$. This contradicts

$$Df(b) > \min\big(Df(0), Df(1)\big).$$

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## References

[1] M.R. Herman. *Conjugaison quasi symétrique des difféomorphismes du cercle à des rotations et applications aux disques singuliers de Siegel*. Manuscrit.[4]

[2] G. Świątek. *Rational rotation numbers for maps of the circle*. Preprint, Univ. Varsovie.[5]

[3] J.C. Yoccoz. *Il n'y a pas de contre-exemple de Denjoy analytique*. CRAS t. 298 (1984), 141–144.

[4] 1986 ?

[5] Published: Comm. Math. Phys., 119 (1988) 109–128.