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# On multivariate minimal polynomials

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*Abstract*

Given compact sets $E$ and $F$ in $\mathbb{C}^n$ $(n \geqslant 1)$ related by $F = q^{-1}(E)$ where $q$ is a polynomial map, we are interested in the general problem of comparing minimal polynomials for $E$ with minimal polynomials for $F$. Let $\alpha$ be an $n$-multi-index of length $d$. We define the classes of polynomials $\mathbb{P}(\alpha) := z^\alpha + \mathbb{C}_{d-1}[z]$ and $\mathscr{P}(\alpha) := \{p\colon p(z) = z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta\}$ where $\prec$ denotes the usual graded lexicographic order. Polynomials in $\mathbb{P}(\alpha)$ or in $\mathscr{P}(\alpha)$ of least deviation from zero on $E$ (with respect to the supremum norm) are called *minimal polynomials* for $E$. We prove that if $q$ is a simple (i.e. $\hat{q}_i(z) = z_i^m$) polynomial mapping of degree $m$ and if $p$ is minimal polynomial for $E$ then $p \circ q$ is a minimal polynomial for $F = q^{-1}(E)$ and, using some algebraic machinery, we can also construct minimal polynomials for $E$ from minimal polynomials for $F$. The result seems to be new even in the one-dimensional case.

## 1. *Introduction*

**1.** Let $d$ be a positive integer and $K$ be a compact set in the complex plane that contains at least $d+1$ points. It is a classical theorem of Tonelli that among the monic polynomials of degree $d$ there exists one and only one polynomial $T_d$ that minimizes the supremum norm on $K$, that is $||T_d||_K = \inf\{||z^d + \sum_{i=0}^{d-1} a_i z^i||_K\}$ where the infimum is taken with respect to $(a_0, a_1, \ldots, a_{d-1}) \in \mathbb{C}^d$ (see [**7**, p. 143]). In other words $T_d - z^d$ is the polynomial (of degree at most $d-1$) of best approximation to $z^d$ on $K$. This polynomial is called *the minimal* (*monic*) *polynomial of degree* $d$. Very few explicit minimal polynomials are known. The most important comes from the Chebyshev polynomials of first kind (those defined by $\nu_d(\cos\theta) = \cos(d\theta), \theta \in [0, 2\pi]$). Indeed $2^{d-1}\nu_d$ is minimal of degree $d$ for $K = [-1, 1]$. In fact minimal polynomials are often called *Chebyshev polynomials*. Apart from their intrinsic interest, they are closely related to the main objects of potential theory. For example, if $\tau_d(K) := ||T_d||_K$ ($\tau_d$ is called the $d$th Chebyshev constant) then, as $d$ goes to $\infty$, $\tau_d(K)^{1/d}$ converges to the