A Determinantal Proof of the Product Formula for the Multivariate Transfinite Diameter

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Summary. We give an elementary proof of the product formula for the multivariate transfinite diameter using multivariate Leja sequences and an identity on vandermondiens.

1. Introduction and statement of the result. Let $\mathbb{N}_d^n$ denote the set of $n$-indices of length at most $d$ endowed with the graded lexicographic order ($\prec$). The cardinality of $\mathbb{N}_d^n$, denoted by $N_d^n$, is equal to $\binom{n+d}{d}$. A vandermon- dian (of order $d$) is the determinant of an $N_d^n \times N_d^n$ matrix of the form $(z_\alpha^\beta)$ where $z_\alpha \in \mathbb{C}^n$, $[\cdot]^\beta$ is the usual monomial and the rows and columns are ordered according to $\prec$. Such a determinant is denoted by $\text{VDM}(z)$ where $z := (z_\alpha : \alpha \in \mathbb{N}_d^n)$. It is a polynomial of degree

$$
\ell_d^n := n\binom{n+d}{n+1}
$$

in the $(N_d^n)^n$ coordinates of the $z_\alpha$'s. The $d$th diameter $D_d(K)$ of a compact subset $K$ of $\mathbb{C}^n$ is defined by

$$
D_d(K) = \sup\{|\text{VDM}(z)|^{1/\ell_d^n} : z \in K^{N_d^n}\},
$$

and a collection $z$ for which the supremum is achieved in (1) is called a Fekete system (of order $d$) for $K$. Now, the transfinite diameter $D(K)$ is the limit of $D_d(K)$ as $d$ goes to $\infty$. That such a limit exists is by no means obvious (when $n > 1$). It is a beautiful result of V. Zaharjuta [9] who not only proved the convergence of $(D_d(K))$ but also related its limit to complex polynomial approximation.

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