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COMPLEX ANALYSIS

A Determinantal Proof of the Product Formula for the Multivariate Transfinite Diameter

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Summary. We give an elementary proof of the product formula for the multivariate transfinite diameter using multivariate Leja sequences and an identity on vandermondians.

1. Introduction and statement of the result. Let $\mathbb{N}^n_d$ denote the set of n-indices of length at most d endowed with the graded lexicographic order ($\prec$). The cardinality of $\mathbb{N}^n_d$, denoted by N^n_d, is equal to $\binom{n+d}{d}$. A *vandermondian* (of order d) is the determinant of an $N^n_d \times N^n_d$ matrix of the form (z^β_α) where $z_\alpha \in \mathbb{C}^n$, $[\cdot]^\beta$ is the usual monomial and the rows and columns are ordered according to $\prec$. Such a determinant is denoted by $\mathrm{VDM}(\mathbf{z})$ where $\mathbf{z} := (z_\alpha : \alpha \in \mathbb{N}^n_d)$. It is a polynomial of degree

$$\ell^n_d := n\binom{n+d}{n+1}$$

in the $(N^n_d)^n$ coordinates of the z_α's. The dth *diameter* $D_d(K)$ of a compact subset K of $\mathbb{C}^n$ is defined by

$$D_d(K) = \sup\{|\mathrm{VDM}(\mathbf{z})|^{1/\ell^n_d} : \mathbf{z} \in K^{N^n_d}\}, \tag{1}$$

and a collection $\mathbf{z}$ for which the supremum is achieved in (1) is called a *Fekete system* (of order d) for K. Now, the *transfinite diameter* $D(K)$ is the limit of $D_d(K)$ as d goes to ∞. That such a limit exists is by no means obvious (when $n > 1$). It is a beautiful result of V. Zaharjuta [9] who not only proved the convergence of $(D_d(K))$ but also related its limit to complex polynomial approximation.

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