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# THE DISTRIBUTION OF EXTREMAL POINTS FOR KERGIN INTERPOLATION: REAL CASE

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## 1. Introduction.

The general purpose of this note is to study, in some cases, the sequences of Kergin interpolation operators that are the best (see below) for approximating holomorphic functions. Let $K$ be a $\mathbb{C}$-convex (for the definition see [1], [2] or [3]) compact set in $\mathbb{C}^n$, $n \geq 1$. We say that an infinite triangular array of points in $K$

$$A = \{A_j^d;\ j = 0, 1, \ldots, d; d = 1, 2, \ldots\} \tag{1.1}$$

is *extremal* for Kergin interpolation on $K$ if, for every function $f$ holomorphic on a neighborhood of $K$ (i.e. $f \in H(K)$), the Kergin interpolation polynomial $\mathcal{K}_{A^d} f$ of $f$ with respect to the points $A_0^d, \cdots, A_d^d$ converges to $f$ uniformly on $K$ as $d \to \infty$. If such an array exists, we say that $K$ admits an extremal array. The question of knowing whether a given array $A$ is extremal or not is related, as we shall see, to the study of the distribution of the points, that is to the behavior of the sequence of probability measures

$$\mu_d^A = \mu_d := \frac{1}{d} \sum_{j=0}^{d} [A_j^d], \qquad (d = 1, 2, \ldots) \tag{1.2}$$

where $[x]$ stands for the Dirac measure of the point $x$.

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Recently, examples of extremal arrays have been found in the case of circular convex sets (see [4] and below). Here, we shall study the case of (totally) real sets, the definition of which follows.

One says that a real subspace $V$ of $\mathbb{C}^n$ is *totally real* if $V \cap iV = \{0\}$. A compact set is said to be totally real if it is contained in a translate of a totally real subspace, in particular its interior as a subset of $\mathbb{C}^n$ is empty. A compact set of the form

$$E = \{a + r\cos\theta e_1 + r\sin\theta e_2,\ 0 \le r \le 1,\ \theta \in [0, 2\pi]\} \tag{1.3}$$

is said to be a (totally real) *ellipse* if the space $V := \mathrm{vect}_{\mathbb{R}}(e_1, e_2)$ is a totally real plane. The measure $d\sigma_E$ is then defined, by $\int_E f d\sigma_E = \dfrac{1}{2\pi}\int_0^{2\pi} f(\cos\theta e_1 + \sin\theta e_2)d\theta$ for all functions $f$ continuous on $E$. $d\sigma_E$ is supported on the boundary of $E$ as a subset of $V$. In fact, if $A$ is an affine automorphism from $\mathbb{R}^2$ to $V$ that maps the unit disc of center 0 onto $E$, then the measure $d\sigma_E$ is only the image by $A$ of the standard $\dfrac{1}{2\pi}d\theta$ measure on the unit circle. A segment $E = \{a + te_1, t \in [-1, 1]\}$ (not reduced to one point) is said to be a degenerate ellipse, the measure $d\sigma_E$ is defined by $\int f d\sigma_E = \dfrac{1}{\pi}\int_0^{\pi} f(a + \cos\theta e_1)d\theta$. Thus $d\sigma_E$ is the image of the arcsin distribution on $[-1, 1]$ by the map $t \to a + te_1$.

The main result of this paper is the characterization of those totally real compact convex sets which admit an extremal array.

THEOREM 1. — *Let $n \ge 1$. A totally real convex compact set $K$ in $\mathbb{C}^n$ (not reduced to one point) admits an extremal array if and only if it is a (possibly degenerate) ellipse. Furthermore, in this case, an array $A$ is extremal for $K$ if and only if the sequence $\mu_d^A$ converges weakly to $d\sigma_K$.*

Using basic properties of Kergin interpolation, we shall easily reduce the statement to the simpler

THEOREM 2. — *Let $K$ be a convex compact set in $\mathbb{R}^n \subset \mathbb{C}^n$ of non void interior (as a subset of $\mathbb{R}^n$).*

(1) *If $n = 1$, every $K$ (which must be an interval) admits extremal arrays.*

(2) *If $n = 2$, $K$ admits extremal arrays if and only if it is an ellipse.*

(3) *If $n > 2$, there is no extremal array in $K$.*

*In the first two cases, an array $A$ is extremal if and only if $\mu_d^A$ converges weakly to $d\sigma_K$ as $d \to \infty$.*