Contributions to polynomial interpolation of functions in one and several variables

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Outline

Some notations

Sets of numbers

- Field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$;
- $D = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disk.

Space of functions

- $\mathcal{P}_d(\mathbb{K}^N)$: The space of all polynomials in $\mathbb{K}^N$ of degree $\leq d$;
- $m_d := \dim \mathcal{P}_d(\mathbb{K}^N) = \binom{N+d}{d}$;
- K compact in $\mathbb{R}^N$, $C^d(K)$ denotes the space of all d-times continuously differentiable functions in open neighborhoods of K;
- For $f \in C^d(K)$, let $\mathrm{D}^d f(x)$ denote the d-th total derivative of f at $x \in K$.

Regular sets

Definition

The set $A = \{a_1, \ldots, a_{m_d}\}$ in $\mathbb{K}^N$ is said to be **regular** of degree d if it is not included in the zero set of a nonzero polynomial of degree not greater than d.

Observation

A is a regular set of degree d if and only if $\mathrm{VDM}(A) \neq 0$.

Comment. Ordering the elements of A, we may regard it as an element of $\mathbb{K}^{N \cdot m_d}$. If we set

$$\mathcal{NR} = \{A \subset \mathbb{K}^{N \cdot m_d} : A \text{ is not regular set of degree } d\},$$

then the Lebesgue measure of $\mathcal{NR}$ is zero. But it is not easy to check whether a particular set A is a regular set.

Vandermonde determinants

Definition

The Vandermonde determinant of $A = \{a_1, \dots, a_{m_d}\} \subset \mathbb{K}^N$, say $\mathrm{VDM}(a_1, \dots, a_{m_d})$, is the determinant of the matrix whose entries are the monomials $z \mapsto z^\alpha$ of degree $\leq d$ evaluated at the a_j, where $m_d = \binom{N+d}{d}$ and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ with $z = (z_1, \dots, z_N)$, $\alpha = (\alpha_1, \dots, \alpha_N)$.

- If $N = 1$ and $A = \{a_1, \dots, a_{d+1}\}$, then

$$\mathrm{VDM}(A) = \det(a_j^k) = \prod_{1 \leq j < k \leq d+1} (a_k - a_j);$$

- Multivariate vandermondians are difficulty to deal with.

Examples of regular sets

- $d + 1$ distinct points in $\mathbb{K}$, a regular set of degree d;
- Chung-Yao lattices constructed from hyperplanes in general position in $\mathbb{K}^N$;
- Padua points in $[-1, 1]^2$: The double points of the curve $\big(\cos(n\theta), \cos((n+1)\theta)\big)$ together with the points on the boundary $[-1, 1]^2$. They form a regular set of degree n.

Biermann intertwining

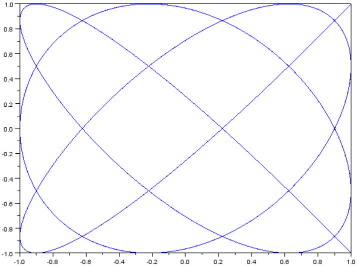
Given $A = (a_0, \ldots, a_d)$ and $B = (b_0, \ldots, b_d)$ in $\mathbb{K}$. The Biermann intertwining of A and B is

$$A \oplus B = ((a_i, b_j) : i + j \leq d).$$

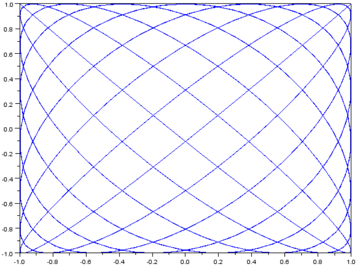
Theorem

If both A and B consist of $d + 1$ distinct points, then $A \oplus B$ is a bi-dimensional regular set of degree d.

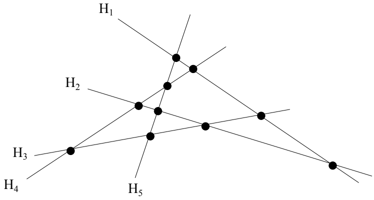
Using graded lexicographic order of $\mathbb{N}^N$ to order (block) regular sets in $\mathbb{K}^N$, we can define the intertwining of two regular tuples (sets) in $\mathbb{K}^N$ and $\mathbb{K}^M$. A result of Calvi points out that the resulting tuple is also regular.



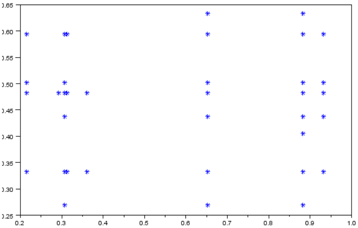
$\mathrm{Pad}_6, n = 6$



$\mathrm{Pad}_{13}, n = 13$



A Chung-Yao lattice



A Biermann intertwining

Lagrange interpolation polynomials

Definition

Let $A = \{a_1, \ldots, a_{m_d}\}$ be a regular set of degree d. Then for any function f defined on A, there exists a unique polynomial in $\mathcal{P}_d(\mathbb{K}^N)$ denoted by $\mathbf{L}[A; f]$ and called the Lagrange interpolation polynomial such that $\mathbf{L}[A; f] = f$ on A. We have

$$\mathbf{L}[A; f](x) = \sum_{j=1}^{m_d} f(a_j)\mathbf{l}(A, a_j; x), \quad \textit{where}$$

$$\mathbf{l}(A, a_j; x) = \frac{\mathrm{VDM}(\{a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_{m_d}\})}{\mathrm{VDM}(\{a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_{m_d}\}))}, \ 1 \leq j \leq m_d.$$

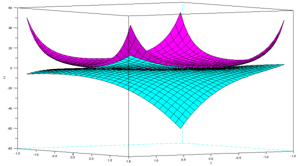
Observation: $\mathbf{l}(A, a_j; a_k) = \delta_{jk}$ for $1 \leq j, k \leq m_d$.

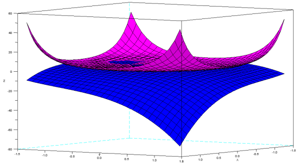
Lagrange interpolation polynomials (to be continued)

When $N = 1$ and $A = \{a_1, \ldots, a_{d+1}\}$, then

$$\mathbf{L}[A; f](x) = \sum_{j=1}^{d+1} f(a_j) \prod_{k=1, k \neq j}^{d+1} \frac{x - a_k}{a_j - a_k}.$$

But when $N > 1$, there is no simplification. This is the origin of the difficulty of multivariate interpolation.





The graphs of the function $(x, y) \mapsto e^{x^2+y^2}$ and its Lagrange polynomial at 10 points.

Natural problems

1. Given a class $\mathcal{F}$ of functions, find regular sets $A^{(d)}$ such that $\mathbf{L}[A^{(d)}; f]$ correctly approximate f as $d \to \infty$ for any $f \in \mathcal{F}$;
2. Conversely, given regular sets $A^{(d)}$, find functions that are correctly approximated by its Lagrange interpolation polynomial at $A^{(d)}$.

Comment: We solve the problems from a constructive point of views:

- We want to construct explicit points that are good for approximation by interpolation polynomials;
- We do not want to give properties which are equivalent to or stronger than to be good points.

One dimensional case

Theorem (Hermite)

Let Γ be a closed contour that contain $A = \{a_0, \ldots, a_d\}$. Then, for any function f analytic on and inside Γ, we have

$$f(z) - \mathbf{L}[A; f](z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\eta) p_n(z) d\eta}{(\eta - z) p_n(\eta)}, \text{ where } p_n(z) = \prod_{j=0}^{d} (z - a_j).$$

We need to study the behaviour of potentials like $\sum_{j=0}^{d} \log |z - a_j|$.

Theorem

Let K be a regular polynomially convex compact set in $\mathbb{C}$ and $A^{(d)} = \{a_0^{(d)}, \ldots, a_d^{(d)}\} \subset K$ such that $(1/(d+1)) \sum_{j=0}^{d} [a_j^{(d)}]$ tends weakly to the equilibrium measure of K. Then for every $f \in H(K)$ we have $\mathbf{L}[A^{(d)}; f]$ converges to f uniformly on K.

Lebesgue inequalities

Definition

Let K be a compact subset in $\mathbb{K}^N$ containing A. The Lebesgue constant $\Delta(A)$ or $\Delta(A \,|\, K)$ is defined as the norm on $C(K)$ of the interpolation operator $\mathbf{L}[A;\cdot]$. We have

$$\Delta(A) = \max_{x \in K} \sum_{j=1}^{m_d} |\mathbf{l}(A, a_j; x)|.$$

Theorem

We have

$$\max_{x\in K} \left| f(x) - \mathbf{L}[A;f](x) \right| \leq \big(1+\Delta(A)\big) \mathrm{dist}_K\big(f, \mathcal{P}_d(\mathbb{K}^N)\big), \;\; f \in C(K),$$

where

$\mathrm{dist}_K\big(f, \mathcal{P}_d(\mathbb{K}^N)\big) = \inf\{\max_{x\in K} |f(x) - p(x)| : p \in \mathcal{P}_d(\mathbb{K}^N)\}.$

The growth of $\text{dist}_K\big(f, \mathcal{P}_d(\mathbb{K}^N)\big)$

- Theorem of Jackson: If $f \in C^m([-1,1])$, then

$$\text{dist}_K\big(f, \mathcal{P}_d(\mathbb{R})\big) = o(\frac{1}{d^m}).$$

- A generalization Jackson theorem: If $K \subset \mathbb{R}^N$ satisfying some geometric conditions and $f \in C^m(K)$, then

$$\text{dist}_K\big(f, \mathcal{P}_d(\mathbb{R}^N)\big) = O(\frac{1}{d^m}).$$

- Theorem of Siciak: If f is a holomorphic function in a neighborhood of a polynomially convex set K in $\mathbb{C}^N$, then

$$\text{dist}_K\big(f, \mathcal{P}_d(\mathbb{C}^N)\big) = O(\frac{1}{\rho^d}) \quad \text{for some} \quad \rho > 1.$$

Questions on the growth of Lebesgue constants

Questions

Find a sequence of regular sets $A^{(d)}$ consisting of m_d points in a compact set $K \subset \mathbb{K}^N$ such that $\Delta(A^{(d)})$ grows

- sub-polynomially of d as $d \to \infty$?
- like a polynomial of d as $d \to \infty$?
- sub-exponentially, that is $\left(\Delta(A^{(d)})\right)^{\frac{1}{d}} = 1$?

Comment: A classical theorem of Bernstein points out that $\Delta(A^{(d)}|[-1,1])$ grows at least like $\log d$, i.e.,

$$\liminf_{d\to\infty} \frac{\Delta(A^{(d)}|[-1,1])}{\log d} > 0.$$

Known results on Lebesgue constants

- Chebyshev points
$\mathcal{T}_{d-1} = \{\cos \frac{(2k-1)\pi}{2d} : k = 1, \ldots, d\} \subset [-1, 1]$:
$\Delta(\mathcal{T}_{d-1}) = O(\log d)$;
- Chebyshev-Lobatto points
$\mathcal{L}_d = \{\cos \frac{j\pi}{d} : j = 0, \ldots, d\} \subset [-1, 1]$: $\Delta(\mathcal{L}_d) = O(\log d)$;
- The d-th roots of unity
$R_d = \{e^{2\pi ki/d} : k = 0, \ldots, d-1\} \subset D$: $\Delta(R_d) = O(\log d)$;
- If K is a plane compact set whose boundary is a C^2 Jordan curve and $Z = (z_n : n \in \mathbb{N})$ is a Leja sequence for K, then $\Delta(Z_d)^{\frac{1}{d}} \to 1$ as $d \to \infty$.
- Padua points Pad_n generated by the curve
$\big(\cos(n\theta), \cos((n+1)\theta)\big)$ (a regular set of degree n):
$\Delta(\mathrm{Pad}_n | [-1, 1]^2) = O(\log^2 n)$.

Lebesgue constants under intertwining process

Theorem (Calvi)

Let $A = (a_0, \ldots, a_d)$ (resp. $B = (b_0, \ldots, b_d)$) be distinct points on compact set $K_1 \subset \mathbb{K}$ (resp. $K_2 \subset \mathbb{K}$). Then

$$\Delta(A \oplus B|K) \leq 4\binom{d+2}{d} \sum_{i+j\leq d} \Delta\left(A^{[i]}|K_1\right) \cdot \Delta\left(B^{[j]}|K_2\right),$$

where $K = K_1 \times K_2$, $A^{[i]} = (a_0, \ldots, a_i)$ and $B^{[j]} = (b_0, \ldots, b_j)$.

Observations:

- The same estimate holds true in the case of multivariate regular points.
- To use the theorem we need to estimate the Lebesgue constants of all blocks $A^{[i]}$ and $B^{[j]}$;
- We need to find sequences of points with nice Lebesgue constants.

Leja sequences

Definition

Let K be a compact in $\mathbb{C}$. One says that $Z = (z_n : n \in \mathbb{N}) \subset K$ is a Leja sequence for K if

$$|z_d - z_0| \cdots |z_d - z_{d-1}| = \max_{z \in K} |z - z_0| \cdots |z - z_{d-1}|, \quad d \geq 1.$$

- For $K = D$, we only consider a Leja sequence $E = (e_n : n \in \mathbb{N})$ with $e_0 = 1$;
- A tuple $E_d = (e_0, \ldots, e_{d-1})$ is called an d-Leja section.

The structure of Leja sequences

Theorem (Białas-Cież and Calvi)

A Leja sequence $E = (e_n : n \in \mathbb{N})$ for D is characterized by the following two properties.

- The set $\{e_0, e_1, \dots, e_{2^n-1}\}$ is formed of the 2^n-th roots of 1.
- $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$, i.e.,

$$(e_0, \dots, e_{2^{n+1}-1}) = (e_0, \dots, e_{2^n-1}, \rho u_0, \dots, \rho u_{2^n-1}),$$

  where ρ is a 2^n-root of -1 and U_{2^n} is a 2^n-Leja section.

An example

A simple Leja sequence is defined by the following rules

$$\begin{cases} E_2 = (1, -1) \\ E_{2^{n+1}} = (E_{2^n}, e^{i\pi/2^n} E_{2^n}), \quad n \geq 1 \end{cases}$$

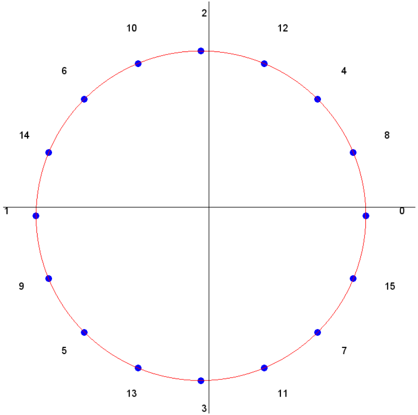


Figure: The first 16 points of a Leja sequence

$\Re$-Leja sequences

Definition

A sequence X (in $[-1,1]$) is said to be a $\Re$-Leja sequence if there exists a Leja sequence $E = (e_k : k \in \mathbb{N})$ such that X is obtained by eliminating repetitions in $\Re(e_k : k \in \mathbb{N})$. Here, we mean that the entry $\Re(e_j)$ is eliminated whenever there exists $i < j$ such that $\Re(e_j) = \Re(e_i)$. We write $X = X(E)$.

STRATEGY : To study $\Re$-Leja sequences, we decompose them into the union of simpler finite sequences from the interpolation theory the point of view.

The structure of $\Re$-Leja sequences

Theorem

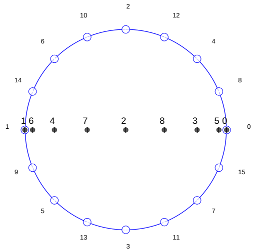
A sequence $X = (x_k \,:\, k \in \mathbb{N})$ is a $\Re$-Leja sequence if and only if there exists a Leja sequence $E = (e_k \,:\, k \in \mathbb{N})$ such that

$$X = (1, -1) \;\wedge\; \bigwedge_{j=1}^{\infty} \Re\Big(E(2^j : 2^j + 2^{j-1} - 1)\Big).$$

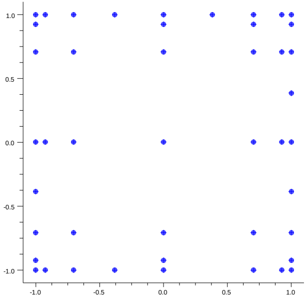
Equivalently, $x_k = \Re(e_{\phi(k)})$, $k \in \mathbb{N}$, with $\phi(0) = 0$, $\phi(1) = 1$ and

$$\phi(k) = \begin{cases} \frac{3k}{2} - 1 & k = 2^n \\ 2^{\lfloor \log_2(k) \rfloor} + k - 1 & k \neq 2^n \end{cases}, \quad k \geq 2,$$

where $\lfloor \cdot \rfloor$ is used for the ordinary floor function, the sign $\wedge$ concatenates tuples, $E(j : k) = (e_j, e_{j+1}, \ldots, e_k)$, $\Re\big(E(j : k)\big) = \big(\Re(e_j), \Re(e_{j+1}), \ldots, \Re(e_k)\big)$.



(I) First 9 points of a $\Re$-Leja sequence.



(II) 45 interpolation points obtained as the intertwining of the points in (I) with themselves.

Figure: Points from a $\Re$-Leja sequence and the intertwining.

Lebesgue constants of Leja sequences

Theorem

If $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D, then $\Delta(E_d) = O(d \log d)$, $d \to \infty$, where $E_d = (e_0, e_1, \ldots, e_{d-1})$.

Theorem

For every Leja sequence E and every $n \in \mathbb{N}^\star$ we have $\Delta(E_{2^n-1}) = 2^n - 1$.

Open question

Is it true that $\Delta(E_d) \leq d$ for all $d \geq 1$?

The idea of the proof

Step 1. showing that if $2^n + 1 \leq d \leq 2^{n+1} - 1$, then

$$\Delta(E_d) \leq C_d \cdot \Delta(E_{2^n}) + \Delta(U_{d-2^n}),$$

where U_{d-2^n} is a $(d-2^n)$-Leja section and $\Delta(E_{2^n}) = O(n)$;

Step 2. Getting an upper bound of C_d, $C_d \leq 2^n$. To prove this we must prove a trigonometric inequality,

$$\prod_{j=0}^{s-1} \left|\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)\right| \geq 1/2^n,$$

where $n - 1 \geq n_1 > \cdots > n_s \geq 0$ and $\theta_j = (2t_j + 1)/2^{n_j}$, $t_j \in \mathbb{N}$, $0 \leq j \leq n-1$;

Step 3. Using Steps 1, 2 and binary expansion of d, we can prove the theorem.

Lebesgue constants of $\Re$-Leja sequences

Theorem

If $X = (x_n : n \in \mathbb{N})$ is a $\Re$-Leja sequence, then
$\Delta(X_d) = O(d^3 \log d), \quad d \to \infty$, where $X_d = (x_0, x_1, \dots, x_{d-1})$.

Proposition

If X is a $\Re$-Leja sequence, then $\Delta(X_{2^n}) \geq 2^n - 2$.

The idea of the proof

Lemma. Let $N = N_0 \cup \cdots \cup N_{s-1}$ where the N_i form a partition of the finite set $N \subset K \subset \mathbb{R}$. We have

$$\Delta(N|K) \leq \sum_{i=0}^{s-1} \max_{x \in K,\, a \in N_i} \left| \frac{w_{N \setminus N_i}(x)}{w_{N \setminus N_i}(a)} \right| \Delta(N_i|K);$$

Decomposition. X_d is decomposed into some modified Chebyshev-Labatto sets whose Lebesgue constant is well estimated,
$\mathcal{T}_d^{(\beta)} = \{\cos \beta_j : \beta_j = \beta + 2j\pi/d, j = 0, \ldots, d\}$.

Application. Using the lemma and the decomposition of X_d, we can prove the upper bound for $\Delta(X_d)$.

Answer to the questions

Univariate case

The following sequences have Lebesgue constants growing like polynomials

- Leja sequences for D;
- Images of Leja sequences under conformal mappings of the exterior of D onto the exteriors of many compact sets K in the complex plane;
- $\Re$-Leja sequences.

Multivariate case

Lebesgue constants of the intertwining of above sequences also grow like polynomials.

Main progress

For the first time, we are able to give the construction of good points in $\mathbb{R}^N$ with $N > 2$.

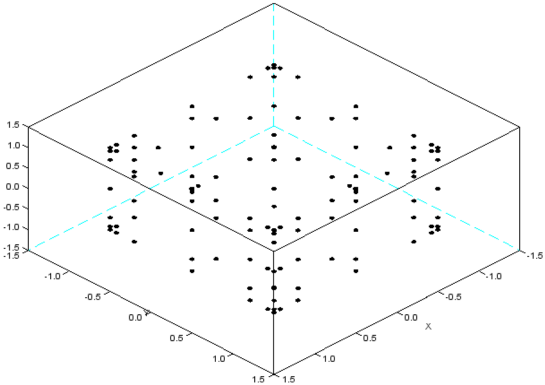


Figure: The intertwining of three $\Re$-Leja sequences

Kergin and Hakopian interpolants

Theorem

Let Ω be an open convex subset of $\mathbb{R}^N$, $A = (a_0, \ldots, a_d)$ be a tuple in Ω and let $k \in \{0, \ldots, d\}$. For every function $f \in C^{d-k}(\Omega)$, there exists a unique polynomial $P \in \mathcal{P}_{d-k}(\mathbb{R}^N)$ such that

$$\int_{[a_0,\ldots,a_{j+k}]} \mathrm{D}^\alpha (f - P) = 0, \quad |\alpha| = j, \quad j = 0, \ldots, d - k,$$

where

$$\int_{[a_0,\ldots,\, a_m]} g := \int_{\Delta_m} g\Big(a_0 + \sum_{j=1}^{m} t_j (a_j - a_0)\Big) \mathrm{d}t, \quad g \in C(\Omega),$$

$$\Delta_m = \{(t_1, t_2, \ldots, t_m) \in [0,\, 1]^m,\ \sum_{j=1}^{m} t_j \leq 1\}.$$

Kergin and Hakopian interpolants (to be continued)

Definition

- When $k = 0$, the polynomial P in the theorem is called the Kergin interpolation and denoted by $\mathcal{K}[A; f]$;
- When $k = N - 1$, the polynomial P in the theorem is called the Kergin interpolation and denoted by $\mathcal{H}[A; f]$.

Extended Kergin (Hakopian) interpolation

When the points of a set $A \subset \mathbb{R}^2$ are in general position, i.e., no three of them are aligned, then the Kergin operator $\mathcal{K}[A; \cdot]$ (resp. $\mathcal{H}[A; \cdot]$) extends to functions of class C^1 (resp. C^0).

A question

We have proved that Leja sequences for D are good for Lagrange interpolation. There arises a natural problem:

Question

Do bi-dimensional Kergin and Hakopian interpolants at Leja sequences for D of sufficiently smooth functions on D converge uniformly to the functions?

Kergin and Hakopian interpolants at the roots of unity

Theorem

Let R_n be the set of n-th roots of unity.

- Theorem of Bos and Calvi: $\mathcal{K}[R_n; f] \to f$, uniformly on D, for every $f \in C^2(D)$;
- Theorem of Liang and Lü: $\mathcal{H}[R_n; f] \to f$, uniformly on D, for every $f \in C^2(D)$.

Kergin interpolation at Leja sequences for D

Theorem

Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D.

1. If $f \in C^4(D)$, then $\mathcal{K}[E_d; f]$ converges uniformly to f on D;
2. If $f \in C^\infty(D)$, then $\mathrm{D}^\beta\big(\mathcal{K}[E_d; f]\big)$ converges uniformly to $\mathrm{D}^\beta f$ on D, for every two-dimensional index β.

Corollary

For every $f \in C^\infty(D)$, the series

$$\sum_{d=0}^{\infty} \int_{[e_0,\ldots,e_d]} \mathrm{D}^d f(\cdot, x - e_0, \ldots, x - e_{d-1})$$

converges to f uniformly on D. Moreover, the convergence extends to all derivatives.

Hakopian interpolation at Leja sequences for D

Theorem

Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D.

1. If $f \in C^5(D)$, then $\mathcal{H}[E_d; f]$ converges uniformly to f on D;
2. If $f \in C^\infty(D)$, then $\mathrm{D}^\beta\big(\mathcal{H}[E_d; f]\big)$ converges uniformly to $\mathrm{D}^\beta f$ on D, for every two-dimensional index β.

Corollary

For every $f \in C^\infty(D)$, the series

$$\sum_{d=1}^{\infty} \sum_{0 \le j_1 < j_2 < \cdots < j_{d-1} \le d-1} \int_{[e_0,\ldots,e_d]} \mathrm{D}^d f(\cdot, x - e_{j_1}, \ldots, x - e_{j_{d-1}}),$$

converges to f uniformly on D. Moreover, the convergence extends to all derivatives.

Thank you for your attention!