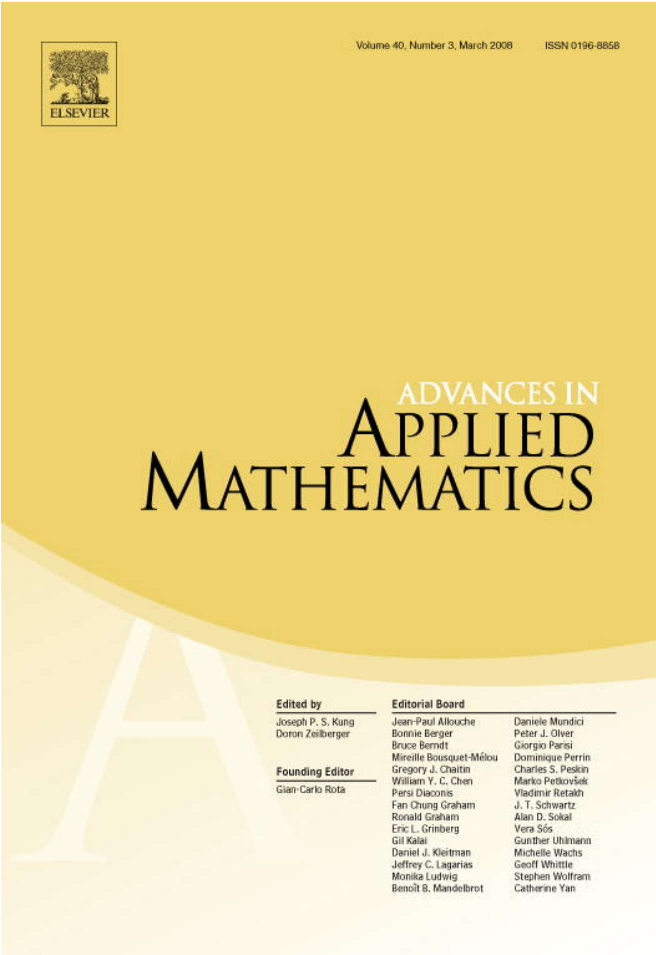
Volume 40, Number 3, March 2008
ISSN 0196-8858
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APPLIED
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Advances in Applied Mathematics 40 (2008) 309–329
ADVANCES IN Applied Mathematics
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Infinite magmatic bialgebras

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Received 4 September 2006; accepted 8 December 2006

Available online 7 June 2007

Abstract

An infinite magmatic bialgebra is a vector space endowed with n-ary operations, and n-ary cooperations, for each n, verifying some compatibility relations. We prove an analogue of the Hopf–Borel theorem for infinite magmatic bialgebras. We show that any connected infinite magmatic bialgebra is of the form $Mag^{\infty}(\mathrm{Prim}\, H)$, where $Mag^{\infty}(V)$ is the free infinite magmatic algebra over the vector space V.

Keywords: Bialgebra; Hopf algebra; Cartier–Milnor–Moore; Poincaré–Birkhoff–Witt; Operad

1. Introduction

The Hopf–Borel theorem is a rigidity theorem for connected bialgebras which are both commutative and cocommutative. It takes the following form in the non-graded case:

Theorem *(Hopf–Borel). Let $\mathcal{H}$ be a commutative and cocommutative bialgebra, over a field $\mathbb{K}$ of characteristic zero. The following are equivalent*:

(1) *$\mathcal{H}$ is connected,*
(2) *$\mathcal{H}$ is isomorphic to $S(\mathrm{Prim}\ \mathcal{H})$.*

Here $S(V)$ is the symmetric algebra over the vector space V, which can also be seen as the polynomial algebra.

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2006.12.005

This theorem has already been generalised to other types of bialgebras, see for example [6–10]. A particular type of bialgebras, verifying a theorem analogue to the Hopf–Borel one, are magmatic bialgebras, see [1]. They are vector spaces endowed with an unitary binary operation and a counitary binary co-operation related by a magmatic compatibility relation. We generalise them to bialgebras endowed with unitary n-ary operations for each $n \geqslant 2$, co-unitary n-ary co-operations, Δ_n for each $n \geqslant 2$, related by some infinite magmatic compatibility relation. We denote $Mag^{\infty}(V)$ the free infinite magmatic algebra over a vector space V.

We define the primitive part of such a bialgebra $\mathcal{H}$ to be:

$$\operatorname{Prim}\mathcal{H} := \bigcap_{n\geqslant 2}\bigl\{x \in \mathcal{H} \bigm| \bar{\Delta}_n(x) = 0\bigr\},$$

where

$$\overline{\Delta}_n(x) := \Delta_n(x) - \sum_{m=1}^{n-1}\sum_{\sigma\in Sh(m,n-m)} \sigma \circ \bigl(\overline{\Delta}_m(x), 1^{\otimes m-n}\bigr),$$

and, $Sh(m, n-m)$ are the $(m, n-m)$-shuffles.

The rigidity theorem for infinite magmatic bialgebras is as follows:

Theorem 25. *Let $\mathcal{H}$ be an infinite magmatic bialgebra over a field $\mathbb{K}$ of any characteristic. The following are equivalent*:

(1) *$\mathcal{H}$ is connected,*
(2) *$\mathcal{H}$ is isomorphic to $Mag^{\infty}(\operatorname{Prim}\mathcal{H})$.*

The proof is based on the construction of an idempotent projector from the bialgebra to its primitive part, as in [1,6,10]. Other examples of type of algebra satisfying an analogue of the Hopf–Borel theorem can be found in [5] but our example is the first one which has generating operations and co-operations in any degree.

2. Infinite magmatic algebra

Definition 1. An *infinite magmatic algebra A* is a vector space endowed with one n-ary unitary operation μ_n for all $n \geqslant 2$ (one for each n) such that: every μ_n admits the same unit, denoted by 1, and that,

$$\mu_n(x_1, \ldots, x_n) = \mu_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad \text{where } x_i = 1 \text{ and } x_j \in A,\ \forall j.$$

Diagrammatically this condition is the commutativity of:

$$\begin{tikzcd}
A^{\otimes n} \arrow[d, "\mu_n"'] & A^{\otimes i} \otimes \mathbb{K} \otimes A^{\otimes n-i-1} \arrow[l, "\mathrm{Id}\otimes\cdots\otimes u\otimes\cdots\otimes\mathrm{Id}"'] \arrow[r, equal] & A^{\otimes n-1} \arrow[dll, "\mu_{n-1}"] \\
A. & &
\end{tikzcd}$$

where $u : \mathbb{K} \to A$ is the unit map.

2.1. *Free infinite magmatic algebra*

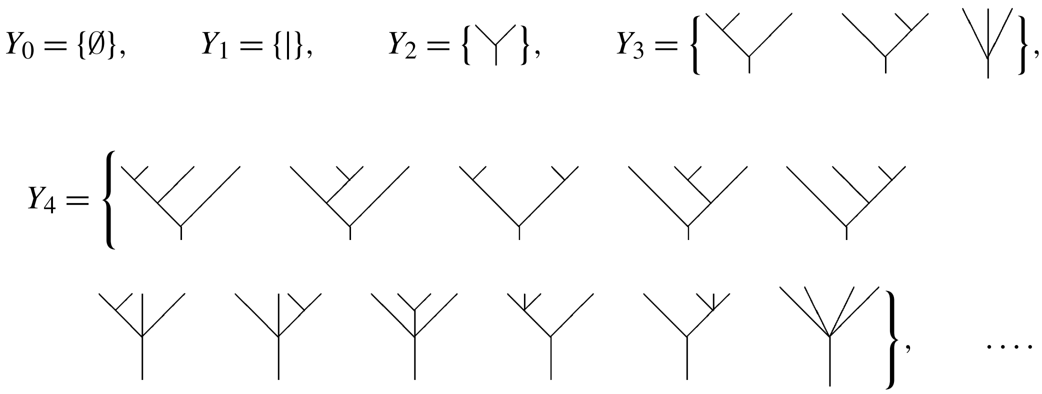
Definition 2. An infinite magmatic algebra A_0 is said to be *free over the vector space* V, if it satisfies the following universal property. Any linear map $f : V \to A$, where A is any infinite magmatic algebra, extends to a unique morphism of algebras $\tilde{f} : A_0 \to A$:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & A_0 \arrow[d, "\tilde{f}"] \\ & A. \end{tikzcd}$$

2.1.1. *Planar trees and n-ary products*

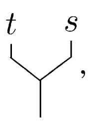
A *planar tree* T is a planar graph which is assumed to be simple (no loops nor multiple edges), connected, rooted and reduced (no vertices with only one outgoing edge). We denote by Y_n the set of planar trees with n leaves. In low dimensions one gets:

$Y_0 = \{\emptyset\}, \qquad Y_1 = \{|\},$



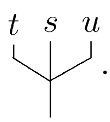
The *n-grafting* of n trees is the gluing of the root of each tree on a new root. For example the 2-grafting of the two trees t and s is

$$\bigvee_2 (t, s) :=$$



the 3-grafting of three trees t, s and u is

$$\bigvee_3 (t, s, u) :=$$

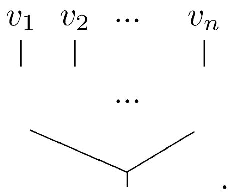


Remark 3. From our definition of a non-empty planar tree, any $t \in Y_n$ is of the form

$$t = \bigvee_k (t_1, \ldots, t_k)$$

for uniquely determined trees $t_1, \ldots, t_k$.

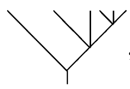
Let V be a vector space. A *labelled tree of degree n*, $n \geqslant 1$, denoted by $(t, v_1 \cdots v_n)$, is a tree t endowed with the labelling of the leaves by the elements $v_1, \ldots, v_n$, represented as:



Moreover one can define the n-grafting of labelled trees by the n-grafting of the trees, where one keeps the labellings on the leaves.

2.1.2. Construction of the free infinite magmatic algebra

Definition 4. Let (A, μ_n) be an infinite magmatic algebra. We define a map φ from the set of trees to the vector space of operations of A as follows: to the nth corolla $t_n = \bigvee_n(\underbrace{|, \ldots, |}_{n})$ we associate the operation μ_n. As the other trees can be seen as a grafting of corollas of degree k, their image by φ is the composition of the respective operations μ_k.

For example, consider the tree $t =$ , we have

$$t \mapsto \mu^t = \mu_2 \circ (\mathrm{Id} \otimes \mu_3) \circ (\mathrm{Id}^{\otimes 3} \otimes \mu_3).$$

Observe that for any trees $t_1, \ldots, t_n$, we have $\mu_n \circ (\mu^{t_1} \otimes \cdots \otimes \mu^{t_n}) = \mu^{(\vee_n(t_1, \ldots, t_n))}$ by definition.

We denote by $Mag^\infty(V) := \bigoplus_{n=0}^{\infty} Mag_n^\infty \otimes V^{\otimes n}$ the free infinite magmatic algebra.

Proposition 5. *Let V be a vector space and $\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n}$ be the vector space spanned by the labelled planar trees endowed with the n-grafting of labelled trees, for all $n \geqslant 2$, and with unit $\emptyset$. The linear extension of φ to $\varphi : \bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n} \to Mag^\infty(V)$ is an infinite magmatic algebra isomorphism*:

$$\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n} \cong Mag^\infty(V).$$

We omit the proof as we will give it in the coalgebraic case, see Proposition 12.

3. Infinite magmatic coalgebra

Definition 6. An *infinite magmatic coalgebra* C is a vector space endowed with one n-ary co-operation $\Delta_n : C \to C^{\otimes n}$ for all $n \geqslant 2$ which is co-unitary: every Δ_n admits the same co-unit $c : C \longrightarrow \mathbb{K}$ and that the following diagram is commutative:

$$\begin{tikzcd} C^{\otimes n} \arrow[r, "\mathrm{Id}\otimes\cdots\otimes c\otimes\cdots\otimes\mathrm{Id}"] & C^{\otimes i}\otimes\mathbb{K}\otimes C^{\otimes n-i-1} \arrow[r, equal] & C^{\otimes n-1} \\ C. \arrow[u, "\Delta_n"] \arrow[urr, "\Delta_{n-1}"'] & & \end{tikzcd}$$

3.1. Construction of the connected cofree infinite magmatic coalgebra

We denote $Sh(p,q)$ the set of (p,q)-shuffles. It is a permutation of $(1,\dots,p,p+1,\dots,p+q)$ such that the image of the elements 1 to p and of the elements $p+1$ to $p+q$ are in increasing order.

We define

$$\overline{\Delta}_1(x) := x,$$

$$\overline{\Delta}_n(x) := \Delta_n(x) - \sum_{m=1}^{n-1} \sum_{\sigma\in Sh(m,n-m)} \sigma\circ\left(\overline{\Delta}_m(x), 1^{\otimes m-n}\right).$$

Let T_n denote the n-corolla. Then $\overline{\Delta}_m(T_n)=0$ for all $m\neq n$ and $\overline{\Delta}_n(T_n) = |^{\otimes n}$.

Definition 7. An infinite magmatic co-augmented coalgebra is *connected* if it verifies the following property:

$$\mathcal{H} = \bigcup_{r\geqslant 0} F_r\mathcal{H} \quad \text{where } F_0\mathcal{H} := \mathbb{K}1$$

and, by induction,

$$F_r\mathcal{H} := \mathbb{K}1 \oplus \left\{x\in\mathcal{H} \;\middle|\; \delta_n(x) = 0 \text{ for any reduced } n\text{-ary cooperation where } n > r\right\}.$$

Remark that connectedness only depends on the unit and co-operations.

We define the primitive part of $\mathcal{H}$ as $\operatorname{Prim}\mathcal{H} := \bigcap_{n\geqslant 2}\{x\in\mathcal{H} \mid \overline{\Delta}_n(x) = 0\}$.

Definition 8. An infinite magmatic coalgebra C_0 is *cofree on the vector space V* if there exists a linear map $p : C_0 \to V$ satisfying the following universal property: any linear map $\phi : C \to V$, where C is any connected infinite magmatic coalgebra such that $\phi(1) = 0$, extends in a unique coalgebra morphism $\tilde{\phi} : C \to C_0$:

$$\begin{tikzcd} C \arrow[d, "\tilde{\phi}"'] \arrow[dr, "\phi"] & \\ C_0 \arrow[r, "p"] & V. \end{tikzcd}$$

3.1.1. Planar trees and n-ary coproducts

We endow the vector space of planar trees with the following n-ary co-operations, for $n \geqslant 2$: for any planar tree t we define:

$$\Delta_n(t) := \sum t_1 \otimes \cdots \otimes t_n$$

where the sum is extended on all the ways to write t as $\bigvee_n(t_1, \ldots, t_n)$, where t_i may be $\emptyset$. It can be explicited, as follows, for $t = \bigvee_n(t_1, \ldots, t_n)$, where $t_i \neq \emptyset$ for all i:

$$\Delta_n(t) := \left(\overset{t_1}{\Big|} \otimes \cdots \otimes \overset{t_n}{\Big|} \right) + \sum_{i=0}^{n-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes n-i-1},$$

$$\Delta_m(t) := \begin{cases} \sum_{i=0}^{m-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes m-i-1}, & \text{if } m < n, \\ \sum_{i=0}^{m-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes m-i-1} & \\ \quad + \sum_{i_1+\cdots+i_{n+1}=m-n} \emptyset^{\otimes i_1} \otimes t_1 \otimes \emptyset^{\otimes i_2} \otimes \cdots \otimes t_n \otimes \emptyset^{\otimes i_{n+1}}, & \text{if } m > n, \end{cases}$$

$$\Delta_n(|) := \sum_{i=0}^{n-1} \emptyset^{\otimes i} \otimes | \otimes \emptyset^{n-i-1},$$

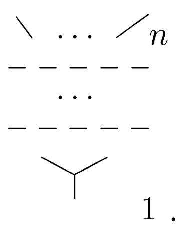
$$\Delta_n(\emptyset) := \emptyset^{\otimes n}.$$

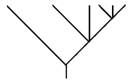
As in the preceding section one can define the n-ungrafting of labelled trees by the n-ungrafting of planar trees and keeping the labelling on the leaves.

Remark that the empty tree $\emptyset$ plays here the role of the unit, it can then be denoted by $1 := \emptyset$. To illustrate the above definition, we compute the image of the n-corolla t_n by $\overline{\Delta}_m$. Since we have $t_n := \bigvee_n(\underbrace{|, \ldots, |}_{n \text{ times}})$, we get $\overline{\Delta}_m(t_n) = 0$ for all $m \neq n$ and $\overline{\Delta}_n(t_n) = |^{\otimes n}$.

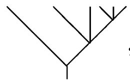
3.1.2. Construction of the cofree connected infinite magmatic coalgebra

Definition 9. The *height* of a planar tree T is the maximal number of inner vertices one can meet when going through all the paths starting from the root to a leaf



Example 10. The n-corolla is of height 1. The tree  is of height 3.

Definition 11. Let (C, Δ_n) be an infinite magmatic coalgebra. We define a map ψ from the set of trees to the vector space of co-operations of C as follows: to the nth corolla $t_n = \bigvee_n(\underbrace{|, \ldots, |}_{n})$ we associate the co-operation Δ_n. As the other trees can be seen as a grafting of corollas of degree k, their image by φ is the composition of the respective co-operations Δ_k.

For example, consider the tree $t =$ , we have

$$\Delta^t = \left(\mathrm{Id}^{\otimes 3} \otimes \Delta_3\right) \circ (\mathrm{Id} \otimes \Delta_3) \circ \Delta_2.$$

Observe that for any tree $t_1, \ldots, t_n$ we have: $(\Delta^{t_1} \otimes \cdots \otimes \Delta^{t_n}) \circ \Delta_n = \Delta^{(\vee_n(t_1 \otimes \cdots \otimes t_n))}$ by definition.

We denote by $Mag^{\infty\ c}(V) := \bigoplus_{n=0}^{\infty} Mag_n^{\infty\ c} \otimes V^{\otimes n}$ the free infinite magmatic coalgebra.

Proposition 12. *Let V be a vector space and $\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n}$ be the vector space spanned by the labelled planar trees endowed with the n-ungrafting of labelled trees, for all $n \geqslant 2$. The linear extension of ψ to $\psi : \bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n} \to Mag^{\infty\ c}(V)$ is an infinite connected magmatic coalgebra isomorphism*:

$$\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n} \cong Mag^{\infty\ c}(V).$$

It is equivalent to say that $\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n}$ is cofree over V among the connected infinite magmatic coalgebras.

Proof. We could prove this proposition by dualising Proposition 5, but since it is important for our arguments later on, we will write this proof completely.

For conveniency of the proof, we denote here $\bigoplus_{n=0}^{\infty} \mathbb{K}[Y_n] \otimes V^{\otimes n}$ by $C^{\infty}(V) := \bigoplus_{n=0}^{\infty} C_n^{\infty}(V)$.

The co-operations are counital by definition, so C^{∞} is a magmatic coalgebra.

Then we verify the connectedness of $C^{\infty}(V)$. It comes naturally that:

$$F_1 C^{\infty}(V) = C_0^{\infty}(V) \oplus C_1^{\infty}(V),$$

$$F_2 C^{\infty}(V) = C_0^{\infty}(V) \oplus C_1^{\infty}(V) \oplus C_2^{\infty}(V).$$

Then,

$$F_r C^{\infty}(V) = \bigoplus_{m=0}^{m=r} \mathbb{K}[\text{trees with } m \text{ leaves}].$$

The proof is done by contradiction. Indeed, let us consider the tree $T \in C^{\infty}(V)$ with p leaves and suppose that there exists a reduced k-ary cooperation such that $k > p$ and $\delta_k(T) \neq 0$. The cooperation δ_k is of the form:

$$(\delta_{i_1} \otimes \cdots \otimes \delta_{i_m}) \circ \overline{\Delta}_m$$

for i_j-ary reduced cooperations δ_{i_j} such that $i_1 + \cdots + i_m = k$. Moreover the tree $T \in \mathbb{K}[Y_k]$ can be seen as the n-grafting of other trees. Therefore we have:

$$\begin{aligned}\delta_k(T) &= \Delta_m \circ \mu_n(T_1 \otimes \cdots \otimes T_n)\\ &= \begin{cases} 0, & \text{if } m \neq n,\\ \delta_{i_1}(T_1) \otimes \cdots \otimes \delta_{i_n}(T_n), & \text{if } m = n.\end{cases}\end{aligned}$$

Then, by descending induction on the height of the tree, any reduced cooperation such that $\delta_k(T) \neq 0$ is of arity at most p, which is in contradiction with $k > p$. So we get the expected filtration. It is clear that $\bigcup_n F_n C^{\infty}(V) = C^{\infty}(V)$.

Let C be a connected magmatic coalgebra and $\phi : C \to V$ a linear map. To prove the cofreeness of the coalgebra, it is sufficient to construct a unique extension of ϕ as a infinite magmatic coalgebra morphism. That is to say, prove the commutativity of the following diagram:

$$\begin{tikzcd} C \arrow[r, dashed, "\tilde{\phi}"] \arrow[dr, "\phi"'] & C^{\infty}(V) = \bigoplus_{n \geqslant 0} C_n^{\infty} \otimes V^{\otimes n} \arrow[d] \\ & V. \end{tikzcd} \tag{1}$$

The map $\tilde{\phi}$ can be decomposed into its homogeneous components as follows:

$$\tilde{\phi}(c) = \tilde{\phi}(c)_{(1)} + \tilde{\phi}(c)_{(2)} + \tilde{\phi}(c)_{(3)} + \cdots. \tag{2}$$

By induction on n, one can determine the homogeneous components of $\tilde{\phi}$. As the map $\tilde{\phi}$ is a coalgebra morphism defined on $\bar{C}$, one defines $\tilde{\phi}(1) = 1$.

The commutativity of the diagram (1) gives the following equality:

$$\tilde{\phi}(c)_1 = \big(|, \phi(c)\big). \tag{3}$$

By definition of $C_2^{\infty}(V)$:

$$\tilde{\phi}(c)_2 = \sum\big(\curlyvee, a_1 a_2\big).$$

We adopt the following notation $\bar{\Delta}(c) = \Sigma c_1 \otimes c_2$. And we compute:

$$\tilde{\phi}_1 \otimes \tilde{\phi}_1 \circ \bar{\Delta}(c) = \sum \tilde{\phi}_1(c_1) \otimes \tilde{\phi}_1(c_2) = \sum\big(|, \phi(c_1)\big) \otimes \big(|, \phi(c_2)\big) \quad \text{thanks to (3).}$$

But

$$\Delta \circ \tilde{\phi}(c)_2 = \sum (|, a_1) \otimes (|, a_2) = \sum \big(|, \phi(c_1)\big) \otimes \big(|, \phi(c_2)\big).$$

Therefore,

$$\tilde{\phi}(c)_2 = \sum \big(\curlyvee, \phi(c_1)\phi(c_2)\big).$$

Recall that, by Definition 11, any tree t determines a co-operation that we denote by Δ^t. So, for a tree t_i of degree n, we have:

$$\begin{aligned} \tilde{\phi}_n(c) &= \sum (t, a_1 \cdots a_n), \\ \bar{\Delta}^{t_i} \tilde{\phi}(c) &= \sum \big(|, a_1^i\big) \otimes \cdots \otimes \big(|, a_n^i\big). \\ \text{Denote:} \quad \bar{\Delta}^{t_i}(c) &= \sum c_1^i \otimes \cdots \otimes c_n^i, \\ \tilde{\phi}_1^{\otimes n} \circ \bar{\Delta}^{t_i}(c) &= \sum \big(|, \phi\big(c_1^i\big)\big) \otimes \cdots \otimes \big(|, \phi\big(c_n^i\big)\big), \end{aligned}$$

which gives us:

$$\tilde{\phi}_n^i(c) = \sum \big(t_i, \phi\big(c_1^i\big) \cdots \phi\big(c_n^i\big)\big).$$

Going through all the trees of degree n, we have:

$$\tilde{\phi}_n(c) = \sum_{t_i \text{ of degree } n} \big(t_i, \phi\big(c_1^i\big) \cdots \phi\big(c_n^i\big)\big)$$

(though we denote $\tilde{\phi}^i$, t_i, we do not assume that there must be an order on the trees, this notation is only used to distinguish the trees with same degree).

Therefore one has:

$$\begin{aligned} \tilde{\phi}(c) = \big(|, \phi(c)\big) &+ \sum \big(\curlyvee, \phi(c_1)\phi(c_2)\big) + \sum \Big(\curlyvee\!\!\curlyvee, \phi\big(c_1^1\big)\phi\big(c_2^1\big)\phi\big(c_3^1\big)\Big) \\ &+ \sum \Big(\curlyvee\!\!\curlyvee, \phi\big(c_1^2\big)\phi\big(c_2^2\big)\phi\big(c_3^2\big)\Big) + \sum \Big(\Psi, \phi\big(c_1^3\big)\phi\big(c_2^3\big)\phi\big(c_3^3\big)\Big) + \cdots. \end{aligned}$$

By construction $\tilde{\phi}$ is a morphism of connected infinite magmatic coalgebras which is unique, since we have no other choice to have the commutativity of diagram (1) and the coalgebra morphism property. Therefore $C^\infty(V) \cong Mag^{\infty\ c}(V)$. $\square$

Remark 13. The primitive part of the cofree infinite magmatic coalgebra is:

$$\operatorname{Prim} Mag^{\infty\ c}(V) = \mathbb{K}\{|\} \otimes V\ .$$

4. Infinite magmatic bialgebra

Definition 14. An *infinite magmatic bialgebra* $(\mathcal{H}, \mu_n, \Delta_n)$ is a vector space $\mathcal{H} = \bar{\mathcal{H}} \oplus \mathbb{K}1$ such that:

(1) $\mathcal{H}$ admits an infinite magmatic algebra structure with n-ary operations denoted μ_n,
(2) $\mathcal{H}$ admits an infinite magmatic coalgebra structure with n-ary co-operations denoted Δ_n,
(3) $\mathcal{H}$ satisfies the following compatibility relation called the "infinite magmatic compatibility":

$$\Delta_n \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_n + \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes n-i-1},$$

$$\Delta_m \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = \begin{cases} \sum_{i=0}^{m-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes m-i-1}, & \text{if } m < n, \\ \sum_{i=0}^{m-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes m-i-1} & \\ \quad + \sum_{i_1+\cdots+i_n+1=m-n} 1^{\otimes i_1} \otimes x_1 \otimes 1^{\otimes i_2} \otimes \cdots \otimes x_n \otimes 1^{\otimes i_{n+1}} & \text{if } m > n, \end{cases} \tag{4}$$

where $\underline{x} := \mu_n(x_1 \otimes \cdots \otimes x_n)$ and $x_1, \ldots, x_n \in \bar{\mathcal{H}}$.

A fundamental example in our context is the following:

Proposition 15. *Let V be a vector space. The space $(Mag^\infty(V), \bigvee_n, \Delta_n)$, where the operations $\bigvee_n$ (respectively the co-operations Δ_n) are defined in Sections* 2.1.1 *and* 3.1.1*, is an infinite magmatic connected bialgebra.*

Proof. Any tree can be seen as the n-grafting of n trees, except the empty tree and the tree reduced to the root. Therefore the m-ungrafting of a tree can be viewed as the m-ungrafting of the n-grafting of n trees. This observation gives the compatibility relation. □

5. The main theorem

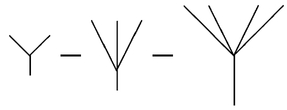
Definition 16. The *completed infinite magmatic algebra*, denoted by $Mag^\infty(\mathbb{K})^\wedge$, is defined by

$$Mag^\infty(\mathbb{K})^\wedge = \prod_{n \geqslant 0} Mag_n^\infty,$$

where the first generator | is denoted by t. This definition allows us to define formal power series of trees in $Mag^\infty(\mathbb{K})^\wedge$.

Remark 17. Recall that any non-empty tree can be seen as the grafting of trees. By induction any non-empty tree can be seen as a composition of graftings of the generator t. This is why some authors adopt the notation $\mathbb{K}\{\{t\}\}_\infty$ for $Mag^\infty(\mathbb{K})^\wedge$ (cf. [2]).

Lemma 18. *The following two formal power series, g and f, are inverse for composition in $Mag^{\infty}(\mathbb{K})^{\wedge}$:*

$g(|) := | -$  $- \cdots, \qquad f(|) := \sum T,$

where the sum is extended to all non-empty planar trees T.

Here the tree T stands for the element $T(t) := T(t, \dots, t)$, where $t = |$ the generator. The composition of $T_1 \circ T_2$ is defined as $T_1 \circ T_2(t) := T_1 \circ T_2(t, \dots, t) = T_1(T_2(t, \dots, t), \dots, T_2(t, \dots, t))$.

Proof. First, we show that $g \circ f = |$, that is to say:

$$\sum T - \sum_{T_1, T_2} \bigvee_2 (T_1 \otimes T_2) - \cdots - \sum_{T_1, \dots, T_n} \bigvee_n (T_1 \otimes \cdots \otimes T_n) - \cdots = |,$$

equivalently:

$$\sum_{T_1, T_2} \bigvee_2 (T_1 \otimes T_2) + \cdots + \sum_{T_1, \dots, T_n} \bigvee_n (T_1 \otimes \cdots \otimes T_n) + \cdots = \sum T - |.$$

It is immediate, as every tree can be seen as the n-grafting of n trees for a certain n, except $|$.

Then one verifies that, as in the associative case, a left inverse is also a right inverse. Let f^{-1} denote the right inverse of f. Then:

$$f^{-1} = (g \circ f) \circ f^{-1} = g \circ \left(f^{-1} \circ f\right) = g.$$

Remark that we have associativity of composition even in the infinite magmatic context. Therefore one has $f \circ g = Id$ and $g \circ f = Id$. $\square$

Definition 19. The *n-convolution* of n infinite magmatic algebra morphisms $f_1, \dots, f_n$ is a linear application defined by:

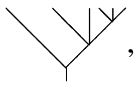
$$\star_n (f_1 \cdots f_n) := \mu_n \circ (f_1 \otimes \cdots \otimes f_n) \circ \Delta_n.$$

Observe that these operations are unitary.

Definition 20. Let $(\mathcal{H}, \mu_n, \Delta_n)$ be an infinite magmatic bialgebra. Let $t \in Y_n$ be a tree. Any tree t determines a composition of convolutions that we denote by $\star_t$. Let $f_1, \dots, f_n$ be infinite magmatic bialgebra morphisms. We define $\star_t(f_1, \dots, f_n)$ by:

$$\star_t (f_1, \dots, f_n) := \mu^t \circ (f_1, \dots, f_n) \circ \Delta^t.$$

Definition 21. Let $(\mathcal{H}, \mu_n, \Delta_n)$ be an infinite magmatic bialgebra. We define a map χ from the set of trees to the vector space of operations of $\mathcal{H}$: to the nth corolla $t_n = \bigvee_n(\underbrace{|, \ldots, |}_{n})$ we associate the operation $\star_n$. As any other tree can be seen as a grafting of corollas of degree k, it is associated with the composition of the respective operations $\star_k$.

For example, consider the tree $t =$ , we have

$$t \mapsto \chi(t) = \star_2 \circ (\mathrm{Id} \otimes \star_3) \circ \left(\mathrm{Id}^{\otimes 3} \otimes \star_3\right).$$

Observe that for any trees $t_1, \ldots, t_n$, we have $\star_n \circ (\star_{t_1} \otimes \cdots \otimes \star_{t_n}) = \star_{(\vee_n(t_1,\ldots,t_n))}$ by definition.

Proposition 22. *Let $(\mathcal{H}, \mu_n, \Delta_n)$ be an infinite magmatic bialgebra and $t \in Y_n$ be a tree. Then we have:*

$$\chi(t) = \star_t.$$

It is convenient to introduce the following notation:

Notation 23. Let $T \in Y_n$ we define

$$\star_T(J) : \mathcal{H} \longrightarrow \mathcal{H} : x \mapsto \star_T(J \cdots J)\left(x^n\right).$$

Observe that $\star_n \circ (\star_{T_1} \otimes \cdots \otimes \star_{T_n})(J) = \star_{(\vee_n(T_1 \otimes \cdots \otimes T_n))}(J)$ by definition.

Lemma 24. *Let $(\mathcal{H}, \mu_n, \Delta_n)$ be a connected infinite magmatic bialgebra. The linear map $e : \mathcal{H} \to \mathcal{H}$ defined as*:

$$e := J - \star_2 \circ J^{\otimes 2} - \star_3 \circ J^{\otimes 3} - \cdots - \star_n \circ J^{\otimes n} - \cdots$$

where $J = Id - uc$, u the unit of the operations, c the co-unit of the co-operations, has the following properties:

(1) $\operatorname{Im} e = \operatorname{Prim} \mathcal{H}$,
(2) *for all $x_1, \ldots, x_n \in \bar{\mathcal{H}}$ one has $e \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = 0$,*
(3) *the linear map e is an idempotent,*
(4) *for $\mathcal{H} = (Mag^{\infty}(V), \mu_n, \Delta_n)$ defined above, e is the identity on $V = Mag_1(V)$ and trivial on the other components.*

Proof. In this proof, we adopt the following notation: $Id := Id_{\bar{\mathcal{H}}}$, and for all $x \in \bar{\mathcal{H}}$, $\bar{\Delta}_n(x) := x_1 \otimes \cdots \otimes x_n$.

(1) Proof of $\operatorname{Im} e = \operatorname{Prim} \mathcal{H}$.

$$\overline{\Delta}_n\left(e(x)\right) = \overline{\Delta}_n(x) - \sum_m \overline{\Delta_n} \circ \mu_m \circ \overline{\Delta_m}(x)$$

$$= x_1 \otimes \cdots \otimes x_n - \overline{\Delta}_n \circ \mu_n(x_1 \otimes \cdots \otimes x_n) - \sum_{m \neq n} \underbrace{\overline{\Delta}_n \circ \mu_m \circ \overline{\Delta_m}(x)}_{=0}$$
$$= 0.$$

(2) Proof that for all $x_1, \ldots, x_n \in \bar{\mathcal{H}}$ one has $e \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = 0$. Indeed,

$$\begin{aligned} e \circ \mu_n(x_1 \otimes \cdots \otimes x_n) &= \mu_n(x_1 \otimes \cdots \otimes x_n) - \sum_m \mu_m \circ \overline{\Delta_m} \circ \mu_m(x_1 \otimes \cdots \otimes x_n) \\ &= \mu_n(x_1 \otimes \cdots \otimes x_n) - \mu_n \circ \overline{\Delta_n} \circ \mu_n(x_1 \otimes \cdots \otimes x_n) \\ &= 0. \end{aligned}$$

(3) Proof that e is an idempotent. We compute:

$$e\big(e(x)\big) = e(x) - \sum_m e\big(\mu_m \circ \overline{\Delta_m}(x)\big) = e(x).$$

(4) Proof that for $\mathcal{H} = (Mag^\infty(V), \mu_n, \Delta_n)$ defined above, e is the identity on $V = Mag_1(V)$ and trivial on the other components.

On $Mag_1(V) = | \otimes V$ we have: $e(| \otimes x) = | \otimes x$. All other trees can be seen as the n-grafting of n trees for a certain n. Then it suffices to apply the second property of the idempotent e to complete the proof. □

Theorem 25. *Let $\mathcal{H}$ be a connected infinite magmatic bialgebra over a field $\mathbb{K}$ of any characteristic, then the following are equivalent*:

(1) *$\mathcal{H}$ is connected,*
(2) *$\mathcal{H} \cong Mag^\infty(\mathrm{Prim}\ \mathcal{H})$.*

Proof. As the implication (2) to (1) is tautological, we focus on the proof of the converse implication.

Suppose that $\mathcal{H}$ is connected, we prove the isomorphism $\mathcal{H} \cong Mag^\infty(\mathrm{Prim}\,\mathcal{H})$ by explicitly giving the two inverse maps.

Using Notation 23, we define the infinite magmatic coalgebra morphism

$$G : \bar{\mathcal{H}} \to \overline{Mag^\infty}(\mathrm{Prim}\,\mathcal{H})$$

as the unique extension of the following linear map:

$$x \mapsto J(x) - \star_2 \circ J^{\otimes 2}(x) - \star_3 \circ J^{\otimes 3}(x) - \cdots - \star_n \circ J^{\otimes n}(x) - \cdots,$$

and the infinite magmatic algebra morphism

$$F : \overline{Mag^\infty}(\mathrm{Prim}\,\mathcal{H}) \to \bar{\mathcal{H}}$$

defined as the unique extension of the linear map:

$$x \mapsto \sum \star_T(J)(x),$$

where the sum is extended to all non-empty planar trees T.

Moreover, denote by t the generator of $Mag^{\infty}(\mathbb{K})$, $t := |$, and by $t^n := \bigvee_n \circ t^{\otimes n}$. We define $g(t) := t - t^2 - t^3 - \cdots - t^n - \cdots$, and $f(t) := \sum T$, where the sum is extended to all planar trees T. By Lemma 18, these two preceding maps are inverse, for composition.

These series can be applied to elements of $Hom_{\mathbb{K}}(\mathcal{H}, \mathcal{H})$ sending 1 on 0 using $\star_n$ as a product, thanks to the following morphism:

$$\begin{aligned}
Mag^{\infty}(\operatorname{Prim}\mathcal{H})^{\wedge} &\to Hom_{\mathbb{K}}(\mathcal{H}, \mathcal{H}), \\
t &\mapsto J, \\
\phi(t) = \sum a_n T &\mapsto \phi^{\star}(J) = \Phi = \sum a_n J^{\star T}, \\
\phi \circ \psi(t) &\mapsto (\phi \circ \psi)^{\star}(J) = \Phi \circ \Psi = \phi^{\star}(J) \circ \psi^{\star}(J).
\end{aligned}$$

It is clear that $e = g^{\star}(J)$.

Therefore composing the two formal power series F and G gives as a result:

$$\begin{aligned}
F \circ G &= f^{\star} \circ g^{\star}(J) = (f \circ g)^{\star}(J) = Id^{\star}(J) = J, \\
G \circ F &= g^{\star} \circ f^{\star}(J) = (g \circ f)^{\star}(J) = Id^{\star}(J) = J.
\end{aligned}$$

The proof is complete since $J = Id$ on $\bar{\mathcal{H}}$. Remark that $\mathcal{H} \cong Mag^{\infty}(\operatorname{Prim}\mathcal{H})$ is an isomorphism of infinite magmatic bialgebra. Indeed, we have the two following properties:

$$\begin{aligned}
\mu_n\big(G(x_1), \ldots, G(x_n)\big) &= G \circ F\big(\mu_n\big(G(x_1), \ldots, G(x_n)\big)\big) \\
&= G \circ \mu_n\big(F \circ G(x_1), \ldots, F \circ G(x_n)\big) \\
&= G \circ \mu_n(x_1, \ldots, x_n), \\
\Delta_n\big(F(x)\big) &= \big((F \circ G) \otimes \cdots \otimes (F \circ G)\big) \circ \Delta_n\big(F(x)\big) \\
&= (F \otimes \cdots \otimes F) \circ \Delta_n\big(G \circ F(x)\big) \\
&= (F \otimes \cdots \otimes F) \circ \Delta_n(x),
\end{aligned}$$

which proves that F is moreover an infinite magmatic coalgebra morphism (respectively G is an infinite magmatic algebra morphism) and hence an infinite magmatic bialgebra morphism. □

6. *m*-Magmatic bialgebras

Instead of considering infinite magmatic bialgebras one may consider m-magmatic bialgebras, with $m \geqslant 2$, where the number of operations and co-operations is restricted to m, cf. [3]. Explicitly, we would have:

6.1. m-Magmatic algebra and free m-magmatic algebra

Definition 26. An *m-magmatic algebra* A is a vector space endowed with one n-ary unitary operation μ_n for all $2 \leqslant n \leqslant m$ (one for each n) such that: every μ_n admits the same unit u and that,

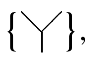
$$\mu_n(x_1, \dots, x_n) = \mu_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \quad \text{where } x_i = 1 \text{ and } x_j \in A,\ \forall j.$$

Diagrammatically this condition is the commutativity of:

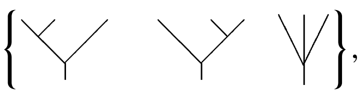
$$\begin{tikzcd}
A^{\otimes n} \arrow[d, "\mu_n"'] & A^{\otimes i} \otimes \mathbb{K} \otimes A^{\otimes n-i-1} \arrow[l, "\mathrm{Id}\otimes\cdots\otimes u\otimes\cdots\otimes\mathrm{Id}"'] \arrow[r, equal] & A^{\otimes n-1} \arrow[dll, "\mu_{n-1}"] \\
A. & &
\end{tikzcd}$$

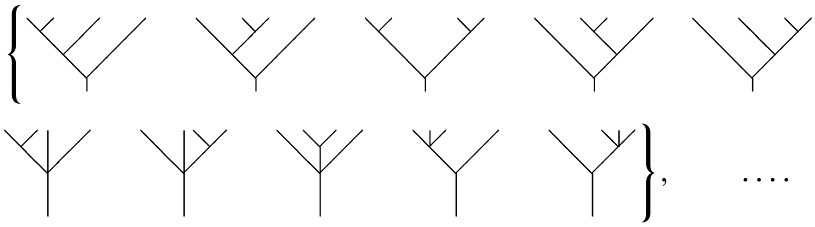
6.1.1. Construction of the free m-magmatic algebra over a vector space V

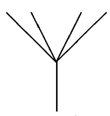
Let us describe the set of m-ary planar trees. An *m-ary planar tree T* is a planar graph which is assumed to be simple (no loops nor multiple edges) and connected, such that the valence of each inner vertex is at most $m + 1$. We denote by Y_n^m the set of m-ary planar trees with n leaves. In low dimensions one gets:

$Y_0^m = \{\emptyset\}$, $Y_1^m = \{|\}$, $Y_2^m =$ , ...

For example if $m = 2$ we have the binary planar trees, see [1]. For $m = 3$, in low dimensions we have:

$Y_3^3 =$ ,

$Y_4^3 =$ , ...

We observe that we have lost the tree  from the planar case.

The *n-grafting*, $2 \leqslant n \leqslant m$, of n trees is the gluing of the root of each tree on a new root, exactly as in the infinite magmatic case.

Remark 27. From our definition of an m-ary planar tree, any $t \in Y_n^m$ is of the form

$$t = \bigvee_k (t_1, \dots, t_k)$$

for uniquely determined trees $t_1, \dots, t_n$.

Note that one can define the n-grafting of labelled trees by the n-grafting of the trees, where one keeps the labellings on the leaves.

We define the vector space, denoted $Mag^m(V)$, as follows:

$$Mag^m := \bigoplus_n Mag_n^m \otimes V^{\otimes n},$$

where $Mag_n^m := \mathbb{K}[Y_n^m]$.

Proposition 28. *Let V be a vector space. The space $Mag^m(V)$ endowed with the n-grafting of labelled trees, for all $2 \leqslant n \leqslant m$, is an m-magmatic algebra. Moreover it is the free m-magmatic algebra over V.*

6.2. m-Magmatic coalgebra and cofree m-magmatic coalgebra

Definition 29. An *m-magmatic coalgebra C* is a vector space endowed with one n-ary co-unitary co-operation $\Delta_n : C \to C^{\otimes n}$ for all $2 \leqslant n \leqslant m$ such that: every Δ_n admits the same co-unit c and that the following diagram is commutative:

$$\begin{tikzcd}
C^{\otimes n} \arrow[r, "\mathrm{Id}\otimes\cdots\otimes c\otimes\cdots\otimes\mathrm{Id}"] & C^{\otimes i} \otimes \mathbb{K} \otimes C^{\otimes n-i-1} \arrow[r, equal] & C^{\otimes n-1} \\
C. \arrow[u, "\Delta_n"] \arrow[urr, "\Delta_{n-1}"'] & &
\end{tikzcd}$$

6.2.1. The cofree m-ary magmatic coalgebra

We endow the vector space of m-planar trees,defined above, with the following n-ary co-operations, for $2 \leqslant n \leqslant m$: for any m-ary planar rooted tree t we define:

$$\Delta_n(t) = \sum t_1 \otimes \cdots \otimes t_n$$

where the sum is extended to all the manner to write $t = \bigvee_n(t_1, \cdot, t_n)$.

Explicitely for $t = \bigvee_n(t_1, \ldots, t_n)$, where all $t_i \neq \emptyset$, we have:

$$\Delta_n(t) := \left(\overset{t_1}{\Big|} \otimes \cdots \otimes \overset{t_n}{\Big|} \right) + \sum_{i=0}^{n-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes n-i-1},$$

$$\Delta_m(t) := \begin{cases} \sum_{i=0}^{m-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes m-i-1}, & \text{if } m < n, \\ \sum_{i=0}^{m-1} \emptyset^{\otimes i} \otimes t \otimes \emptyset^{\otimes m-i-1} & \\ \quad + \sum_{i_1+\cdots+i_{n+1}=m-n} \emptyset^{\otimes i_1} \otimes t_1 \otimes \emptyset^{\otimes i_2} \otimes \cdots \otimes t_n \otimes \emptyset^{\otimes i_n+1}, & \text{if } m > n, \end{cases}$$

$$\Delta_n(|) := \sum_{i=0}^{n-1} \emptyset^{\otimes i} \otimes | \otimes \emptyset^{n-i-1},$$

$$\Delta_n(\emptyset) := \emptyset^{\otimes n}.$$

As in the preceding section one can define the n-ungrafting of labelled trees by the n-ungrafting of planar trees and keeping the labelling on the leaves.

Remark that $\emptyset$ plays here the role of the unit, it can then be denoted by $1 := \emptyset$.

Proposition 30. *Let V be a vector space. The space $Mag^m(V)$ endowed with the n-ungrafting co-operations, $2 \leqslant n \leqslant m$, on labelled trees is a connected infinite magmatic coalgebra. Moreover it is free over V among the connected infinite magmatic coalgebras.*

Proof. It is similar to the proof of cofree infinite magmatic coalgebra. $\square$

6.3. *m-Magmatic bialgebra*

Definition 31. An *m-magmatic bialgebra* $(\mathcal{H}, \mu_n, \Delta_n)$, where $2 \leqslant n \leqslant m$, is a vector space $\mathcal{H} = \bar{\mathcal{H}} \oplus \mathbb{K}1$ such that:

(1) $\mathcal{H}$ admits an m-magmatic algebra structure with n-ary operations denoted μ_n,
(2) $\mathcal{H}$ admits an m-magmatic coalgebra structure with n-ary co-operations denoted Δ_n,
(3) $\mathcal{H}$ satisfies the following "compatibility relation":

$$\Delta_n \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_n + \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes n-i-1},$$

$$\Delta_m \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = \begin{cases} \sum_{i=0}^{m-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes m-i-1}, & \text{if } m < n, \\ \sum_{i=0}^{m-1} 1^{\otimes i} \otimes \underline{x} \otimes 1^{\otimes m-i-1} \\ \quad + \sum_{i_1+\cdots+i_n+1=m-n} 1^{\otimes i_1} \otimes x_1 \otimes 1^{\otimes i_2} \otimes \cdots \otimes x_n \otimes 1^{\otimes i_{n+1}}, & \text{if } m > n, \end{cases} \tag{5}$$

$\forall \underline{x} := \mu_n(x_1 \otimes \cdots \otimes x_n)$ and $x_1, \ldots, x_n \in \bar{\mathcal{H}}$ and $2 \leqslant n \leqslant m$.

Example 32. Let V be a vector space. The space $(Mag^m(V), \bigvee_n, \Delta_n)$, where $2 \leqslant n \leqslant m$ and the operations $\bigvee_n$ (respectively the cooperations Δ_n) are defined in Section 6.1.1 (respectively in Section 6.2.1) is an infinite magmatic connected bialgebra.

6.4. *The rigidity theorem*

We can now state a rigidity theorem:

Theorem 33. *Let $\mathcal{H}$ be a connected m-ary magmatic bialgebra over a field $\mathbb{K}$ of any characteristic, then the following are equivalent*:

(1) *$\mathcal{H}$ is connected,*
(2) $\mathcal{H} \cong Mag^m(\mathrm{Prim}\ \mathcal{H})$.

The proof is very similar to the infinite magmatic case. One has to use the following two definitions and the two lemmas:

Definition 34. The *completed m-ary magmatic algebra*, denoted by $Mag^m(\mathbb{K})^\wedge$, is defined by

$$Mag^m(\mathbb{K})^\wedge = \prod_{n \geqslant 0} Mag^m_n,$$

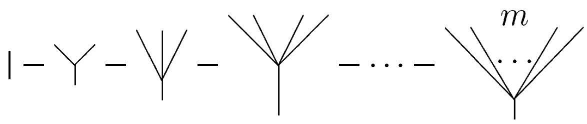
where the first generator | is denoted by t. This definition allows us to define formal power series of trees in $Mag^m(\mathbb{K})^\wedge$.

Definition 35. Let $2 \leqslant n \leqslant m$. The *n-convolution* of n m-ary magmatic algebra morphisms $f_1, \ldots, f_n$ is defined by:

$$\star_n(f_1 \cdots f_n) := \mu_n \circ (f_1 \otimes \cdots \otimes f_n) \circ \Delta_n.$$

Observe that these operations are unitary.

Lemma 36. *The following two formal power series are inverse for composition in* $Mag^m(\mathbb{K})^\wedge$:

$g(|) :=$ 

$, \qquad f(|) := \sum T,$

where the sum is extended to all m-ary planar trees T.

Lemma 37. *Let* $(\mathcal{H}, \mu_n, \Delta_n)$ *be a connected m-ary magmatic bialgebra. The linear map* $e : \mathcal{H} \to \mathcal{H}$ *defined as*:

$$e := J - \star_2 J^{\otimes 2} - \star_3 J^{\otimes 3} - \cdots - \star_m J^{\otimes m}$$

where $J = Id - uc$, *u the unit of the operations, c the co-unit of the co-operations, has the following properties*:

(1) $\operatorname{Im} e = \operatorname{Prim} \mathcal{H}$,
(2) *for all* $x_1, \ldots, x_n \in \bar{\mathcal{H}}$ *one has* $e \circ \mu_n(x_1 \otimes \cdots \otimes x_n) = 0$,
(3) *the linear map e is an idempotent,*
(4) *for* $\mathcal{H} = (Mag^m(V), \mu_n, \Delta_n)$ *defined above, e is the identity on* $V = Mag_1(V)$ *and trivial on the other components.*

Remark 38. Theorems 25, 33, 45 fit into the framework of "triples of operad" of Loday [5]. In our case the triple is $(Mag^m, Mag^m, Vect)$, $m \in \mathbb{N}$ or $m = \infty$. In the case $(Mag^m, Mag^n, \mathfrak{P})$ the operad of the primitives is still to be unraveled.

7. Binary magmatic bialgebras

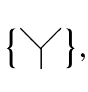
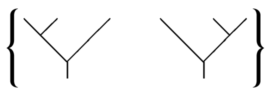
A special case of the m-ary magmatic bialgebra is the binary magmatic bialgebras, known also as magmatic bialgebras [1–4].

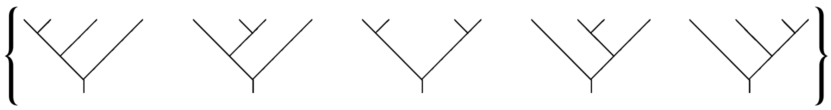
Definition 39. A *binary magmatic algebra A* is a vector space endowed with a binary unitary operation.

Definition 40. A *binary magmatic coalgebra* A is a vector space endowed with a binary counitary cooperation.

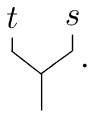
7.1. Planar binary trees equipped with a product and a coproduct

A *planar binary tree* T is a planar connected graph which is assumed to be simple (no loops nor multiple edges), rooted and such that any inner vertices is of valence at most 2. We denote by Y_n^2 the set of planar trees with n leaves. In low dimensions one gets:

$Y_0^2 = \{\emptyset\}$, $Y_1^2 = \{|\}$, $Y_2^2 =$ , $Y_3^2 =$ ,

$Y_4^2 =$ , $\ldots$.

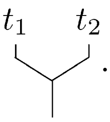
The *grafting* of two trees is the gluing of the root of each tree on a new root. For example, if t and s are two planar binary trees, their grafting is defined as:

$\bigvee_2 (t, s) :=$ .

Remark 41. From our definition of a planar binary tree, any $t \in Y_n^2$ is of the form

$$t = \bigvee(t_1, t_2)$$

for uniquely determined trees t_1, t_2. Diagrammatically it is represented by:

$t =$ .

Thanks to this remark one can define the ungrafting of a tree $t = t_1 \vee t_2$ as:

$$\Delta(t) := \overset{t_1}{\Big|} \otimes \overset{t_2}{\Big|} + t \otimes \emptyset + \emptyset \otimes t.$$

We add:

$$\Delta(|) = | \otimes \emptyset + \emptyset \otimes |,$$
$$\Delta(\emptyset) = \emptyset \otimes \emptyset.$$

Let V be a vector space, one can define the grafting (respectively the ungrafting) of labelled trees by the grafting (respectively the ungrafting) of the trees, where one keeps the labellings on the leaves. Therefore we have defined an operation and a cooperation on the vector space $Mag^2(V) := \bigoplus_{n \geqslant 0} Mag_n^2 \otimes V^{\otimes n}$, where $Mag_n^2 = \mathbb{K}[Y_n^2]$.

7.2. Binary magmatic bialgebras

Definition 42. A *binary magmatic bialgebra* $(\mathcal{H}, \cdot, \Delta)$ is a vector space such that $\mathcal{H} = \bar{\mathcal{H}} \oplus \mathbb{K} \cdot 1$ verifying:

(1) $(\mathcal{H}, \cdot)$ is a binary magmatic algebra,
(2) $(\mathcal{H}, \Delta)$ is a binary magmatic coalgebra,
(3) $\mathcal{H}$ satisfies the following "compatibility relation":

$$\Delta(x \cdot y) = x \cdot y \otimes 1 + x \otimes y + 1 \otimes x \cdot y, \quad \forall x, y \in \mathcal{H}.$$

Definition 43. A binary magmatic coalgebra $\mathcal{H}$ is said to be *connected* if it verifies the following property:

$$\mathcal{H} = \bigcup_{r \geqslant 0} F_r \mathcal{H} \quad \text{where } F_0 := \mathbb{K}1, \text{ and by induction}$$
$$F_r := \{x \in \mathcal{H} \mid \bar{\Delta}(x) \in F_{r-1} \otimes F_{r-1}\},$$

where $\bar{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$.

Note that the connectedness only depends on the unit and the cooperation.

A fundamental example, in this context, is:

Proposition 44. *The space* $(Mag^2(V), \cdot, \Delta)$, *where* $\cdot$ (*respectively* Δ) *was defined in Section* 7.1 *is a connected magmatic bialgebra.*

And, we can state the following theorem, see [1]:

Theorem 45. *If* $\mathcal{H}$ *be a connected binary magmatic bialgebra over a field of any characteristic, then* $\mathcal{H}$ *is isomorphic to the binary magmatic bialgebra* $Mag^2(\operatorname{Prim}\mathcal{H})$.

Acknowledgments

I am debtful to L. Gerritzen who raised out this question in Bochum's seminar. I would like to thank J.-L. Loday for his advisory, D. Guin, R. Holtkamp for a careful reading of a first version and A. Bruguieres for some remarks.

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