ON THE ZEROS AND CRITICAL POINTS OF A RATIONAL MAP.

XAVIER BUFF

ABSTRACT. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d. It is well known that f has d zeros and $2d-2$ critical points counted with multiplicities. In this note, we explain how those zeros and those critical points are related.

In this note, $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map. We denote by $\{\alpha_i\}_{i\in I}$ the set of zeros of f, and by $\{\omega_j\}_{j\in J}$ the set of critical points of f which are not zeros of f (the sets I and J are finite). Moreover, we denote by n_i the multiplicity of α_i as a zero of f and by m_j the multiplicity of ω_j as a critical point of f. The local degree of f at α_i is n_i and the local degree of f at ω_j is $d_j = m_j + 1$. In particular, when $\omega_j \neq \infty$ and $f(\omega_j) \neq \infty$, the point ω_j is a zero of f' of order m_j.

Our goal is to understand the relations that exist between the points α_i and the points ω_j.

Proposition 1. *Given a finite collection of distinct points $\alpha_i \in \mathbb{P}^1$ with multiplicities n_i and $\omega_j \in \mathbb{P}^1$ with multiplicities m_j, there exists a rational map f vanishing exactly at the points α_i with multiplicities n_i and having extra critical points exactly at the points ω_j with multiplicities m_j if and only if*
(1) $\sum(n_i+1) - \sum m_j = 2$, and
(2) for any k such that $\alpha_k \in \mathbb{C}$,

$$\operatorname{res}\left(\frac{\prod_{\omega_j\in\mathbb{C}}(z-\omega_j)^{m_j}}{\prod_{\alpha_i\in\mathbb{C}}(z-\alpha_i)^{n_i+1}}dz, \alpha_k\right) = 0.$$

We will give a geometric interpretation of (2) in the case where α_k is a simple zero of f: working in a coordinate where $\alpha_k = \infty$, the barycentre of the remaining zeros weighted with their multiplicities is equal to the barycentre of the critical points of f weighted with their multiplicities (see proposition 2 below).

Proof. The proof is elementary. It is based on the observation that the 1-forms $d(1/f)$ and

$$\phi = \frac{\prod_{\omega_j\in\mathbb{C}}(z-\omega_j)^{m_j}}{\prod_{\alpha_i\in\mathbb{C}}(z-\alpha_i)^{n_i+1}}dz$$

are proportional. The differential equation $d(1/f) = \phi$ has a rational solution if and only if ϕ is exact, if and only if the residues of ϕ at all finite poles are equal to zero.

Lemma 1. *Let f be a rational map. Denote by α_i its zeros and by n_i their multiplicities. Denote by ω_j the critical points of f which are not multiple zeros of f and by m_j their multiplicities. The zeros of the 1-form $d(1/f)$ are exactly the points ω_j with order m_j and its poles are exactly the points α_i with order $n_i + 1$.*

1991 *Mathematics Subject Classification.* 30C15.

Proof. A singularity of the 1-form $d(1/f) = -df/f^2$ is necessarily a zero or a pole of f, a zero of f', or ∞ (where ϕ is defined by analytic continuation). Considering the Laurent series of f at each of those points, one immediately gets the result. □

Let us now assume that there exists a rational map f with the required properties. Lemma 1 shows that the 1-forms ϕ and $d(1/f)$ have the same poles and the same zeros in $\mathbb{C}$, with the same multiplicities. Hence, their ratio is a rational function which does not vanish in $\mathbb{C}$. Thus, ϕ and $d(1/f)$ are proportional. In particular, ϕ has a singularity at ∞ if and only if $d(1/f)$ has a singularity at ∞ and the singularity is of the same kind for both 1-forms. Since the number of poles minus the number of zeros of any non-zero 1-form on $\mathbb{P}^1$ is equal to 2 (the Euler characteristic of $\mathbb{P}^1$), we see that $\sum(n_i+1)-\sum m_j = 2$ which is precisely condition (1). Besides, since ϕ is exact, it follows that the residues at all the poles α_k vanish and condition (2) is satisfied.

Conversely, the 1-form ϕ has poles of order n_i+1 at the points $\alpha_i \in \mathbb{C}$ and zeros of order m_j at the points $\omega_j \in \mathbb{C}$. Condition (2) implies that ϕ is exact, i.e., there exists a rational map $g:\ \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi = dg$. Since the number of poles of ϕ in $\mathbb{P}^1$ minus the number of zeros of ϕ in $\mathbb{P}^1$ is equal to 2, condition (1) implies that when ∞ is neither a point α_i nor a point ω_j, it is a regular point of ϕ, when $\infty = \alpha_{i_0}$, it is a pole of ϕ of order n_{i_0}, and when $\infty = \omega_{j_0}$, it is a zero of ϕ of order m_{j_0}. Finally, $\phi = d(1/f)$, with $f = 1/g$, and lemma 1 shows that the rational map $f = 1/g$ vanishes exactly at the points α_i with multiplicities n_i and has extra critical points exactly at the points ω_j with multiplicities m_j. ■

We will now give a geometric interpretation of (2) when α_k is a simple zero of f. Let us first work in a coordinate where ∞ is neither one of the points α_i nor a point ω_j. Define

$$R(z) = \frac{\prod_j (z-\omega_j)^{m_j}}{\prod_{i\neq k}(z-\alpha_i)^{n_i+1}}.$$

Then,

$$\mathrm{res}\left(\frac{\prod_j (z-\omega_j)^{m_j}}{\prod_i (z-\alpha_i)^{n_i+1}}dz, \alpha_k\right) = \mathrm{res}\left(\frac{R(z)}{(z-\alpha_k)^2}dz, \alpha_k\right) = R'(\alpha_k).$$

Since $R(\alpha_k) \neq 0$, this residue vanishes if and only if

$$\frac{R'(\alpha_k)}{R(\alpha_k)} = \sum_j \frac{m_j}{\alpha_k-\omega_j} - \sum_{i\neq k}\frac{n_i+1}{\alpha_k-\alpha_i} = 0.$$

Let d be the number of zeros counted with multiplicities, i.e., $d = \sum_i n_i$. The total number of critical points is $2d-2 = \sum_j m_j + \sum_i (n_i-1)$ (the critical points of f are the points ω_j and the multiple zeros of f). Then, the above equation can be rewritten as

$$\frac{1}{2d-2}\left(\sum_j \frac{m_j}{\alpha_k-\omega_j} + \sum_{i\neq k}\frac{n_i-1}{\alpha_k-\alpha_i}\right) = \frac{1}{d-1}\sum_{i\neq k}\frac{n_i}{\alpha_k-\alpha_i}.$$

This last equality can be interpreted in the following way.

Proposition 2. *Assume f is a rational map having a simple zero at ∞. Then, the barycentre of the remaining zeros weighted with their multiplicities is equal to the barycentre of the critical points of f weighted with their multiplicities.*

Remark. One can prove this proposition directly. We may write $f = P/Q$ where

$$P = \sum_{k=0}^{d-1} a_k z^k \quad \text{and} \quad Q = \sum_{k=0}^{d} b_k z^k$$

are co-prime polynomials with $\deg(Q) = \deg(P)+1 = d$. Without loss of generality, we may assume that the barycenter of the zeros of f is equal to 0. In other words, we may assume that P is a centered polynomial, i.e., $a_{d-2} = 0$. A simple calculation shows that

$$P'Q - Q'P = \sum_{k=0}^{2d-2} c_k z^k$$

is a polynomial of degree $2d - 2$ and that $c_{2k-1} = 0$. Therefore, the barycenter of the zeros of $P'Q - Q'P$, i.e., the barycenter of the critical points of f, is equal to 0.

Let us apply this geometric interpretation in order to re-prove two known results. The first corollary is related to the Sendov conjecture (see for example [M] and more particularly section 4). This conjecture asserts that if a polynomial P has all its roots in the closed unit disk, then, for each zero α_i there exists a critical point ω (possibly a multiple zero) such that $|\alpha_i - \omega| \leq 1$.

Corollary 1. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. Assume the zeros of P are all contained in the closed unit disk and $\alpha_0 \in S^1$ is a zero of P. Then, the closed disk of diameter $[0, \alpha_0]$ contains at least one critical point of f.*

Proof. Denote by d the degree of P. If α_0 is a multiple zero of P, then the result is trivial. Thus, assume α_0 is a simple zero of P. Let us work in the coordinate $Z = \alpha_0/(\alpha_0 - z)$. The rational map $f : Z \mapsto P(\alpha_0 - \alpha_0/Z)$ has a simple zero at $Z = \infty$ and the remaining zeros are contained in the half-plane $\{Z \in \mathbb{P}^1 \mid \Re(Z) \geq 1/2\}$. Thus the barycentre β of those zeros satisfies $\Re(\beta) \geq 1/2$. Moreover, f has a critical point of multiplicity d at $Z = 0$. Thus, the barycentre of the d remaining critical points is 2β. Since $\Re(2\beta) \geq 1$, we see that f has at least one critical point ω contained in the half plane $\{Z \in \mathbb{P}^1 \mid \Re(Z) \geq 1\}$. Then, $\alpha_0 - \alpha_0/\omega$ is a critical point of P contained in the closed disk of diameter $[0, \alpha_0]$. ■

The second corollary has been proved by Videnskii [V]. Our result provides an alternate proof.

Corollary 2. *Assume $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map and $\Delta \subset \mathbb{P}^1$ is a closed disk or a closed half-plane containing all the zeros of f. Then, Δ contains at least one critical point of f.*

Proof. Without loss of generality, we may assume that the zeros are simple and that at least one zero, let us say α_0, is on the boundary of Δ. In a coordinate where $\alpha_0 = \infty$, Δ is a closed half-plane. The barycentre of the remaining zeros is contained in this half-plane. Consequently, the barycentre of the critical points is contained in Δ. Thus, Δ contains at least one critical point. ■

Videnskii also proved that this result is optimal in the sense that there exist rational maps of arbitrary degrees with simple zeros contained in a disk Δ but only one critical point in Δ.

References

[M] M. Marden, *Conjectures on the critical points of a polynomial*, Amer. Math. Monthly 90 (1983), no. 4, 267–276.

[V] I. Videnskii, *On the zeros of the derivative of a rational function and invariant subspaces for the backward shift operator on the Bergman space*, in preparation.

E-mail address: buff@picard.ups-tlse.fr

Université Paul Sabatier, Laboratoire Emile Picard, 118, route de Narbonne, 31062 Toulouse Cedex, France