On the size of linearization domains

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Abstract

Assume $f : U \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic map fixing 0 with derivative λ, where $0 < |\lambda| \leq 1$. If λ is not a root of unity, there is a formal power series $\phi_f(z) = z+\mathcal{O}(z^2)$ such that $\phi_f(\lambda z) = f\big(\phi_f(z)\big)$. This power series is unique and we denote by $\mathrm{R}_{\mathrm{conv}}(f) \in [0, +\infty]$ its radius of convergence. We denote by $\mathrm{R}_{\mathrm{geom}}(f)$ the largest radius $r \in \big[0, \mathrm{R}_{\mathrm{conv}}(f)\big]$ such that $\phi_f\big(D(0, r)\big) \subset U$. In this article, we present new elementary techniques for studying the maps $f \mapsto \mathrm{R}_{\mathrm{conv}}(f)$ and $f \mapsto \mathrm{R}_{\mathrm{geom}}(f)$. Contrary to previous approaches, our techniques do not involve studying the arithmetical properties of rotation numbers.

1. *Introduction*

Notations:

- $\mathbb{D} := \{z \in \mathbb{C}\ ;\ |z| < 1\}$,
- $\mathbb{U} := \{z \in \mathbb{C}\ ;\ |z| = 1\}$,
- $\overline{\mathbb{D}} := \{z \in \mathbb{C}\ ;\ |z| \leq 1\}$,
- $\mathbb{D}^* := \{z \in \mathbb{C}\ ;\ 0 < |z| \leq 1\}$,
- $\mathbb{U}^{\circledast} := \{z \in \mathbb{C}\ ;\ |z| = 1$ and z is not a root of unity$\}$,
- $\overline{\mathbb{D}}^{\circledast} := \{z \in \mathbb{C}\ ;\ 0 < |z| \leq 1$ and z is not a root of unity$\}$.

In the whole article, we assume that the map f is defined and holomorphic in a neighborhood of 0 with $f(0) = 0$ and $f'(0) = \lambda \in \overline{\mathbb{D}}^{\circledast}$. It is well known that when $\lambda \in \mathbb{D}^*$, the map f is linearizable: in a neighborhood of the origin, it is analytically conjugate to its linear part

$$L_\lambda : z \mapsto \lambda z.$$

If $\lambda \in \mathbb{U}^{\circledast}$, the map is formally linearizable, but the linearizing power series might fail to be convergent.

Definition 1. *If $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a holomorphic germ fixing 0 with derivative $\lambda \in \overline{\mathbb{D}}^{\circledast}$, we let $\phi_f(Z) \in \mathbb{C}[[Z]]$ be the unique formal power series $\phi_f(Z) = Z + \mathcal{O}(Z^2)$ such that*

$$\phi_f \circ L_\lambda = f \circ \phi_f.$$

We denote by $\mathrm{R}_{\mathrm{conv}}(f) \in [0, +\infty]$ its radius of convergence.

Our first result is the following. We shall prove it in section 2.

Theorem 1. *Suppose $(\lambda_n \in \mathbb{D})$ converges to $\lambda_0 \in \mathbb{U}^{\circledast}$ and $r > 0$. If the sequence $(f_n : D(0, r) \to \mathbb{C})$ with $f_n(0) = 0$ and $f_n'(0) = \lambda_n$ satisfies*

$$\sup_{|z|<r} \bigl|f_n(z) - \lambda_n z\bigr| \underset{n\to+\infty}{=} \mathcal{O}\bigl(1 - |\lambda_n|\bigr),$$

then:

- $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \ge r$ *and*
- *the sequence (ϕ_{f_n}) converges locally uniformly to the identity on $D(0, r)$.*

Definition 2. *Let $Q_\lambda : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial defined by*

$$Q_\lambda(z) = \lambda z(1 + z).$$

Our second result is the following. We shall prove it in section 4.

Theorem 2. *Assume $(\lambda_n \in \overline{\mathbb{D}}^{\circledast})$ converges to $\lambda_0 \in \mathbb{U}^{\circledast}$ with $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n}) > 0$ and $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$ and assume $r > 0$. If the sequence $f_n : D(0, r) \to \mathbb{C}$ with $f_n(0) = 0$ and $f_n'(0) = \lambda_n$ converges to the rotation L_{λ_0} locally uniformly on $D(0, r)$, then:*

$$\liminf_{n\to+\infty} \left(\frac{\mathrm{R}_{\mathrm{conv}}(f_n)}{r} \Big/ \frac{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})}{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0})} \right) \ge 1.$$

Remark. The condition $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n}) > 0$ is automatically satisfied if $\lambda_n \in \mathbb{D}^*$.

Remark. This theorem is a form of extremality property for the quadratic family Q_λ. In this article, we give a very short proof of this optimality, by an enhancement of Yoccoz's methods. In the case $\lambda_0 = \mathrm{e}^{i2\pi\theta_0}$ and $\lambda_n = e^{i2\pi\theta_n}$ with $\theta_0 \in \mathbb{R}$ and $\theta_n \in \mathbb{R}$, this optimality follows from earlier works [**Ri**], [**ABC**] and [**BC1**]. There, as $n \to +\infty$, one bounds $\mathrm{R}_{\mathrm{conv}}(f_n)$ from above and $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})$ from below, in terms of arithmetical properties of θ_0 and θ_n. Together, these results imply the aforementioned optimality. Here, we bypass the arithmetics.

Let us mention a selection of applications of our theorems.

Definition 3. *If $f : U \to \mathbb{C}$ is a holomorphic map defined in a neighborhood U of 0, fixing 0 with derivative $\lambda \in \overline{\mathbb{D}}^{\circledast}$, we set*

$$\mathrm{R}_{\mathrm{geom}}(f) := \sup \bigl\{ r \in [0, \mathrm{R}_{\mathrm{conv}}(f)] \bigm| \phi_f\bigl(D(0, r)\bigr) \subset U \bigr\}$$

and

$$\Delta_f := \phi_f\bigl(D\bigl(0, \mathrm{R}_{\mathrm{geom}}(f)\bigr)\bigr).$$

As we shall see in section 3, the map ϕ_f is univalent on $D\bigl(0, \mathrm{R}_{\mathrm{geom}}(f)\bigr)$, and so, $\mathrm{R}_{\mathrm{geom}}(f)$ is the *conformal radius* of Δ_f.

- If $\lambda\in\mathbb{D}^*$, the origin is an attracting fixed point which has a *basin of attraction* $\Omega_f := \big\{z\in U \,\big|\, f^{\circ n}(z)\to 0 \text{ as } n\to+\infty\big\}$ and $\Delta_f\subset\Omega_f$.
- If $\lambda\in\mathbb{U}^{⊛}$ and if $\mathrm{R}_{\mathrm{conv}}(f)>0$, then Δ_f is the largest domain in U containing 0 on which f is conjugate to the rotation L_λ. It is the *Siegel disk* of f.

Remark. If Δ_f is relatively compact in U, then $\mathrm{R}_{\mathrm{conv}}(f)=\mathrm{R}_{\mathrm{geom}}(f)$. In particular, this equality holds if $f:\mathbb{C}\to\mathbb{C}$ is a polynomial fixing 0 with derivative λ. For example $\mathrm{R}_{\mathrm{conv}}(Q_\lambda)=\mathrm{R}_{\mathrm{geom}}(Q_\lambda)$. Also $\mathrm{R}_{\mathrm{conv}}(f)=\mathrm{R}_{\mathrm{geom}}(f)$ if all boundary points of U are singularities of f.

We are interested in studying the properties of the map $f\mapsto\mathrm{R}_{\mathrm{geom}}(f)$, and for this purpose, it will be useful to study the properties of the map $f\mapsto\mathrm{R}_{\mathrm{conv}}(f)$. If the maps f_n and f_0 belong to an analytic family (as in the case of the family of quadratic polynomials Q_λ), we usually only have the Lipschitz inequality

$$\sup_{z\in K}\big|f_n(z)-f_0(z)\big| \underset{n\to+\infty}{=} \mathcal{O}\big(|\lambda_n-\lambda_0|\big)$$

for all compact subsets K of the domain of definition of f_0. In order to apply theorem 1, we must consider non-tangential limits.

Definition 4. *A sequence $(\lambda_n\in\mathbb{D})$ converges to $\lambda_0\in\mathbb{U}$ non-tangentially if the argument of $1-\lambda_n/\lambda_0$ remains in a compact subset of $]-\pi/2,\pi/2[$. Equivalently,*

$$|\lambda_n-\lambda_0| = \mathcal{O}\big(1-|\lambda_n|\big).$$

Using the later formulation we shall extend this notion to sequences $(\lambda_n\in\overline{\mathbb{D}}^{⊛})$ converging to λ_0. Thus, such sequences are allowed to take the value λ_0.

We have the following corollaries of theorem 1.

Corollary 1. *Suppose $(\lambda_n\in\mathbb{D})$ converges non-tangentially to $\lambda_0\in\mathbb{U}^{⊛}$ and $r>0$. If the sequence $(f_n:D(0,r)\to\mathbb{C})$ with $f_n(0)=0$, $f_n'(0)=\lambda_n$ satisfies*

$$\sup_{|z|<r}\big|f_n(z)-\lambda_0 z\big| \underset{n\to+\infty}{=} \mathcal{O}\big(|\lambda_0-\lambda_n|\big), \tag{1·1}$$

then:

- $\liminf_{n\to+\infty}\mathrm{R}_{\mathrm{conv}}(f_n)\geq r$ *and*
- *the sequence (ϕ_{f_n}) converges to the identity locally uniformly on $D(0,r)$.*

The following result was recently obtained by Gumenuk [**G**] with different techniques.

Corollary 2. *Suppose $(\lambda_n\in\mathbb{D})$ converges non-tangentially to $\lambda_0\in\mathbb{U}^{⊛}$ and suppose $f_0:U\to\mathbb{C}$ with $f_0(0)=0$, $f_0'(0)=\lambda_0$ has a Siegel disk Δ_{f_0}. If the sequence $(f_n:U\to\mathbb{C})$ satisfies $f_n(0)=0$, $f_n'(0)=\lambda_n$ and for every compact set $K\subset\Delta_{f_0}$*

$$\sup_{z\in K}\big|f_n(z)-f_0(z)\big| \underset{n\to+\infty}{=} \mathcal{O}\big(|\lambda_0-\lambda_n|\big), \tag{1·2}$$

then:

- $\lim_{n\to+\infty}\mathrm{R}_{\mathrm{geom}}(f_n)=\mathrm{R}_{\mathrm{geom}}(f_0)$,
- *the sequence (ϕ_{f_n}) converges to ϕ_{f_0} locally uniformly on $D\big(0,\mathrm{R}_{\mathrm{geom}}(f_0)\big)$ and*
- *any compact set $K\subset\Delta_{f_0}$ is contained in Δ_{f_n} for n large enough.*

The case $f_n = Q_{\lambda_n}$ was previously obtained by Yoccoz [**Y**]. Figure 1 shows the sets Δ_{Q_λ} for different values of $\lambda \in \overline{\mathbb{D}}^{\circledast}$. As $(\lambda_n \in \mathbb{D}^*)$ converges non-tangentially to $\lambda_0 \in \mathbb{U}$ with $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$, the sequence of pointed domains $(\Delta_{Q_{\lambda_n}}, 0)$ converges in the sense of Carathéodory to the pointed domain $(\Delta_{Q_{\lambda_0}}, 0)$.

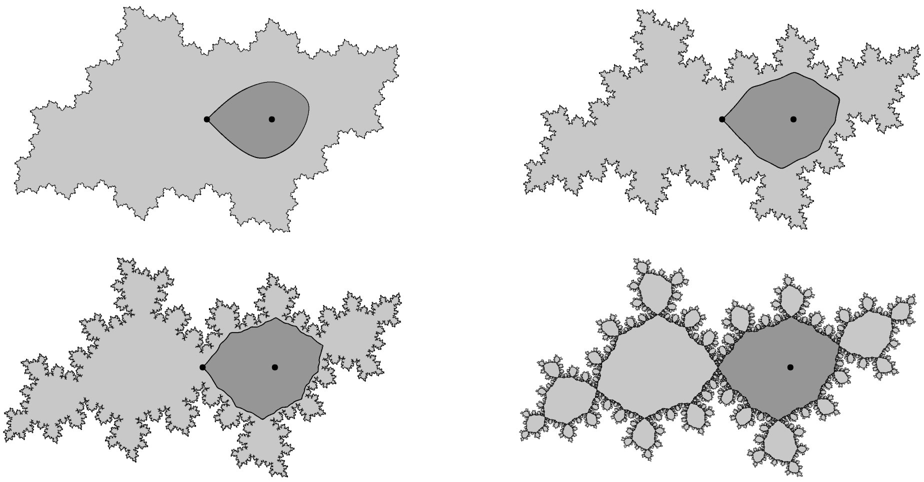

Fig. 1. The filled-in Julia sets of the polynomials $Q_{te^{i2\pi\theta}}$ for $\theta := \sqrt{2}$ and $t = .9$, $t = .99$, $t = .999$ and $t = 1$. We have colored dark grey the sets $\Delta_{Q_{te^{i2\pi\theta}}}$.

Let us now come to the corollaries of theorem 2. First, we recover the following theorem of Yoccoz [**Y**]. This is not surprising since our proof is largely inspired by the proof of Yoccoz.

Theorem (Yoccoz). *If $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$ and if $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ fixing 0 with derivative λ_0, then $\mathrm{R_{conv}}(f) > 0$.*

Second, under the assumption $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$, we can eliminate the conditions (1·1) and (1·2) in Corollaries 1 and 2.

Corollary 3. *Suppose $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$, suppose $(\lambda_n \in \overline{\mathbb{D}}^{\circledast})$ converges non-tangentially to λ_0 and suppose $r > 0$. If the sequence $(f_n : D(0, r) \to \mathbb{C})$ with $f_n(0) = 0$ and $f_n'(0) = \lambda_n$ converges to L_{λ_0} uniformly on $D(0, r)$, then:*

- $\liminf_{n\to+\infty} \mathrm{R_{conv}}(f_n) \geq r$ *and*
- *the sequence (ϕ_{f_n}) converges to the identity locally uniformly on $D(0, r)$.*

Recall that by Yoccoz's theorem, if $\lambda_0 \in \mathbb{U}$ is such that $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$ and if the map $f_0 : U \to \mathbb{C}$ satisfies $f_0(0) = 0$ and $f_0'(0) = \lambda_0$, then $\mathrm{R_{geom}}(f_0) > 0$ and f_0 has a Siegel disk Δ_{f_0}.

Corollary 4. *Suppose $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R_{conv}}(Q_{\lambda_0}) > 0$, suppose $(\lambda_n \in \mathbb{D})$ converges non-tangentially to λ_0 and suppose $f_0 : U \to \mathbb{C}$ satisfies $f_0(0) = 0$ and $f_0'(0) = \lambda_0$. If the sequence $(f_n : U \to \mathbb{C})$ with $f_n(0) = 0$ and $f_n'(0) = \lambda_n$ converges to f_0 locally uniformly in U, then:*

- $\lim_{n\to+\infty} \mathrm{R_{geom}}(f_n) = \mathrm{R_{geom}}(f_0)$,
- *the sequence (ϕ_{f_n}) converges to ϕ_{f_0} locally uniformly on $D\big(0, \mathrm{R_{geom}}(f_0)\big)$ and*

- *any compact set $K \subset \Delta_{f_0}$ is contained in Δ_{f_n} for n large enough.*

Fourth, the following corollary was obtained by Risler [**Ri**] with completely different techniques. His proof relies on a careful study of the arithmetic properties of θ where $\lambda_0 = e^{i2\pi\theta}$. In our proof, we do not use these arithmetic properties.

Corollary 5 (Risler). *Assume $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$ and $f_0 : U \to \mathbb{C}$, $(f_n : U \to \mathbb{C})$ satisfy $f_0(0) = f_n(0) = 0$, $f_0'(0) = f_n'(0) = \lambda_0$. If the sequence (f_n) converges to f_0 locally uniformly in U, then:*

- $\lim\limits_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) = \mathrm{R}_{\mathrm{geom}}(f_0)$,
- *the sequence (ϕ_{f_n}) converges to ϕ_{f_0} locally uniformly on $D\big(0, \mathrm{R}_{\mathrm{geom}}(f_0)\big)$ and*
- *any compact set $K \subset \Delta_{f_0}$ is contained in Δ_{f_n} for n large enough.*

Finally, we have the following consequence which can be found in [**C**].

Corollary 6. *Assume $\lambda_0 \in \mathbb{U}$ and let $\mathrm{Poly}_d(\lambda_0)$ be the set of polynomials of degree d fixing 0 with multiplier λ_0. If $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$, then $\mathrm{R}_{\mathrm{conv}} : \mathrm{Poly}_d(\lambda_0) \to]0, +\infty[$ is continuous.*

2. *A lower bound on the size of Siegel disks*

In this section, we shall prove Theorem 1. So, assume $(\lambda_n \in \mathbb{D})$ converges to $\lambda_0 \in \mathbb{U}^{\circledast}$ and $r > 0$. Let $(f_n : D(0,r) \to \mathbb{C})$ with $f_n(0) = 0$, $f_n'(0) = \lambda_n$ satisfy

$$\sup_{|z|<r} \big|f_n(z) - \lambda_n z\big| \underset{n\to+\infty}{=} \mathcal{O}\big(1 - |\lambda_n|\big).$$

We will prove that for every $\rho < r$, we have $\mathrm{R}_{\mathrm{conv}}(f_n) \geq \rho$ for n large enough.

Lemma 1. *For all $\rho < r$, there are integers k_0 and n_0 such that for all $n \geq n_0$,*

$$f_n^{\circ k_0}\big(D(0,\rho)\big) \subset D(0,\rho).$$

Proof. Set $s_n = 1 - |\lambda_n|$. For n large enough, we can write:

$$f_n(z) = \lambda_n z \exp\big(s_n \cdot u_n(z)\big)$$

with $\big(u_n : D(0,r) \to \mathbb{C}\big)$ a sequence of holomorphic functions defined by:

$$u_n(z) = \frac{1}{s_n} \log\left(1 + \left(\frac{f_n(z)}{\lambda_n z} - 1\right)\right).$$

The branch of logarithm is the one defined on $D(1,1)$, sending 1 to 0, so that $u_n(0) = 0$. In addition, our assumptions imply that the sequence (u_n) is uniformly bounded on $D(0,r)$.

Let us fix $\rho < r$ and choose $\rho' \in]\rho, 1[$. For each integer $k \geq 1$, there exists an integer n_k such that for all $n \geq n_k$, $f_n^{\circ k+1}$ is defined on $D(0,\rho')$. Then, for all $z \in D(0,\rho')$, we have:

$$f_n^{\circ k}(z) = \lambda_n^k z \exp\big(k s_n \cdot u_n^k(z)\big) \quad \text{with} \quad u_n^k = \frac{1}{k}\sum_{j=0}^{k-1} u_n \circ f_n^{\circ j}.$$

Note that $u_n^k(0) = u_n(0) = 0$ and

$$u_n^k \circ f_n = u_n^k + \frac{1}{k}\big(u_n \circ f_n^{\circ k} - u_n\big).$$

The family $\big(u_n^k : D(0,\rho') \to \mathbb{C}\big)_{k\geq 1,\ n\geq n_k}$ is uniformly bounded on $D(0,\rho')$. Let $u : D(0,\rho') \to \mathbb{C}$ be any limit value as $k \to +\infty$. Since $f_n \to L_{\lambda_0}$, we have:

$$u \circ L_{\lambda_0} = u.$$

The function u is constant on orbits under iteration of the rotation L_{λ_0}. Since λ_0 is not a root of unity, these orbits are not discrete in $D(0,\rho')$. Thus, $u : D(0,\rho') \to \mathbb{C}$ is constant. This constant is $u(0) = 0$. Thus, we have proved that as $k \to +\infty$, any limit value of the family $\big(u_n^k : D(0,\rho') \to \mathbb{C}\big)_{k\geq 1,\ n\geq n_k}$ is constant equal to 0.

Observe that $\dfrac{\log|\lambda_n^{-1}|}{s_n} \underset{n\to+\infty}{\longrightarrow} 1$. Thus, there are integers k_0 and n_0 such that

$$(\forall n \geq n_0)\ \big(\forall z \in D(0,\rho)\big) \qquad \big|u_n^{k_0}(z)\big| \leq \frac{1}{2}\frac{\log|\lambda_n^{-1}|}{s_n}.$$

We then easily derive that

$$(\forall n \geq n_0)\ \big(\forall z \in D(0,\rho)\big) \qquad |f_n^{\circ k_0}(z)| \leq |\lambda_n|^{k_0/2} \cdot |z| < |z|.$$

Thus,

$$(\forall n \geq n_0) \qquad f_n^{\circ k_0}\big(D(0,\rho)\big) \subset D(0,\rho).$$

□

Let us fix $\rho < r$ and choose k_0 such that for all sufficiently large n, we have the inclusion $f_n^{\circ k_0}\big(D(0,\rho)\big) \subset D(0,\rho)$. Then, $D(0,\rho)$ belongs to the basin of attraction of 0 and the map

$$\psi_n = \lim_{j\to+\infty} \frac{f_n^{\circ j}}{\lambda_n^j} : D(0,\rho) \to \mathbb{C}$$

linearizes f_n. We have $\psi_n(0) = 0$ and $\psi_n'(0) = 1$. In addition, ψ_n is a limit of univalent maps. It is therefore univalent and the family (ψ_n) is normal. Any limit value ψ linearizes the rotation L_{λ_0} and is therefore the identity. As a consequence, for n sufficiently large, the inverse of $\psi_n : D(0,\rho) \to \mathbb{C}$ is defined on a Euclidean disk centered at 0 with radius arbitrarily close to ρ. This inverse is ϕ_{f_n}. Thus, for all $\rho < r$,

$$\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \geq \rho.$$

The result now follows by letting ρ tend to r. Q.E.D.

3. *Corollaries of theorem* 1

Let us begin with some observations regarding $\mathrm{R}_{\mathrm{geom}}(f)$.

Lemma 2. *If $\lambda \in \overline{\mathbb{D}}^{\circledast}$ and $f : U \to \mathbb{C}$ is a holomorphic map fixing 0 with derivative λ, then $\phi_f : D\big(0, \mathrm{R}_{\mathrm{geom}}(f)\big) \to \Delta_f$ is an isomorphism.*

Proof. Since $\phi_f \circ L_\lambda = f \circ \phi_f$, we see that $f(\Delta_f) \subset \Delta_f$.

If $|\lambda| < 1$, the map $\psi_f := \lim\limits_{k\to+\infty} \lambda^{-k} f^{\circ k}(z)$ is defined on Δ_f, fixes 0 with derivative 1 and satisfies $\psi_f \circ f = L_\lambda \circ \psi_f$.

If $|\lambda| = 1$, the sequence $(\psi_m : \Delta_f \to \mathbb{C})_{m\geq 1}$ defined by $\psi_m := \dfrac{1}{m}\displaystyle\sum_{k=0}^{m-1} \lambda^{-k} f^{\circ k}$ is uniformly bounded. Any limit value $\psi_f : \Delta_f \to \mathbb{C}$ fixes 0 with derivative 1 and satisfies $\psi_f \circ f = L_\lambda \circ \psi_f$.

In both cases, $\psi_f \circ \phi_f$ fixes 0 with derivative 1 and commutes with L_λ. It follows that $\psi_f \circ \phi_f$ is equal to the identity near 0, thus on $D\big(0, \mathrm{R}_{\mathrm{geom}}(f)\big)$ by analytic continuation. So, $\phi_f : D\big(0, \mathrm{R}_{\mathrm{geom}}(f)\big) \to \Delta_f$ is an isomorphism with inverse ψ_f. $\square$

Lemma 3. *Assume* $(f_n : U \to \mathbb{C})$ *converges to* $f_0 : U \to \mathbb{C}$ *locally uniformly, assume* $f_0(0) = f_n(0) = 0$ *and suppose* $\lambda_0 := f_0'(0) \in \overline{\mathbb{D}}^{\circledast}$ *and* $\lambda_n := f_n'(0) \in \overline{\mathbb{D}}^{\circledast}$. *Then,*

$$\limsup_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \le \mathrm{R}_{\mathrm{geom}}(f_0).$$

Proof. Set $\rho := \limsup \mathrm{R}_{\mathrm{geom}}(f_n)$ and assume, extracting a subsequence if necessary, that $\mathrm{R}_{\mathrm{geom}}(f_n) \to \rho$. For any $r < \rho$, if n large enough, the maps ϕ_{f_n} are defined and univalent on $D(0, r)$ and take their values in U. They fix 0 with derivative 1, and thus, form a normal family. Passing to the limit on the linearizing equation $\phi_{f_n} \circ L_{\lambda_n} = f_n \circ \phi_{f_n}$, we see that any limit value $\phi : D(0, r) \to U$ linearizes f_0 :

$$\phi \circ L_{\lambda_0} = f_0 \circ \phi.$$

Since $\phi(0) = 0$ and $\phi'(0) = 1$, we see that ϕ is a restriction of ϕ_{f_0} and since $\phi\big(D(0, r)\big) \subset U$, we see that $r \le \mathrm{R}_{\mathrm{geom}}(f_0)$. Letting r tend to ρ, we see that

$$\limsup_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \le \mathrm{R}_{\mathrm{geom}}(f_0).$$

$\square$

Lemma 4. *Assume* $(f_n : U \to \mathbb{C})$ *converges to* $f_0 : U \to \mathbb{C}$ *locally uniformly, assume* $f_0(0) = f_n(0) = 0$ *and suppose* $\lambda_0 := f_0'(0) \in \mathbb{D}$ *and* $\lambda_n := f_n'(0) \in \mathbb{D}$. *Then,*

$$\lim_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) = \mathrm{R}_{\mathrm{geom}}(f_0).$$

Proof. It suffices to show that $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \ge \mathrm{R}_{\mathrm{geom}}(f_0)$. Let $r < \mathrm{R}_{\mathrm{geom}}(f_0)$ be arbitrary and set $\Delta' := \phi_{f_0}\big(D(0, r)\big)$. If n is large enough, then $f_n(\Delta')$ is relatively compact in Δ' and the linearizing map $\psi_{f_n} := \lim_{k\to+\infty} \lambda_n^{-k} f_n^{\circ k}$ is defined on Δ'. Since $f_0 : \Delta_{f_0} \to \mathbb{C}$ is univalent, for n large enough, the maps $f_n : \Delta' \to \mathbb{C}$ are univalent. As limits of univalent maps, the maps $\psi_{f_n} : \Delta' \to \mathbb{C}$ are also univalent. They fix 0 with derivative 1 and thus, belong to a normal family. Any limit value coincides with $\psi_{f_0} : \Delta' \to D(0, r)$. It follows that $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \ge r$. The result follows by letting r tend to $\mathrm{R}_{\mathrm{geom}}(f_0)$. $\square$

Remark. This establishes the continuity of the map $f \mapsto \mathrm{R}_{\mathrm{geom}}(f)$ at f_0 when f_0 has an attracting fixed point at 0.

We now come to the proof of our corollaries of theorem 1.

Corollary 1. *Suppose* $(\lambda_n \in \mathbb{D})$ *converges non-tangentially to* $\lambda_0 \in \mathbb{U}^{\circledast}$ *and* $r > 0$. *If the sequence* $(f_n : D(0, r) \to \mathbb{C})$ *with* $f_n(0) = 0$, $f_n'(0) = \lambda_n$ *satisfies*

$$\sup_{|z|<r} \big|f_n(z) - \lambda_0 z\big| \underset{n\to+\infty}{=} \mathcal{O}\big(|\lambda_0 - \lambda_n|\big),$$

then:

- $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \geq r$ *and*
- *the sequence* (ϕ_{f_n}) *converges to the identity locally uniformly on* $D(0,r)$.

Proof. If $\lambda_n \to \lambda_0$ non tangentially, then $|\lambda_n - \lambda_0| = \mathcal{O}\big(1-|\lambda_n|\big)$, and thus

$$\sup_{|z|<r}\big|f_n(z)-\lambda_n z\big| = \sup_{|z|<r}\big|f_n(z)-\lambda_0 z\big| + \mathcal{O}\big(|\lambda_0-\lambda_n|\big) = \mathcal{O}\big(1-|\lambda_n|\big).$$

Now, apply theorem 1. □

Corollary 2. *Suppose* $(\lambda_n \in \mathbb{D})$ *converges non-tangentially to* $\lambda_0 \in \mathbb{U}^{\circledast}$ *and suppose* $f_0 : U \to \mathbb{C}$ *with* $f_0(0)=0$, $f_0'(0)=\lambda_0$ *has a Siegel disk* Δ_{f_0}. *If the sequence* $(f_n : U \to \mathbb{C})$ *satisfies* $f_n(0) = f_0(0) = 0$, $f_n'(0) = \lambda_n$ *and for every compact set* $K \subset \Delta_{f_0}$

$$\sup_{z\in K}\big|f_n(z)-f_0(z)\big| \underset{n\to+\infty}{=} \mathcal{O}\big(|\lambda_0-\lambda_n|\big),$$

then:

- $\lim_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) = \mathrm{R}_{\mathrm{geom}}(f_0)$,
- *the sequence* (ϕ_{f_n}) *converges to* ϕ_{f_0} *locally uniformly on* $D\big(0,\mathrm{R}_{\mathrm{geom}}(f_0)\big)$ *and*
- *any compact set* $K \subset \Delta_{f_0}$ *is contained in* Δ_{f_n} *for* n *large enough.*

Proof. By lemma 3, $\limsup_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \leq \mathrm{R}_{\mathrm{geom}}(f_0)$. Thus, it is enough to prove that $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \geq \mathrm{R}_{\mathrm{geom}}(f_0)$. Let $\phi_{f_0} : D\big(0,\mathrm{R}_{\mathrm{geom}}(f_0)\big) \to U$ be the linearizing map of f_0 fixing 0 with derivative 1. Set

$$g_n := \phi_{f_0}^{-1} \circ f_n \circ \phi_{f_0}.$$

For any $r < \mathrm{R}_{\mathrm{geom}}(f_0)$, the map g_n is eventually defined on $D(0,r)$ and the sequence of maps $\big(g_n : D(0,r) \to \mathbb{C}\big)$ converges uniformly to the rotation L_{λ_0}. More precisely,

$$\sup_{z\in D(0,r)}\big|g_n(z)-\lambda_0 z\big| = \sup_{w\in\phi_{f_0}(D(0,r))}\Big|\psi_{f_0}\big(f_n(w)\big)-\psi_{f_0}\big(f_0(w)\big)\Big| = \mathcal{O}\big(|\lambda_0-\lambda_n|\big).$$

By corollary 1 , $\liminf \mathrm{R}_{\mathrm{conv}}(g_n) \geq r$ and the sequence (ϕ_{g_n}) converges uniformly on every compact subset of $D(0,r)$ to the identity. Using $\phi_{f_n} = \phi_{f_0} \circ \phi_{g_n}$ we see that for all $r' < r$, if n is large enough, ϕ_{f_n} is defined on $D(0,r')$ and takes its values in U and thus, $\mathrm{R}_{\mathrm{geom}}(f_n) \geq r'$. In addition, any compact subset of $\phi_{f_0}\big(D(0,r)\big)$ is eventually contained in $\phi_{f_n}\big(D(0,r)\big)$. The result follows easily by letting r tend to $\mathrm{R}_{\mathrm{geom}}(f_0)$. □

4. *Perturbations of a Brjuno rotation*

4·1. *A key proposition*

Recall that a quadratic-like map $g : U \to V$ is a proper map of degree 2 between two simply connected domains U and V with U compactly contained in V (see for example [**DH**]).

Proposition 1. *Assume* $f : \mathbb{D} \to \mathbb{C}$ *and* $g : \mathbb{D} \to \mathbb{C}$ *are holomorphic maps fixing* 0 *with multiplier* $\lambda \in \overline{\mathbb{D}}^{\circledast}$. *If* g *is a quadratic-like map and*

$$\sup_{|z|<1}\big|f(z)-\lambda z\big| < \liminf_{|z|\to 1}\big|g(z)\big| - 1. \tag{4·1}$$

Then, $\mathrm{R}_{\mathrm{conv}}(f) \geq \mathrm{R}_{\mathrm{conv}}(g)$.

Proof. The proof to which the rest of this subsection is devoted, is a modification of a proof exposed in [**BC2**], which itself is a modification of a proof of Yoccoz [**Y**]. The essential steps are formulated as lemmas imbedded in the proof.

Without loss of generality, we may assume that $\mathrm{R}_{\mathrm{conv}}(g) > 0$ since otherwise, the result is trivial. Consider the one-parameter families of maps

$$\big\{f_a : D\big(0, \min(1, |a|^{-1})\big) \to \mathbb{C}\big\}_{a\in\mathbb{C}} \quad \text{and} \quad \big\{g_b : D\big(0, \min(1, |b|^{-1})\big) \to \mathbb{C}\big\}_{b\in\mathbb{C}}$$

defined by:

$$f_a(z) = f(z) - \lambda z + \frac{1}{a}g(az) \quad \text{and} \quad g_b(w) = \frac{1}{b}f_{1/b}(bw) = \frac{1}{b}f(bw) - \lambda w + g(w).$$

The family f_a extends analytically at $a = 0$ by $f_0 = f$ and the family g_b extends analytically at $b = 0$ by $g_0 = g$. We have:

$$(\forall b \in \mathbb{C}^*) \quad \mathrm{R}_{\mathrm{conv}}(g_b) = \frac{1}{|b|}\,\mathrm{R}_{\mathrm{conv}}(f_{1/b}).$$

Set

$$U_b := \big\{z \in \mathbb{D} \,\big|\, g_b(z) \in \mathbb{D}\big\}.$$

Lemma 5. *When $|b| \le 1$, the restriction $g_b : U_b \to \mathbb{D}$ is quadratic-like.*

Proof. Since $g_0 : \mathbb{D} \to \mathbb{C}$ is quadratic-like, $U_0 = g_0^{-1}(\mathbb{D})$ is compactly contained in $\mathbb{D}$ and every point in $\mathbb{D}$ has exactly two preimages by g_0 in $\mathbb{D}$, counting multiplicities. Schwarz's lemma and Rouché's theorem implies that for $|b| \le 1$, U_b is compactly contained in $\mathbb{D}$ and every point in $\mathbb{D}$ has exactly two preimages by g_b in $\mathbb{D}$, counting multiplicities. It follows that $g_b : U_b \to \mathbb{D}$ is a proper map of degree 2. Note that U_b is connected since otherwise, the component of U_b containing 0 would be mapped biholomorphically to $\mathbb{D}$, which, by Schwarz's lemma, is not possible since $|\lambda| \le 1$. □

This lemma has the following consequence.

Lemma 6. *The map $b \mapsto \log \mathrm{R}_{\mathrm{conv}}(g_b)$ is harmonic in a neighborhood of $\overline{\mathbb{D}}$.*

Proof. If $|\lambda| < 1$, we may argue as follows. We can find $r_0 > 1$ so that when $|b| < r_0$, the map $g_b : U_b \to \mathbb{D}$ is a quadratic-like map. It has an attracting fixed point at 0 with multiplier λ. Let ω_b be the unique critical point of g_b in U_b, let Ω_b be the basin of attraction of 0 and let $\psi_b : \Omega_b \to \mathbb{C}$ be the linearizing map defined by: $\psi_b = \lim_{n\to+\infty} \lambda^{-n} g_b^{\circ n}$. Then, $\phi_{g_b} : D\big(0, \mathrm{R}_{\mathrm{conv}}(g_b)\big) \to \mathbb{D}$ is univalent with inverse ψ_b and $\mathrm{R}_{\mathrm{conv}}(g_b) = \big|\psi_b(\omega_b)\big|$. The map $b \mapsto \psi_b(\omega_b)$ is holomorphic in $D(0, r_0)$ and so, the map $b \mapsto \log \mathrm{R}_{\mathrm{conv}}(g_b)$ is harmonic in $D(0, r_0)$.

If $|\lambda| = 1$, we may argue as follows. We can find $r_0 > 1$ and $t_0 < 1$ so that when $t_0 < t \le 1$ and $|b| < r_0$, the map $g_{b,t} : U_b \to D(0,t)$ defined by

$$g_{b,t}(w) = t \cdot g_b(w)$$

is a quadratic-like map. This quadratic-like map has an attracting fixed point at 0 with multiplier $t\lambda$ converging non tangentially to λ as $t \to 1$. As previously,

$$\mathrm{R}_{\mathrm{conv}}(g_{b,t}) = \big|\psi_{b,t}(\omega_{b,t})\big|.$$

with $\omega_{b,t}$ the unique critical point of $g_{b,t}$ in U_b and $\psi_{b,t} : \Omega_{b,t} \to \mathbb{C}$ the linearizing map defined by $\psi_{b,t} = \lim_{n\to+\infty} (t\lambda)^{-n} g_{b,t}^{\circ n}$.

Consider the family of holomorphic maps

$$\left\{u_t : D(0,r_0) \to \mathbb{D}^*\right\}_{t_0<t<1}$$

defined by:

$$u_t(b) = \psi_{b,t}(\omega_{b,t}).$$

This family is normal and there is a sequence $t_n \in\ ]t_0,1[$ converging to 1 such that the sequence (u_{t_n}) converges uniformly on every compact subset of $D(0,r_0)$ to a limit $u_1 : D(0,r_0) \to \overline{\mathbb{D}}$ which either identically vanishes, or does not vanish.

By Corollary 2 and Lemma 4, we have[1]

$$\forall b \in D(0,r_0) \quad \log \mathrm{R}_{\mathrm{conv}}(g_b) = \log\bigl|u_1(b)\bigr|.$$

Since we assumed that $\mathrm{R}_{\mathrm{conv}}(g_0) = \mathrm{R}_{\mathrm{geom}}(g) > 0$, u_1 does not vanish on $D(0,r_0)$ and $b \mapsto \log \mathrm{R}_{\mathrm{conv}}(g_b)$ is harmonic in $D(0,r_0)$. $\square$

Let $\operatorname*{avg}_{|a|=r} m(a)$ denote the average of the function $m(a)$ on the circle $|a| = r$ (with respect to the Lebesgue measure on the circle). As an immediate consequence of lemma 6, we have the following equality:

$$\log \mathrm{R}_{\mathrm{conv}}(g) = \operatorname*{avg}_{|b|=1} \log \mathrm{R}_{\mathrm{conv}}(g_b) = \operatorname*{avg}_{|a|=1} \log \mathrm{R}_{\mathrm{conv}}(f_a). \tag{4·2}$$

The following result is extracted from [**BC2**]. We include it here for completeness.

Lemma 7. $\log \mathrm{R}_{\mathrm{conv}}(f) \geq \operatorname*{avg}_{|a|=1} \log \mathrm{R}_{\mathrm{conv}}(f_a)$

Proof. Look at the formal linearizing power series of f_a:

$$\phi_{f_a}(Z) = Z + \sum_{n=2}^{+\infty} b_n(a) Z^n.$$

By Hadamard's theorem,

$$\frac{1}{\mathrm{R}_{\mathrm{conv}}(f_a)} = \limsup_{n\to+\infty} \sqrt[n]{|b_n(a)|}.$$

The coefficients $b_n(a)$ are polynomials in a. Thus

$$\frac{1}{n}\log\bigl|b_n(0)\bigr| \leq \operatorname*{avg}_{|a|=1} \frac{1}{n}\log\bigl|b_n(a)\bigr|.$$

By lemma 5, for $|a| = 1$, the map f_a has a quadratic-like restriction $f_a : \frac{1}{a}U_{1/a} \to \mathbb{D}$. In that case, the linearizing map ϕ_{f_a} takes its values in $\mathbb{D}$ and it follows from the Cauchy inequalities that

$$\bigl|b_n(a)\bigr| \leq \frac{1}{\bigl(\mathrm{R}_{\mathrm{conv}}(f_a)\bigr)^n}.$$

Recall that $b = 1/a$. We have seen that $b \mapsto \mathrm{R}_{\mathrm{conv}}(g_b)$ is continuous and non-vanishing on $\overline{\mathbb{D}}$. Thus, when $|a| = 1$, $\mathrm{R}_{\mathrm{conv}}(f_a) = \mathrm{R}_{\mathrm{conv}}(g_b)$ reaches a minimum $c > 0$ and

$$\frac{1}{n}\log\bigl|b_n(a)\bigr| \leq \log\frac{1}{\mathrm{R}_{\mathrm{conv}}(f_a)} \leq \log\frac{1}{c}.$$

[1] Note that since the maps $g_b : U_b \to \mathbb{D}$ and $g_{b,t} : U_b \to D(0,t)$ are quadratic-like, we have $\mathrm{R}_{\mathrm{conv}}(g_b) = \mathrm{R}_{\mathrm{geom}}(g_b)$ and $\mathrm{R}_{\mathrm{conv}}(g_{b,t}) = \mathrm{R}_{\mathrm{geom}}(g_{b,t})$.

This uniform upper bound allows us to apply Fatou's lemma:

$$\begin{aligned}-\log \mathrm{R}_{\mathrm{conv}}(f) &= \limsup_{n\to+\infty} \frac{1}{n}\log|b_n(0)| \le \limsup_{n\to+\infty} \operatorname*{avg}_{|a|=1} \frac{1}{n}\log|b_n(a)|\\ &\le \operatorname*{avg}_{|a|=1} \limsup_{n\to+\infty} \frac{1}{n}\log|b_n(a)| = -\operatorname*{avg}_{|a|=1} \log \mathrm{R}_{\mathrm{conv}}(f_a).\end{aligned}$$

□

Equality 4·2 and lemma 7 yield:

$$\log \mathrm{R}_{\mathrm{conv}}(f) \ge \operatorname*{avg}_{|a|=1} \log \mathrm{R}_{\mathrm{conv}}(f_a) = \log \mathrm{R}_{\mathrm{conv}}(g),$$

whence $\mathrm{R}_{\mathrm{conv}}(f) \ge \mathrm{R}_{\mathrm{conv}}(g)$. This completes the proof of Proposition 1. □

4·2. *Proof of theorem* 2

Our standing assumption is that $\lambda_0 \in \mathbb{U}$ is such that $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$. In that case, we say that the rotation L_{λ_0} is a Brjuno rotation.[2]

We assume $\big(f_n : D(0,r) \to \mathbb{C}\big)_{n\ge 0}$ is a sequence of maps such that

- $f_n(0) = 0$,
- $f_n'(0) = \lambda_n \in \overline{\mathbb{D}}^{\circledast}$ and
- the sequence (f_n) converges to the rotation L_{λ_0} uniformly on every compact subset of $D(0,r)$.

We want to show that

$$\liminf_{n\to+\infty} \left(\frac{\mathrm{R}_{\mathrm{conv}}(f_n)}{r} \Big/ \frac{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})}{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0})} \right) \ge 1.$$

Let K_λ be the filled-in Julia set of Q_λ and $\mathrm{böt}_\lambda : \mathbb{C}\setminus K_\lambda \to \mathbb{C}\setminus\overline{\mathbb{D}}$ be the Böttcher coordinate conjugating Q_λ to $z\mapsto z^2$. Let $G_\lambda : \mathbb{C} \to [0,+\infty[$ be the Green function of Q_λ:

$$G_\lambda(z) := \begin{cases} 0 & \text{if } z \in K_\lambda \\ \log\big|\mathrm{böt}_\lambda(z)\big| & \text{if } z \in \mathbb{C}\setminus K_\lambda. \end{cases}$$

For $\eta > 0$, set

$$U_{\lambda,\eta} := \big\{ z \in \mathbb{C} \,\big|\, G_\lambda(z) < \eta \big\}.$$

By Schwarz's reflection principle, any conformal representation $\chi_{\lambda,\eta} : U_{\lambda,\eta} \to \mathbb{D}$ sending 0 to 0 extends univalently to $U_{\lambda,2\eta}$. Indeed, if $\mathrm{e}^\eta \le |z| < \mathrm{e}^{2\eta}$, then $1 < \big|\mathrm{e}^{2\eta}/\bar z\big| \le \mathrm{e}^\eta$ and we can set

$$\chi_{\lambda,\eta}\left(\mathrm{böt}_\lambda^{-1}(z)\right) = s \circ \chi_{\lambda,\eta} \circ \mathrm{böt}_\lambda^{-1}\left(\frac{\mathrm{e}^{2\eta}}{\bar z}\right) \quad \text{with} \quad s(z) = \frac{1}{\bar z}.$$

Denote by $V_{\lambda,\eta}$ the image of the extended map $\chi_{\lambda,\eta} : U_{\lambda,2\eta} \to \mathbb{C}$. The isomorphism $\chi_{\lambda,\eta} : U_{\lambda,2\eta} \to V_{\lambda,\eta}$ conjugates $Q_\lambda : U_{\lambda,\eta} \to U_{\lambda,2\eta}$ to a quadratic-like map $g_{\lambda,\eta} : \mathbb{D} \to V_{\lambda,\eta}$.

[2] Yoccoz [**Y**] proved that $Q_{\mathrm{e}^{i2\pi\theta}}$ is linearizable if and only if θ satisfies an arithmetic condition known as the Brjuno condition.

The modulus of the annulus $V_{\lambda,\eta} \setminus \overline{\mathbb{D}}$ is equal to $\eta/(2\pi)$ and thus, there is a constant $c(\eta) > 0$ such that

$$(\forall \lambda \in \overline{\mathbb{D}}) \quad \liminf_{|z|\to 1} \left|g_{\lambda,\eta}(z)\right| > 1 + c(\eta).$$

If $U \subset \mathbb{C}$ is a simply connected domain containing 0, we denote by $\mathrm{Rad}(U)$ its conformal radius at 0, i.e. $\mathrm{Rad}(U) = \left|1/\chi'(0)\right|$ for any isomorphism $\chi : U \to \mathbb{D}$ fixing 0.

Now, given $\rho < r$, if n is large enough, the map $F_n : z \mapsto \rho^{-1} f_n(\rho z)$ is defined and holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and

$$\sup_{|z|=1} \left|F_n(z) - \lambda_n z\right| < c(\eta).$$

By proposition 1, we see that for all $\eta > 0$ and all $\rho < r$, if n is large enough,

$$\frac{\mathrm{R}_{\mathrm{conv}}(f_n)}{\rho} = \mathrm{R}_{\mathrm{conv}}(F_n) \geq \mathrm{R}_{\mathrm{conv}}(g_{\lambda_n,\eta}) = \frac{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})}{\mathrm{Rad}(U_{\lambda_n,\eta})}.$$

As n tends to infinity, $\mathrm{Rad}(U_{\lambda_n,\eta})$ tends to $\mathrm{Rad}(U_{\lambda_0,\eta})$. So, this can be rewritten as

$$(\forall \eta > 0)\ (\forall \rho < r) \quad \liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \cdot \frac{\mathrm{Rad}(U_{\lambda_0,\eta})}{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})} \geq \rho.$$

As η tends to 0, $\mathrm{Rad}(U_{\lambda_0,\eta})$ tends to $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0})$. So, letting η tend to 0 and ρ tend to r, we obtain the required result:

$$\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \cdot \frac{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0})}{\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n})} \geq r.$$

This completes the proof of theorem 2.

5. *Corollaries of theorem* 2

Let us begin with Yoccoz's theorem.

Theorem (Yoccoz). *If $\lambda_0 \in \mathbb{U}$ is such that $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$ and if $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ fixing 0 with derivative λ_0, then $\mathrm{R}_{\mathrm{conv}}(f) > 0$.*

Proof. Choose $\lambda_n = \lambda_0$ and define $f_n : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ by

$$f_n(z) = \frac{1}{r_n} f(r_n z)$$

where (r_n) is a sequence converging to 0. Then, (f_n) converges locally uniformly in $\mathbb{C}$ to the rotation L_{λ_0}. Therefore,

$$\liminf_{n\to+\infty} \frac{\mathrm{R}_{\mathrm{conv}}(f)}{r_n} = \liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) = +\infty.$$

Thus, $\mathrm{R}_{\mathrm{conv}}(f) > 0$. □

Let us now come to the proofs of our corollaries of theorem 2.

Corollary 3. *Suppose $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$, suppose $(\lambda_n \in \overline{\mathbb{D}}^{\circledast})$ converges non-tangentially to λ_0 and supose $r > 0$. If the sequence $(f_n : D(0, r) \to \mathbb{C})$ with $f_n(0) = 0$ and $f'_n(0) = \lambda_n$ converges to L_{λ_0} uniformly on $D(0, r)$, then:*

- $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \geq r$ *and*

- *the sequence (ϕ_{f_n}) converges to the identity locally uniformly on $D(0,r)$.*

Proof. Assume $(\lambda_n \in \overline{\mathbb{D}}^{\circledast})$ converges non tangentially to λ_0. According to Corollary 2, $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_n}) = \mathrm{R}_{\mathrm{geom}}(Q_{\lambda_n})$ converges to $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) = \mathrm{R}_{\mathrm{geom}}(Q_{\lambda_0})$. Thus, by Theorem 2, $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{conv}}(f_n) \geq r$.

The space of univalent maps on $D(0,r)$, fixing the origin with derivative 1, is compact. The map ϕ_{f_n} is univalent on $D\big(0, \mathrm{R}_{\mathrm{geom}}(f_n)\big)$ and fixes the origin with derivative 1. We will show that $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \geq r$. Then, any limit map of the sequence (ϕ_{f_n}) is the identity on $D(0,r)$, because it conjugates L_{λ_0} to itself. Thus, the whole sequence (ϕ_{f_n}) converges to the identity locally uniformly on $D(0,r)$.

To see $\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \geq r$, choose a function g holomorphic in $D(0,r)$ such that all points of modulus r are singularities of g. By upper semicontinuity of $\mathrm{R}_{\mathrm{geom}}$ (Lemma 3), if ε is small enough, then $\mathrm{R}_{\mathrm{geom}}(f_n + \varepsilon z^2 g) \leq \mathrm{R}_{\mathrm{geom}}(f_n) + 1/n$ and for all but countably many values of ε all points of modulus r are singularities of $f_n + \varepsilon z^2 g$. Thus we can choose a sequence (ε_n) converging to 0 such that $g_n = f_n + \varepsilon_n z^2 g$ satisfies:

(i) $g_n \to L_\lambda$ locally uniformly in $D(0,r)$.
(ii) $\mathrm{R}_{\mathrm{geom}}(g_n) = \mathrm{R}_{\mathrm{conv}}(g_n)$ by the Remark following Definition 3.
(iii) $\mathrm{R}_{\mathrm{geom}}(g_n) \leq \mathrm{R}_{\mathrm{geom}}(f_n) + 1/n$.

Hence

$$\liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) \geq \liminf_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(g_n) = r,$$

where the inequality follows from (iii) and the equality follows by combining Theorem 2 and (ii). □

Corollary 4. *Suppose $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$, suppose $(\lambda_n \in \mathbb{D})$ converges non-tangentially to λ_0 and suppose $f_0 : U \to \mathbb{C}$ satisfies $f_0(0) = 0$ and $f_0'(0) = \lambda_0$. If the sequence $(f_n : U \to \mathbb{C})$ with $f_n(0) = 0$ and $f_n'(0) = \lambda_n$ converges to f_0 locally uniformly in U, then:*

- $\lim_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) = \mathrm{R}_{\mathrm{geom}}(f_0)$,
- *the sequence (ϕ_{f_n}) converges to ϕ_{f_0} locally uniformly on $D\big(0, \mathrm{R}_{\mathrm{geom}}(f_0)\big)$ and*
- *any compact set $K \subset \Delta_{f_0}$ is contained in Δ_{f_n} for n large enough.*

Proof. Similar to the proof of Corollary 2, Corollary 1 being replaced by Corollary 3. □

Corollary 5 (Risler). *Assume $\lambda_0 \in \mathbb{U}$ satisfies $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$ and $f_0 : U \to \mathbb{C}$, $(f_n : U \to \mathbb{C})$ satisfy $f_0(0) = f_n(0) = 0$, $f_0'(0) = f_n'(0) = \lambda_0$. If the sequence (f_n) converges to f_0 locally uniformly in U, then:*

- $\lim_{n\to+\infty} \mathrm{R}_{\mathrm{geom}}(f_n) = \mathrm{R}_{\mathrm{geom}}(f_0)$,
- *the sequence (ϕ_{f_n}) converges to ϕ_{f_0} locally uniformly on $D\big(0, \mathrm{R}_{\mathrm{geom}}(f_0)\big)$ and*
- *any compact set $K \subset \Delta_{f_0}$ is contained in Δ_{f_n} for n large enough.*

Proof. Similar to the proof of Corollary 4.

Corollary 6. *Assume $\lambda_0 \in \mathbb{U}$ and let $\mathrm{Poly}_d(\lambda_0)$ be the set of polynomials of degree d fixing 0 with multiplier λ_0. If $\mathrm{R}_{\mathrm{conv}}(Q_{\lambda_0}) > 0$, then $\mathrm{R}_{\mathrm{conv}} : \mathrm{Poly}_d(\lambda_0) \to]0, +\infty[$ is continuous.*

Proof. By Yoccoz's theorem, $\mathrm{R}_{\mathrm{conv}}(P) > 0$ for all $P \in \mathrm{Poly}_d(\lambda_0)$. The linearizing maps $\phi_P : D\bigl(0, \mathrm{R}_{\mathrm{conv}}(P)\bigr) \to \mathbb{C}$ are univalent and take their values in the filled-in Julia set of P: they are conformal isomorphisms between the disk $D\bigl(0, \mathrm{R}_{\mathrm{conv}}(P)\bigr)$ and the Siegel disk Δ_P. It follows that $\mathrm{R}_{\mathrm{conv}}(P) < +\infty$ for all $P \in \mathrm{Poly}_d(\lambda_0)$. The result now follows from corollary 5 since $\mathrm{R}_{\mathrm{conv}}(P) = \mathrm{R}_{\mathrm{geom}}(P)$. $\square$

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