ON KÖNIG'S ROOT-FINDING ALGORITHMS.

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Abstract. In this article, we first recall the definition of a family of root-finding algorithms known as König's algorithms. We establish some local and some global properties of those algorithms. We give a characterization of rational maps which arise as König's methods of polynomials with simple roots. We then estimate the number of non-repelling cycles König's methods of polynomials may have. We finally study the geometry of the Julia sets of König's methods of polynomials and produce pictures of parameter spaces for König's methods of cubic polynomials.

Introduction.

In this article, we study a family of root-finding algorithms known as König's algorithms. To our understanding, those algorithms are actually due to Schröder [Sch].

Definition 1. *Assume $f : U \subset \mathbb{C} \to \mathbb{P}^1$ is a meromorphic map and $\sigma \geq 2$ is an integer. The König's method of f of order σ is the meromorphic map $K_{f,\sigma} : U \to \mathbb{P}^1$ defined by the formula:*

$$K_{f,\sigma} = \mathrm{Id} + (\sigma - 1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}},$$

where $(1/f)^{[k]}$ is the k-th derivative of $1/f$.

For $\sigma = 2$, the map $K_{f,\sigma}$ is the Newton's method of f and for $\sigma = 3$, the map $K_{f,\sigma}$ is the Halley's method of f – this is a projectively natural Newton's method (see [Sa], [ST] or [YB] for example). We will see that for each integer $\sigma \geq 2$, each root of f is a (super)attracting fixed point of $K_{f,\sigma}$ and that generically, the local degree of $K_{f,\sigma}$ at such a fixed point α is σ. One says that $K_{f,\sigma}$ is a root-finding algorithm of order σ.

In Sect. 1, we collect some preliminary results. Some are known, some are new (compare with [ABD], [Dr], [Kn], [Ho1] Sect. 4.4, [Sch], [Ste] or [VG]). We first study the nature of the fixed points of König's methods.

Proposition 1. *Assume $f : U \subset \mathbb{C} \to \mathbb{P}^1$ is a meromorphic map. Denote by α_i its zeros and by n_i their multiplicities. Then, for any integer $\sigma \geq 2$, the fixed points of the König's method $K_{f,\sigma} : U \to \mathbb{P}^1$ are either (super)attracting or repelling.*

The (super)attracting fixed points are exactly the zeros α_i and their multipliers are $1 - (\sigma - 1)/(n_i + \sigma - 2)$. When $n_i = 1$, the local degree of $K_{f,\sigma}$ at α_i is at least equal to σ.

The extraneous fixed points of $K_{f,\sigma}$ are exactly the zeros of $(1/f)^{[\sigma-2]}$. If β_j is a zero of $(1/f)^{[\sigma-2]}$ with multiplicity m_j, then it is a repelling fixed point of $K_{f,\sigma}$ with multiplier $1 + (\sigma - 1)/m_j$.

This research was supported by a Marie-Curie Fellowship.

We then prove a semi-global result essentially due to Schröder.

Proposition 2. *Assume f is meromorphic in a Euclidean disk D centered at z_0 and has a unique zero (possibly multiple) $\alpha \in D$. Then, there exists a neighborhood of z_0 on which the sequence $K_{f,\sigma}$ converges uniformly to the constant function α, as σ tends to $+\infty$.*

Remark. In this proposition, there is no iteration involved.

In Sect. 2, we restrict our study to the case where $f : \mathbb{C} \to \mathbb{C}$ is a polynomial.

Proposition 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d. Then, for any $\sigma \geq 2$,*

(a) *the König's method $K_{f,\sigma}$ is a rational map, it has a repelling fixed point at ∞ with multiplier is $1 + (\sigma - 1)/(d - 1)$;*
(b) *if f has N distinct roots, then $K_{f,\sigma}$ has at most $(\sigma - 2)(N - 1)$ repelling fixed points in $\mathbb{C}$ and its degree is at most $(\sigma - 1)(N - 1) + 1$.*

For a generic polynomial of degree d,

(c) *the local degree of $K_{f,\sigma}$ at the roots of f is exactly σ;*
(d) *the rational map $K_{f,\sigma}$ has $(\sigma - 2)(d - 1)$ repelling fixed points in $\mathbb{C}$ and their multipliers are all equal to σ;*
(e) *the rational map $K_{f,\sigma}$ has degree $(\sigma - 1)(d - 1) + 1$.*

We also show that rational maps which arise as König's methods of polynomials with simple roots are completely characterized by the nature of their fixed points. The problem was brought to our attention by Dierk Schleicher. The characterization was known for Newton's method, i.e., for $\sigma = 2$ (see [He], Prop. 2.1.2). For applications in the setting of Newton's methods, see for example [He], [R] or [Ta]. We do not know how to characterize rational maps which arise as König's methods of polynomials with multiple roots.

Proposition 4. *Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map, whose fixed points are either superattracting or repelling. Assume the local degree of h at every superattracting fixed point α_i is at least σ and the multiplier at every repelling fixed point β_j is of the form $1 + (\sigma - 1)/m_j$, where $m_j \geq 1$ is an integer. Finally, assume ∞ is a repelling fixed point of h with multiplier $1 + (\sigma - 1)/(d - 1)$, where d is the number of superattracting fixed points of h. Then, $h = K_{f,\sigma}$, where $f(z) = \prod_i (z - \alpha_i)$.*

In Sect. 3, we try to determine how many extraneous non-repelling cycles König's methods of polynomials may have. Those extraneous cycles are interesting since they may give rise to open sets of points whose orbits under iteration of $K_{f,\sigma}$ do not converge to a root of f.

Definition 2. *A periodic cycle of $K_{f,\sigma}$ is called an* extraneous cycle *if it does not coincide with a root of f. For any integers $d \geq 2$ and $\sigma \geq 2$, we define the integer $N(d, \sigma)$ as the maximum number of non-repelling cycles of $K_{f,\sigma}$ for polynomials f of degree d, i.e.,*

$$N(d, \sigma) = \max_{f \in \mathrm{Poly}_d} \#\{\text{extraneous non} - \text{repelling cycles of } K_{f,\sigma}\}.$$

Remark. Prop.s 1 and 3 show that when f is a polynomial, the extraneous fixed points of $K_{f,\sigma}$ are always repelling.

Proposition 5. *For any integers $d \geq 2$ and $\sigma \geq 2$, we have*

$$(d - 2) \leq N(d, \sigma) \leq (\sigma - 1)(d - 2).$$

Moreover, if $\sigma \geq d > 2$,

$$N(d,\sigma) \leq (\sigma-1)(d-2)-1.$$

This proposition shows that in the case of quadratic polynomials ($d = 2$), $N(2,\sigma) = 0$, in the case of Newton's methods ($\sigma = 2$), $N(d,2) = d-2$ and in the case of Halley's methods of cubic polynomials ($d = \sigma = 3$), $N(3,3) = 1$. In fact, the result $N(d,2) = d-2$ has already been proved by Hurley [Hur]. His arguments are different from ours.

The lower bound $N(d,\sigma) \geq (d-2)$ follows from McMullen's results on algebraic families of rational maps (see [McM1] Thm. 2.2). For the upper bound, we will see that there are $(\sigma-1)(d-2)$ *free critical points*. It follows from results by Fatou and improvements by Douady-Hubbard [DH] and Shishikura [Shi] that $N(d,\sigma) \leq (\sigma-1)(d-2)$. To improve this inequality in the case $\sigma \geq d > 2$, we will use extremal-length arguments.

In Sect. 4, we study Julia sets of König's methods and bifurcation loci.

Definition 3. *The Julia set* $\mathcal{J}(K_{f,\sigma})$ *can be defined as the closure of the set of repelling cycles of* $K_{f,\sigma}$ *or as the boundary of any basin of attraction of* $K_{f,\sigma}$.

Remark. The Julia set is relevant when the degree of $K_{f,\sigma}$ is greater than 1, i.e., when f has at least two distinct roots.

Definition 4. *Let* $(f_\lambda)_{\lambda\in\Lambda}$ *be a family of rational maps analytically parametrized by* Λ. *The bifurcation locus is the set of parameters that admit no neighborhood on which the period of superattracting cycles is bounded.*

Remark. In this definition, it is more customary to replace *superattracting* by *attracting* but the definitions are equivalent.

As an application of Prop. 2, we prove that the Julia sets $\mathcal{J}(K_{f,\sigma})$ have remarkable limits as σ tends to $+\infty$.

Definition 5. *Let us define the bisecting locus of a set* $\mathcal{S} \subset \mathbb{C}$ *as the set of points* $z \in \mathbb{C}$ *where the distance function* $z \mapsto d(z,\mathcal{S})$ *is not differentiable.*

Proposition 6. *As* σ *tends to* ∞, *the sequence of Julia sets* $\mathcal{J}(K_{f,\sigma})$ *converges, in the Hausdorff topology on compact subsets of* $\mathbb{P}^1$, *to the union of* $\{\infty\}$ *and the bisecting locus of the set of roots of* f.

We finally produce pictures of bifurcation loci for König's methods of cubic polynomials. The reason why we restrict our study to cubic polynomials is that for higher degree polynomials, the dimension of the parameter spaces are greater than 1, and thus, we could only draw slices. We do not know which slices would be relevant.

1. Preliminary results.

1.1. **The nature of fixed points.** Let us prove Prop. 1 step by step.

Lemma 1. *If a germ* $f : (\mathbb{C},\alpha) \to (\mathbb{C},0)$ *has a zero* α *of multiplicity* n, *then* $K_{f,\sigma} : (\mathbb{C},\alpha) \to (\mathbb{C},\alpha)$ *has a (super)attracting fixed point at* α *and its multiplier is* $1-(\sigma-1)/(n+\sigma-2)$.

Proof. The map $1/f$ has a pole of order n at α. Therefore, for any $j \geq 0$, the j-th derivative of $1/f$ has a pole of order $n+j$ at α. Thus, there exists $\lambda \in \mathbb{C}^*$ such that

$$\left(\frac{1}{f}\right)^{[\sigma-2]}(z) = \frac{\lambda}{(z-\alpha)^{n+\sigma-2}}\Big[1+\mathcal{O}(|z-\alpha|)\Big]$$

and

$$\left(\frac{1}{f}\right)^{[\sigma-1]}(z) = -\frac{(n+\sigma-2)\lambda}{(z-\alpha)^{n+\sigma-1}}\Big[1+\mathcal{O}(|z-\alpha|)\Big].$$

As a consequence,

$$K_{f,\sigma}(z) = \alpha + (z-\alpha)\left(1 - \frac{\sigma-1}{n+\sigma-2}\right) + \mathcal{O}(|z-\alpha|^2).$$

□

This result can be improved when α is a simple zero of f.

Lemma 2. *If a germ $f : (\mathbb{C},\alpha) \to (\mathbb{C},0)$ has a simple zero at α, then α is a superattracting fixed point of the König's method $K_{f,\sigma} : (\mathbb{C},\alpha) \to (\mathbb{C},\alpha)$ and the local degree of $K_{f,\sigma}$ at α is at least equal to σ.*

Proof. Since f has a simple zero at α, $1/f$ has a simple pole at α. Thus,

$$\frac{1}{f(z)} = \frac{\lambda}{z-\alpha} + f_\alpha(z),$$

where f_α is holomorphic in a neighborhood of α. It follows that

$$\left(\frac{1}{f}\right)^{[\sigma-2]} = \frac{\mu}{(z-\alpha)^{\sigma-1}} + \mathcal{O}(1) \quad \text{and} \quad \left(\frac{1}{f}\right)^{[\sigma-1]} = -(\sigma-1)\frac{\mu}{(z-\alpha)^{\sigma}} + \mathcal{O}(1)$$

for a suitable constant $\mu \in \mathbb{C}^*$. Thus,

$$\begin{aligned} K_{f,\sigma}(z) &= \alpha + (z-\alpha) + (\sigma-1)\frac{\dfrac{\mu}{(z-\alpha)^{\sigma-1}}(1+\mathcal{O}(|z-\alpha|^{\sigma-1}))}{-(\sigma-1)\dfrac{\mu}{(z-\alpha)^{\sigma}}(1+\mathcal{O}(|z-\alpha|^{\sigma}))} \\ &= \alpha + \mathcal{O}(|z-\alpha|^{\sigma}). \end{aligned}$$

□

The following result appears in a weaker form in [Dr], [Kn] and [VG].

Lemma 3. *If $f : U \to \mathbb{P}^1$ is a meromorphic map, the extraneous fixed points of $K_{f,\sigma}$ are exactly the zeros of $(1/f)^{[\sigma-2]}$. If β is a zero of $(1/f)^{[\sigma-2]}$ with multiplicity m, then it is a repelling fixed point of $K_{f,\sigma}$ with multiplier $1+(\sigma-1)/m$.*

Proof. By definition, we have

$$K_{f,\sigma}(z) = z + (\sigma-1)\frac{g(z)}{g'(z)} \quad \text{where} \quad g = (1/f)^{[\sigma-2]}.$$

The fixed points of $K_{f,\sigma}$ are the zeros and the poles of g. On the one hand, the poles of g are the zeros of f, which have already been studied. On the other hand, if β is a zero of g of order m, then there exists a $\lambda \in \mathbb{C}^*$ such that

$$g(z) = \lambda(z-\beta)^m + \mathcal{O}(|z-\beta|^{m+1}) \quad \text{and} \quad g'(z) = \lambda m(z-\beta)^{m-1} + \mathcal{O}(|z-\beta|^m).$$

Thus,

$$K_{f,\sigma}(z) = \beta + (z-\beta)\left(1 + \frac{\sigma-1}{m}\right) + \mathcal{O}(|z-\beta|^2).$$

Therefore, β is a repelling fixed point of $K_{f,\sigma}$ and its multiplier is $1+(\sigma-1)/m$.
□

Remark. When $\sigma = 2$, there are no extraneous fixed points, and when $\sigma = 3$, the extraneous fixed points are the critical points of f which are not zeros of f.

1.2. **Schröder's approach.** Prop. 2 seems to be known when α is a simple root of f and follows from a theorem due to König [Kö]. This theorem asserts that if an analytic function has a single singularity at the radius of convergence of its power series, and if this singularity is a simple pole, then the ratios of the successive coefficients of its power series converge to that pole. We will prove that this result still holds when the pole is multiple. Prop. 2 follows immediately by setting $g = 1/f$ and $n = \sigma - 2$.

Proposition 7. *Let D be a Euclidean disk centered at z_0 and α be a point of D. Assume g is meromorphic on D and has a unique pole (possibly multiple) at α. Then, there exists a neighborhood of z_0 on which the sequence of functions $(n+1)g^{[n]}/g^{[n+1]}$ converges uniformly to the function $z \mapsto \alpha - z$.*

Proof. Restricting D slightly if necessary, we may assume that g is continuous on ∂D. Let r be the radius of D and M be the supremum of $|g(z)|$ on the boundary of D. Besides, let $V \subset D$ be a neighborhood of z_0 for which there exists a real number $\lambda < 1$ such that

$$(\forall z \in V) \quad |z - \alpha| \leq \lambda d(z, \partial D).$$

The following lemma implies that when $z \in V$,

$$\frac{g^{[n]}(z)}{n!} = \frac{(-1)^m a_{-m}}{(\alpha - z)^{n+m}} \frac{(n+m-1)!}{n!(m-1)!}(1 + o(1)),$$

with a uniform control on the term $o(1)$. It follows that for any $z \in V$,

$$(n+1)\frac{g^{[n]}(z)}{g^{[n+1]}(z)} \underset{n \to +\infty}{\longrightarrow} \alpha - z,$$

the convergence being uniform on V. ■

Lemma 4. *Let*

$$g(z) = \sum_{k=-m}^{+\infty} a_k (z - \alpha)^k$$

be the Laurent series of g at α. Then, for any $z \in V$, we have

$$\left| \frac{g^{[n]}(z)}{n!} + \frac{1}{(\alpha - z)^{n+1}} \sum_{k=1}^{m} \frac{a_{-k}}{(z-\alpha)^{k-1}} \frac{(n+k-1)!}{n!(k-1)!} \right| \leq \frac{\lambda^{n+1} M r}{|z - \alpha|^{n+1}}.$$

Proof. By the residue theorem, we have

$$\frac{1}{2i\pi} \int_{\partial D} \frac{g(w)}{(w-z)^{n+1}} dw = \frac{g^{[n]}(z)}{n!} + \text{residue}\left(\frac{g(w)}{(w-z)^{n+1}}, \alpha \right).$$

Now, for any $z \in V$,

$$\left| \frac{1}{2i\pi} \int_{\partial D} \frac{g(w)}{(w-z)^{n+1}} dw \right| \leq \frac{\lambda^{n+1} M r}{|z - \alpha|^{n+1}}$$

and an easy computation gives

$$\text{residue}\left(\frac{g(w)}{(w-z)^{n+1}}, \alpha \right) = \frac{1}{(\alpha - z)^{n+1}} \sum_{k=1}^{m} \frac{a_{-k}}{(z-\alpha)^{k-1}} \frac{(n+k-1)!}{n!(k-1)!}.$$

□

2. König's methods of polynomials.

From now on, $f : \mathbb{C} \to \mathbb{C}$ will always be a polynomial.

2.1. **Proof of Proposition 3.** Everything we have proved up to now is valid for any meromorphic map $f : U \subset \mathbb{C} \to \mathbb{P}^1$. Let us now restrict our study to the case where $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $d \geq 2$. In that case, the König's method $K_{f,\sigma}$ extends to a rational map $K_{f,\sigma} : \mathbb{P}^1 \to \mathbb{P}^1$.

Figure 1 shows two pictures of $\mathbb{P}^1$. The points of the Riemann sphere are colored according to which root of f their orbits under iteration of $K_{f,\sigma}$ converge. The roots of the polynomial are on the hidden face of the Riemann sphere. One can observe the self-similarity of the basins near ∞, which indicates that ∞ is a repelling fixed point of $K_{f,\sigma}$.

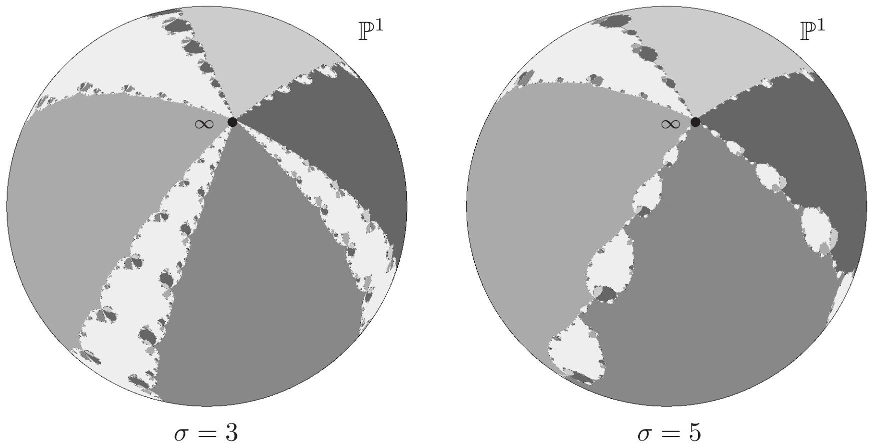

Figure 1. The basins of attraction of $K_{f,\sigma}$ for the polynomial $f(z) = z(z-1)(z-i)(z+i)(z+1-i)$, for $\sigma = 3$ and $\sigma = 5$.

Proof of (a). When $|z|$ tends to ∞, we have

$$f(z) = \lambda z^d \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right).$$

Therefore,

$$\left(\frac{1}{f}\right)^{[\sigma-2]}(z) = \frac{\mu}{z^{d+\sigma-2}}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right)$$

and

$$\left(\frac{1}{f}\right)^{[\sigma-1]}(z) = -(d+\sigma-2)\frac{\mu}{z^{d+\sigma-1}}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right)$$

for a suitable constant μ. Consequently,

$$K_{f,\sigma}(z) = \left(1 - \frac{\sigma - 1}{d + \sigma - 2}\right) z + \mathcal{O}(1) = \frac{d-1}{d+\sigma-2} z + \mathcal{O}(1).$$

It follows that ∞ is a fixed point of $K_{f,\sigma}$ with multiplier $(d + \sigma - 2)/(d - 1) = 1 + (\sigma - 1)/(d - 1)$. This concludes the proof of (a).

Proof of (b). Let α_i, $i = 1, \ldots, N$, be the zeros of f and n_i be their multiplicities. Since f is a polynomial of degree d, we have

$$\sum_{i=1}^{N} n_i = d.$$

We have seen that the fixed points of $K_{f,\sigma}$ are ∞, the points α_i and the zeros of the rational map $g = (1/f)^{[\sigma-2]}$ in $\mathbb{C}$. For any rational map, the number of zeros in $\mathbb{P}^1$ is equal to the number of poles in $\mathbb{P}^1$. The poles of g are the points α_i, with multiplicity $n_i + \sigma - 2$, and g has a zero of order $d + \sigma - 2$ at ∞. Thus, g has

$$\sum_{i=1}^{N} (n_i + \sigma - 2) = d + N(\sigma - 2)$$

poles counted with multiplicities. It follows that g has

$$d + N(\sigma - 2) - (d + \sigma - 2) = (N - 1)(\sigma - 2)$$

zeros in $\mathbb{C}$, counting multiplicities. Consequently, $K_{f,\sigma}$ has at most $(N - 1)(\sigma - 2)$ repelling fixed points in $\mathbb{C}$.

For any rational map, the number of fixed points counted with multiplicities is equal to the degree plus one. The fixed points of $K_{f,\sigma}$ are simple (a fixed point is multiple if and only if its multiplier is equal to 1). There are N (super)attracting fixed points, one repelling fixed point at infinity and at most $(N-1)(\sigma-2)$ repelling fixed points in $\mathbb{C}$. Therefore, the degree of $K_{f,\sigma}$ is at most

$$[N + 1 + (N - 1)(\sigma - 2)] - 1 = (N - 1)(\sigma - 1) + 1.$$

This concludes the proof of (b).

Remark. The definition of $K_{f,\sigma}$ only involves the $\sigma - 1$ first derivatives of f. It follows from known results (see for example [Tr] Theorem 5.3) that for a generic analytic map having a zero α, the local degree of $K_{f,\sigma}$ at α is at most σ. However, the set of polynomials is not generic within the set of analytic mappings, and thus, the proof of (c) does not follow from those known results.

Proof of (c), (d) and (e). If (c) does not hold, then either the local degree of $K_{f,\sigma}$ at one of the roots of f is less than σ and in that case, f has a multiple root, or the local degree of $K_{f,\sigma}$ at one of the roots of f is larger than σ.

If f and $g = (1/f)^{[\sigma-2]}$ only have simple zeros in $\mathbb{C}$ we can redo the proof of (b) with $N = d$. Then, g has exactly $(d - 1)(\sigma - 2)$ simple zeros, which correspond to exactly $(d - 1)(\sigma - 2)$ repelling fixed point of $K_{f,\sigma}$ with multiplier σ. Moreover, the degree of $K_{f,\sigma}$ is exactly $(d - 1)(\sigma - 1) + 1$. Therefore, if (d) or (e) does not hold, either f has a multiple root, or $g = (1/f)^{[\sigma-2]}$ has a multiple zero in $\mathbb{C}$.

In summary, assume f does not satisfy one of the properties (c), (d) or (e). Then, either f has a multiple root, or $g = (1/f)^{[\sigma-2]}$ has a multiple zero in $\mathbb{C}$, or the local degree of $K_{f,\sigma}$ at one of the roots of f is larger than σ.

Let us first show that these conditions are algebraic.

- The polynomial f has a multiple root if and only if its discriminant vanishes.
- Define the polynomial

$$P_\sigma = f^{\sigma-1} \cdot \left(\frac{1}{f}\right)^{[\sigma-2]}.$$

  Supposing f has only simple roots then $(1/f)^{[\sigma-2]}$ has a multiple zero in $\mathbb{C}$ if and only if the discriminant of P_σ vanishes.
- Observe that the local degree of $K_{f,\sigma}$ at a simple root of f exceeds σ if and only if $K_{f,\sigma}^{[\sigma]}$ vanishes at the root. Now

$$K_{f,\sigma} = \frac{z \cdot P_{\sigma+1} + (\sigma-1)f \cdot P_\sigma}{P_{\sigma+1}},$$

  so letting

$$Q_\sigma = (P_{\sigma+1})^{\sigma+1} \cdot (K_{f,\sigma})^{[\sigma]},$$

  we obtain a polynomial Q_σ. If f has only simple roots, the partial fractional expansion of $1/f$ shows that $P_{\sigma+1}$ does not vanish at any root of f. Hence supposing all the roots of f are simple, the local degree of $K_{f,\sigma}$ at one of them is larger than σ if and only if the resultant of f and Q_σ vanishes.

Since the coefficients of P_σ and Q_σ are polynomials in the coefficients of f, those three conditions are algebraic in the coefficients of f. As a consequence, it is enough to show that properties (c), (d) and (e) are satisfied by some polynomial of degree d in order to show that they are satisfied generically by polynomials of degree d.

Let us now prove the existence of such a polynomial. We will give a proof by induction on the degree d (in the rest of the proof, σ is fixed). There may be a more elementary (or more algebraic) proof than the one we present here. The inductive hypothesis is the following.

($\mathcal{H}_d$). *There exists a polynomial f of degree d with d distinct roots, such that $K_{f,\sigma}$ has (at least) $(\sigma-1)(d-2)$ critical points, counting multiplicities, which do not coincide with the roots of f.*

Let us see that a polynomial f satisfying ($\mathcal{H}_d$) automatically satisfies properties (c), (d) and (e). Let us first show that the local degree of $K_{f,\sigma}$ at the roots of f is exactly equal to σ. We know that the local degree of $K_{f,\sigma}$ at the roots of f is at least equal to σ. Thus, $K_{f,\sigma}$ has at least $d(\sigma-1)+(d-2)(\sigma-1) = 2(d-1)(\sigma-1)$ critical points. By the Riemann-Hurwitz formula, a rational map of degree k has $2k-2$ critical points counted with multiplicities. Therefore, the degree of $K_{f,\sigma}$ is at least $(d-1)(\sigma-1)+1$. Statement (b) above asserts that the degree is at most $(d-1)(\sigma-1)+1$. Hence, the degree of $K_{f,\sigma}$ is exactly equal to $(d-1)(\sigma-1)+1$ and the local degree of $K_{f,\sigma}$ at the roots of f is exactly equal to σ.

Let us now show that the zeros of $g = (1/f)^{[\sigma-2]}$ are simple. We have seen above that g has $(d-1)(\sigma-2)$ zeros in $\mathbb{C}$, counting multiplicities. Therefore, it is sufficient to show that g has $(d-1)(\sigma-2)$ distinct zeros in $\mathbb{C}$, or equivalently that $K_{f,\sigma}$ has $(d-1)(\sigma-2)$ repelling fixed points in $\mathbb{C}$. A rational map of degree k has $k+1$ fixed points counted with multiplicities and the fixed points of $K_{f,\sigma}$ are all simple. Since the degree of $K_{f,\sigma}$ is $(d-1)(\sigma-1)+1$, it has $(d-1)(\sigma-1)+2$ fixed points. One of these is at ∞, and d of these are at the roots of f, so there are $(d-1)(\sigma-2)$ repelling fixed points in $\mathbb{C}$. Using that the zeros of g all are simple, Prop. 1 shows that the multiplier equals σ at each repelling fixed point of $K_{f,\sigma}$.

Proof of $(\mathcal{H}_d)$**.** The inductive hypothesis $(\mathcal{H}_2)$ is clearly true: for each quadratic polynomial f with distinct roots, there is at least 0 critical points which do not coincide with the roots of f. Let us now assume that f is a polynomial of degree d which satisfies $(\mathcal{H}_d)$. We will show that for $\alpha \in \mathbb{C}$ with $|\alpha|$ large enough, the polynomial f_α defined by

$$f_\alpha(z) = (z-\alpha)f(z)$$

satisfies $(\mathcal{H}_{d+1})$.

The key point is the following lemma.

Lemma 5. *Assume $(f_n)_{n\in\mathbb{N}}$ is a sequence of polynomials converging uniformly to a polynomial f on every compact subset of $\mathbb{C}$. Denote by Fix_σ the set of fixed points of $K_{f,\sigma}$. Then, the sequence of rational maps $K_{f_n,\sigma}$ converges uniformly to $K_{f,\sigma}$ on every compact subset of $\mathbb{P}^1 \setminus \mathrm{Fix}_\sigma$, for the spherical metric on $\mathbb{P}^1$.*

Proof. Let us define $U = \mathbb{C} \setminus \mathrm{Fix}_\sigma$. Since the set of zeros of f is contained in Fix_σ, we see that $1/f_n$ converges uniformly (for the Euclidean metric on $\mathbb{C}$) to $1/f$ on every compact subset of U. It follows that the sequences $g_n = (1/f_n)^{[\sigma-2]}$ and $g_n' = (1/f_n)^{[\sigma-1]}$ converge uniformly on every compact subset of U to $g = (1/f)^{[\sigma-2]}$ and $g' = (1/f)^{[\sigma-1]}$ (again for the Euclidean metric on $\mathbb{C}$). Next, the set of zeros of g_n is contained in a neighborhood of Fix_σ. Thus, g_n'/g_n converges uniformly on every compact subset of U to g'/g (still for the Euclidean metric on $\mathbb{C}$). Therefore, g_n/g_n' and $K_{f_n,\sigma} = z + (\sigma-1)g_n/g_n'$ both converge uniformly on every compact subset of U to g/g' and $K_{f,\sigma} = z + (\sigma-1)g/g'$ (this time, for the spherical metric on $\mathbb{P}^1$). □

Choose $R > 0$ large enough, so that the disk $\mathbb{D}_R = \{z \in \mathbb{C} \mid R > |z|\}$ contains all the fixed points of $K_{f,\sigma}$. When $|\alpha| > R$, the roots of f_α are the roots of f (they are contained in $\mathbb{D}_R$) and α. Thus, they are simple. Denote by $\mathrm{Fix}_{f,\sigma} \subset \mathbb{D}_R$ the set of fixed points of $K_{f,\sigma}$. Observe that $-\frac{1}{\alpha}f_\alpha$ converges uniformly to f on every compact subset of $\mathbb{C}$ when $|\alpha|$ tends to ∞. Since $K_{f_\alpha,\sigma} = K_{-\frac{1}{\alpha}f_\alpha,\sigma}$, the previous lemma asserts that $K_{f_\alpha,\sigma}$ converges uniformly to $K_{f,\sigma}$ on every compact subset of $\mathbb{D}_R \setminus \mathrm{Fix}_{f,\sigma}$. Thus, when α is large enough, $K_{f_\alpha,\sigma}$ has $(\sigma-1)(d-2)$ critical points in $\mathbb{D}_R$ which do not coincide with the roots of f_α.

We will now show that when $|\alpha|$ is large enough, $K_{f_\alpha,\sigma}$ has $\sigma - 1$ critical points outside $\mathbb{D}_R$ that differ from α. It then follows that f_α satisfies $(\mathcal{H}_{d+1})$. Define

$$F_\alpha(z) = \frac{1}{\alpha^{d+1}} f_\alpha(\alpha z) = (z-1)\frac{f(\alpha z)}{\alpha^d}.$$

It is possible to show (see Lemma 8 below) that

$$K_{F_\alpha,\sigma}(z) = \frac{1}{\alpha} K_{f_\alpha,\sigma}(\alpha z).$$

Therefore, the critical points of $K_{f_\alpha,\sigma}$ are the images of the critical points of $K_{F_\alpha,\sigma}$ by the scaling map $z \mapsto \alpha z$. In particular, it is sufficient to prove that when $|\alpha|$ is large enough, $K_{F_\alpha,\sigma}$ has $\sigma - 1$ critical points in $\mathbb{C} \setminus (\mathbb{D}_{R/|\alpha|} \cup \{1\})$. When $|\alpha|$ tends to ∞, F_α converges uniformly to $F : z \mapsto \lambda(z-1)z^d$ on every compact subset of $\mathbb{C}$, where $\lambda \neq 0$ is the leading coefficient of f.

Lemma 6. *Let F be the polynomial $z \mapsto \lambda(z-1)z^d$ and denote by $\mathrm{Fix}_{F,\sigma} \subset \mathbb{D}_R$ the set of fixed points of $K_{F,\sigma}$. Then, $K_{F,\sigma}$ has $\sigma - 1$ critical points in $\mathbb{C} \setminus \mathrm{Fix}_{F,\sigma}$.*

Proof. We know that $K_{F,\sigma}$ has a superattracting fixed point with local degree at least σ at 1. Hence, its degree is at least σ. Since F has only $N = 2$ roots, statement (b) above asserts that the degree of $K_{F,\sigma}$ is at most σ. Therefore, this degree is exactly equal to σ. A rational map of degree σ has $2(\sigma-1)$ critical points, and such a critical point has multiplicity at most $\sigma-1$ (with equality if and only if the rational map is conjugate to a polynomial). As a consequence, $K_{F,\sigma}$ has $\sigma-1$ critical points, counting multiplicities, in $\mathbb{P}^1 \setminus \{1\}$. Since the remaining fixed points of $K_{F,\sigma}$ are not superattracting, those critical points belong to $\mathbb{C} \setminus \mathrm{Fix}_{F,\sigma}$. □

Remark. This discussion shows that $K_{F,\sigma}$ is a polynomial of degree σ in the coordinate $w = 1/(z-1)$.

Let us now finish the proof of Prop. 3. We know that when $|\alpha|$ tends to ∞, F_α converges uniformly to $F : z \mapsto \lambda(z-1)z^d$ on every compact subset of $\mathbb{C}$. We know that $K_{F,\sigma}$ has $\sigma-1$ critical points in $\mathbb{C} \setminus \mathrm{Fix}_{F,\sigma} \subset \mathbb{C} \setminus \{0,1\}$. It follows immediately that when $|\alpha|$ is large enough, $K_{F_\alpha,\sigma}$ has $\sigma-1$ critical points in $\mathbb{C} \setminus (\mathbb{D}_{R/|\alpha|} \cup \{1\})$.

2.2. **König's methods for quadratic polynomials.** As a trivial application of Prop. 3, let us recall that the case of quadratic polynomials is completely understood (compare with [VG] and [Dr]):

- if f has a double root $a \in \mathbb{C}$, the König's method $K_{f,\sigma}$ is the affine map $z \mapsto a + (z-a)/\sigma$;
- the König's method of a quadratic polynomial $f(z) = \lambda(z-a)(z-b)$, $\lambda \in \mathbb{C}^*$ and $a \neq b$, is conjugate, in the coordinate $w = (z-a)/(z-b)$, to the rational map $w \mapsto w^\sigma$.

In both cases, the result may be obtained by a direct calculation. One may also argue that in the case of a double root, the König's method has to be a rational map of degree at most 1. This rational map must have an attracting fixed point with multiplier $1/\sigma$ at a and a repelling fixed point with multiplier σ at ∞. In the case of two distinct roots $a \neq b$, Prop. 3 asserts that the degree of $K_{f,\sigma}$ is at most $(2-1)(\sigma-1)+1 = \sigma$. This rational map has two superattracting fixed points at a and b with local degree at least σ. As a consequence, the degree of $K_{f,\sigma}$ is at least σ. Therefore, it is equal to σ and the local degree of $K_{f,\sigma}$ at a and b is exactly σ. We know that $K_{f,\sigma}$ has a repelling fixed point at ∞. Therefore, in the coordinate $w = (z-a)/(z-b)$, $K_{f,\sigma}$ is conjugate to a rational map of degree σ having superattracting fixed points at 0 and ∞ and a repelling fixed point at 1. This rational map is $w \mapsto w^\sigma$.

2.3. **Comments on the degree of $K_{f,\sigma}$.** According to McMullen [McM1], a purely iterative root-finding algorithm is a rational map $T : f \mapsto T_f$ from the space Poly_d of polynomials of degree d into the space Rat_k of rational map of degree k, such that each root of f is a (super)attracting fixed point of T_f. As a set-theoretic mapping, T may be undefined on an algebraic sub-variety of its domain. We say that the algorithm T is of order σ if, outside an algebraic sub-variety of codimension at least 1, the local degree of T_f at the roots of f is equal to σ.

Prop. 3 shows that the König's algorithm $f \mapsto K_{f,\sigma}$ from the set Poly_d of polynomials of degree d to the set $\mathrm{Rat}_{(\sigma-1)(d-1)+1}$ of rational maps of degree $(\sigma-1)(d-1)+1$ is a purely iterative root-finding algorithm of order σ.

Problem. Given two integers $d \geq 2$ and $\sigma \geq 2$, what is the smallest integer $k = k(d,\sigma)$ for a purely iterative root-finding algorithm $T : \mathrm{Poly}_d \to \mathrm{Rat}_k$ to be of order σ?

It is clear that generically, the rational map T_f has at least $d(\sigma-1)$ critical points at the roots of f. Then, the Riemann-Hurwitz formula implies that

$$k(d,\sigma) \geq \frac{d(\sigma-1)}{2}+1.$$

The previous discussion shows that $k(d,\sigma) \leq (\sigma-1)(d-1)+1$.

Using McMullen's result on rigidity of algebraic families of rational maps (see [McM1] Thm. 2.2), one may prove that there must be at least $d-3$ extra critical points (see the remark at the end of Sect. 3.2). Thus,

$$k(d,\sigma) \geq \frac{d(\sigma-1)+d-3}{2}+1 = \frac{d\sigma-1}{2},$$

and

$$\frac{d\sigma-1}{2} \leq k(d,\sigma) \leq (d-1)(\sigma-1)+1.$$

This result is clearly optimal for $\sigma = 2$.

2.4. **Proof of Prop. 4.** Prop. 1 shows that when $f(z) = \prod_i (z-\alpha_i)^{n_i}$, the König's method $K_{f,\sigma}$ satisfies the following properties:

- the points α_i are (super)attracting fixed points with multipliers $(n_i-1)/(\sigma+n_i-2)$ and when $n_i = 1$, the local degree at α_i is at least σ;
- the point ∞ is a repelling fixed point with multiplier $1+(\sigma-1)/(d-1)$;
- the remaining fixed points β_j are repelling with multipliers of the form $1+(\sigma-1)/m_j$.

On the one hand, the following example shows that those conditions are not sufficient to characterize rational maps which are König's methods of polynomials with possibly multiple roots.

Example. The rational map $h : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$h(z) = z - 2\frac{(z-1)(z+1)(z-2)}{5z^2-12z+1}$$

has attracting fixed points at ± 1 with multiplier $1/3 = (2-1)/(3+2-2)$, a repelling fixed point at 2 with multiplier $3 = 1+(3-1)/1$ and a repelling fixed point at ∞ with multiplier $5/3 = 1+(3-1)/(4-1)$. It is a good candidate to be the König's method of order 3, i.e., the Halley's method, for the quartic polynomial $f(z) = (z-1)^2(z+1)^2$. However,

$$K_{f,3}(z) = z - 2\frac{z(z-1)(z+1)}{5z^2+1} \neq h.$$

On the other hand, Prop. 4 asserts that those conditions are sufficient to characterize rational maps which arise as König's methods of polynomials with simple roots. We will now prove this proposition.

Let us recall the problem. We have a rational map $h : \mathbb{P}^1 \to \mathbb{P}^1$, whose fixed points are either superattracting or repelling. The local degree of h at any superattracting fixed point α_i is at least σ and the multiplier at any repelling fixed point β_j is of the form $1+(\sigma-1)/m_j$, where $m_j \geq 1$ is an integer. Finally, ∞ is a repelling fixed point of h with multiplier $1+(\sigma-1)/(d-1)$, where d is the number of superattracting fixed points of h. We must show that $h = K_{f,\sigma}$, where $f(z) = \prod_i (z-\alpha_i)$.

Step 1. We claim that $h = \mathrm{Id} + (\sigma-1)g/g'$, where $g : \mathbb{P}^1 \to \mathbb{P}^1$ is the rational map defined by

$$g(z) = \frac{\prod_j (z-\beta_j)^{m_j}}{\prod_i (z-\alpha_i)^{\sigma-1}}.$$

Indeed, let us define $h_1 = \mathrm{Id} + (\sigma-1)g/g'$. The fixed points in $\mathbb{C}$ of the rational map h_1 are exactly the zeros and poles of g. One can easily check that the multiplier at α_i is 0 and the multiplier at β_j is $1 + (\sigma-1)/m_j$. Thus, the two rational maps h and h_1 have the same fixed points in $\mathbb{C}$ with the same multipliers.

If α is a fixed point of a rational map f, one can define a quantity $\iota(f,\alpha) \in \mathbb{C}^*$ which is called the residue fixed point index of f at α. If the multiplier λ of f at α differs from 1, we have

$$\iota(f,\alpha) = \frac{1}{1-\lambda}.$$

The holomorphic fixed point formula (see [M] Sect. 10) asserts that the sum of the residue fixed point indices of a rational map at all its fixed points is always equal to 1.

By assumption, ∞ is a fixed point of h. Since h and h_1 have the same fixed point in $\mathbb{C}$ with the same multipliers, and since those multipliers differ from 1, it follows from the holomorphic fixed point formula that h_1 fixes ∞ (the sum of the residue fixed point indices at the fixed points in $\mathbb{C}$ is not equal to 1) and

$$\iota(h_1,\infty) = \iota(h,\infty).$$

Note that

$$\frac{g'(z)}{g(z)} = \sum_j \frac{m_j}{z-\beta_j} - \sum_i \frac{\sigma-1}{z-\alpha_i} \underset{z\to\infty}{\longrightarrow} 0.$$

Therefore, $h_1(z)$ is not equivalent to z as $z \to \infty$, and so, the multiplier of h_1 at ∞ differs from 1. Since the holomorphic fixed point residues of h_1 and h at ∞ coincide, the multipliers also coincide. As a consequence, the rational maps h and h_1 have the same fixed points. Their multipliers are equal and all differ from 1. The following lemma shows that $h = h_1$.

Lemma 7. *Assume that two rational maps h_1 and h_2 have the same fixed points β_i with the same multipliers $\lambda_i \neq 1$. Then, the two rational maps are equal.*

Proof. Let us work in a coordinate where ∞ is not one of the fixed points β_i. The rational function $1/(z-h_1)$ tends to 0 as z tends to ∞. Moreover, the poles of $1/(z-h_1)$ are the fixed points β_i. Since $\lambda_i \neq 1$ these are all simple poles, and the residue of $1/(z-h_1)$ at β_i is $1/(1-\lambda_i)$. The same is true for $1/(z-h_2)$. Thus, we see that the rational map $1/(z-h_1) - 1/(z-h_2)$ has no pole in $\mathbb{C}$ and tends to 0 as z tends to ∞. Thus, this rational map is equal to 0. This proves that the rational maps h_1 and h_2 are equal. □

Remark. It is necessary to forbid multiple fixed points. For example, the polynomials $z \mapsto z + z^2$ and $z \mapsto z + 2z^2$ have the same fixed points, with the same multipliers (and even the same holomorphic indices) but they are not equal. Also, note that we are not explicitely assuming that the maps have the same degree, but this follows since there are no multiple fixed points.

Step 2. We next claim that there exist $\lambda_i \in \mathbb{C}^*$ such that

$$g(z) = \sum_i \frac{\lambda_i}{(z-\alpha_i)^{\sigma-1}}. \tag{1}$$

In other words, we claim that there is only one term in the polar part of the Laurent series of g at any of its poles α_i. It is clear that the points α_i are poles of order $\sigma - 1$. Let us write

$$g(z) = \frac{\lambda_i}{(z-\alpha_i)^{\sigma-1}}\Big(1 + \mu(z-\alpha_i)^j + o(|z-\alpha_i|^j)\Big),$$

with $\mu \neq 0$. We must prove that $j \geq \sigma - 1$. Observe that

$$g'(z) = -\frac{(\sigma-1)\lambda_i}{(z-\alpha_i)^\sigma}\left(1 - \mu\frac{j-\sigma+1}{\sigma-1}(z-\alpha_i)^j + o(|z-\alpha_i|^j)\right).$$

Therefore,

$$z + (\sigma-1)\frac{g(z)}{g'(z)} = \alpha_i - \mu\frac{j}{\sigma-1}(z-\alpha_i)^{j+1} + o(|z-\alpha_i|^{j+1}).$$

Since α_i is superattracting with order at least σ, it follows that $j \geq \sigma - 1$.

Step 3. Let us now consider the rational map

$$G(z) = \sum_i \frac{\lambda_i}{z-\alpha_i},$$

where the λ_i's are given by equation 1. We claim that G has a zero of order d at ∞. Indeed, $G(z)$ tends to 0 as z tends to ∞ and thus, $G(z) \sim a/z^n$ for some integer n and some complex number $a \in \mathbb{C}^*$. Moreover, the $(\sigma-2)^{\text{th}}$ derivative of G is

$$G^{[\sigma-2]} = (-1)^{\sigma-2}(\sigma-2)!g.$$

Hence, $g(z) \sim b/z^{n+\sigma-2}$ for some complex number $b \in \mathbb{C}^*$. Observe that

$$z + (\sigma-1)\frac{g(z)}{g'(z)} \sim \left(1 - \frac{\sigma-1}{n+\sigma-2}\right)z.$$

Since $h = \mathrm{Id} + (\sigma-1)g/g'$ has a repelling fixed point with multiplier $1+(\sigma-1)/(d-1)$ at ∞, it follows that $1/(1-(\sigma-1)/(n+\sigma-2)) = 1 + (\sigma-1)/(d-1)$. After simplifications, this gives $n = d$.

Step 4. The zeros of the rational map $1/G$ are exactly the points α_i. They are simple zeros. Thus, $1/G$ is a rational function of degree d. Moreover, $1/G$ has a pole of order d at ∞. Thus, $1/G$ is equal to the polynomial $f(z) = \prod_i (z-\alpha_i)$ up to multiplication by a non-zero constant. The rational maps g and $(1/f)^{[\sigma-2]}$ are equal up to multiplication by a non-zero constant. Therefore,

$$h = \mathrm{Id} + (\sigma-1)\frac{g}{g'} = \mathrm{Id} + (\sigma-1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}} = K_{f,\sigma}.$$

This concludes the proof of Prop. 4.

3. The number of extraneous non-repelling cycles.

Before studying the number of extraneous non-repelling cycles of König's methods, we need to determine which König's methods might be conjugate.

3.1. **Affine conjugacy classes.** We will first prove that König's methods behave nicely under affine changes of coordinates. This result has been proved by Plaza (see [Pl] Thm. 6).

Lemma 8. *Let $A : \mathbb{C} \to \mathbb{C}$ be an affine automorphism and $\lambda \in \mathbb{C}^*$ be any non zero constant. For any meromorphic map $f : U \to \mathbb{P}^1$, define $g = \lambda f \circ A : A^{-1}(U) \to \mathbb{P}^1$. Then, for any integer $\sigma \geq 2$, the König's methods $K_{f,\sigma} : U \to \mathbb{P}^1$ and $K_{g,\sigma} : A^{-1}(U) \to \mathbb{P}^1$ are affine conjugate:*

$$K_{f,\sigma} \circ A = A \circ K_{g,\sigma}.$$

Proof. The proof is elementary. Let us write $A(z) = az + b$ with $a \in \mathbb{C}^*$ and let us assume that $g = \lambda f \circ A$. Then,

$$\left(\frac{1}{g}\right)^{[j]} = \frac{a^j}{\lambda}\left(\frac{1}{f}\right)^{[j]} \circ A,$$

and thus

$$A \circ K_{g,\sigma}(z) = az + b + (\sigma - 1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}} \circ A(z) = K_{f,\sigma} \circ A(z).$$

■

We also prove that the König's methods of two polynomials are affine conjugate if and only if the polynomials are equal up to pre-composition with an affine automorphism of $\mathbb{C}$ and up to post-composition with a scaling map.

Lemma 9. *Let f and g be two polynomials, and $A : \mathbb{C} \to \mathbb{C}$ an arbitrary affine automorphism. Then A conjugates $K_{g,\sigma}$ to $K_{f,\sigma}$ (i.e., $K_{f,\sigma} \circ A = A \circ K_{g,\sigma}$) if and only if there exists $\lambda \in \mathbb{C}^*$ such that $g = \lambda f \circ A$.*

Remark. In particular, this shows that the König's methods of two polynomials f and g are equal if and only if f/g is a non-zero constant.

Proof. The proof is again elementary. Lemma 8 gives the reverse implication, so assume $K_{f,\sigma} \circ A = A \circ K_{g,\sigma}$ Define $h = f \circ A$. Using Lemma 8, we see that $K_{g,\sigma} = K_{h,\sigma}$. Since the roots of a polynomial and their multiplicities are determined by the König's method of the polynomial (see Prop. 1), the polynomials g and h have the same roots with the same multiplicities. Thus, there exists a non-zero constant $\lambda \in \mathbb{C}^*$ such that $g = \lambda \circ h$. □

Those two results are very useful in order to decrease the number of dimensions when studying parameter spaces of König's methods (see for example Sect. 4.2.1).

3.2. **The lower bound on $N(d, \sigma)$.** Let us fix $d \geq 2$ and $\sigma \geq 2$ and denote by $\mathcal{P}_{d,\sigma}$ the subset of $\text{Poly}_d \simeq \mathbb{C}^{d+1}$ whose elements are polynomials f with simple roots such that $(1/f)^{[\sigma-2]}$ has simple roots. This set is open for the Zariski topology (see Prop. 3) and its dimension is $d + 1$.

Definition 6. *We say that an analytic family of rational maps $\{f_\lambda,\ \lambda \in \Lambda\}$, is stable if there is a uniform bound on the period of superattracting cycles of the rational maps f_λ. We say the family is trivial if all of its members are conjugate by Moebius transformations.*

McMullen [McM1] defines a family to be stable if there is a uniform bound on the period of attracting cycles of the rational maps occuring in the family. Our definition is equivalent (see [McM3] for example).

Lemma 10. *If $\{K_{f,\sigma},\ f \in \Lambda \subset \mathcal{P}_{d,\sigma}\}$ is a trivial family of König's methods, then the dimension of Λ is at most* 3.

Proof. For all polynomial $f \in \mathcal{P}_{d,\sigma}$, ∞ is the only repelling fixed point of $K_{f,\sigma}$ with multiplier $1+(\sigma-1)/(d-1)$ (for $d=2$, ∞ is the only repelling fixed point and by proposition 3 (d), for $d>2$ and $f \in \mathcal{P}_{d,\sigma}$, the other repelling fixed points have multiplier $\sigma > 1+(\sigma-1)/(d-1)$). By definition, all the König's methods $K_{f,\sigma}$, $f \in \Lambda$ are conjugate via Moebius transformations. Those Moebius transformations must fix ∞. Therefore, they are affine maps. Lemma 9 implies that the family $\{K_{f,\sigma},\ f \in \Lambda\}$ is contained in the family $\{\lambda f_0(az+b),\ (\lambda,a,b) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}\}$, f_0 being any polynomial in Λ. □

Lemma 11. *For any (quasiprojective) variety $X \subset \mathcal{P}_{d,\sigma}$ of dimension greater than 3, there is no uniform bound on the period of superattracting cycles of the König's methods $K_{f,\sigma}$, $f \in X$.*

Proof. McMullen's rigidity result (see [McM1] Thm. 2.2) asserts that a stable algebraic family of rational maps is either trivial or its members are Lattès examples. Lattès examples do not have attracting cycles and thus, König's methods cannot be Lattès examples. □

Lemma 12. *For any degree $d \geq 2$ and any integer $\sigma \geq 2$, there exists a polynomial f of degree d such that $K_{f,\sigma}$ has at least $d-2$ extraneous superattracting cycles.*

Proof. If $d=2$, there is nothing to be proved. If $d>2$, the dimension of $\mathcal{P}_{d,\sigma}$ is $(d+1)>3$ and Lemma 11 asserts that one can find a polynomial $f_1 \in \mathcal{P}_{d,\sigma}$ such that $K_{f_1,\sigma}$ has a periodic critical point ω_1 which is periodic with some period $n_1>1$. If $d=3$, the proof is completed: $K_{f_1,\sigma}$ has at least one extraneous superattracting cycle.

If $d>3$ we can repeat this argument. For each polynomial $f \in \mathcal{P}_{d,\sigma}$, a point $z \in \mathbb{C}$ is a periodic point of $K_{f,\sigma}$ with period n_1 if and only if z is a root of some polynomial $P_f(z)$ whose coefficients are rational functions in the coefficients of f. For each polynomial $f \in \mathcal{P}_{d,\sigma}$, a point $z \in \mathbb{C}$ is a critical point of $K_{f,\sigma}$ if and only if z is a root of some polynomial $Q_f(z)$ whose coefficients are rational functions in the coefficients of f. The König's method, $K_{f,\sigma}$, has a periodic critical point with period n_1 if and only if the resultant of P_f and Q_f vanishes. Therefore the subset of $\mathcal{P}_{d,\sigma}$ for which $K_{f,\sigma}$ has a critical point which is periodic with period n_1, is an algebraic subset of $\mathcal{P}_{d,\sigma}$. The polynomial f_1 obtained above belongs to this subset. We define X_1 to be the irreducible component containing f_1. It is an irreducible quasiprojective variety contained in $\mathcal{P}_{d,\sigma}$. By the dimension theorem, the dimension of X_1 is at least $(d+1)-1=d$.

When $d>3$, McMullen's rigidity result implies that there exists a polynomial $f_2 \in X_1$ such that $K_{f_2,\sigma}$ has a critical point ω_2, periodic with period $n_2>n_1$. The subset of X_1 for which $K_{f,\sigma}$ has such a critical point is an algebraic subset, and the irreducible component X_2 containing f_2 has dimension at least $d-1$.

We can repeat this procedure, until we exhibit a quasiprojective variety $X_{d-2} \subset \mathcal{P}_{d,\sigma}$ of dimension at least $(d+1)-(d-2)=3$ and whose elements are polynomials f for which $K_{f,\sigma}$ has $d-2$ critical points $\omega_1,\omega_2,\ldots,\omega_{d-2}$, periodic with periods $n_1<n_2<\ldots<n_{d-2}$. □

Remark. The above argument generalizes to arbitrary purely iterative root-finding algorithms $T : \mathrm{Poly}_d \to \mathrm{Rat}_k$. The algorithm T is defined on an open subset of

Poly_d which has dimension $d+1$. If two rational maps $T(f)$ and $T(g)$ are conjugate by a Moebius transformation, the (super)attracting fixed points of $T(f)$ and the (super)attracting fixed points of $T(g)$ are related by the Moebius transformation. Since cross-ratios between those points must be preserved, and since there are at least $d-3$ such cross-ratios, the dimension of conjugacy classes is at most 4. The above argument implies that there exists a polynomial f of degree d such that $T(f)$ has at least $d-3$ periodic critical points.

3.3. **The upper bound on $N(d,\sigma)$ in the general case.** In this subsection, we show that for any $d \geq 2$ and any $\sigma \geq 2$, $N(d,\sigma) \leq (\sigma-1)(d-2)$. This inequality is obtained by counting the number of critical points of $K_{f,\sigma}$ which do not coincide with the roots of f.

Lemma 13. *Assume f is a polynomial and $\sigma \geq 2$ is an integer such that $K_{f,\sigma}$ has k extraneous non-repelling cycles. Then, there must be at least k critical points of $K_{f,\sigma}$ outside the basins of attractions of the roots of f.*

Proof. This lemma is essentially due to Fatou who proved that for any rational map,

- the immediate basin of attraction of an attracting or parabolic cycle always contains at least one critical point;
- the boundary of a cycle of Siegel disks is always accumulated by a critical orbit;
- a non-linearizable indifferent cycle is always contained in the closure of a critical orbit.

In other words, each non-repelling cycle must “trap” the orbit of at least one critical point. The only problem is that two distinct indifferent cycles need not trap two distinct critical points. However, using quasiconformal surgery, Douady-Hubbard [DH] and Shishikura [Shi] overcame this difficulty.

Roughly speaking, one can find a rational h which may be thought as a perturbation of $K_{f,\sigma}$ and which has the following properties:

- every non-repelling cycle of h is attracting;
- there is a bijection between the attracting cycles of h and the non-repelling cycles of $K_{f,\sigma}$;
- there is a bijection between the set of critical points of h and the set of critical points of $K_{f,\sigma}$;
- the critical points attracted by the roots of f under iteration of $K_{f,\sigma}$ correspond to critical points attracted by the attracting fixed points of h under iteration of h.

By construction, such a map h has k attracting cycles which correspond to the k extraneous non-repelling cycles of $K_{f,\sigma}$. Those attracting cycles attract k critical points of h by Fatou's theorem. Those critical points are in bijection with k critical points of $K_{f,\sigma}$ which are not attracted by the roots of f under iteration of $K_{f,\sigma}$. □

Definition 7. *A critical point of $K_{f,\sigma}$ which does not coincide with a root of f is called a* free critical point *of $K_{f,\sigma}$.*

Lemma 14. *Let f be a polynomial of degree $d \geq 2$. For any integer $\sigma \geq 2$, the König's method $K_{f,\sigma}$ has at most $(\sigma-1)(d-2)$ free critical points.*

Remark. Generically, the number of free critical points of $K_{f,\sigma}$ is exactly $(\sigma-1)(d-2)$.

Proof. Assume f has N distinct roots, N_1 of them being simple roots, and N_2 of them being multiple roots. Then, we have $N = N_1 + N_2$ and $N_1 + 2N_2 \le d$. Prop. 3 (b) asserts that the degree of $K_{f,\sigma}$ is at most $(\sigma-1)(N-1)+1$. Thus, the Riemann-Hurwitz formula implies that the number of critical points of $K_{f,\sigma}$ is at most $2(\sigma-1)(N-1)$. Moreover, $N_1(\sigma-1)$ critical points of $K_{f,\sigma}$ coincide with the roots of f. Therefore, there are at most

$$2(\sigma-1)(N-1) - N_1(\sigma-1) = (\sigma-1)(N_1+2N_2-2) \le (\sigma-1)(d-2)$$

free critical points. □

3.4. **The upper bound on $N(d,\sigma)$ in the case $2 < d \le \sigma$.** We will now show that when $\sigma \ge d > 2$, the König's method $K_{f,\sigma}$ has at most $(\sigma-1)(d-2)-1$ non-repelling cycles. Let N_1 denote the number of simple zeros of f and N_2 the number of multiple zeros. As above, we have $N_1+2N_2 \le d$. Let k denote the degree of $K_{f,\sigma}$; then $k \le (\sigma-1)(N_1+N_2-1)+1$, by Prop. 3 (b).

Counting multiplicity, the number of critical points of $K_{f,\sigma}$ is $2k-2$. Of these at least $(\sigma-1)N_1$ are trapped in basins corresponding to simple roots and N_2 in basins corresponding to multiple roots. Using Lemma 13 we get

$$\begin{aligned} N(d,\sigma) &\le 2k-2-(\sigma-1)N_1 - N_2 \le (\sigma-1)(N_1+2N_2-2) - N_2 \\ &\le (\sigma-1)(d-2) - N_2 \le (\sigma-1)(d-2). \end{aligned}$$

Equality can only be achieved if all the roots of f are simple ($N_2 = 0$), if each basin corresponding to such a root contains only $\sigma-1$ critical points of $K_{f,\sigma}$, and if $k = (\sigma-1)(N_1+N_2-1)+1 = (\sigma-1)(d-1)+1$. In this case properties (c), (d) and (e) of Prop. 3 are satisfied. Hence the bound holds for trivial reasons except in this case, which we settle by the following lemma.

Lemma 15. *Assume $\sigma \ge d > 2$ and f is a polynomial of degree d whose König's method $K_{f,\sigma}$ satisfies properties (c), (d) and (e) of Prop. 3. Then, there is at least one free critical point of $K_{f,\sigma}$ in the immediate basin of one superattracting fixed point of $K_{f,\sigma}$.*

Sketch of proof. Denote by $\alpha_1, \ldots, \alpha_d$, the d roots of f, by $\Omega_1, \ldots, \Omega_d$, the d immediate basins of those points under iteration of $K_{f,\sigma}$ and assume by contradiction that none of those basins contains a free critical point of $K_{f,\sigma}$.

Step 1. Then, each immediate basin Ω_i is simply connected (the argument is exactly the same as for proving that the basin of ∞ for a polynomial is simply connected if and only if the only critical point in this basin is ∞). By assumption, the local degree of $K_{f,\sigma}$ at each point α_i is exactly σ. It follows that there exists a conformal representation $\varphi_i : \Omega_i \to \mathbb{D}$ conjugating $K_{f,\sigma} : \Omega_i \to \Omega_i$ to the mapping $Q_\sigma : \mathbb{D} \to \mathbb{D}$ with $Q_\sigma(z) = z^\sigma$. We can define the internal ray $\mathcal{R}_i(\theta)$ by

$$\mathcal{R}_i(\theta) = \varphi_i^{-1}\{re^{2i\pi\theta} \ : \ r \in \,]0,1[\}.$$

We say that the ray $\mathcal{R}_i(\theta)$ lands at a point $\beta \in \mathbb{P}^1$ if

$$\lim_{r\to 1, r<1} \varphi_i^{-1}\left(re^{2i\pi\theta}\right) = \beta.$$

The König's method $K_{f,\sigma}$ maps the ray $\mathcal{R}_i(\theta)$, $\theta \in \mathbb{R}/\mathbb{Z}$, to the ray $\mathcal{R}_i(\sigma\theta)$. In particular, if $\sigma\theta = \theta$ mod(1), the ray $\mathcal{R}_i(\theta)$ is fixed and classical arguments that go back to Fatou imply that these fixed rays must land at fixed points of $K_{f,\sigma}$ which, in our case, must be repelling (see Hubbard-Schleicher-Sutherland [HSS], proof of Prop. 6, for more details). Figure 2 illustrates this situation. It shows an immediate basin where $K_{f,\sigma}$ is conjugate to Q_σ in the case $\sigma = 3$. There are two fixed rays landing at two repelling fixed points.

Step 2. We claim that if $\beta \in \mathbb{P}^1$ is a repelling fixed point of $K_{f,\sigma}$ with multiplier λ, the number of fixed internal rays that can land at β is strictly less than $2\log\sigma/\log\lambda$. This result is a version of the Pommerenke-Levin-Yoccoz inequality (see for example Hubbard [Hub], Levin [L], Petersen [Pe] or Pommerenke [Po]). We will just give the sketch of the proof. For more details, the reader is invited to consult Hubbard-Schleicher-Sutherland [HSS], Prop. 7.

In a neighborhood V_β of β, the dynamics of $K_{f,\sigma}$ is conjugate to $z \mapsto \lambda z$. Therefore, the quotient V_β modulo $K_{f,\sigma}$ is a torus $\mathbf{T}$ isomorphic to $\mathbb{C}^*$ modulo multiplication by λ. The Julia set $\mathcal{J}(K_{f,\sigma})$ projects to a set $\mathbf{J} \subset \mathbf{T}$ in this torus.

Each fixed internal ray that lands at β does so in a connected component of $V_\beta \cap \Omega_i$, where Ω_i is the immediate basin containing the ray. By hypothesis, the dynamics of $K_{f,\sigma}$ in $V_\beta \cap \Omega_i$ is conjugate to the dynamics of $z \mapsto z^\sigma$ in the unit disk intersected with a neighborhood of 1. Figure 2 illustrates this situation. It shows fundamental domains for the dynamics of the maps $K_{f,\sigma}$ and Q_σ in a neighborhood of the landing points of the two fixed rays.

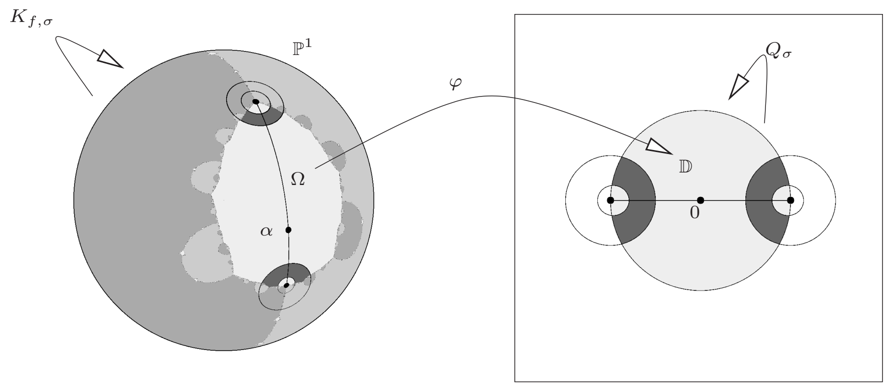

FIGURE 2. The conformal representation $\varphi : (\Omega, \alpha) \to (\mathbb{D}, 0)$ between the immediate basin Ω of $\alpha = 4$ and the unit disk $\mathbb{D}$, conjugates the Halley's method $K_{f,\sigma}$, for $f(z) = (z-4)(z^2+1)$ and $\sigma = 3$, to $Q_\sigma : z \mapsto z^\sigma$. There are two fixed rays landing at two repelling fixed points. Fundamental domains for the dynamics of $K_{f,\sigma}$ and Q_σ near those fixed points are drawn.

As a consequence, one easily proves that the fixed external rays that land at β project to simple closed curves in $\mathbf{T}$ and that the connected components of $\mathbf{T} \setminus \mathbf{J}$ that contain those simple closed curves are disjoint annuli with moduli all equal to $\pi/\log\sigma$.

Then, a classical length-area inequality asserts that the sum of moduli of the disjoint annuli is not greater than $2\pi/\log\lambda$. Moreover, if equality were achieved, the annuli would be round annuli and the torus $\mathbf{T}$ would be equal to the union of the closed annuli. As a consequence, the set $\mathbf{J} \subset \mathbf{T}$ would be a union of circles. Thus, the intersection $\mathcal{J}(K_{f,\sigma}) \cap V_\beta$ would consist in finitely many analytic arcs. Therefore, any point of $\mathcal{J}(K_{f,\sigma}) \cap V_\beta$ would be in the boundary of at most two Fatou components. However, when $d > 2$, $K_{f,\sigma}$ has at least 3 basins of attraction and any point of $\mathcal{J}(K_{f,\sigma})$ must be in the boundary of all the basins of attraction. Thus, when $d > 2$, the sum of moduli of the disjoint annuli is strictly less than $2\pi/\log\lambda$.

Step 3. By assumption, the König's method $K_{f,\sigma}$ has $(\sigma-2)(d-1)$ repelling fixed points in $\mathbb{C}$ and their multipliers are all equal to σ. We just saw that each of these is the landing point of at most one internal ray. In each basin, there are $\sigma - 1$ fixed rays. Thus, there are at least $d(\sigma-1) - (\sigma-2)(d-1) = d + \sigma - 2$ fixed internal rays that land at ∞.

The multiplier of $K_{f,\sigma}$ at ∞ is $1 + (\sigma-1)/(d-1)$. Thus, we have the inequality

$$d + \sigma - 2 < 2\frac{\log\sigma}{\log\left(1 + \dfrac{\sigma-1}{d-1}\right)}.$$

By concavity of the logarithm, we also have

$$\log\left(1 + \frac{1}{d-1}(\sigma-1)\right) \geq \log 1 + \frac{1}{d-1}(\log\sigma - \log 1) = \frac{\log\sigma}{d-1}.$$

Thus, we get

$$d + \sigma - 2 < 2\frac{\log\sigma}{\log\left(1 + \dfrac{\sigma-1}{d-1}\right)} \leq 2(d-1).$$

We have a contradiction since $d \leq \sigma$. □

4. The Julia sets and bifurcation loci.

4.1. **The limit of $\mathcal{J}(K_{f,\sigma})$ as σ tends to $+\infty$.** Figure 3 shows the Julia sets of the König's methods $K_{f,\sigma}$ for $f(z) = z(z-1)(z-i)(z+i)(z+1-i)$ and $\sigma = 2$, 5, 10 and 100. The pixels are colored according to which root of f their orbit under iteration of $K_{f,\sigma}$ converge. The Julia set appears in black.

We will now prove Prop. 6: as σ tends to ∞, the sequence of Julia sets $\mathcal{J}(K_{f,\sigma})$ converges, for the Hausdorff topology on compact subsets of $\mathbb{P}^1$, to $\{\infty\}$ union the bisecting locus of the set of roots of f.

Let f be a polynomial with at least 2 distinct roots and let $\mathcal{B}$ denote the union of $\{\infty\}$ and the bisecting locus of the set of roots of f. For any point $z \in \mathbb{P}^1 \setminus \mathcal{B}$, there is a unique zero α of f closest to z. By Prop. 2, there exists a neighborhood V_z on which the sequence $K_{f,\sigma}$ converges uniformly to α as σ tends to $+\infty$.

It follows that for each root α, $K_{f,\sigma}(V_\alpha) \subset V_\alpha$ for all sufficiently large σ. Thus, V_α is contained in the immediate basin of attraction of α for $K_{f,\sigma}$, as soon as σ is sufficiently large. Similarly, if z does not belong to $\mathcal{B}$ and α is the root of f closest to z, $K_{f,\sigma}(V_z) \subset V_\alpha$ for all sufficiently large σ. As a consequence, V_z is contained in the immediate basin of α for $K_{f,\sigma}$, as soon as σ is sufficiently large.

Consequently, any spherical ε-neighborhood of $\mathcal{B}$ contains the Julia set $\mathcal{J}(K_{f,\sigma})$ for all sufficiently large σ.

To finish the proof we must also show that $\mathcal{B}$ is contained in a spherical ε-neighborhood of $\mathcal{J}(K_{f,\sigma})$ for all σ sufficiently large. By compactness it is enough

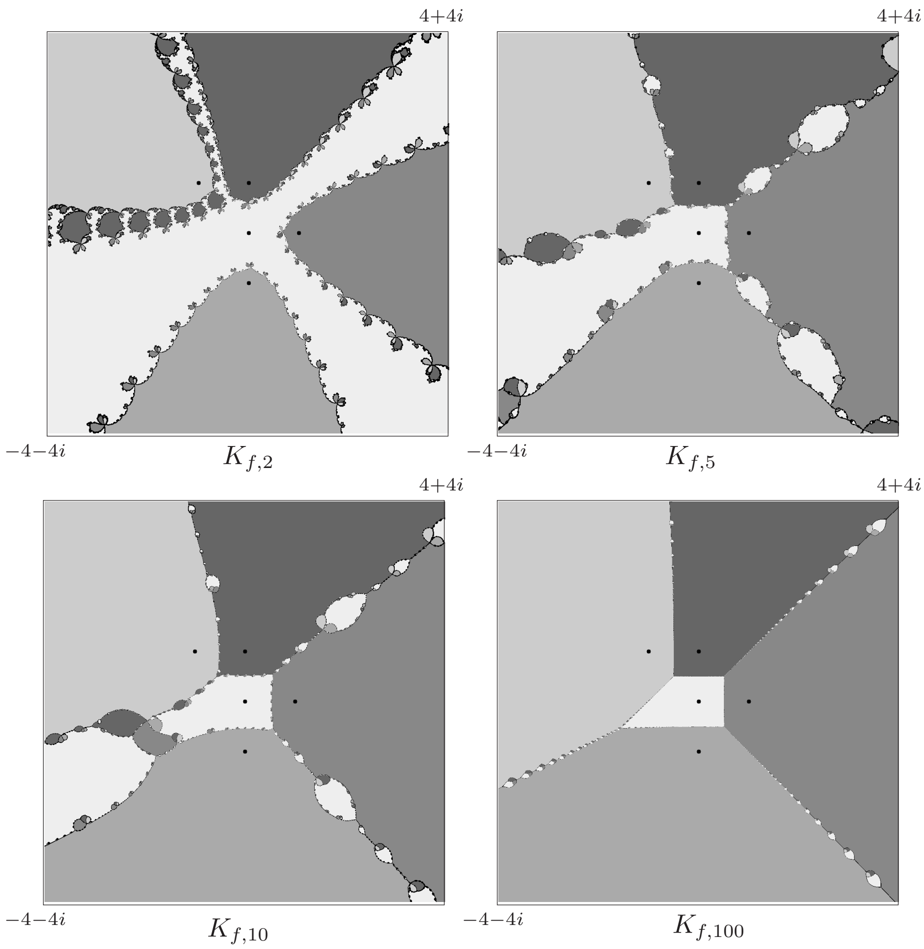

FIGURE 3. The Julia sets of the König's methods $K_{f,\sigma}$ for the polynomial $f(z) = z(z-1)(z-i)(z+i)(z+1-i)$ and $\sigma = 2$, 5, 10 and 100.

to show this for each point $z \in \mathcal{B}$. Let D denote a spherical ball of radius ε centered at z. Take $z_1, z_2 \in D$ in different components of the complement of $\mathcal{B}$. This is possible since f has more than one root. We have just seen that the points z_1 and z_2 will be in different basins of attraction of $K_{f,\sigma}$ for all sufficiently large σ. This shows that $\mathcal{J}(K_{f,\sigma})$ intersects D for all sufficiently large σ.

Remark. Let f be a degree d polynomial with a unique root α closest to 0, which is allowed to be multiple. Setting $x_n = K_{n+2,f}(0)$, then x_n converges to α by Prop. 2. A computation shows that

$$x_{n+1} = \frac{-a_0}{a_1 + a_2 x_n + a_3 x_n x_{n-1} + \cdots + a_d x_n \cdots x_{n-d+2}},$$

where we have set $x_{-1} = x_{-2} = \ldots = x_{-d+1} = 0$. So this is a simple root finding algorithm that works for almost every polynomial. The algorithm is known (see for example [Ho2]), and convergence is generally slow. Notice that this is not a purely iterative algorithm in the sense defined previously since it requires keeping track of $d-1$ successive approximations at each step.

4.2. **Pictures of parameter spaces for König's methods of cubic polynomials.**

4.2.1. *A possible parametrization.* The parameter space of cubic polynomials $f(z) = a_3z^3+a_2z^2+a_1z+a_0$ may be identified with $\mathbb{C}^4$. Prop. 9 tells us that when two polynomials f_1 and f_2 are equal up to pre-composition with an affine automorphism of $\mathbb{C}$ and up to post-composition with a scaling map, then the König's methods $K_{f_1,\sigma}$ and $K_{f_2,\sigma}$ are conjugate. Therefore, we may restrict our study to centered polynomials (i.e., the barycentre of the roots is equal to 0 or equivalently, $a_2 = 0$). We may also assume that the polynomial is monic (i.e., $a_3 = 1$). We are still allowed to normalize our polynomials by pre and post-composing with a scaling map (keeping the property of being monic). This will reduce the study to a one dimensional parameter space. There is no canonical way of doing this reduction. One possibility is to ask that one of the roots be equal to 1. In this way, we omit the polynomial $z \mapsto z^3$. The König's method of this polynomial is $K_{f,\sigma}(z) = 2z/(\sigma+1)$. However, with this normalization, we still have different polynomials with conjugate König's methods.

We prefer to normalize our polynomials by setting $a_0 = a_1$. Indeed, if $a_0 \neq 0$ and $a_1 \neq 0$, then setting $\lambda = a_1/a_0 \in \mathbb{C}^*$ and $a = a_0^2/a_1 \in \mathbb{C}^*$, we have $f(\lambda z)/\lambda^3 = z^3 + az + a$. For $a = 0$, we have the polynomial z^3. Therefore, up to conjugacy by an affine map, the only omitted König's methods are those of the polynomials $z^3 + a_0$ and $z^3 + a_1 z$. They are respectively conjugate to the König's method of the polynomials $z^3 - 1$ and $z(z^2-1)$. The advantage of this normalization is the following.

Lemma 16. *Let $f_1(z) = z^3 + a_1z + a_1$ and $f_2(z) = z^3 + a_2z + a_2$ be two cubic polynomials. The two König's methods $K_{f_1,\sigma}$ and $K_{f_2,\sigma}$ are affine conjugate if and only if $a_1 = a_2$.*

Proof. Let us write $A(z) = bz + c$. Prop. 9 asserts that $K_{f_1,\sigma}$ and $K_{f_2,\sigma}$ are affine conjugate if and only if there exists $\lambda \in \mathbb{C}^*$ such that

$$z^3 + a_1z + a_1 = \lambda\Big((bz+c)^3 + a_2(bz+c) + a_2\Big).$$

After simplifications, this gives $\lambda = 1$, $b = 1$, $c = 0$ and $a_1 = a_2$. □

4.2.2. *The pictures.* We will now investigate parameter spaces of $K_{f_a,\sigma}$ for various values of σ.

As we have seen previously, the König's method $K_{f_a,\sigma}$ has generically $(\sigma-1)$ free critical points. A relevant picture consists in coloring the parameters $a \in \mathbb{C}$ according to the number of critical points attracted by each root of f_a. The bifurcation locus $\mathcal{B}_\sigma$ appears in black on each picture (see definition 4 for the definition of bifurcation locus). A theorem by Mañe, Sad and Sullivan [MSS] asserts that this set has empty interior. The bifurcation locus has been defined as the set of parameters where the period of superattracting cycles is not locally bounded. It is possible to prove that it is also the closure of the set of parameters where the Julia set does not vary continuously (see for example McMullen's book [McM2]).

The bifurcation locus also coincides with the set of parameters where the behaviour of the free critical points under iteration is not stable.

Observe that the polynomial f_a has multiple roots as soon as the discriminant $\Delta(f_a) = -a^2(4a+27)$ vanishes, i.e., for $a=0$ and $a=-27/4$ (we plotted those parameters on the parameter pictures). Thus, when $a \in \mathbb{C} \setminus \{-27/4, 0\}$, f_a has three distinct roots α_1, α_2 and α_3. In that case, an easy computation shows that the free critical points of $K_{f_a,\sigma}$ are α_1, α_2, α_3 and the roots of the polynomial:

$$Q_{a,\sigma}(z) = \frac{(z-\alpha_1)^{\sigma+1}}{f_a'(\alpha_1)} + \frac{(z-\alpha_2)^{\sigma+1}}{f_a'(\alpha_2)} + \frac{(z-\alpha_3)^{\sigma+1}}{f_a'(\alpha_3)}.$$

In fact, the degree of $Q_{a,\sigma}$ is generically $\sigma - 1$ (there are simplifications). When $\sigma \le 5$, the equation $Q_{a,\sigma} = 0$ is solvable by radicals. However, as soon as $\sigma \ge 5$, one gets into trouble. To overcome this difficulty, one may try and find a good approximation of the roots of $Q_{a,\sigma}$ using Newton's method. The problem consists in finding good initial starting points to which apply Newton's method. One solution is provided by Hubbard-Schleicher-Sutherland [HSS].

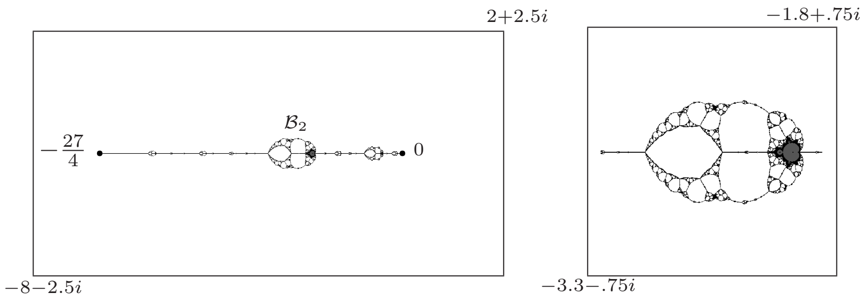

FIGURE 4. The parameter space for the family $K_{f_a,2}$, $a \in \mathbb{C}$.

For $\sigma = 2$, König's method is the same as Newton's method. When $a \notin \{0, -27/4\}$, the König map $K_{f_a,2}$ has one free critical point: 0. Figure 4 shows the set of parameters $a \in \mathbb{C}$ for which this critical point is not in the basin of attraction of one of the roots of f_a. The boundary of this set is the bifurcation locus $\mathcal{B}_2$. It appears in black on Figure 4. The unbounded connected component of $\mathbb{C} \setminus \mathcal{B}_2$ corresponds to parameters a for which 0 is in the immediate basin of one of the roots of f_a. The picture to the right is an enlargement on a copy of the Mandelbrot set. The existence of such a copy is not surprising: McMullen [McM3] showed that copies of the Mandelbrot set (or generalizations known as multibrot sets) are dense in the bifurcation locus of any analytic family of rational maps.

Figure 5 shows the parameter spaces for the families of König's methods of order 3 and 4. Generically, the König's methods $K_{f_a,3}$ have two free critical points. The set of parameters for which the two free critical points are in the basin of attraction of the same root of f_a is colored white. The set of parameters for which the two free critical points are in the basin of attraction of a root of f_a, but not in the same basin is colored light grey. The main components Ω_0 and Ω_1 correspond to

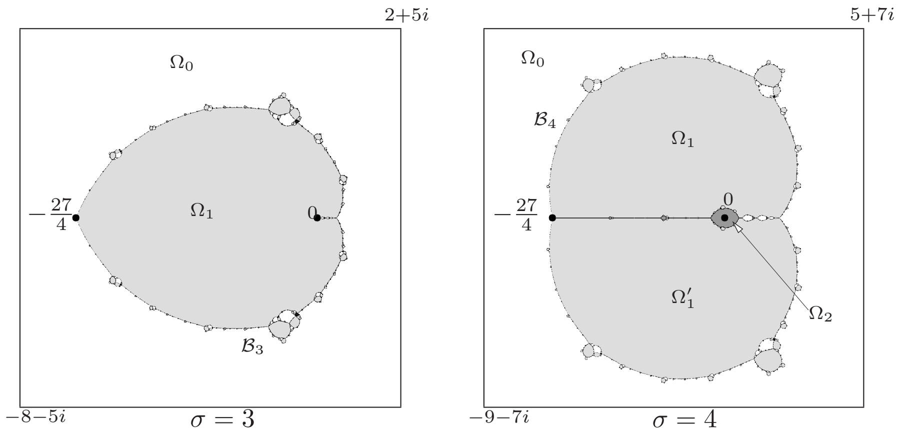

FIGURE 5. Left: the parameter space for König's methods of order 3. Right: the parameter space for König's methods of order 4.

parameters for which the two free critical points are in the immediate basins of the roots of f_a. The bifurcation locus $\mathcal{B}_3$ is colored in black. Again, one may find tiny copies of the Mandelbrot set by zooming on the bifurcation locus.

Generically, the König's methods $K_{f_a,4}$ have three free critical points. The coloring of the parameter pictures depends on whether the three free critical points lie in the same basins or not: white when they all belong to the same basin (for example in Ω_0), light grey when two of them are in the same basin and the third is in another basin (for example in ω_1 or ω_1'), dark grey when the three critical points are in three distinct basins (for example in Ω_2). The components Ω_0, Ω_1, Ω_1' and Ω_2 correspond to parameters for which the three free critical points are in the immediate basins of the roots.

Figure 6 shows a similar picture for König's methods of order 5. The two top-right pictures show an enlargement onto a copy of the Mandelbrot set. Generically, the König's methods $K_{f_a,5}$ have four free critical points. The coloring of the parameter pictures depends on whether the four free critical points lie in the same basins or not: white when they all are in the same basin (for example in Ω_0), light grey when three of them are in the same basin (for example in Ω_1 and Ω_1'), darker grey when one of the basin contains no free critical point and the other two contains two free critical points (for example in Ω_2), and the darkest grey when two basins contain one free critical point and the remaining basin contains two free critical points (for example in Ω_3). Figure 6 also shows some dynamical pictures which illustrate each of these cases. The free critical points ω_i, $i = 1..4$, of the corresponding König's methods are plotted. On the bottom-right picture, one can observe that the free critical point ω_4 is not in the basin of a root of f_a. There is one extraneous attracting cycle.

Figure 7 shows parameter pictures for higher order König's methods of cubic polynomials. The size of the screen is fixed. There does not seem to be any obvious limit behaviour of the bifurcation locus as σ increases.

While drawing all those pictures, we looked for cubic polynomials whose König's methods have several extraneous cycles. For this purpose, while scanning the screen, we counted the number of free critical points which were outside the basins of the

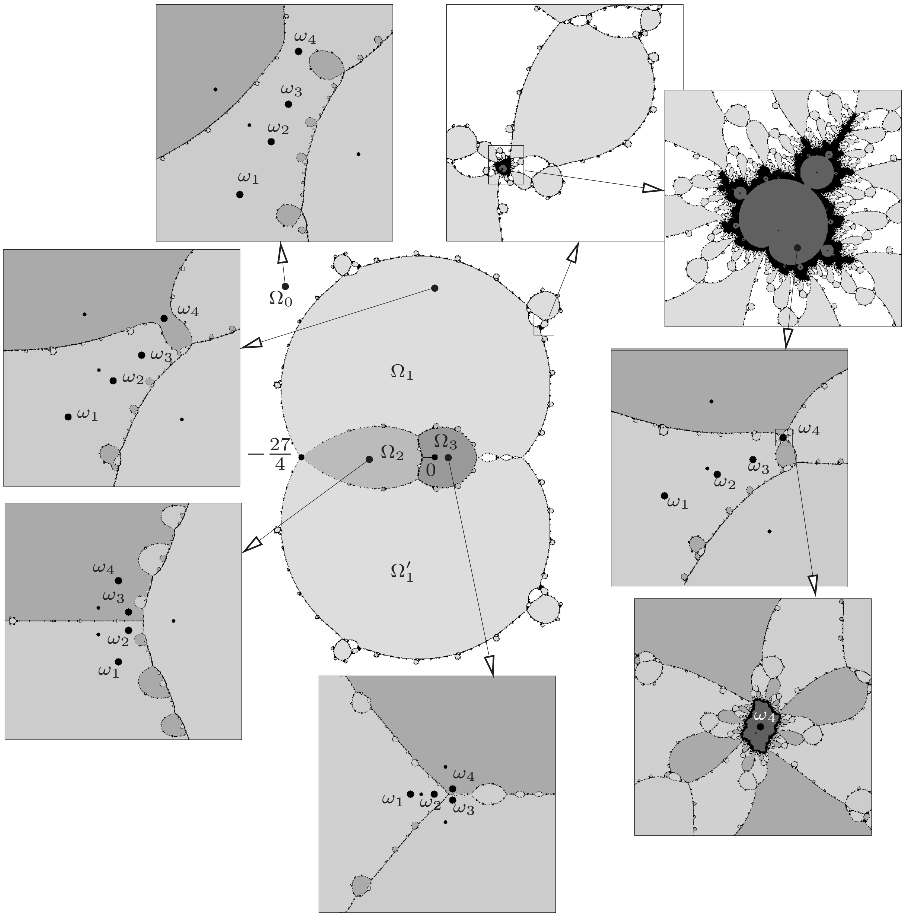

FIGURE 6. The parameter space for the family $K_{f_a,5}$, $a \in \mathbb{C}$, with enlargements on a copy of the Mandelbrot set and some dynamical pictures.

roots of f_a. We have not been able to locate any König's method of cubic polynomial with more than one extraneous cycle.

5. Appendix. On the number of distinct zeros of some rational maps.

In this section, we will use some dynamical results studied above in order to obtain lower bounds on the number of distinct zeros of some rational maps. This idea already appears in the work of Bergweiler and Eremenko [BE] and in the work of Epstein [E].

Proposition 8. *Assume $P : \mathbb{C} \to \mathbb{C}$ is a polynomial having at least 2 distinct zeros and n multiple zeros. Then, the nonlinearity*

$$\mathfrak{N}_P = \frac{P''}{P'}$$

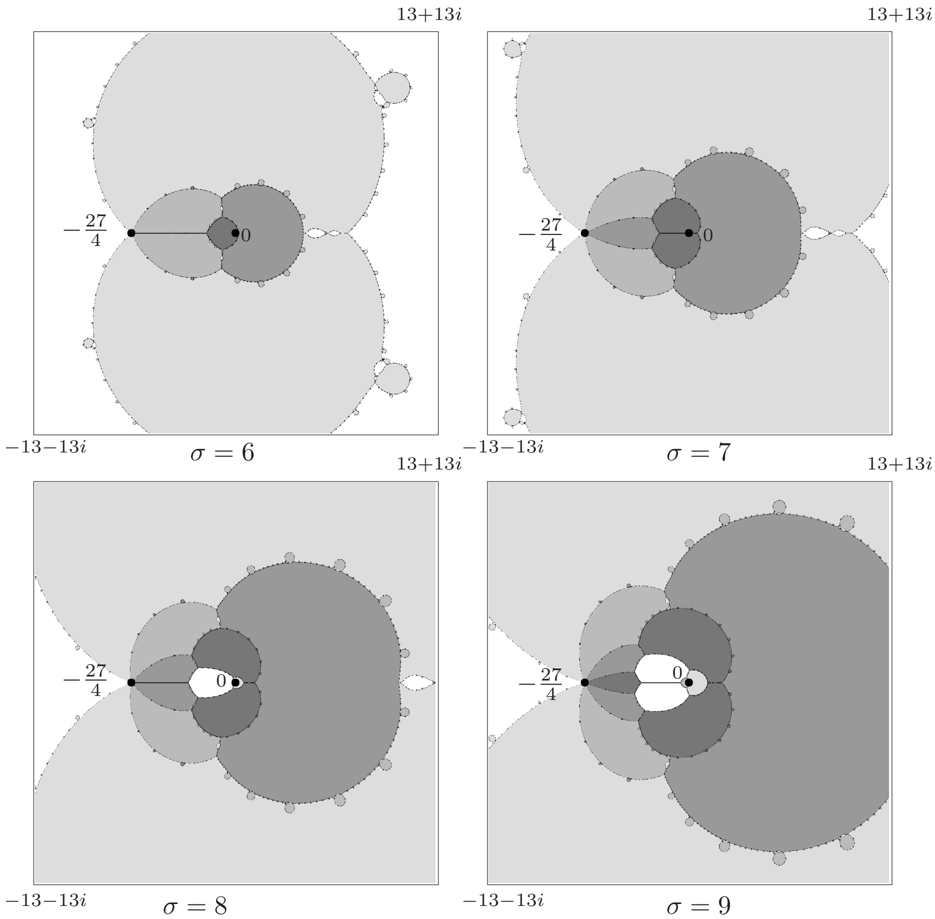

FIGURE 7. The parameter spaces for the family $K_{f_a,\sigma}$, for $a \in \mathbb{C}$ and $\sigma = 6$, 7, 8 and 9.

has at least n distinct zeros which do not coincide with the zeros of P.

Proof. The proof is elementary. Each multiple zero of P gives rise to an attracting (but not superattracting) fixed point of its Newton's method $N_P = K_{P,2}$. Therefore, N_P has n attracting (but not superattracting) fixed points. The hypothesis that P has at least two distinct zeors guaranties that N_P has degree 2 or more. By Fatou's theorem, the basin of attraction of each such fixed point must contain a critical point of N_P. The critical points of N_P are the simple zeros of P and the zeros of P'' which are not zeros of P'. Thus, each of the n basins of attraction contains a zero of P'' which is neither a zero of P nor a zero of P'. ∎

This result is sharp. Indeed, for any integer $n \geq 2$, the polynomial $P : z \mapsto (z^n - 1)^2$ has n multiple zeros (the n^{th} roots of unity). Its nonlinearity

$$\mathfrak{N}_P = \frac{(2n-1)z^n - (n-1)}{z(z^n - 1)}$$

has exactly n distinct zeros (the n^{th} root of $(n-1)/(2n-1)$).

The points where the second derivative of P vanishes are the points where P is best approximated by affine maps. When considering a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$, it may be more natural to consider the points where the rational maps is best approximated by Moebius transformations. Those are the points where the Schwarzian derivative of f vanishes.

Proposition 9. *Assume $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map having at least 2 distinct zeros and n multiple zeros. Then, the Schwarzian derivative*

$$\mathfrak{S}_f = \mathfrak{N}_f' - \frac{1}{2}\mathfrak{N}_f^2$$

has at least n distinct zeros which do not coincide with the zeros of f.

Proof. The proof is almost the same as above. Instead of considering Newton's method, we will consider Halley's method $H_f = K_{f,3}$. Again, the multiple zeros of H_f give rise to attracting (but not superattracting) fixed points of H_f. The critical points of H_f are the simple zeros of f and the zeros of the Schwarzian derivative. The proposition follows as above. ■

Remark. In this situation, we do not know whether this result is optimal or not.

We have seen that the number of distinct zeros of $(1/f)^{[\sigma-2]}$ is equal to the number of repelling fixed points of $K_{f,\sigma}$ in $\mathbb{C}$. This leads us to the following result.

Proposition 10. *Let $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, d$ be d distinct complex numbers, $\lambda_i \in \mathbb{C}^*$ be arbitrary complex numbers, and $n \geq 1$ be an integer. Define $g : \mathbb{P}^1 \to \mathbb{P}^1$ to be the rational map*

$$g(z) = \sum_{i=1}^{d} \frac{\lambda_i}{(z - \alpha_i)^n}.$$

Let N be the number of zeros of g in $\mathbb{C}$ counted with multiplicities, and M be the number of distinct zeros. Then, we have

$$(d-1)(n-1) + 1 \leq N \leq (d-1)n \quad \text{and} \quad \frac{dn}{2} - d + 1 \leq M \leq (d-1)n.$$

Proof. The bounds on N are obtained as follows. The rational map g is proportional to the $(n-1)$-th derivative of the rational map

$$\sum_{i=1}^{d} \frac{\lambda_i}{z - \alpha_i}.$$

This rational map has a zero at infinity of order at least 1 and at most d (in which case it is the reciprocal of a polynomial of degree d). Thus g has a zero at infinity of order at least $1 + (n-1)$ and at most $d + (n-1)$. Since g has dn poles, counting multiplicities, it follows that it has dn zeros in $\mathbb{P}^1$, counting multiplicities, and the bounds on N follow.

We now have to prove the lower bound on M. Let us introduce the rational map $h = \mathrm{Id} + ng/g'$. This rational map has superattracting fixed points with local degree at least $n+1$ at the points α_i. Therefore, its degree is at least $dn/2+1$, again by Fatou's theorem. Consequently, it has at least $dn/2+2$ fixed points in $\mathbb{P}^1$, and thus, at least $dn/2+1-d$ distinct repelling fixed points in $\mathbb{C}$. Those fixed points are distinct zeros of g. ■

6. Acknowledgement

The authors thank the referees for their many useful comments.

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