DYNAMICAL CONVERGENCE AND POLYNOMIAL VECTOR FIELDS

Xavier Buff & Tan Lei

Abstract

Let $f_n \to f_0$ be a convergent sequence of rational maps, preserving critical relations, and $f_0$ be geometrically finite with parabolic points. It is known that for some unlucky choices of sequences $f_n$, the Julia sets $J(f_n)$ and their Hausdorff dimensions may fail to converge as $n \to \infty$. Our main result here is to prove the convergence of $J(f_n)$ and $H\dim J(f_n)$ for generic sequences $f_n$. The same conclusion was obtained earlier, with stronger hypotheses on the sequence $f_n$, by Bodart-Zinsmeister and then by McMullen. We characterize those choices of $f_n$ by means of flows of appropriate polynomial vector fields (following Douady-Estrada-Sentenac). We first prove an independent result about the ($s$-dimensional) length of separatrices of such flows, and then use it to estimate tails of Poincaré series. This, together with existing techniques, provides the desired control of conformal densities and Hausdorff dimensions. Our method may be applied to other problems related to parabolic perturbations.

1. Introduction

We say that a sequence of rational maps $f_n$ converges to $f_0$ algebraically if $\deg f_n = \deg f_0$ and the coefficients of $f_n$ (as a ratio of polynomials) can be chosen to converge to those of $f_0$. Algebraic convergence is equivalent to uniform convergence for the spherical metric on $\mathbb{P}^1$.

Assume $f_n \to f_0$ algebraically. Let $J(f_n)$ be the Julia set of $f_n$ and $H\dim J(f_n)$ be its Hausdorff dimension. The question that interests us here is: do we have

$$J(f_n) \underset{n \to \infty}{\longrightarrow} J(f_0) \quad \text{and} \quad H\dim J(f_n) \underset{n \to \infty}{\longrightarrow} H\dim J(f_0)?$$

(The limit of Julia sets is for the Hausdorff topology on compact subsets of $\mathbb{P}^1$.)

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If $f_0$ is a hyperbolic rational map, the answer is yes. But in general, one only has

$$J(f_0) \subset \liminf J(f_n) \quad \text{and} \quad \text{H.d.} \ J(f_0) \leq \liminf \text{H.d.} \ J(f_n).$$

The typical example where those inequalities are strict is given by $f_n(z) = \lambda_n z + z^2$ with $\lambda_n \to 1$ (and $f_0(z) = z + z^2$). On the one hand, if $\operatorname{Re}(1/(1 - \lambda_n))$ remains bounded (i.e., $\lambda_n \to 1$ avoiding two disks $D(1 + \varepsilon, \varepsilon)$ and $D(1 - \varepsilon, \varepsilon)$ with $\varepsilon > 0$), then $J(f_0) \subset \liminf J(f_n)$ (see Douady [D], 1994) and $\text{H.d.} \ J(f_0) < \liminf \text{H.d.} \ J(f_n)$ (see Douady-Sentinac-Zinsmeister [DSZ], 1997). On the other hand, if $\lambda_n \to 1 \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$, then $J(f_n) \to J(f_0)$ and $\text{H.d.} \ J(f_n) \to \text{H.d.} \ J(f_0)$ (see Bodart-Zinsmeister [BZ], 1996).

This last result was generalized by McMullen [McM], 2000, who proved the following result. Let $f_0$ be a geometrically finite rational map (i.e. every critical point in $J(f)$ has a finite forward orbit) and let $f_n \to f_0$ algebraically, preserving the critical relations (see the statement of Theorem A for a precise definition). For each parabolic point $\beta \in J(f_0)$, let $j$ be the least integer such that $f_0^j$ fixes $\beta$ with multiplier 1, let $p$ be the number of petals of $f_0$ at $\beta$. Assume in addition

(a) $f_0^j$ has a fixed point $\beta_n$ converging to $\beta$ with multiplier $\lambda_n$, and $p$ simple fixed points which are symmetrically placed at the vertices of an almost regular $p$-mirror centered at $\beta_n$ (see definition 2.16 for a more rigorous statement).

Then, $J(f_n) \to J(f_0)$ and $\text{H.d.} \ J(f_n) \to \text{H.d.} \ J(f_0)$ as long as

(b) $\lambda_n - 1 \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$. 

In this article, we will prove the same result in its full generality, namely without the extra assumption (a). Therefore the perturbed fixed points do not have to present any symmetry, and may very well fail to be simple. In this case, the characterization of good perturbations, namely condition (b), is no longer valid. We will replace it by the notion of 'stable perturbations' in terms of some polynomial vector fields. This will take us some time to describe.

(In [McM], McMullen shows that condition (b) can be replaced by condition $\text{H.d.} \ J(f_0) > 2p(f_0)/(p(f_0) + 1)$ and $\operatorname{Re}(1/(1 - \lambda_n)) \to \infty$, where $p(f_0)$ denotes the maximum number of petals at a parabolic point of $f_0$ or one of its preimages. We do not generalize this result.)

Set $D(r) = \{z : |z| < r\}$. Let $f_0 : D(r) \to \mathbb{C}$ be a holomorphic map in the form $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ with $p \geq 1$, in other words $f_0$ has a multiple fixed point at 0. Most of our work consists in studying the dynamics of holomorphic maps $f : D(r) \to \mathbb{C}$ which are small perturbations of $f_0$, and in finding out which perturbations are dynamically stable with respect to $f_0$.

If $p = 1$, $f_0(z) = z + z^2 + \mathcal{O}(z^3)$ and if $f(z) = \lambda z + \mathcal{O}(z^2)$, $\lambda \neq 1$, is a small perturbation of $f_0$, then $f$ has two simple fixed points close to 0:
0 itself and \( \sigma \). The classical method is to use the Möbius transformation \( z \mapsto w = z/(z - \sigma) \) to pull apart the two fixed points. And then, in a suitably normalized log-coordinate of \( w \), our map \( f \) is conjugate to a map close to the translation \( Z \mapsto Z + 1 \). In such a way, we obtain the so-called approximate Fatou coordinates and may analyse the dynamics in these coordinates.

When \( p > 1 \), there are too many fixed points to be pulled-apart by a Möbius transformation. The advantage of Assumption (a) is that one may quotient out the symmetry and reduce the situation to the case with two fixed points. Without this assumption, we will have to take a different approach. The key idea, developed by Douady, Epstein, Oudkerk and Shishikura (among others), is to approximate \( f \) by the time-one map of the flow of \( \dot{z} = P_f(z) \), or of an appropriate polynomial differential equation. More precisely, a small perturbation \( f \) of \( f_0 \) has \( p + 1 \) fixed points close to 0, counting multiplicities. Let \( P_f \) be the monic polynomial of degree \( p + 1 \) which vanishes at those points. Then one can prove that the time-one map of the flow of \( \dot{z} = P_f(z) \) gives a good approximation of short term and sometimes long term iterates of \( f \); further, the complex time coordinates of the differential equation provide excellent approximate Fatou coordinates for \( f \) (see Lemma 5.1).

One may then expect to describe different types of perturbations in terms of the corresponding flows. Following [DES], we say that a maximal real-time solution \( \psi : [t_{\text{min}}, t_{\text{max}}] \to \mathbb{C} \) for the polynomial differential equation \( \dot{z} = P(z) \) is a homoclinic connection (at infinity) if \( t_{\text{min}} \) and \( t_{\text{max}} \) are finite and \( \lim_{t \to t_{\text{min}}} \psi(t) = \lim_{t \to t_{\text{max}}} \psi(t) = \infty \). For \( \alpha \in ]0, \frac{\pi}{2}[ \), the polynomial \( P \) is called \( \alpha \)-stable if the differential equation of the rotated vector field, \( \dot{z} = e^{i\theta}P(z) \), has no homoclinic connection, for any \( \theta \in [\alpha, \alpha] \).

Let \( (f_n : D(r) \to \mathbb{C})_{n \geq 1} \) be a sequence of holomorphic maps converging locally uniformly to a holomorphic map \( f_0 : D(r) \to \mathbb{C} \) with \( f_0(z) = z + z^{p+1} + O(z^{p+2}) \), \( p \geq 1 \). We may now define that the convergence \( f_n \to f_0 \) is stable at 0 if for \( n \) sufficiently large, the corresponding monic polynomials \( P_{f_n} \) of degree \( p + 1 \) are \( \alpha \)-stable for some uniform \( \alpha \in ]0, \frac{\pi}{2}[ \). And more generally, an algebraically convergent sequence of rational maps \( f_n \to f_0 \) is stable if for each parabolic point of \( f_0 \), there are suitable local coordinates in which the convergence is stable.

From results in [DES] we know already two important properties of this concept:

- First, stable perturbations are “generic”. For instance, in an analytically parameterized family

  \[
  (f_\lambda : D(r) \to \mathbb{C})_{\lambda \in \mathbb{D}} \quad \text{with} \quad f_0(z) = z + z^{p+1} + O(z^{p+2}),
  \]

  there exists a finite set of directions (the implosive directions), such that the convergence \( f_\lambda \to f_0 \) is stable at 0 as soon as \( \lambda \to 0 \) avoiding
a sector neighborhood of those directions. For example, in the family
\((1 - \lambda)z + z^2\), there are exactly two implosive directions, namely \(i\mathbb{R}^+\) and \(i\mathbb{R}^-\).

- Second, stable convergence implies convergence of Julia sets. More precisely, let \(f_n\) be a sequence of rational maps converging algebraically to \(f_0\), such that \(f_0\) has neither Siegel disks nor Herman rings. If convergence is stable then \(J(f_n) \to J(f_0)\).

We may now state our main result in this paper:

**Theorem A.** Assume \(f_0\) is a geometrically finite rational map (i.e. every critical point in \(J(f_0)\) has a finite forward orbit), and \(f_n \to f_0\) algebraically, preserving the critical relations on \(J(f_0)\) (i.e. for every critical point \(b \in J(f_0)\) satisfying \(f_0^{\circ j}(b) = f_0^{\circ j}(b)\), there are critical points \(b_n \to b\) for \(f_n\), with the same multiplicity as \(b\), also satisfying \(f_n^{\circ j}(b_n) = f_n^{\circ j}(b_n)\)). If the convergence is stable, then for \(n\) large enough, \(f_n\) is geometrically finite, \(J(f_n) \to J(f_0)\) and \(\text{H.dim } J(f_n) \to \text{H.dim } J(f_0)\).

(We will provide a self-contained proof, namely independent of results in [DES]).

The reason why \(f_0\) is assumed to be geometrically finite is that in that case, there is a unique non-atomic \(f_0\)-invariant conformal measure \(\mu_{f_0}\) supported on \(J(f_0)\), and the dimension of \(\mu_{f_0}\) is equal to \(\text{H.dim } J(f_0)\) (an \(f_0\)-invariant conformal measure of dimension \(\delta > 0\) is a probability measure \(\mu\) on \(\mathbb{P}^1\) such that \(\mu(E) = \int_E |f_0'(x)|^\delta\, d\mu\) whenever \(f_0|E\) is injective). This conformal measure is called the **canonical conformal measure**.

Once we know that for \(n\) large enough, \(f_n\) is geometrically finite and that \(J(f_n) \to J(f_0)\), we see that any weak accumulation point \(\nu\) of the canonical conformal measures \(\mu_{f_n}\) is an \(f_0\)-invariant conformal measure and is supported on \(J(f_0)\). In order to prove that

\[
\text{H.dim } J(f_n) \to \text{H.dim } J(f_0),
\]

it is therefore enough to show that \(\nu\) is non-atomic.

The proofs of Bodart-Zinsmeister and of McMullen can be both decomposed into two parts: the first one is to use appropriate Fatou coordinates to establish the convergence of tails of Poincaré series (in the terminology of McMullen), and the second is to show that this intermediate convergence implies the non-atomicity of limits of the canonical conformal measures.

The second part, as is stated in [McM], is still valid in our more general setting. Our only task is to prove the first part. We will first prove an independent result which concerns only polynomial vector fields.

A maximal real-time solution \(\psi(t)\) of \(\dot{z} = P(z)\) (for a polynomial \(P\), with defining interval of the form \([0, t_0]\) or \([-t_0, 0]\), and with \(\lim_{t \to 0} \psi(t) = \infty\), is called a **separatrix**. We have \(t_0 = \infty\) in case \(P\)
is $\alpha$-stable for some $\alpha > 0$. We have (a more precise version will be given in Corollary 2.10):

**Proposition 1.1.** Assume $(P_n)_{n \geq 1}$ is a sequence of $\alpha$-stable polynomials converging algebraically to the polynomial $P_0(z) = z^{p+1}$. Assume $\psi_n : [0, +\infty[ \to \mathbb{C}$ is a separatrix of $\dot{z} = P_n(z)$. Then, for any $\eta > p/(p + 1)$, we have

$$\int_{t_n}^{t+\infty} |\psi_n'(u)|^\eta \, du \longrightarrow 0,$$

as $t, n \to +\infty$.

**Remark.** There is an analogous result for a sequence of separatrices defined on $]-\infty, 0[$.

The main tool in proving this is to take Hausdorff limits of closures of invariant trajectories, to use appropriate renormalizations to get normal families (an idea of C. Petersen [P], see also [PT, T] for other applications), and to apply inequalities from hyperbolic geometry.

Once this is done, we will establish a lemma connecting the flow of $\dot{z} = P(z)$ to the iteration of $z + P(z)(1 + s(z))$ with $s(z)$ small (various forms of the lemma can be found in [DES, E, O]). We then use hyperbolic geometry and bounded Koebe distortion to translate Proposition 1.1 into a control of tails of Poincaré series, establishing thus their convergence and consequently Theorem A.

Most of our intermediate results will in fact require only partial stability of a polynomial vector field, and provide partial stability of the flow as well as of the dynamics.

The paper is organized as follows: In §2.1 we define stable polynomial vector fields and restate Proposition 1.1. Its proof is completed in §4. In §2.2 we define stable convergence of iterated maps. §3 contains criteria of stability of polynomial vector fields. §5.1 contains the linking lemma from the discrete dynamics to the time-one map of the flow of some polynomial vector fields. In §5.2 we prove that a stable perturbation of $z + z^{p+1}$ is well approximated by the corresponding time-one maps. We then prove in §§6.2-6.3 the absence of implosion and the continuity of Julia sets, and in §6.4 the uniform convergence of tails of Poincaré series. In §7 we recall known results about conformal measures and their relations to Hausdorff dimension of Julia sets, and prove Theorem A together with its corollary.

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2. Definitions and statements

2.1. Stable polynomial differential equations. Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. Consider the holomorphic vector field $P(z) \cdot \mathbf{T}$. It is associated to an autonomous ordinary differential equation $\dot{z} = P(z)$ and a meromorphic 1-form $\frac{dz}{P(z)}$. The following lemma and definition single out certain particular solutions of the differential equation, which will play important roles in the sequel of our study.

**Lemma 2.1** (coordinates at $\infty$).
Assume $p \geq 1$ and $P : z \mapsto A z^{p+1} + \mathcal{O}(z^p)$ is a polynomial of degree $p + 1$. Then, the anti-derivative $\Phi_P(z) = \int_\infty^z \frac{du}{P(u)}$ is well defined and holomorphic in a neighborhood of $\infty$. As $z \to \infty$, $\Phi_P(z) \sim -1/(pAz^p)$. If $P_n \to P_0$ algebraically, then $\Phi_{P_n} \to \Phi_{P_0}$ uniformly in some neighborhood of $\infty$.

The proof is elementary. See [DES, E] for details. Any local inverse $\Psi$ of $\Phi_P$ satisfies $\Psi'(w) = P(\Psi(w))$, thus is a solution of the equation $\frac{dz}{dw} = P(z)$, with $w$ complex. It follows that $\dot{z} = P(z)$ has exactly $p$ germs of forward real-time trajectories with initial point $\infty$ (i.e., solutions $\gamma : ]0, \varepsilon[ \to \mathbb{C}$ such that $\gamma(t) \to \infty$ as $t \searrow 0$), and $p$ germs of backward real-time trajectories with initial point $\infty$ (i.e., solutions $\gamma : ]-\varepsilon, 0[ \to \mathbb{C}$ such that $\gamma(t) \to \infty$ as $t \nearrow 0$). We call them outgoing, respectively incoming, $\infty$-germs. See Figure 1.

![Figure 1. ∞-germs of a polynomial differential equation (with $p = 2$).](image)

When $P$ is monic of degree $p + 1$, there is a natural numeration of these germs by $\{\gamma_k, k \in \mathbb{Z}/2p\mathbb{Z}\}$, so that $\gamma_k$ is tangential to $e^{2\pi i \frac{k}{2p}} \cdot \mathbb{R}^+$ at $\infty$. The germ $\gamma_k$ is of outgoing (resp. incoming) type if $k$ is odd (resp. even).
Definition 2.2. For a (polynomial, \(\infty\)-germ) pair \((P, \gamma)\), define \(\Psi_{P, \gamma}\) to be the inverse branch of \(\Phi_P\) in a sector neighborhood of 0 as follows:

- for \(\gamma^+\) an outgoing \(\infty\)-germ, \(\Psi_{P, \gamma^+}\) is defined on \(D(\varepsilon) \setminus \mathbb{R}^-\) and coincides with \(\gamma^+\) on \([0, \varepsilon]\);
- for \(\gamma^-\) an incoming \(\infty\)-germ, \(\Psi_{P, \gamma^-}\) is defined on \(D(\varepsilon) \setminus \mathbb{R}^+\) and co-incides with \(\gamma^-\) on \([-\varepsilon, 0]\).

Example 0. Let \(P_0(z) = z^{p+1}\). We have \(\Phi_{P_0}(z) = -\frac{1}{pz^p} = w\) and
\[
\Psi_{P_0, \gamma_k}(w) = \frac{1}{(-pw)^{\frac{1}{p}}} = \frac{1}{w},
\] where the \(p\)-th root is chosen so that

- if \(k\) is odd, \(\Psi_{P_0, \gamma_k}\) extends analytically to \(\mathbb{C} \setminus \mathbb{R}^-\) with \(\Psi_{P_0, \gamma_k}(\mathbb{R}^+) = e^{2\pi i \frac{k}{p}} \cdot \mathbb{R}^+\);
- if \(k\) is even, \(\Psi_{P_0, \gamma_k}\) extends analytically to \(\mathbb{C} \setminus \mathbb{R}^+\) with \(\Psi_{P_0, \gamma_k}(\mathbb{R}^-) = e^{2\pi i \frac{k}{p}} \cdot \mathbb{R}^-\).

These local solutions have analytic extensions as more global solutions of the differential equation \(\dot{z} = P(z)\). In this article, we are mainly interested by the solutions with real-time. In that case, the maximal solutions are defined on real open intervals of the form \([t_{min}, t_{max}]\). We do need however complex-time solutions on suitable neighborhoods of \([t_{min}, t_{max}]\) in order to control perturbed trajectories.

Definition 2.3 (separatrices and homoclinic connections). For a polynomial differential equation \(\dot{z} = P(z)\), a trajectory (or an orbit) is a maximal solution \(\psi : [t_{min}, t_{max}] \to \mathbb{C}\). The maximal solution of an \(\infty\)-germ \(\gamma\) is called the \(\gamma\)-separatrix. A homoclinic connection is a maximal solution \(\psi(t)\) with \(|t_{min}|, |t_{max}| < \infty\) and with \(\lim_{t \to t_{min}} \psi(t) = \lim_{t \to t_{max}} \psi(t) = \infty\).

We now come to the definition of \(\alpha\)-stability. For \(\alpha \in ]0, \frac{\pi}{2}[,\) let us define a sector neighborhood of \(\mathbb{R}^\pm\) by
\[
S^+(\alpha) = \{w \in \mathbb{C}^*; \arg(w) < \alpha\}
\] and
\[
S^-\alpha \alpha = \{w \in \mathbb{C}^*; \pi - \arg(w) < \alpha\}.
\]
When there is no possible confusion, we will use the notation \(S(\alpha)\) instead of \(S^+(\alpha)\) or \(S^-\alpha\). The following is to be compared with the notion of tolerant angles in [DES]:

Definition 2.4 (\(\alpha\)-stability). Given a polynomial \(P\) and an \(\infty\)-germ \(\gamma\), we say that \(P\) is \(\gamma\)-implosive if the \(\gamma\)-separatrix is a homoclinic connection.

For \(\alpha \in ]0, \frac{\pi}{2}[,\) we say that \(P\) is \((\alpha, \gamma)\)-stable if \(\Psi_{P, \gamma}\) extends holomorphically to the entire sector \(S^+(\alpha)\) (if \(\gamma\) is an outgoing germ), or \(S^-\alpha\) (if \(\gamma\) is an incoming germ). We will denote by \(\Psi_{P, \gamma} : S^\pm(\alpha) \to \mathbb{C}\) this extension.
We say that $P$ is (globally) $\alpha$-stable if it is $(\alpha, \gamma)$-stable for all $\infty$-germs $\gamma$.

It is proved in [DES] that $P$ is not $\gamma$-implosive implies that it is $(\alpha, \gamma)$-stable for some $\alpha > 0$ (see also Proposition 3.2 below). Note that when $P$ is $(\alpha, \gamma)$-stable, the $\gamma$-separatrix coincides with $\Psi_{P,\gamma}(S(\alpha) \cap \mathbb{R})$ and is not a homoclinic connection, and the set $\Psi_{P,\gamma}(S(\alpha))$ may be considered as a protecting neighborhood of it. Criteria of $\alpha$-stability will be given in §3.2. For instance it is enough to require $P$ to be $(\alpha, \gamma)$-stable for every outgoing germ (or every incoming germ).

Example 1. If $P_\lambda(z) = z(z^p - \lambda)$, we have

$$\Phi_{P_\lambda}(z) = \frac{1}{p\lambda} \log \left(1 - \frac{\lambda}{z^p}\right),$$

with formal inverse

$$\Psi_{P_\lambda}(w) = \left(\frac{\lambda}{1 - e^{p\lambda w}}\right)^{\frac{1}{p}}.$$

No matter which $p$-th root we take, the map $\Psi_{P_\lambda}$ has singularities at $\{w = \frac{2m\pi i}{p\lambda}, m \in \mathbb{Z}\}$. If $\lambda \notin i\mathbb{R}$, the polynomial $P_\lambda$ is $\alpha$-stable, as soon as $0 < \alpha < \min \{|\arg(i/\lambda)|, |\pi - \arg(i/\lambda)|\}$, where $\arg(i/\lambda) \in ]-\pi/2, 3\pi/2[$. However, if $\lambda \in i\mathbb{R}$, $P_\lambda$ is $\gamma$-implosive for every $\infty$-germ $\gamma$.

![Figure 2. $\alpha$-stability for $P_\lambda(z) = z(z - \lambda)$](image)

Example 2. If $P_\lambda(z) = z(z - \lambda)(z - 2\lambda)$, we have

$$p = 2, \quad \Phi_{P_\lambda}(z) = \frac{1}{2\lambda^2} \log \frac{1 - 2\lambda/z}{(1 - \lambda/z)^2}, \quad \Psi_{P_\lambda}(w) = \lambda \pm \left(\frac{\lambda^2}{1 - e^{2\lambda^2 w}}\right)^{1/2}.$$

The singularities of $\Psi_{P_\lambda}$ are at $\{w = \frac{m\pi i}{\lambda^2}, m \in \mathbb{Z}\}$. If $\lambda^2 \in i\mathbb{R}$, $P_\lambda$ is $\gamma$-implosive for every germ $\gamma$.

We now state a list of results regarding $\alpha$-stable polynomial vector fields (the first of them is contained in [DES], but with a different proof). We will provide self-contained proofs (independent of [DES]) in §4.
Proposition 2.5 ([DES]). Assume that a polynomial $P$ is $(\alpha', \gamma)$-stable for some $\infty$-germ $\gamma$ and some $0 < \alpha' < \frac{\pi}{2}$. Then

$\Psi_{P, \gamma}(S(\alpha')) \subset \mathbb{C} \setminus \{\text{zeros of } P\}$

and there exists a zero $a_{P, \gamma}$ of $P$ such that $\Psi_{P, \gamma}(w) \to a_{P, \gamma}$ uniformly as $w \to \infty$ within any sector $S(\alpha)$ with $\alpha < \alpha'$.

In particular, the $\gamma$-separatrix starts at $\infty$ and lands at $a_{P, \gamma}$ (called the landing point of the $\gamma$-separatrix).

Note that due to the continuity of $\Phi_P$ with respect to $P$ near $\infty$ (Lemma 2.1), an algebraic convergence of polynomials $P_n \to P_0$ induces necessarily the uniform and bijective convergence of the set of $\infty$-germs of $P_n$ (restricted to some common time-interval) to those of $P_0$. Thus it makes sense to talk about the convergence of pairs $(P_n, \gamma_n) \to (P_0, \gamma_0)$.

In this case, $\gamma_0$ and all $\gamma_n$ for $n$ sufficiently large, are of the same (outgoing or incoming) type.

Proposition 2.6. Let $(P_n, \gamma_n)$ be a sequence of pairs (polynomial, $\infty$-germ) converging to a pair $(P_0, \gamma_0)$. Assume $P_n$ are $(\alpha', \gamma_n)$ stable for some common $\alpha' \in \left[0, \frac{\pi}{2}\right]$. Then,

• $P_0$ is $(\alpha', \gamma_0)$-stable and
• for $\alpha < \alpha'$, we have $\Psi_{P_n, \gamma_n} \to \Psi_{P_0, \gamma_0}$ uniformly (not just locally uniformly) on $S(\alpha)$.

In particular, $a_{P_n, \gamma_n} \to a_{P_0, \gamma_0}$.

Definition 2.7 ($\eta$-length of separatrices). Let $P$ be a polynomial and $\gamma$ be an $\infty$-germ such that $P$ is not $\gamma$-implusive. If $\gamma$ is an outgoing (resp. incoming) $\infty$-germ, let $\psi : [0, +\infty[ \to \mathbb{C}$ (resp. $\psi : ]-\infty, 0[ \to \mathbb{C}$) be the $\gamma$-separatrix. For $\eta > 0$ and $t > 0$, set

$\ell_{\eta}(P, \gamma, t) = \int_{-t}^{+\infty} |\psi'(u)|^\eta \, du \quad \text{(resp. } \ell_{\eta}(P, \gamma, t) = \int_{-\infty}^{-t} |\psi'(u)|^\eta \, du \text{).}$

They should be considered as the $\eta$-dimensional length of a portion of the underlying separatrix.

Proposition 2.8. Assume $P$ is a polynomial of degree $p + 1$ which is $(\alpha, \gamma)$-stable for some $\infty$-germ $\gamma$ and some $0 < \alpha < \frac{\pi}{2}$. Then, for all $\eta > p/(p + 1)$ and all $t > 0$, the $\eta$-length $\ell_{\eta}(P, \gamma, t)$ is finite. It decreases with respect to $t$ and tends to 0 as $t$ tends to $+\infty$.

Our main result about flows of polynomial vector fields is the following.

Proposition 2.9 (length stability of separatrices). Let $(P_n, \gamma_n)$ be a sequence of pairs (polynomial, $\infty$-germ) converging to a pair $(P_0, \gamma_0)$. Assume there is an $\alpha > 0$ independent of $n$ such that $P_n$ are $(\alpha, \gamma_n)$-stable. Finally, assume $t_n \to t_0 > 0$ as $n \to \infty$. Then for $\eta > p/(p + 1)$, we have $\ell_{\eta}(P_n, \gamma_n, t_n) \to \ell_{\eta}(P_0, \gamma_0, t_0)$ as $n \to \infty$. 
Corollary 2.10. Let \((P_n, \gamma_n)\) be a sequence of pairs \((\text{polynomial, } \infty\text{-germ})\) converging to a pair \((P_0, \gamma_0)\) with \(P_0(z) = z^{p+1}\) and assume that the \(P_n\) are \((\alpha, \gamma_n)\)-stable for some common \(\alpha \in ]0, \frac{\pi}{2}\) and all \(n\) sufficiently large. Then, for \(\eta > p/(p+1)\) and for all \(\varepsilon > 0\), there exist \(t_0\) and \(n_0\) such that for all \(t \geq t_0\) and all \(n \geq n_0\), we have \(\ell_\eta(P_n, \gamma_n, t) < \varepsilon\).

2.2. Stable convergence for analytic maps. Assume the sequence \((f_n : D(r) \to \mathbb{C})_{n \geq 1}\) converges locally uniformly to \(f_0 : D(r) \to \mathbb{C}\) with \(f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})\), \(p \geq 1\). As \(n \to +\infty\), \(f_n\) has \(p + 1\) fixed points, counting multiplicities, converging to 0. Let \(P_n\) be the monic polynomials of degree \(p + 1\) which vanish at those \(p + 1\) fixed points of \(f_n\).

Let us now fix \(k \in \mathbb{Z}/2p\mathbb{Z}\) odd and let \(K \subset D(r)\) be a compact set such that for all \(z \in K\),

- \(f_0^{\delta j}(z)\) is defined for all \(j \geq 0\) and \(f_0^{\delta j}(z) \xrightarrow{j \to +\infty} 0\) tangentially to the direction \(e^{2i\pi \frac{k}{p}}\).

Let \(\gamma_n\) be the \(\infty\)-germ tangent to \(\mathbb{R}^+ \cdot e^{2i\pi \frac{k}{p}}\) at \(\infty\). Assume there is an \(\alpha \in ]0, \frac{\pi}{2}\) such that \(P_n\) is \((\alpha, \gamma_n)\)-stable for all sufficiently large \(n\). By Proposition 2.5, we know that the \(\gamma_n\)-separatrix lands at a fixed point \(a_n\) of \(f_n\). The following result is essentially contained in [DES]. We will reprove it in §4.

Proposition 2.11. Under the previous assumptions, for \(n\) large enough, \(K\) is contained in the basin of attraction of \(a_n\). In other words, for all \(z \in K\),

- \(f_0^{\delta j}(z)\) is defined for all \(j \geq 0\) and \(f_0^{\delta j}(z) \xrightarrow{j \to +\infty} a_n\).

The next result, concerning tails of Poincaré series, can be considered as a discrete version of Corollary 2.10. Replacing \(f_0\) with \(f_0^{-1}\) and \(f_n\) with \(f_n^{-1}\), we obtain similar results with \(k\) odd replaced by \(k\) even and forward iterations replaced by backward iterations.

Proposition 2.12. Under the previous assumptions, if \(\delta_0 > p/(p+1)\) and \(\varepsilon > 0\), there exist \(m_0\) and \(n_0\) such that for all \(z \in K\), all \(\delta \in ]\delta_0, 2]\), all \(m \geq m_0\) and all \(n \geq n_0\), we have

\[
S_\delta(f_n, z, m) := \sum_{j=m}^{+\infty} |(f_n^{\delta j})'(z)|^\delta < \varepsilon.
\]

Definition 2.13 (stable convergence). We say that the convergence \(f_n \to f_0\) is stable at 0 if and only if there is an \(\alpha \in ]0, \frac{\pi}{2}\) such that for \(n\) large enough, the polynomials \(P_n\) are \(\alpha\)-stable.

We will prove that this notion is invariant under coordinate changes, but with probably a different \(\alpha\) (see Lemma 4.2).
If $\lambda$ is a parabolic point of $\beta$ then $\beta^{-1}(0)$ is a fixed point of $\beta_0$. We say that the convergence $f_n \to f_0$ is stable at 0 if and only if there is an $\alpha \in ]0, \frac{\pi}{2}[$ such that the polynomials $P_{\lambda_n}(z) := z(z^p - \lambda_n)$ are $\alpha$-stable for $n$ large enough, if and only if $\lambda_n \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$. Furthermore, we say that the convergence $f_n \to f_0$ is stable if and only if $\beta \in X$.

**Example 1** (continued). Set $f_\lambda(z) = z + z(z^p - \lambda)(1 + s_\lambda(z))$, with $(\lambda, z) \mapsto s_\lambda(z)$ holomorphic satisfying $s_\lambda(0) = 0$. Then the convergence $f_{\lambda_n} \to f_0$ is stable at 0 if and only if there is an $\alpha \in ]0, \frac{\pi}{2}[$ such that the polynomials $P_{\lambda_n}(z) := z(z^p - \lambda_n)$ are $\alpha$-stable for $n$ large enough, if and only if $\lambda_n \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$. 

**Definition 2.14.** If $(f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n \geq 1}$ is a sequence of rational maps converging algebraically to a rational map $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$ and if $\beta$ is a parabolic point of $f_0$, we say that the convergence is stable at $\beta$ if we can find a local coordinate sending $\beta_0$ to 0 such that $f_0^{n+1}(z) = z + z^{p+1} + O(z^{p+2})$ and such that the convergence $f_n^{n+1} \to f_0^{n+1}$ is stable at 0 (j is the least integer such that $f_0^{n+1}$ fixes $\beta$ with multiplier 1 and $p$ is the number of petals at $\beta$).

We say that the convergence $f_n \to f_0$ is stable if and only if it is stable at all the parabolic points of $f_0$.

Theorem A claims essentially that modulo technical assumptions stable convergence implies convergence of dimensions. Combining Propositions 2.11 and 2.12 to results concerning Poincaré series and conformal measures in [McM], we obtain easily Theorem A.

**Definition 2.15.** We say that $(f_n, \beta_n) \to (f_0, \beta_0)$ if $\beta_n$ (respectively $\beta_0$) is a fixed point of $f_n$ (respectively $f_0$), if $\beta_n \to \beta_0$ and if $f_n \to f_0$ uniformly in some neighborhood of $\beta_0$.

**Definition 2.16 (following [McM]).** Assume $f_0$ has a multiple fixed point at $\beta_0$ with $p$ petals. We say that the convergence $(f_n, \beta_n) \to (f_0, \beta_0)$ is dominant if there exists an $M$ such that

$$|f_n^{(i)}(\beta_n)| \leq M|f_n'(\beta_n) - 1| \quad \text{for } 1 < i < p + 1.$$ 

Furthermore, we say that the convergence $(f_n, \beta_n) \to (f_0, \beta_0)$ is radial if in addition $f_n'(\beta_n) - 1$ tends to 0 avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$. 

We will show in Lemma 3.12 that radial convergence implies stable convergence. Therefore, our Theorem A recovers as a corollary the following:

**Theorem (McMullen, [McM])** Assume $f_0$ is a geometrically finite rational map and $f_n \to f_0$ algebraically, preserving critical relations. For each parabolic point $\beta_0 \in J(f_0)$, let $j$ be the least integer such that $f_0^{j+1}$ fixes $\beta_0$ with multiplier 1 and assume $f_n^{j+1}$ has a fixed point $\beta_n$ converging to $\beta_0$ such that $(f_n^{j+1}, \beta_n) \to (f_0^{j+1}, \beta_0)$ radially. Then, $J(f_n) \to J(f_0)$ and $H.d\dim J(f_n) \to H.d\dim J(f_0)$.

Combining criteria of $\alpha$-stability given in [DES] (see Proposition 3.11 below) with Theorem A, we get another important corollary.
Corollary 2.17. Assume \( \{ f_\lambda \}_{\lambda \in \mathbb{D}} \) is an analytic family of rational maps such that \( f_0 \) is geometrically finite and such that the critical orbit relations in \( J(f_0) \) are persistent. Then, there exists a set \( L \subset \mathbb{D} \) composed of finitely many rays \( e^{i\theta} \cdot ]0,1[ \) (called implosive directions), such that if \( \lambda_n \to 0 \) avoiding a sector neighborhood of the set \( L \), then for \( n \) large enough, \( f_{\lambda_n} \) is geometrically finite, \( J(f_{\lambda_n}) \to J(f_0) \) and \( \text{H.dim } J(f_{\lambda_n}) \to \text{H.dim } J(f_0) \).

Remark. The case \( f_\lambda(z) = (1-\lambda)z+z^2 \) is proved by Bodart-Zinsmeister [BZ], 1996. In this case the implosive directions correspond to \( \arg(\lambda) = \pm \pi/2 \).

Theorem A and Corollary 2.17 will be proved in \( \S 7 \).

3. Polynomial differential equations

In \( \S 3.1 \) and \( \S 3.2 \), we recall results mostly contained in [DES], together with some easy consequences. In \( \S 3.3 \) we connect radial convergence to stable convergence.

3.1. Time criterion for a separatrix. Given a polynomial \( P : \mathbb{C} \to \mathbb{C} \) and a point \( z_0 \in \mathbb{C} \), we will denote by \( \Psi_{P,z_0} \) the solution of the differential equation \( \dot{z} = P(z) \) such that \( \Psi_{P,z_0}(0) = z_0 \). By uniqueness of solutions, two such solutions coincide in a neighborhood of \( 0 \), thus define the same germ at \( 0 \).

Lemma 3.1. Let \( Q \) be a holomorphic germ.

a) Let \( \psi(w) \) be a non-constant holomorphic solution of the equation \( \dot{z} = Q(z) \), defined on a disc \( D(R) \) with \( R < \infty \). Then \( \psi(D(R)) \) does not meet the zeros of \( Q \).

b) Let \( \psi(t) \) be a non-constant real-time solution of the equation \( \dot{z} = Q(z) \), defined (at least) on a bounded interval \( ]a,b[ \). Then \( \psi(]a,b[) \) does not meet the zeros of \( Q \).

Proof. a) We may assume by contradiction that \( Q(0) = 0 \) and \( 0 \in \psi(D(R)) \). Choose a sequence \( w_n \in D(R) \) tending to \( w' \) such that \( \psi(w_n) \to 0 \). However, using the local study in [DES] on the various (sink, source or multiple) types of zeros of \( Q \), one deduces easily that for \( n \) sufficiently large, \( \psi \) has an analytic extension to \( w_n + D(1) \) with \( \psi(w_n + D(1)) \) avoiding the zeros of \( Q \). In particular \( \psi \) has a continuous extension at \( w' \) with \( \psi(w') \neq 0 \). This is a contradiction.

The proof of b) is similar. q.e.d.

Proposition 3.2. Assume \( P \) is a polynomial and \( \psi : ]t_{\min},t_{\max}[ \) is a maximal solution of \( \dot{z} = P(z) \). Then,

(a) \( \psi(t) \to \infty \) as \( t \searrow t_{\min} \) (respectively as \( t \nearrow t_{\max} \)) if and only if \( t_{\min} > -\infty \) (respectively \( t_{\max} < +\infty \)). In that case, \( t \mapsto \psi(t+t_{\min}) \) (respectively \( t \mapsto \psi(t+t_{\max}) \)) is the \( \gamma \)-separatrix for some outgoing (respectively incoming) germ \( \gamma \).
(b) if $t_{\text{min}} = -\infty$ (respectively $t_{\text{max}} = +\infty$) then either
- $\psi$ is periodic and the trajectory $\psi(\mathbb{R})$ is a topological circle, or
- $\psi(t)$ tends a zero of $P$ as $t \to -\infty$ (respectively as $t \to +\infty$).
(c) If $P$ is not $\gamma$-implasive for some $\infty$-germ $\gamma$, then $P$ is $(\alpha, \gamma)$-stable for some $\alpha > 0$.

Please refer to [DES] for a proof. We do not need this result in the proof of Theorem A.

**Corollary 3.3.** A maximal solution $\psi : [t_{\text{min}}, t_{\text{max}}]$ of $\dot{z} = P(z)$ is a homoclinic connection iff both $t_{\text{min}}$ and $t_{\text{max}}$ are finite, iff it contains an incoming $\infty$-germ on the one end and an outgoing $\infty$-germ on the other end.

An example of homoclinic connection is provided by the real axis for any real polynomial which does not vanish on $\mathbb{R}$.

### 3.2. Criteria of $\alpha$-stability of polynomial vector fields (following [DES]).

Recall that under the change of variables $z = h(u)$, the equation $\dot{z} = P(z)$ is transformed into the equation $\dot{u} = P \circ h(u)/h'(u)$.

The proof of the following sequence of lemmas are fairly elementary and can be easily supplied by the reader. Details are to be found in [DES].

**Lemma 3.4** (affine conjugacies). Assume $P$ and $Q$ are related by $Q(u) = P(au + b)/a$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Then, $P$ is $\alpha$-stable if and only if $Q$ is $\alpha$-stable. In other words, affine conjugacies preserve $\alpha$-stability.

**Lemma 3.5** (semi-conjugacies). Assume $P$ and $Q$ are two polynomials which vanish at $0$ and are related by $Q(u) = \frac{1}{mu^{m-1}}P(u^m)$ for some integer $m \geq 2$. Then, $P$ is $\alpha$-stable if and only if $Q$ is $\alpha$-stable. In other words, semi-conjugacies $u \mapsto u^m$ preserve $\alpha$-stability.

**Lemma 3.6.** If $\lambda \in \mathbb{C}^*$ and if $\psi(w)$ is a complex solution of $\dot{z} = P(z)$ then $\psi(\lambda w)$ is a complex solution of the vector field $\dot{z} = \lambda P(z)$.

In particular, if $k \in \mathbb{R}^*$, the trajectories of the vector field $\dot{z} = kP(z)$ and $\dot{z} = P(z)$ are the same up to re-parameterization. It follows that $P$ is $\alpha$-stable if and only if $kP$ is $\alpha$-stable.

In addition, if $\psi(w)$ is a solution of $\dot{z} = P(z)$, then $\psi(e^{i\theta}w)$ is a solution of the differential equation of the rotated vector field, $\dot{z} = e^{i\theta}P(z)$.

**Lemma 3.7.** Assume $\alpha \in ]0, \frac{\pi}{2}[$. Then, $P$ is $\alpha$-stable if and only if $\dot{z} = e^{i\theta}P(z)$ does not have homoclinic connections for $\theta \in ]-\alpha, \alpha[$.

We now come to a characterization of stability in terms of residues of $1/P$:
Lemma 3.8. Assume $P$ has a homoclinic connection $\psi : \{t_{\min}, t_{\max}\} \to \mathbb{C}$. Then, there is a subset $X$ of the set of zeros of $P$ such that

$$2\pi i \sum_{x \in X} \text{Res} \left( \frac{1}{P}, x \right) = t_{\max} - t_{\min} \in \mathbb{R}^+.$$ 

Lemma 3.9. Assume $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $P$ is not $\alpha$-stable. Then, there exists $R > 0$, $\theta \in [-\alpha, \alpha]$ and a subset $X$ of the set of zeros of $P$ such that the equation $\dot{z} = e^{i\theta}P(z)$ has a homoclinic connection $\psi : \{0, R\} \to \mathbb{C}$, and $2\pi i \sum_{x \in X} \text{Res} \left( \frac{1}{P}, x \right) = e^{i\theta}R \in S^{+}(\alpha).

Consequently, if $2\pi i \sum_{x \in X} \text{Res} \left( \frac{1}{P}, x \right) \not\in S^{+}(\alpha)$ for any subset $X$ of zeros of $P$, then $P$ is $\alpha$-stable.

Lemma 3.10. If $P$ is $(\alpha, \gamma)$-stable for every outgoing $\infty$-germ $\gamma$, then $P$ is $\alpha$-stable. Idem if we replace outgoing germs by incoming germs.

We will now give a result in a slightly more general form than the original result of [DES] (for instance we allow $P_\lambda$ to have multiple zeros), but with essentially the same proof.

Proposition 3.11 ([DES]). Assume $(P_\lambda)_{\lambda \in \mathbb{D}}$ is an analytic family of polynomials of degree $p+1$ such that $P_0(z) = z^{p+1}$. Then, there exists a set $L \subset \mathbb{D}$ composed of finitely many rays $e^{i\theta} \cdot \{0, 1\}$ such that for every closed sector $S$ avoiding $L$, there exists $\alpha > 0$ and $\varepsilon > 0$ such that for all $\lambda \in S \cap D(\varepsilon)$, $P_\lambda$ is $\alpha$-stable.

Proof. Without loss of generality, re-parameterizing by $\lambda^{1/n}$ for some integer $n$ if necessary, we may assume that we can follow holomorphically all the zeros of $P_\lambda$ in a neighborhood of $\lambda = 0$. In that case, we can follow holomorphically all the possible sums of residues of $1/P_\lambda$ in some punctured disc $\mathbb{D}^*$. There are only finitely many such sums. We denote by $(\sigma_j : \mathbb{D}^* \to \mathbb{C})_{j \in J}$ those sums of residues. The functions $\sigma_j$ extend meromorphically at 0. Some of them might be constant. We denote by $J^*$ the set of indices for which $\sigma_j$ is not constant. And we choose $T_0 > \max\{2\pi \sigma_j, \sigma_j \text{ is constant}\}$.

We claim that taking $\varepsilon$ smaller if necessary, we may assume that for all $\lambda \in \mathbb{D}_\varepsilon$, and all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, all the separatrices of $\dot{z} = e^{i\theta}P_\lambda(z)$ with initial point at $\infty$ are defined for a time larger than this $T_0$. It is enough to show that for $\lambda$ close enough to 0, $\Phi_{P_\lambda}$ is a ramified covering above the disk $D(T_0)$, ramified only above 0. This easily follows form the fact that $\Phi_{P_\lambda} \to \Phi_{P_0}$ as $\lambda \to 0$ and that $\Phi_{P_0}(z) = -1/pz^p$.

Let us now fix $\alpha > 0$ and $\lambda \in \mathbb{D}^*_\varepsilon$ and assume $P_\lambda$ is not $\alpha$-stable. Then by Lemma 3.9 there are $j \in J$, $R > 0$ and $|\theta'| < \alpha$ such that $Re^{i\theta'} = 2\pi i \sigma_j(\lambda)$ and $\psi : \{0, R\} \to \mathbb{C}$ is a homoclinic connection for the differential equation of the rotated vector field, $\dot{z} = e^{i\theta}P(z)$.
By assumption, \( R > T_0 \). Hence \( \sigma_j \) is not constant and \( j \in J^\ast \). The proposition follows easily by choosing for \( L \) the union of all the rays in \( \mathbb{D}^\ast \) which are tangent at 0 to a connected component of the curve \( \{ \lambda \mid 2\pi i \sigma_j(\lambda) \in \mathbb{R}^+ \} \) for some \( j \in J^\ast \). Indeed, if \( S \) is a closed sector avoiding \( L \), then we can find \( \alpha > 0 \) and \( \varepsilon > 0 \) such that for all \( j \in J^\ast \) and all \( \lambda \in D^\ast \cap S \), \( |\arg(2\pi i \sigma_j(\lambda))| > \alpha \). It follows from the above discussion that the polynomial \( P_\lambda \) is \( \alpha \)-stable.

q.e.d.

3.3. Radial convergence implies stable convergence.

**Lemma 3.12.** Assume \( f_0 \) has a multiple fixed point with \( p \) petals at \( \beta_0 \) and assume \( (f_n, \beta_n) \to (f_0, \beta_0) \) radially. Then, there exists a local coordinate sending \( \beta_0 \) to 0 such that \( f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}) \) and such that the convergence \( f_n \to f_0 \) is stable at 0.

**Proof.** Using an affine change of coordinate, we may assume that \( \beta_0 = 0 \) and that \( f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}) \). We assume that the convergence \( (f_n, \beta_n) \to (f_0, 0) \) is radial (this property is clearly preserved by affine changes of coordinates). We will show that for \( n \) sufficiently large, the monic polynomials \( P_{f_n} \) of degree \( p+1 \) which vanish at the \( p+1 \) fixed points of \( f_n \) close to 0 are \( \alpha \)-stable for some uniform \( \alpha \).

We can write \( f_0(z) = z + z^{p+1}(1 + s_0(z)) \) and \( f_n(z) = z + P_{f_n}(z)(1 + s_n(z)) \) with \( s_0(0) = 0 \) and \( s_n(z) \to s_0(z) \). Let \( x_n \) be a zero of \( P_{f_n} \), i.e., a fixed point of \( f_n \). Then, denoting \( A_n \to B_n \) if \( A_n = B_nC_n \) and \( \lim_{n \to \infty} C_n = 1 \),

\[
\text{Res} \left( \frac{1}{P_{f_n}(z)}, x_n \right) = \text{Res} \left( \frac{1 + s_n(z)}{f_n(z) - z}, x_n \right) \sim \text{Res} \left( \frac{1}{f_n(z) - z}, x_n \right).
\]

Indeed, \( 1 + s_n(x_n) \to 1 + s_0(0) = 1 \). We will prove below that

\[
\begin{cases}
\text{Res} \left( \frac{1}{f_n(z) - z}, x_n \right) = \frac{1}{f_n'(x_n)} & \text{if } x_n = \beta_n \\
\text{Res} \left( \frac{1}{f_n(z) - z}, x_n \right) \sim \frac{1}{f_n'(x_n)} & \text{if } x_n \neq \beta_n
\end{cases}
\]

It follows immediately that when \( 1 - f_n'(\beta_n) \to 0 \) avoiding a sector neighborhood of the imaginary axis, there exists an \( \alpha \in \left[ 0, \frac{\pi}{2} \right] \) such that for all \( n \) sufficiently large and for any subset \( X_n \) of the set of zeros of \( P_{f_n} \), we have

\[
2i\pi \sum_{x \in X_n} \text{Res} \left( \frac{1}{P_{f_n}(z)}, x \right) \notin S^+(\alpha).
\]

It follows from Lemma 3.9 that \( P_{f_n} \) is \( \alpha \)-stable for \( n \) large enough.

Let us now prove (3.1). Set \( a_n = f_n'(\beta_n) \). Part \( x_n = \beta_n \) is trivial. Assume now \( x_n \neq \beta_n \). According to McMullen ([McM] Proposition 7.2), and taking a subsequence if necessary, we can find maps \( \phi_n \to \phi_0 \) univalent in a common neighborhood of 0, sending \( \beta_n \) to 0 and such that the maps \( y_n = \phi_n \circ f_n \circ \phi_n^{-1} \) are of the form \( g_n(z) = a_n z + z^{p+1} + \mathcal{O}(z^{p+2}) \). If \( y_n \neq 0 \) is a fixed point of \( g_n \), then \( y_n = g_n(y_n) = a_n y_n + y_n^{p+1} + \mathcal{O}(y_n^{p+2}) \).
so that \( y_n^0 \sim (1-a_n) \), and \( g_n'(y_n) - 1 = a_n - 1 + (p+1)y_n^0 + \mathcal{O}(y_n^{p+1}) \sim p(1-a_n) \). The fixed points of \( g_n \) are simple, so are those of \( f_n \) and
\[
\text{Res} \left( \frac{1}{f_n(z) - z}, x_n \right) = \frac{1}{f_n'(x_n) - 1} = \frac{1}{g_n'(\phi_n(x_n)) - 1} \underset{n \to \infty}{\sim} \frac{1}{p(1-a_n)},
\]
q.e.d.

3.4. Lifting via \( z = u^m \). The following lemma will not be used before the end of §7. Recall that the notation \( (f_n, 0) \to (f_0, 0) \) means that \( f_n \) (respectively \( f_0 \)) is holomorphic in a neighborhood of \( 0 \) and fixes \( 0 \) and \( f_n \to f_0 \) uniformly in a neighborhood of \( 0 \).

**Lemma 3.13.** Assume \( (f_n, 0) \to (f_0, 0) \) and \( (g_n, 0) \to (g_0, 0) \) with \( g_n(z)^m = f_n(z^m) \). If the convergence \( f_n \to f_0 \) is stable, then the convergence \( g_n \to g_0 \) is stable.

**Proof.** Without loss of generality, conjugating with a scaling map if necessary, we may assume that \( f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}) \). Set \( z = u^m \). Then, \( g_0(u) = u + \frac{1}{m}u^{p+1} + \mathcal{O}(u^{p+2}) \). Let us work in the coordinate \( v = u/m^{1/p} \). Then, \( g_n \) is conjugate to \( h_n \) and \( h_n \to h_0 \) with \( h_0(v) = v + v^{p+1} + \mathcal{O}(v^{p+2}) \). Let \( P_n \) (respectively \( Q_n \)) be the monic polynomials which vanish at the fixed points of \( f_n \) (respectively \( h_n \)) close to \( 0 \). We assume \( P_n \) are \( \alpha \)-stable for some uniform \( \alpha \). One easily checks that
\[
Q_n(v) = \frac{P_n(m^m/p^m)}{m^{m(p+1)/p^m-1}} = \frac{1}{m^{m-1}} \cdot \frac{P_n(m^m/p^m)}{m^{m/p \cdot m^m-1}} =: \frac{1}{m^{m-1}} \cdot R_n(v).
\]
It follows easily from lemmas 3.4 and 3.5 that \( R_n(v) = \frac{P_n(m^m/p^m)}{m^{m/p \cdot m^m-1}} \) is \( \alpha \)-stable. Since for all \( \theta \), \( e^{i\theta} Q_n \) is a real multiple of \( e^{i\theta} R_n \), we do not change the trajectories, and \( e^{i\theta} Q_n \) has a homoclinic connection if and only if \( e^{i\theta} R_n \) has a homoclinic connection. Thus, \( Q_n \) is \( \alpha \)-stable. q.e.d.

4. \((\alpha, \gamma)\)-stability and length of the \( \gamma \)-separatrix

In this section, we will prove the propositions stated in §2.1 and some refinements. Our proof will be self-contained, in particular independent of Proposition 3.2 above. The key idea is to take Hausdorff limits of closures of invariant arcs and renormalize the map appropriately to get a normal family. We will only do the proofs for outgoing \( \infty \)-germs \( \gamma_n \) and \( \gamma_0 \). The proofs for incoming \( \infty \)-germs are similar or can be obtained by replacing \( P \) by \( e^{-i\pi/p} P \), which has the effects of changing the orientation on trajectories.

**4.1. Proof of Proposition 2.5.** Assume that \( P \) is \((\alpha', \gamma)\)-stable for some outgoing \( \infty \)-germ \( \gamma \) and some \( \alpha' > 0 \). We will prove that
\[
\Psi_{P, \gamma}(S(\alpha')) \subset \mathbb{C} \setminus \{\text{zeros of } P\}
\]
and there exists a zero \(a_{P,\gamma}\) of \(P\) such that \(\Psi_{P,\gamma}(w) \to a_{P,\gamma}\) as \(w \to \infty\) within any sector \(S(\alpha)\) with \(0 < \alpha < \alpha'\).

By assumption \(\Psi := \Psi_P\) is defined and holomorphic on \(S(\alpha')\).

**Claim A.** \(\Psi(S(\alpha'))\) intersects neither the zeros nor the incoming \(\infty\)-germs of \(P\).

**Proof.** The first part is because it requires an infinite time to reach a zero (see Lemma 3.1). For the second part, assume \(\Psi(w_0) \in \gamma^-\) for some incoming \(\infty\)-germ \(\gamma^-\) and some \(w_0 \in S(\alpha)\). Then, by definition of incoming \(\infty\)-germs, the trajectory with initial point \(\Psi(w_0)\) reaches \(\infty\) at some positive finite time \(t_0\). However, by uniqueness of solutions, this trajectory coincides with \(\Psi(w_0 + t)\). The fact that \(\Psi\) is defined on a neighborhood of \(w_0 + t_0\) implies that \(\Psi(w_0 + t_0) \neq \infty\). This leads to a contradiction. q.e.d.

**Claim B.** Assume \(w_n \in S(\alpha)\) with \(w_n \to \infty\) and \(\Psi(w_n) \to z_0\). Then either \(z_0 = \infty\) or \(z_0\) is a zero of \(P\).

**Proof.** Assume by contradiction that \(z_0 \neq \infty\) and \(P(z_0) \neq 0\).

Set \(L_n(w) = \Psi(w_n + w)\). Those maps are defined on the translated sectors \(T_{-w_n}S(\alpha')\) (which eventually contain any compact set of \(\mathbb{C}\) for \(n\) large enough), whose images coincide with \(\Psi(S(\alpha'))\). Therefore, they form a normal family as they avoid the incoming \(\infty\)-germs of \(P\) by Claim A. So, we may take a subsequence and assume \(L_n \to L_0\) locally uniformly in \(\mathbb{C}\), and the limit \(L_0\) is an entire function. As \(z_0 \neq \infty\) and \(P(z_0) \neq 0\), the flow of \(\dot{z} = P(z)\) with initial point \(z_0\) is well defined and non-constant. But \(L_n(w)\) is the flow of \(\dot{z} = P(z)\) with initial point \(z_n = \Psi(w_n)\). Therefore \(L_0\) is the flow of \(\dot{z} = P(z)\) with initial point \(z_0\), and thus non-constant. This contradicts Picard’s Theorem since \(L_0(\mathbb{C})\) avoids incoming \(\infty\)-germs of \(P\).

Recall that \(0 < \alpha < \alpha' < \frac{\pi}{2}\). Set

\[S(\alpha)_R = S(\alpha) \cap \{w \mid |\text{Re}(w)| > R\}\]

**Claim C.** There is a zero \(a(P)\) of \(P\) such that

\[
\lim_{R \to +\infty} \Psi(S(\alpha)_R) = a(P).
\]

**Proof.** For each \(R > 0\), the set \(\Psi(S(\alpha)_R)\) is connected and therefore has connected closure in \(\overline{\mathbb{C}}\). As the intersection of nested continua is again a continuum, \(\bigcap_{R > 0} \Psi(S(\alpha)_R)\) is a continuum. But it is contained in the finite set \(\{\infty\} \cup \{\text{zeros of } P\}\) by Claim B. So it reduces to a single point \(a(P)\). In other words \(\lim_{R \to +\infty} \Psi(S(\alpha)_R) = a(P)\). In particular \(\lim_{t \to +\infty, t \in \mathbb{R}^+} \Psi(t) = a(P)\). This implies that \(a(P) \neq \infty\), since no real-time trajectory converges to \(\infty\) in infinite time. So \(a(P)\) is a zero of \(P\).

This ends the proof of Proposition 2.5.
4.2. Proof of Proposition 2.6. We must show that if \((P_n, \gamma_n) \to (P_0, \gamma_0)\) for outgoing \(\infty\)-germs \(\gamma_n \to \gamma_0\) and if the \(P_n\) are all \((\alpha', \gamma_n)\) stable, then,

- \(P_0\) is \((\alpha', \gamma_0)\)-stable and \(\Psi_{P_n, \gamma_n} \to \Psi_{P_0, \gamma_0}\) uniformly on \(S(\alpha)\) for any \(\alpha < \alpha'\).

Set \(\Psi_n = \Psi_{P_n, \gamma_n}|_{S(\alpha')}\) for simplicity. Observe that the family \(\{\Psi_n : S(\alpha') \to \mathbb{C}\}\) is a normal family. Indeed, since \(P_n \to P_0\) uniformly, for any incoming \(\infty\)-germ \(\gamma^-\) of \(P_0\), there is a unique sequence \(\gamma^-_n\) of incoming \(\infty\)-germs of \(P_n\) converging to \(\gamma^-\). Normality of the family \(\{\Psi_n : S(\alpha') \to \mathbb{C}\}\) follows from the fact (given in Claim A) that \(\Psi_n(S(\alpha'))\) avoids \(\gamma^-([-\varepsilon, 0]) \to \gamma^-([-\varepsilon, 0])\).

Next, the maps \(\Psi_n\) converge to \(\Psi_0\) locally uniformly in \(S(\alpha') \cap D(\varepsilon)\) for some \(\varepsilon > 0\). So, any limit function of the sequence \(\Psi_n\) must coincide with \(\Psi_0\) on an open set. By analytic continuation, there is only one such limit function \(\Psi : S(\alpha') \to \mathbb{C}\) and it coincides with \(\Psi_0\) on \(S(\alpha') \cap D(\varepsilon)\). It follows that \(P_0\) is \((\alpha, \gamma_0)\)-stable and we have the local uniform convergence of the entire sequence \(\Psi_n : S(\alpha') \to \mathbb{C}\) to \(\Psi_0 : S(\alpha') \to \mathbb{C}\).

We must now promote this local uniform convergence on \(S(\alpha')\) to a global uniform convergence on \(S(\alpha), \alpha < \alpha'\).

**Claim D.** Assume \(w_n \in S(\alpha)\) with \(w_n \to \infty\) and \(\Psi_n(w_n) \to z_0\). We claim that either \(z_0 = \infty\) or \(z_0\) is a zero of \(P_0\).

**Proof.** This is very similar to the proof of Claim B. Assume by contradiction that \(z_0 \not= \infty\) and \(P_0(z_0) \not= 0\). Set \(L_n(w) = \Psi_n(w_n + w)\). Those maps are defined on \(T_{-w_n}S(\alpha')\) which eventually contain any compact set of \(\mathbb{C}\) for \(n\) large enough. They form, as \((\Psi_n)\), a normal family. So, we may take a further subsequence and assume \(L_n \to L_0\) locally uniformly in \(\mathbb{C}\), and the limit \(L_0\) is an entire function. As \(z_0 \not= \infty\) and \(P_0(z_0) \not= 0\), the flow of \(\dot{z} = P_0(z)\) with initial point \(z_0\) is well defined and non-constant. But \(L_n(w)\) is the flow of \(\dot{z} = P_0(z)\) with initial point \(z_n\). By local uniform convergence of \(P_n \to P_0\) we conclude that \(L_0\) is the flow of \(\dot{z} = P_0(z)\) with initial point \(z_0\), and thus non-constant. This contradicts Picard’s Theorem since \(L_0(\mathbb{C})\) avoids incoming \(\infty\)-germs of \(P_0\). q.e.d.

**Claim E.** Set \(\Gamma_n = \overline{\Psi_n(\mathbb{R}^+)}\) and \(\Gamma_0 = \overline{\Psi_0(\mathbb{R}^+)}\) (the closures are taken in \(\mathbb{P}^1\)). We claim that \(\Gamma_n \to \Gamma_0\) in the Hausdorff topology, and \(a(P_n) \to a(P_0)\).

**Proof.** Taking a subsequence if necessary we may assume \(\Gamma_n \to \Gamma\). By local uniform convergence of \(\Psi_n\) to \(\Psi_0\), we have, for any \(R \in \mathbb{R}^+\), \(\Psi_0(R) = \lim_{n \to \infty} \Psi_n(R) \in \Gamma\). So \(\Gamma_0 \subset \Gamma\).

Now choose \(z_0 \in \Gamma\). There is a sequence \(\Psi_n(R_n) \in \Gamma_n\) tending to \(z_0\). If \(R_n \to R \in [0, +\infty]\), we have \(z_0 = \Psi_0(R) \in \Gamma_0\). If \(R_n \to +\infty\) by Claim D the point \(z_0\) is either at \(\infty\) (which is in \(\Gamma_0\)) or at a zero of \(P_0\).
Consequently $\Gamma \subset \Gamma_0 \cup \{\text{zeros of } P_0\}$. But $\Gamma$ is compact and connected. We conclude that $\Gamma = \Gamma_0$.

Now any accumulation point of $a(P_n)$ is in $\Gamma_0$ and must be a zero of $P_0$, by global uniform convergence of $P_n$ to $P_0$. So $a(P_n) \rightarrow a(P_0)$. q.e.d.

We will now show

\[(4.1) \quad (\forall \varepsilon > 0) \ (\exists R > 0) \ (\forall n) \ (\forall w \in S(\alpha)_R) \ |\Psi_n(w) - a(P_n)| \leq \varepsilon.\]

Assume this is not the case. Then for any $j \in \mathbb{N}$, there is $n_j$ and $w_j \in S(\alpha)_j$ such that $|\Psi_{n_j}(w_j) - a(P_{n_j})| \geq \varepsilon_0 > 0$. By Proposition 2.5, this implies that $n_j \rightarrow \infty$ as $j \rightarrow \infty$. Set $w_j = R e^{i\theta_j}$. We have $R_j \rightarrow \infty$.

We will prove that $\Psi_{n_j}(w_j) \rightarrow a(P_0)$ and thus get a contradiction. Set $\Lambda_0 = \Psi_{n_j}(w_j + [0, +\infty]) \cup \{a(P_{n_j})\}$. They are compact connected. Let $\Lambda_0$ be a limit set of a subsequence. It is again connected, containing $a(P_0)$ by Claim E and is contained in a finite set by Claim D, therefore reduces to $\{a(P_0)\}$. It follows that for any $\varepsilon > 0$, there are $R > 0, N > 0$ such that for all $n \geq N$ and all $w \in S(\alpha)_R$, we have

$$|\Psi_n(w) - \Psi_0(w)| \leq |\Psi_n(w) - a(P_n)| + |a(P_n) - a(P_0)| + |a(P_0) - \Psi_0(w)| < 3\varepsilon.$$ Uniform convergence of $\Phi_{P_n}$ to $\Phi_{P_0}$ in a neighborhood of $\infty$ yields uniform convergence of $\Psi_n$ to $\Psi_0$ in $S(\alpha) \cap D(\varepsilon)$. We have local uniform convergence of $\Psi_n$ to $\Psi_0$ in $S(\alpha)$, thus, uniform convergence on $S(\alpha) \setminus (D(\varepsilon) \cup S(\alpha)_R)$. So, if $n$ is sufficiently large, for all $w \in S(\alpha)$, the spherical distance between $\Psi_n(w)$ and $\Psi_0(w)$ is less than $3\varepsilon$.

This ends the proof of Proposition 2.6.

4.3. Non-algebraic convergence.

**Proposition 4.1.** Assume $P_n$ is a sequence of polynomials of degree $p + 1$ which converge locally uniformly in $\mathbb{C}$ to some polynomial $P_0$ (not necessarily of degree $p + 1$). Assume $P_n(0) = P_n(1) = 0$. Assume $\Psi_n$ are holomorphic maps defined on $S(\alpha)$ such that

$$\Psi_n(S(\alpha)) \subset \mathbb{C} \setminus \{0, 1\}, \quad \Psi_n = P_n \circ \Psi_n,$$

$$\lim_{t \in \mathbb{R}, t \rightarrow 0} \Psi_n(t) = \infty \quad \text{and} \quad \lim_{t \in \mathbb{R}, t \rightarrow -\infty} \Psi_n(t) = 0.$$

Then, any limit function $\Psi_0$ of the normal family $\{\Psi_n\}$ satisfies

$$\Psi_0(S(\alpha)) \subset \mathbb{C} \setminus \{0, 1\}, \quad \Psi_0 = P_0 \circ \Psi_0,$$

$$\lim_{t \in \mathbb{R}, t \rightarrow 0} \Psi_0(t) = \infty \quad \text{and} \quad \lim_{t \in \mathbb{R}, t \rightarrow -\infty} \Psi_0(t) = 0.$$

Furthermore, the Hausdorff limit of the separatrices $\Gamma_n = \Psi_n([0, \infty]) \cup \{0, \infty\}$ is the separatrix $\Gamma_0 = \Psi_0([0, \infty]) \cup \{0, \infty\}$. In particular, the separatrices $\Gamma_n$ remain uniformly bounded away from 1.
Proof. As we no longer assume global uniform convergence $P_n \to P_0$ in $\mathbb{P}^1$, we cannot conclude as above that $\Psi_n$ converges to some $\Psi_{P_0, \gamma}$ locally uniformly in $S(\alpha')$ without arguing further.

Since the maps $\Psi_n$ take their values in $\mathbb{C} \setminus \{0, 1\}$, they form a normal family. So, extracting a subsequence if necessary, we may assume that the maps $\Psi_n$ converge to a map $\Psi_0$ locally uniformly in $S(\alpha')$. A priori, the map $\Psi_0$ could be constant (even constantly equal to 0, 1 or $\infty$).

Claim D'. If $\lim_{t_n \to +\infty} \Psi_n(t_n) = z_0$, then either $z_0 = \infty$ or $z_0$ is a zero of $P_0$.

This is proved as Claim D. Let us now consider the opposite case $t_n \in \mathbb{R}^+$, $t_n \searrow 0$.

Claim F. If $\lim_{t_n \to 0} \Psi_n(t_n) = z_0$ then $z_0 = \infty$.

Proof. We proceed by contradiction and assume that $z_0 \neq \infty$. Then, starting at $z_0$, we can follow the real-time flow of the differential equation $\dot{z} = P_0(z)$ backwards at least during time $2\varepsilon$. By local uniform convergence $P_n \to P_0$, we know that for $n$ large enough, starting at $z_n = \Psi_n(t_n)$, we can also follow the real-time flow of the differential equation $\dot{z} = P_n(z)$ backwards at least during time $\varepsilon$. It follows that $t_n \geq \varepsilon$ for $n$ large enough.

Set $\Gamma_n = \Psi_n([0, +\infty]) \cup \{0, \infty\}$. It is a continuum. Extracting a further subsequence if necessary, we may assume that $\Gamma_n \to \Gamma$ for the Hausdorff topology on compact subsets of $\mathbb{P}^1$. The set $\Gamma$ is connected and contains 0 and $\infty$. There are therefore infinitely many points in $\Gamma$ which are neither zeros of $P_0$, nor $\infty$.

Claim G. If $z_0 \in \Gamma$, is neither a zero of $P_0$, nor $\infty$, then, it is the image by $\Psi_0$ of some point $t_0 \in [0, \infty]$.

Proof. Since $z_0 \in \Gamma$, it is a limit of points $z_n = \Psi_n(t_n) \in \Gamma_n$. By Claims D' and F, we see that the sequence $t_n$ is bounded away from 0 and $\infty$. Extracting a subsequence if necessary, we may assume that $t_n \to t_0 \in [0, +\infty[$.

Thus, we now know that $\Psi_0$ is not constantly equal to $\infty$ or to a zero of $P_0$ (in particular, it takes its values in $\mathbb{C} \setminus \{0, 1\}$). Passing to the limit on the equation $\Psi_0' = P_0 \circ \Psi_n$, we get $\Psi_0' = P_0 \circ \Psi_0$.

Let us now prove $\lim_{t \to 0} \Psi_0(t) = \infty$. Let $t_j$ be a sequence which tends to 0 and choose $n_j$ large enough so that $|\Psi_n(t_j) - \Psi_0(t_j)| < 1/j$. Then, the two sequences have the same limit as $j \to \infty$, and by Claim F, this limit is $\infty$.

Let us finally prove $\lim_{t \to +\infty} \Psi_0(t) = 0$. Apply Proposition 2.5 we know that as $t \searrow +\infty$, $\Psi_0(t)$ has a limit $A$ which is a zero of $P_0$. Since $\Gamma$ connects 0 to $\infty$, we can find points $z_j \in \Gamma$ which are not zeros of $P_0$ with $\lim z_j = 0$. By Claim G, we can find a sequence $t_j \in [0, \infty]$ such that $\Psi_0(t_j) = z_j \to 0$. We necessarily have $t_j \to +\infty$ since $\Psi_0([0, \infty])$ avoids the zeros of $P_0$ and since $\Psi_0(t) \to \infty$ as $t \to 0$. Therefore,
\( A = 0 \) and \( \lim_{t \to +\infty} \Psi_0(t) = 0 \). Finally, the exact same argument shows that any point on \( \Gamma \) is a limit of points \( \Psi_0(t_j) \) with \( t_j \in [0, \infty] \) and so, \( \Gamma = \Psi_0([0, \infty]) \cup \{0, \infty\} \). This completes the proof of Proposition 4.1.

q.e.d.

4.4. Stable convergence and change of coordinates. Assume

\[
\begin{align*}
  f_0(z) &= z + z^{p+1} + O(z^{p+2}) \\
  g_0(z) &= z + z^{p+1} + O(z^{p+2})
\end{align*}
\]

are conjugate by a change of coordinates \( h_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \), i.e.,

\[
  g_0 = h_0 \circ f_0 \circ h_0^{-1}, \text{ with } h_0(0) = 1.
\]

We will say that \( g_n \to g_0 \) is related to \( f_n \to f_0 \) by a coordinate change if there are univalent maps \( h_n \to h_0 \) such that the new sequence is obtained from the old one by conjugation, that is: \( g_n = h_n \circ f_n \circ h_n^{-1} \).

Lemma 4.2. Stable convergence is preserved by a coordinate change.

Proof. Assume \( g_n \to g_0 \) is related to \( f_n \to f_0 \) by a coordinate change. Let \( h_n \to h_0 \) be the coordinate change conjugating \( f_n \) to \( g_n \). We must show that the convergence \( g_n \to g_0 \) is stable at 0 if and only if the convergence \( f_n \to f_0 \) is stable at 0.

Let \( P_n \) (respectively \( Q_n \)) be the monic polynomials vanishing at the \( p+1 \) fixed points of \( f_n \) (respectively \( g_n \)) close to 0 and assume \( Q_n \) are all \( \alpha \)-stable for some uniform \( \alpha \). We will show that for \( n \) sufficiently large, the polynomials \( P_n \) are all \( \alpha/3 \)-stable. Note that \( P_n \to P_0 \) and \( Q_n \to Q_0 \) algebraically, with \( P_0(z) = Q_0(z) = z^{p+1} \). It is enough to prove that for any sequence of outgoing \( \infty \)-germs \( \gamma_n \) for \( \dot{u} = P_n(u) \), which converge to an outgoing \( \infty \)-germ \( \gamma_0 \) for \( \dot{u} = P_0(u) \), the polynomials \( P_n \) are (\( \alpha/3 \), \( \gamma_n \))-stable.

Let \( \xi_0(u) := Q_0(h_0(u))/h_0'(u) \). Let us choose \( \varepsilon > 0 \) sufficiently small so that the \( h_n \) are all defined on \( D(2\varepsilon) \) and so that \( \xi_0(u) \) on \( D(\varepsilon) \) makes an angle less than \( \alpha/5 \) with \( P_0(u) \). This is possible since \( h_0(u) = u \cdot (1 + o(1)) \) and thus, as \( u \) tends to 0, \( \xi_0(u) = u^{p+1} \cdot (1 + o(1)) = P_0(u) \cdot (1 + o(1)) \).

Next, set \( \xi_n(u) := Q_n(h_n(u))/h_n'(u) \). The differential equation \( \dot{u} = \xi_n(u) \) has the same set of zeros (counting with multiplicities) as the differential equation \( \dot{u} = P_n(u) \) (i.e. the set of fixed points of \( f_n \) on \( D(2\varepsilon) \)), so \( \xi_n(u)/P_n(u) \) is holomorphic on \( D(2\varepsilon) \). On the circle \( C(0, \varepsilon) \), we have \( \xi_n(u)/P_n(u) \to \xi_0(u)/P_0(u) \) as \( n \to \infty \). Thus, by the maximum modulus principle, for \( n \) large enough and for all \( u \in D(\varepsilon) \), the vector \( \xi_n(u) \) makes an angle less than \( \alpha/4 \) with the vector \( P_n(u) \).

Denote by \( \eta_0 = \gamma_0 \) the \( \infty \)-germ of \( Q_0 = P_0 \). Denote by \( \eta_n \) the corresponding sequence of \( \infty \)-germs of \( Q_n \) tending to \( \eta_0 \).

Increasing \( n_0 \) and choosing \( R \) sufficiently large, we may assume that for all \( n \geq n_0 \) we have \( V_n := \Psi_{Q_n, \eta_n}(S^+(3\alpha/4)_R) \subset h_n(D(\varepsilon)) \). Indeed, by Proposition 2.6, as \( n \to \infty \), \( \Psi_{Q_n, \eta_n} : S^+(3\alpha/4) \to \mathbb{C} \) converges uniformly to \( \Psi_{Q_0, \eta_0} : S^+(3\alpha/4) \to \mathbb{C} \). Note that at each point \( z \in \partial V_n \), the vector \( Q_n(z) \) makes an angle \( \geq 3\alpha/4 \) with \( \partial V_n \).
Set $U_n := h_n^{-1}(V_n)$. Then, for every $u \in \partial U_n$, the vector $\xi_n(u)$ points towards the interior of $U_n$ and makes an angle $\geq 3\alpha/4$ with $\partial U_n$ (this is because $h_n$ is conformal and $\dot{u} = \xi_n(u)$ is the pullback of $\dot{z} = Q_n(z)$ via $z = h_n(u)$). Since the vector $\xi_n(u)$ makes an angle less than $\alpha/4$ with the vector $P_n(u)$, we see that at each point $u \in \partial U_n$, the vector $P_n(u)$ points towards the interior of $U_n$ and makes an angle $\geq \alpha/2$ with $\partial U_n$.

Choose $\theta \in [-\alpha/2, \alpha/2]$ and consider the differential equation of the rotated vector field, $\dot{u} = R_n(u) := e^{i\theta}P_n(u)$. Then, at every point $u \in \partial U_n$, the vector $R_n(u)$ points towards the interior of $U_n$, and thus, every orbit for $\dot{u} = R_n(u)$ which enters $U_n$ remains in $U_n$ and cannot form a homoclinic connection.

Fix now $t_0$ sufficiently large so that $h_0(\Psi_{P_0,\gamma_0}(t_0)) \in V_0$ and therefore $\Psi_{P_0,\gamma_0}(t_0) \in U_0$ (such a $t_0$ exists always by a local study of the pushed forward field of $P_0$ by $h_0$, see [DES]). It follows that for $n$ sufficiently large, $\Psi_{P_n,\gamma_n}$ extends analytically to $t_0 + S^+(\alpha/2)$ (and maps it into $U_n$). On the other hand, as $P_n \to P_0$ uniformly, for $n$ sufficiently large the maps $\Psi_{P_n,\gamma_n}$ extends analytically to a large slit disc $D(\hat{R}) \setminus \mathbb{R}^-$ containing $S^+(\alpha/3) \setminus (t_0 + S^+(\alpha/2))$. As a consequence, for $n$ sufficiently large, $\Psi_{P_n,\gamma_n}$ extends analytically to the whole sector $S^+(\alpha/3)$. q.e.d.


Assume that $P$ is a polynomial of degree $p+1$, that $\gamma$ is an outgoing $\infty$-germ and that $P$ is $(\alpha, \gamma)$-stable for some $\alpha > 0$. Let $\psi : [0, +\infty[ \to \mathbb{C}$ be the $\gamma$-separatrix. We must show that for $\eta > p/(p+1)$ and $t > 0$, the $\eta$-dimensional length $\ell_\eta(P, \gamma, t) := \int_t^{+\infty} |\psi'(u)|^\eta \, du$ is finite. It is then clear that it decreases with respect to $t$ and tends to 0 as $t$ tends to $+\infty$.

By Proposition 2.5, the $\gamma$-separatrix lands at a zero $a$ of $P$. If $a$ is a simple zero of $P$, then we have $\frac{1}{P(z)} \sim_a \frac{1}{P'(a)(z-a)}$. It follows that for $t_0$ large enough, we have

$$ t = t_0 + \int_{\psi(t_0)}^{\psi(t)} \frac{1}{P'(a)} \, dz = \frac{1}{P'(a)} \log(|\psi(t) - a|) + C + o(1) $$

as $t \to +\infty$. It follows that

$$ \psi(t) - a \sim K e^{P'(a)t} \quad \text{and} \quad \psi'(t) \sim K P'(a) e^{P'(a)t} $$

as $t \to +\infty$. Thus $\int_t^{+\infty} |\psi'(u)|^\eta \, du < \infty$ for $t > 0$.

If $a$ is a multiple zero of $P$ with multiplicity $p'+1 \leq p+1$, then

$$ \frac{1}{P(z)} \sim_a \frac{C}{(z-a)^{p'+1}}, \quad \psi(t) - a \sim \frac{C'}{t^{1/p'}} \quad \text{and} \quad \psi'(t) \sim \frac{C''}{(\eta^{p'+1})/(p')} $$

Since $p' \leq p$, we have $\eta^{p'+1}/p' \geq \eta^{p+1}/p > 1$ and $\int_t^{+\infty} |\psi'(u)|^\eta \, du < \infty$ for $t > 0$. q.e.d.
Lemma 4.3 (a uniform constant). For all \( p \geq 1 \) and \( \alpha \in ]0, \frac{\pi}{2} [ \), there exists a constant \( C_\alpha \), such that for any degree \( p + 1 \) polynomial \( P \) which is \( (\alpha, \gamma) \)-stable, any point \( z \) on the \( \gamma \)-separatrix and any zero \( a_j \) of \( P \), we have \( |z - a_j| \geq C_\alpha|z - a_0| \), where \( a_0 \) is the zero of \( P \) which is the ending point of the \( \gamma \)-separatrix.

Proof. We will proceed by contradiction. Without loss of generality, conjugating with a translation if necessary, we may assume that \( a_0 = 0 \) (see Lemma 3.4). Since multiplying \( P \) by a positive real \( k \) only changes the parametrization of the real trajectories (not their images), we may assume that the leading coefficient of \( P \) is of modulus 1.

We assume that we can find a sequence of \((\alpha, \gamma_n)\)-stable polynomials \( P_n \) of degree \( p + 1 \) with the \( \gamma_n \)-separatrix \( \Gamma_n \) ending at 0, points \( z_n \in \Gamma_n \setminus \{0\} \) and \( a_n \neq 0 \) with of \( P_n(a_n) = 0 \) such that \( \lim_{n \to \infty} \frac{z_n - a_n}{z_n} = 0 \). We will show that this is not possible thanks to Proposition 4.1. Note that \( \Gamma_n = \Psi_n([0, \infty[) \) for some map \( \Psi_n : S(\alpha) \to \mathbb{C} \setminus \{0, a_n\} \) satisfying \( \Psi_n = P_n \circ \Psi_n, \lim_{t \to 0} \Psi_n(t) = \infty \) and \( \lim_{t \to +\infty} \Psi_n(t) = 0 \).

We first need to re-scale the situation. We factorize \( P_n \) into

\[
P_n(z) = A_n(z - a_n) \prod_{j=2}^{p}(z - a_{j,n}) \quad \text{with} \quad |A_n| = 1.
\]

Extracting a subsequence if necessary, we may assume that as \( n \) tends to \( \infty \), the ratios \( \frac{a_j}{a_n} \) have limits in \( \mathbb{P}^1 \) and we let \( J \) be the set of indices \( j \in \{2, \cdots, p\} \) for which the limit is finite. We then define

\[
\rho_n := |a_n|^p \cdot \prod_{j \in \{2, \cdots, p\} \setminus J} \left| \frac{a_{j,n}}{a_n} \right| \quad \text{(} \rho_n := |a_n|^p \text{ if } J = \{2, \cdots, p\}. \)
\]

Set

\[
Q_n(z) := \frac{P_n(a_n z)}{a_n \rho_n} \quad \text{and} \quad \Xi_n(w) := \frac{1}{a_n} \Psi_n \left( \frac{w}{\rho_n} \right).
\]

Then \( Q_n \) is again a polynomial and

\[
\Xi'_n(w) = \frac{1}{a_n \rho_n} \Psi'_n \left( \frac{w}{\rho_n} \right) = \frac{1}{a_n \rho_n} P_n \circ \Psi_n \left( \frac{w}{\rho_n} \right) = Q_n \circ \Xi_n(w).
\]

Taking a subsequence if necessary, as \( n \to \infty \), the polynomials \( Q_n \) converges locally uniformly in \( \mathbb{C} \) to

\[
Q_0(z) = \mu \cdot z(z-1) \prod_{j \in J} \left( z - \lim_{n \to \infty} \frac{a_{j,n}}{a_n} \right) \quad \text{or} \quad Q_0(z) = \mu \cdot z(z-1) \text{ if } J = \emptyset
\]

with \( |\mu| = 1 \).

Since \( \rho_n > 0 \), the map \( \Xi_n \) is also defined on \( S(\alpha) \), and \( \Lambda_n := \Xi_n([0, +\infty[) \) is equal to \( \Gamma_n/a_n \). By Proposition 4.1, we know that \( \Lambda_n \) remains bounded away from 1 as \( n \) tends to \( \infty \). This shows that \( z_n/a_n \in \Lambda_n \) cannot tend to 1, which gives a contradiction. q.e.d.
Proof of Proposition 2.9. We assume that \((P_n, \gamma_n)\) is a sequence of pairs (polynomial, outgoing \(\infty\)-germ) converging to a pair \((P_0, \gamma_0)\) and that the \(P_n\) are \((\alpha, \gamma_n)\)-stable. We let \(\psi_n : \mathbb{C} \to \mathbb{C}\) be the \(\gamma_n\)-separatrices. Given \(\eta > p/(p + 1)\) and \(t_n \to t_0 > 0\), we must show that

\[
\int_{t_n}^{+\infty} |\psi'_n(u)|^\eta \, du - \int_{t_0}^{+\infty} |\psi'_0(u)|^\eta \, du.
\]

We will do this by using Lebesgue dominated convergence theorem.

By Proposition 2.6, we know that \(\psi'_n \to \psi'_0\) uniformly on \(\mathbb{R}^+\). So, we only have to find an integrable function that dominates \(|\psi'_n(t)|^\eta\) for \(n \geq n_0\) and \(t \geq R\).

Let \(a_n\) be the landing point of the \(\gamma_n\)-separatrix \(\psi_n\). By Proposition 2.6, we know that \(a_n\) tends to \(a_0\) as \(n \to +\infty\). Thus, conjugating with the translations \(z \mapsto z - a_n\) (this will not change the stability, by Lemma 3.4), we may assume that \(a_n = 0\), so that \(\psi_n(t) \to 0\) as \(t \to +\infty\) and \(P_n(0) = 0\).

It follows from Lemma 4.3 (with \(a_0(P_n) = 0\)) that for all \(n \geq 0\) and all \(t > 0\), we have

\[
|\psi'_n(t)| = |P_n(\psi_n(t))| = |A_n| |\psi_n(t) - 0| \prod_{j=1}^{p} |\psi_n(t) - a_{j,n}| \geq \frac{C_n}{2} |\psi_n(t)|^{p+1},
\]

where \(a_{j,n}\) are the other zeros of \(P_n\), and \(A_n\) is the leading coefficient of \(P_n\) (with \(A_n \to A_0\)).

Set \(\Psi_n := \Psi_{P_n, \gamma_n}\) (note that \(\psi_n = \Psi_n|_{\mathbb{R}^+}\)). By Proposition 2.5, for \(\alpha' < \alpha\), if \(R\) is sufficiently large, \(\Psi_0(S(\alpha')_R) \subset D(1/2) \setminus \{0\}\) and by Proposition 2.6, if \(n_0\) is sufficiently large, \(\Psi_n(S(\alpha')_R) \subset D^s\) for all \(n \geq n_0\). By Schwarz lemma, with \(d_\mathbb{D}\) denoting the hyperbolic metric of \(V\), we have, \((\forall n \geq n_0)\) \((\forall w, w' \in S(\alpha')_R)\),

\[
d_\mathbb{D}(\Psi_n(w), \Psi_n(w')) \leq d_{S(\alpha')_R}(w, w') \leq d_{S(\alpha)}(w, w').
\]

The coefficient of the hyperbolic metric on \(D^s\) is \(\frac{|dz|}{|z| \log \frac{1}{|z|}}\), and the coefficient at \(t \in \mathbb{R}\) in \(S(\alpha)\) is bounded from above by \(C_1/t\). So,

\[
(\forall n \geq n_0) \quad (\forall t > R) \quad \frac{|\psi'_n(t)|}{|\psi_n(t)| \log \frac{1}{|\psi_n(t)|}} \leq \frac{C_1}{t}.
\]

Recall that \(\eta > \frac{p}{p + 1}\). Choose \(\varepsilon > 0\) small so that \(\eta^{\frac{p+1}{p+1+\varepsilon}} > 1\). There is a constant \(C_2\) such that

\[
(\forall x \in [0, 1]) \quad \log \frac{1}{x} \leq \frac{C_2}{x^{\varepsilon}}.
\]
So, for all $n \geq N_0$ and all $t > R$, we have
\begin{equation}
\frac{C_p}{2} |\psi_n(t)|^{p+1} \leq |\psi_n(t)|^{p+1} \leq \frac{C_1 |\psi_n(t)| \log \frac{1}{|\psi_n(t)|}}{t} \leq \frac{C_1 C_2 |\psi_n(t)|^{1-\varepsilon}}{t}.
\end{equation}
Hence, $|\psi_n(t)|^{p+\varepsilon} \leq C_4 t$. Thus,
\begin{equation}
|\psi_n(t)|^{p+\varepsilon} \leq \left( \frac{C_1 C_2 |\psi_n(t)|^{1-\varepsilon}}{t} \right)^{\eta} = \left( \frac{C_4}{t^{p+1}} \right)^{\eta}.
\end{equation}

The right hand side function is integrable since $\eta^{p+1} > 1$. q.e.d.

5. Relating iterated orbits to trajectories

5.1. Case of a single map $f(z)$ close to $z + z^{p+1}$. For a holomorphic map $g : D(r) \to \mathbb{C}$, we define $\|g\| := \sup_{z \in D(r)} |g(z)|$.

Let $f(z) = z + g(z) = z + (1 + s(z))Q(z)$. We do not assume that $Q$ is a polynomial. The goal of the following lemma is to show that once $Q(z)$ is close to $z^{p+1}$, the time-one iterate of $f$, and sometimes the long term iterates of $f$, are well approximated by the flow-time map of the flow of $\dot{z} = Q(z)$.

**Lemma 5.1.** Let $g, Q, s, f : D(r) \to \mathbb{C}$ be four holomorphic functions with the following relations: $g(z) = (1 + s(z))Q(z)$ and $f(z) = z + g(z) = z + (1 + s(z))Q(z)$. Let $\psi(w)$ be a local solution of $\dot{z} = Q(z)$ (the time $w$ is considered to be complex). Set
\[ \varepsilon_f' := \max \{ \|Q\|, \|g\|, \|s\| \} \quad \text{and} \quad \varepsilon_f := \max \{ \|g\|, \|Q\| \}. \]

(A) (very much inspired by the Main Lemma in [O] and a similar estimate in [DES], see also [E], Lemma 3). Assume
\[ \varepsilon_f' \leq \varepsilon < \frac{r}{4}, \quad \varepsilon_f \leq \varepsilon < \frac{r}{4} \quad \text{and} \quad \psi(w_0) \in D(r-4\varepsilon) \setminus \{ \text{zeros of } Q \}. \]

Then $\psi(w)$ is defined at least on $w_0 + D(4)$, and this last disk contains two points $F^\pm(w_0)$ satisfying $\psi(F^\pm(w_0)) = f^\pm(\psi(w_0))$, $|F^\pm(w_0) - (w_0 \pm 1)| < 5\varepsilon'$ and $|f^\pm(\psi(w_0)) - \psi(\pm 1)| \leq 5\varepsilon'$.

**B.** Set $S(\alpha) = S^+(\alpha)$. Assume $\frac{\alpha}{2} > \alpha > 0$ and $\psi$ is defined on an open set $W$. Assume further
\begin{equation}
\varepsilon_f' \leq \varepsilon' < \sin \frac{\alpha}{5}, \quad \varepsilon_f \leq \varepsilon < \frac{r}{4};
\end{equation}
$W + S(\alpha) \subset W$ and $\psi(W) \subset D(r-4\varepsilon) \setminus \{ \text{zeros of } Q \}$. 
(Note that such a map \( \psi \) is often globally non-univalent). Then \( \psi \) has a local inverse map \( \phi \) and \( F(w) := \phi(f(\psi(w))) \) is a globally well defined and holomorphic function on \( W \), mapping \( W \) into \( W \), and satisfying:
\[
(5.2) \quad \psi(F(w)) = f(\psi(w)) \quad \text{and} \quad \sup_{j \in \mathbb{N}, w \in W} |F^j(w) - (w + j)| \leq 5\varepsilon'.
\]

If \( W \) is convex the map \( F \) is also univalent. There is a similar statement for \( S^- (\alpha) \) replacing the triple \( (F, f, w + j) \) by \( (F^{-1}, f^{-1}, w - j) \).

One may consider \( w \) as an approximate Fatou coordinate, in which the dynamics is close to the translation by 1.

**Proof.**

**Part A.** Set \( \varepsilon_1 = \|s\|_r \). We have \( \varepsilon_1 < \frac{1}{5} < \frac{1}{4} \).

\( (I) \) \( f \) is univalent on \( D(r) \), its image contains \( D(r - \varepsilon) \) and for all \( z \in D(r - \varepsilon) \), we have \( f^{\pm 1}(z) \in D(z, \varepsilon) \) and \( |f(z) - z| \leq \frac{5}{4}|Q(z)| \), \( |f^{-1}(z) - z| \leq \frac{5}{3}|Q(z)| \).

**Proof.** Note that \( D(r) \) is convex. For \( a, b \in D(r) \), \( a \neq b \), we have
\[
(5.3) \quad f(b) - f(a) = b - a + \int_{[a,b]} (f'(z) - 1)dz \quad \text{and} \quad |f(b) - f(a)| \geq \frac{4}{5}|b - a|
\]
as \( |f'(z) - 1| \leq \varepsilon' < \frac{1}{5} \). So \( f \) is univalent. Evidently \( \frac{|f(z) - z|}{|Q(z)|} = \frac{|g(z)|}{|Q(z)|} = |1 + s(z)| \leq 1 + \varepsilon_1 < 1 + \varepsilon_1 < 1 + \varepsilon_1 < \frac{5}{4} \).

Let \( b \in D(r - \varepsilon) \). For \( z \) on the circle \( |z - b| = \varepsilon \), we have \( |g(z)| < \varepsilon = |z - b| \). So, by Rouché’s theorem, \( z \mapsto z - b \) and \( z \mapsto z - b + g(z) \) have the same number of roots in \( D(b, \varepsilon) \). This shows that \( f^{-1}(b) \) is well defined and \( f^{-1}(b) \in D(b, \varepsilon) \). Set \( z = f^{-1}(b) \). We have
\[
\frac{|f^{-1}(b) - b|}{|Q(b)|} = \frac{|z - f(z)|}{|Q(f(z))|} = \frac{|g(z)|}{|Q(z)|} \cdot \frac{|Q(z)|}{|Q(f(z))|} < \frac{5}{4} \cdot \frac{|Q(z)|}{|Q(f(z))|}.
\]

On the other hand,
\[
\frac{|Q(z + g(z)) - Q(z)|}{|Q(z)|} \leq \sup_{r \in [z, z + g(z)]} |Q'(r)| \cdot \frac{|g(z)|}{|Q(z)|} < \varepsilon'|1 + s(z)| < \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{4}.
\]

So \( \frac{|Q(f(z))|}{|Q(z)|} \geq 1 - \frac{1}{4} = \frac{3}{4} \) and \( 2 \cdot \frac{5}{4} \cdot \frac{|Q(z)|}{|Q(f(z))|} \leq \frac{5}{4} \cdot \frac{4}{3} = \frac{5}{3} \). This proves (I).

Let us now assume that \( w_0 = 0 \) and set \( z_0 = \psi(w_0) = \psi(0) \). By assumption, \( z_0 \in D(r - 4\varepsilon) \setminus \{ \text{zeros of } Q \} \).

**II** The map \( \psi \) is defined at least on \( D(4) = \{|w| \leq 4\} \) with images contained in \( D(r) \).

**Proof.** Let \( D(s) \) be the maximal disc on which \( \psi \) is defined. Since \( Q \) is holomorphic on \( D(r) \), the set \( \partial \psi(D(s)) \) meets either the zeros of \( Q \)
Thus, the former case does not occur, due to Lemma 3.1. Hence there is \( w \) with \( |w| < s \) such that \( |\psi(w)| > 4\varepsilon + |z_0| \). Therefore
\[
4\varepsilon < |\psi(w) - z_0| = |\psi(w) - \psi(0)| \leq \sup_{\tau \in D(s)} |\psi'(\tau)| \cdot |w| = \sup_{\tau \in D(s)} |Q(\psi(\tau))| \cdot |w| \leq \varepsilon s.
\]
So \( s > 4 \).

Now set \( \Delta := D(z_0, \frac{5}{3}|Q(z_0)|) \subset D(r) \). We have the following property.

**Proof.** The fact \( f^\pm(z_0) \in \Delta \) follows from (I). By assumption, \( |s(z)| < \frac{1}{4} \), thus, \( |1 + s(z)| > \frac{3}{4} \) and so, \( |g(z)| > \frac{3}{4}|Q(z)| \). In particular, \( g(z) \) has the same set of zeros as \( Q(z) \). By assumption \( Q(z_0) \neq 0 \). For \( z \in \Delta \), we have
\[
|Q(z) - Q(z_0)| \leq \varepsilon'|z - z_0| \leq \varepsilon' \cdot \frac{5}{3}|Q(z_0)| \leq \frac{1}{5} \cdot \frac{5}{3}|Q(z_0)| = \frac{1}{3}|Q(z_0)|.
\]
So,
\[
\text{(5.4)} \quad |Q(z)| \geq \frac{2}{3}|Q(z_0)|
\]
and
\[
|g(z)| > \frac{3}{4}|Q(z)| \geq \frac{3}{4} \cdot \frac{2}{3}|Q(z_0)| = \frac{1}{2}|Q(z_0)| > 0.
\]

Now \( \phi(z) \) is just a primitive of \( \frac{1}{Q(z)} \) on \( \Delta \) with \( \phi(z_0) = 0 \). For \( z \in \Delta \) we have
\[
|\phi(z) - \phi(z_0)| \leq \sup_{\tau \in \Delta} |\phi'(\tau)| \cdot |z - z_0| \leq \sup_{\tau \in \Delta} |\frac{1}{Q(\tau)}| \cdot \frac{5}{3}|Q(z_0)| \leq \frac{3}{2}|Q(z_0)| \cdot \frac{5}{3}|Q(z_0)| < 3.
\]
Thus, \( \phi(\Delta) \subset D(3) \). Now \( \psi \circ \phi \) is holomorphic on \( \Delta \) and is locally the identity map. So it is identity on \( \Delta \), and \( \phi|_\Delta \) is univalent. q.e.d.

Finally we have the following property.

**Proof.** Set \( w^\pm = F^\pm(0) \). To compare \( w^\pm \) with \( \pm 1 \) we need the help of the second derivative of \( \phi \). We have \( \phi' = 1/Q \) and so, for \( z \in \Delta \),
\[
\text{(5.5)} \quad |\phi''(z)| = \left| -\frac{Q'(z)}{Q(z)^2} \right| \leq \frac{9\varepsilon'}{4|Q(z_0)|^2}
\]
and
\[
|w^+ - (1 + s(z_0))| = |\phi(f(z_0)) - \phi(z_0) - \phi'(z_0)g(z_0)| \\
\leq \frac{|f(z_0) - z_0|^2}{2} \sup_{\tau \in \Delta} |\phi''(\tau)| \\
\leq (l) (5.5) \left( \frac{5}{4} \right)^2 |Q(\hat{z})|^2 \frac{9\epsilon'}{4|Q(z_0)|^2} < 2\epsilon'.
\]

So
\[
|w^+ - 1| \leq |w^+ - (1 + s(z_0))| + \epsilon_1 \leq 2\epsilon' + \epsilon_1 \leq 5\epsilon'.
\]

Similarly,
\[
|w^- + (1 + s(f^{-1}(z_0)))| = |\phi(z_0) - \phi(f^{-1}(z_0)) - \phi'(f^{-1}(z_0))g(f^{-1}(z_0))| \\
\leq (l) \left( \frac{5}{4} \right)^2 |Q(\hat{z})|^2 \frac{9\epsilon'}{4|Q(z_0)|^2} < 4\epsilon'.
\]

Therefore, \(|w^- + 1| \leq 4\epsilon' + \epsilon_1 \leq 5\epsilon'.\)

**Part B.** This part is an easy corollary of Part A. For any \(w \in W\) we have \(\psi(w) \in D(r - 4\epsilon) \setminus \{\text{zeros of } Q\},\) \(\epsilon' < \frac{1}{5}\) and \(\epsilon < \frac{r}{5}\). So \(F(w)\) is well defined by Part A. To check that \(F(w) \in W\), we note that \(|F(w) - (w + 1)| < 5\epsilon' < \sin \alpha\). So \(F(w) \in w + S(\alpha) \subset W\). The fact that \(F\) is holomorphic follows from the functional equation \(\psi(F(w)) = f(\psi(w))\). The inequality about \(F^j(w)\) is proved by induction. The univalence of \(F\) follows similarly as in (5.3), by checking \(|F^j(w)-1| < 1\).

In fact:
\[
|F'(w) - 1| \leq \left\| \left( \frac{Q(z) - Q(f(z))f'(z) + Q(f(z))(f'(z) - 1)}{Q(f(z))} \right) \right\|_r \\
\leq (1 + \epsilon') \left\| \frac{Q(z) - Q(f(z))}{Q(f(z))} \right\|_r + \epsilon' \\
\leq (1 + \epsilon') \epsilon' \left\| \frac{f(z) - z}{Qf(z)} \right\|_r + \epsilon' \\
\leq (l) (1 + \epsilon') \epsilon' \cdot \frac{5}{4} + \epsilon' \frac{\epsilon' \epsilon}{5} < \frac{3}{5} < 1.
\]

The case \(S(\alpha) = S^-(\alpha)\) is similar.

**5.2. Stable perturbations of \(z + z^{p+1}\) imply well approximation by flow.**

**Definition 5.2.** If \(V \subset \mathbb{C}\) is a hyperbolic subset, we denote by \(d_V(\cdot, \cdot)\) the hyperbolic distance in \(V\). By convention, we set \(d_V(a, b) = +\infty\) if one or two of \(a, b\) do not belong to \(V\).

Let us now assume \(r < 1\) and that the sequence \((f_n : D(r) \to \mathbb{C})_{n \geq 1}\) converges locally uniformly to \(f_0 : D(r) \to \mathbb{C}\) with \(f_0(z) = z + z^{p+1} + O(z^{p+2})\). Let \(P_n\) be the monic polynomials of degree \(p + 1\) which vanish at the \(p + 1\) fixed points of \(f_n\) close to 0. Fix \(k \in \mathbb{Z}/2p\mathbb{Z}\). Let \(\gamma_n\) be the
∞-germ tangent to $\mathbb{R}^+ e^{2\pi i \frac{k}{2p}}$ at $\infty$ and assume $P_n$ is $(\alpha', \gamma_n)$-stable for some $\alpha' > \alpha$ and all $n$ sufficiently large. Set $\Psi_n := \Psi_{P_n, \gamma_n} : S(\alpha') \to \mathbb{C}$. Finally, for $z \in \mathbb{C}$, let $\psi_n(z, \cdot) : [t^-(z), t^+(z)] \to \mathbb{C}$ be the maximal real-time solution of $\dot{z} = P_n(z)$ with initial condition $\psi_n(z, 0) = z$.

**Lemma 5.3** (long term approximation by flow). There are $n_0 > 0$ and $R_0 > 0$ such that for all $n \geq n_0$,

(a) $\Psi_n(S(\alpha)_{R_0}) \subset D(r) \setminus \{\text{fixed points of } f_n\}$ and $\Psi'_n(S(\alpha)_{R_0}) \subset \mathbb{D}^*$;

(b) there is a univalent map $F_n : S(\alpha)_{R_0} \to S(\alpha)_{R_0}$ satisfying $\Psi_n \circ F_n = f_n \circ \Psi_n$;

(c) $F_n \to F_0$ uniformly on $S(\alpha)_{R_0}$;

(d) \[ \left\{ \begin{array}{l} \sup_{n > N, j \in \mathbb{N}, w \in S(\alpha)_{R}} d_{S(\alpha)_{R}} (F^o_j (w), w + j) \xrightarrow{N, R \to \infty} 0 \\ \sup_{n > N, j \in \mathbb{N}, z \in \Psi_n(S(\alpha)_{R})} d_{D(r) \setminus \{\text{fixed points of } f_n\}} (f^o_j (z), \psi_n(z, j)) \xrightarrow{N, R \to \infty} 0 \\ \sup_{n > N, j \in \mathbb{N}, z \in \Psi_n(S(\alpha)_{R})} d_{\mathbb{D}^*} (P_n(f^o_j (z)), P_n(\psi_n(z, j))) \xrightarrow{N, R \to \infty} 0, \end{array} \right. \]

**Remark.** When $P_n(0) = 0$, one may use $d_{\mathbb{D}^*}$ in the second limit of (5.6).

**Figure 3.** The lifted dynamics $F_n$ is close to translation by 1.

**Proof.** We will do the proof for outgoing $\infty$-germs ($k$ is even). The proof for incoming ones is similar.

We will find $W$ such that $f_n$, $\Psi_n$ and $W$ satisfy the hypothesis (5.1) of Lemma 5.1.B for $n$ large enough.

**Proof of (a).** Since $P_n(z) \to z^{p+1}$, if $\varepsilon \in ]0, r[$ is small enough and $n$ is sufficiently large, $P_n(D(\varepsilon)) \subset \mathbb{D}$. By Proposition 2.6 with $P_0(z) = z^{p+1}$,
we can find $N_0 > 0$ and $R_0 > 0$ such that for any $n \geq N_0$, we have
$\Psi_n(S(\alpha)_R) \subset D(\varepsilon) \setminus \{\text{zeros of } P_n\}$. Since $\Psi'_n = P_n \circ \Psi_n$ we have $\Psi'_n(S(\alpha)_R) \subset \mathbb{D}^*$. By definition, the zeros of $P_n$ are the fixed points of $f_n$.

**Proof of (b).** For any $N \geq N_0$ and $R \geq R_0$, there is a constant $\rho_{N,R} > 0$ as $N, R \to \infty$ such that,

\[(5.7) \quad \Psi_n(S(\alpha)_R) \subset D(\rho_{N,R}) \setminus \{\text{zeros of } P_n\}, \forall n \geq N.\]

Set
\[(5.8) \quad r(N,R) := 5 \cdot \rho_{N,R} \quad \text{and} \quad \rho'_{N,R} := 2 \max\{\|P'_0\|_{r(N,R)}, \|g'_0\|_{r(N,R)}, \|s_0\|_{r(N,R)}\}.\]

As $P'_0(0) = g'_0(0) = s_0(0) = 0$ and $r(N,R) \to 0$ as $N, R \to \infty$, we have $\rho'_{N,R} \to 0$ as $N, R \to \infty$. We may increase $N_0, R_0$ if necessary so that

\[(5.9) \quad \rho'_{N,R} \leq \rho'_{N_0, R_0} < \frac{\sin \alpha}{5} < \frac{1}{5}.\]

Set $\varepsilon_{f_0|D(r(N,R))} := \max\{\|g_n\|_{r(N,R)}, \|P_n\|_{r(N,R)}\}$ and
\[\varepsilon'_{f_0|D(r(N,R))} := \max\{\|P'_0\|_{r(N,R)}, \|g'_0\|_{r(N,R)}, \|s_0\|_{r(N,R)}\}\]

so that $\varepsilon'_{f_0|D(r(N,R))} = \rho'_{N,R}/2$. We know that $P_n, g_n, s_n$ and their derivatives converge uniformly to $P_0, g_0, s_0$ and their derivatives in some neighborhood of $0$ as $n \to \infty$. So,

\[\varepsilon'_{f_n|D(r(N,R))} \xrightarrow{n \to \infty} \varepsilon'_{f_0|D(r(N,R))}, \quad \varepsilon_{f_n|D(r(N,R))} \xrightarrow{n \to \infty} \varepsilon_{f_0|D(r(N,R))}.\]

There is therefore $n_0(N,R) > N$ such that for $n \geq n_0(N,R)$,

\[(5.10) \quad \varepsilon'_{f_n|D(r(N,R))} < 2 \cdot \varepsilon'_{f_0|D(r(N,R))} = \rho'_{N,R} \leq \rho'_{N_0, R_0} < \frac{\sin \alpha}{5}.\]

in particular $\varepsilon'_{f_0|D(r(N,R))} < \frac{1}{10}$, and

\[(5.11) \quad \varepsilon_{f_n|D(r(N,R))} < 2 \varepsilon_{f_0|D(r(N,R))} \leq 2 \cdot \frac{r(N,R)}{10} = \frac{r(N,R)}{5} \leq \rho_{N,R} \quad \text{and} \quad \frac{r(N,R)}{5} \leq \rho_{N,R}.\]
Proof of (c). The uniform convergence of $F_n$ towards $F_0$ on $S(\alpha)_{R_0}$ follows from that of $f_n$ (by assumption) and of $\Psi_n$ (by Proposition 2.6).

Proof of (d). Given any $N \geq n_0$ and $R \geq R_0$, and for $n \geq N$, the $F_n$ defined on $S(\alpha)_{R}$ is the restriction of the $F_n$ defined on $S(\alpha)_{R_0}$. So by \((5.2)\), and setting $N' = n_0(N, R)$,

$$\sup_{n > N', j \in \mathbb{N}, w \in S(\alpha)_{R}} \frac{|F_n^{\circ j}(w) - (w + j)|}{5j} \leq \rho_{N, R}' .$$

Replacing now the Euclidean metric by the hyperbolic metric, we can find $s_{N,R}$ depending in fact only on $\rho_{N, R}'$, with $s_{N,R} \to 0$ as $\rho_{N, R}' \to 0$ (therefore as $N, R \to \infty$), such that

$$\sup d_{S(\alpha)_{R_0}}(F_n^{\circ j}(w), w + j) \leq \sup d_{S(\alpha)_{R}}(F_n^{\circ j}(w), w + j) \leq s_{N,R}$$

where both sup are taken over the set \(\{n > N', j \in \mathbb{N}, w \in S(\alpha)_{R}\}\).

Now $f_n^{\circ j} \circ \Psi_n = \Psi_n \circ F_n^{\circ j}$, and

$$\Psi_n : S(\alpha)_{R_0} \to \mathbb{D} \setminus \{\text{fixed points of } f_n\} =: U_n$$

is holomorphic. So, by Schwarz Lemma:

$$d_{U_n}(f_n^{\circ j} \circ \Psi_n(w), \Psi_n(w + j)) = d_{U_n}(\Psi_n(F_n^{\circ j}(w)), \Psi_n(w + j))$$

$$\leq d_{S(\alpha)_{R_0}}(F_n^{\circ j}(w), w + j) \leq s_{N,R} .$$

So

$$\sup_{n > N', j \in \mathbb{N}, z \in \Psi_n(S(\alpha))} d_{U_n}(f_n^{\circ j} \psi_n(z), j) \leq s_{N,R} .$$

From this one derives easily the first two limits in \((5.6)\). The remaining limit in \((5.6)\) is obtained similarly, by composing with $P_n$, and by using $P_n \circ \Psi_n = \Psi'_n$.

q.e.d.

6. Continuity of Julia sets and Poincaré series

6.1. Proof of Proposition 2.11. Assume the sequence

$$(f_n : D(r) \to \mathbb{C})_{n \geq 1}$$

converges locally uniformly to $f_0 : D(r) \to \mathbb{C}$ with

$$f_0(z) = z + z^{p+1} + O(z^{p+2}), \quad p \geq 1.$$ 

Let $P_n$ be the monic polynomials of degree $p + 1$ which vanish at the $p + 1$ fixed points of $f_n$ close to 0. Let us fix $k \in \mathbb{Z} / 2p\mathbb{Z}$ odd and let $\gamma_n$ be the $\infty$-germ for $z = P_n(z)$, tangent to $e^{2\pi i k \frac{1}{2p}}$ at $\infty$.

Definition 6.1. We say that the convergence $f_n \to f_0$ is \((\alpha, k)\)-stable if and only if for $n$ sufficiently large, the polynomials $P_n$ are \((\alpha', \gamma_n)\)-stable for some $\alpha' > \alpha$. 

Now, let $K \subset D(r)$ be a compact set such that for all $z \in K$,  

- $f^{o_j}_0(z)$ is defined for all $j \geq 0$ and $f^{o_j}_0(z) \xrightarrow[j\to\infty]{} 0$ tangentially to the direction $e^{2\pi i k/s}$.

Let $a_n$ be the landing point of the $\gamma_n$-separatrix of $\hat{z} = P_n(z)$. We will now show that for $n$ large enough, $K$ is contained in the basin of attraction of $a_n$, i.e., for all $z \in K$,  

- $f^{o_j}_n(z)$ is defined for all $j \geq 0$ and $f^{o_j}_n(z) \xrightarrow[j\to\infty]{} a_n$.

Set $\Psi_n := \Psi_{P_n,\gamma_n} : S(\alpha') \to \mathbb{C}$.

**Lemma 6.2.** For any $R > 0$, there are an open set $L$ relatively compact in $S(\alpha)_{R}$, a compact $K' \subset \Psi_0(L)$, $n_0 > 0$ and $j_0 > 0$ such that for all $n \geq n_0$,  

$$f^{o_j}_n(K) \subset K' \subset \Psi_n(L).$$

**Proof.** Note that $\Psi_0(S(\alpha)_{R})$ is a sector neighborhood of 0 around the $e^{2\pi i k/s} \mathbb{R}^+$ of opening angle $\alpha/p$ (see Example 0).

By the classical theory of Fatou flowers around parabolic points (which can be easily reproved using Lemma 5.1), $f^{o_j}_0(K)$ will tend to 0 within a cusp region bounded by two curves tangential to $\mathbb{R}^+ \cdot e^{2\pi i k/s}$.

Thus we can find $j_0$ and an open set $L$ compactly contained in $S(\alpha)_R$, such that $f^{o_j}_0(K) \subset \Psi_0(L)$. Since $f_n \to f_0$ uniformly and $\Psi_n \to \Psi_0$ uniformly on every compact subset of $S(\alpha')$, we see that for $n$ sufficiently large, $f^{o_j}_n(K) \subset K' \subset \Psi_n(L)$ for some compact set $K'$. q.e.d.

**Proof of Proposition 2.11.** By Lemma 5.3, for $n$ and $R$ large enough,  

$$\Psi_n(S(\alpha)_{R}) \subset D(0,r) \setminus \{\text{fixed points of } f_n\}$$

and there is a holomorphic map $F_n : S(\alpha)_{R} \to S(\alpha)_{R}$ such that $\Psi_n \circ F_n = f_n \circ \Psi_n$. Moreover, for all $w \in S(\alpha)_{R}$, the hyperbolic distance in $S(\alpha)_{R}$ between $F^{o_j}_n(w)$ and $w + j$ is bounded. Thus, $F^{o_j}_n(w)$ tends to $\infty$ as $j$ tends to $\infty$, and so,  

$$f^{o_j}_n(\Psi_n(w)) = \Psi_n(F^{o_j}_n(w)) \to a_n \text{ as } j \to \infty.$$  

The previous lemma asserts that for this $R$ and for $n$ large enough,  

$$f^{o_j}_n(K) \subset \Psi_n(L) \text{ with } L \subset S(\alpha)_{R}.$$  

q.e.d.

**6.2. Partial continuity of Fatou components.** Let us now assume that $\beta$ is a parabolic periodic point of a rational map $f_0$. Let $l$ be the period of $\beta$. Then, $f^{2\pi l}_{0}(\beta) = \beta$ and $[f^{2\pi l}_{0}]'(\beta) = e^{2\pi i r/s}$ for some integers $r \in \mathbb{Z}$ and $s \geq 1$ co-prime. Then, $[f^{2\pi l_s}_{0}]'(\beta) = 1$ and conjugating $f_0$ with a Moebius transformation (non uniquely determined), we may assume that $\beta = 0$ and $f^{o_s}_{0}(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ with $p = ms$ a multiple of $s$. The number $p$ is called the number of petals of $f_0$ at $\beta$.

It is known that there exists $p$ attracting petals and $p$ repelling petals $\mathcal{P}_k$, $k \in \mathbb{Z}/2p\mathbb{Z}$, contained in a neighborhood of 0 in which $f_0$ is univalent, such that

- $\mathcal{P}_k$ contains $\{r e^{i \theta} : 0 < r < \varepsilon \text{ and } |\theta - \frac{k}{2p}| < \frac{1}{4p}\}$ for $\varepsilon$ small enough,
• \( f_0^k(\mathcal{P}_k) \subset \mathcal{P}_{k+2mr} \) if \( k \) is odd; and \( f_0^n(\mathcal{P}_k) \supset \mathcal{P}_{k+2mr} \) if \( k \) is even.

The repelling petals are those for \( k \) even and the attracting petals are those for \( k \) odd. Under iteration of \( f_0^k \), the orbit of every point contained in an attracting petal converges to the parabolic fixed point.

Let us fix \( k \in \mathbb{Z}/2p\mathbb{Z} \) odd and let \( \mathcal{F}_k \) the the set of points whose forward orbit under iteration of \( f_0 \) intersects \( \mathcal{P}_k \). The set \( \mathcal{F}_k \) is a union of Fatou components contained in the attracting basin of \( \beta \).

**Proposition 6.3** (partial stability of Fatou components). Assume \((f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n \geq 1}\) is a sequence of rational maps converging algebraically to \( f_0 : \mathbb{P}^1 \to \mathbb{P}^1 \). Let \( \beta \) be a parabolic periodic point of \( f_0 \), let \( l \) be the period of \( \beta \), \( e^{2i\pi r/s} \) be the multiplier of \( f_0 \) at \( \beta \) and choose a coordinate on \( \mathbb{P}^1 \) such that \( \beta = 0 \) and \( f^{ls}(z) = z + z^{p+1} + O(z^{p+2}) \). If the convergence \( f_n \to f_0 \) is \((\alpha, k)\)-stable for some odd \( k \in \mathbb{Z}/2p\mathbb{Z} \), then the set \( \mathcal{F}_k \) contains no limit point of \( J_{f_n} \).

**Proof.** Let \( Q \) be an open subset of \( \mathcal{F}_k \), relatively compact in \( \mathcal{F}_k \). Choose \( j_0 \) large enough so that \( f_0^{j_0}(Q) \) is contained in \( \mathcal{P}_k \). Choose \( n_0 \) sufficiently large so that for \( n \geq n_0 \), \( f_n^{j_0}(Q) \subset \mathcal{P}_k \). And set

\[
K := \bigcup_{n \geq n_0} f_n^{j_0}(Q).
\]

By Proposition 2.11, for all \( n \) sufficiently large, \( K \) is in the attracting basin of some fixed point \( a_n \) of \( f_n^{sls} \) close to 0. Thus, \( Q \) is contained in the Fatou set of \( f_n \) and \( Q \cap J(f_n) = \emptyset \).

**q.e.d.**

6.3. Continuity of Julia sets.

**Theorem 6.4** (see also \[DES\]). (a) Let \((f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n \geq 1}\) be a sequence of rational maps converging algebraically to a rational map \( f_0 : \mathbb{P}^1 \to \mathbb{P}^1 \). Then \( J(f_n) \to J(f_0) \) assuming

• the convergence \( f_n \to f_0 \) is stable at each parabolic point of \( f_0 \),

• for each irrationally indifferent periodic point \( \beta_0 \) of \( f_0 \) with multiplier \( e^{2i\pi \alpha_0} \) and period \( l_0 \), either
  - \( f_0 \) is not linearizable at \( \beta_0 \), or
  - \( \alpha_0 \) is a Brjuno number and \( f_n \) has a \( l_0 \)-periodic point \( \beta_n \) converging to \( \beta_0 \) with multiplier \( e^{2i\pi \alpha_0} \), and

• \( f_0 \) does not have Herman rings.

(b) If in addition \( f_0 \) is geometrically finite and \( f_n \to f_0 \) preserving critical relations, then, for \( n \) sufficiently large, \( f_n \) is geometrically finite.

**Remark.** If \( f_0 \) is geometrically finite, it does not have irrationally indifferent cycles nor Herman rings.

**Proof.** (a) Let \( J' \) be a limit of a subsequence of \( J(f_n) \). We know that \( J' \supset J(f_0) \) due to the density of repelling periodic points in \( J(f_0) \) and their stability. We just need to prove \( J' \subset J(f_0) \).
According to the non-wandering theorem and the classification theorem of Sullivan, the Fatou set \( \mathbb{P}^1 \setminus J(f_0) \) of \( f_0 \) has four types of components: components of attracting basins, components of parabolic basins, preimages of Siegel discs and preimages of Herman rings. By assumption \( f_0 \) has no Herman rings.

Assume at first that \( B \) is a component of an attracting basin. Let \( K \subset B \) be a compact connected subset. Then for \( n \) large enough, \( K \) is contained in the attracting basin of a nearby attracting cycle for \( f_n \). Therefore \( B \cap J' = \emptyset \).

Now assume that \( B \) is a component of a parabolic basin. Then it is contained in \( \mathcal{F}_k \) for some odd \( k \), in the setting of Proposition 6.3. As the convergence is \((\alpha, k)\)-stable at every parabolic periodic point and for every \( k \), Proposition 6.3 implies that \( B \cap J' = \emptyset \).

Finally assume that \( B \) is a preimage of a Siegel disk. Let \( K \subset B \) be compact. Our assumption that the rotation number is a Brjuno number allows us to apply results in Risler [R]: for large \( n \), we have \( K \subset B_n \), for \( B_n \) the corresponding preimage of the perturbed Siegel disk of \( f_n \). Therefore \( K \cap J' = \emptyset \). Consequently \( B \cap J' = \emptyset \).

These cases together prove that \( J' \subset J(f_0) \).

(b) In this more particular setting, we have \( J(f_n) \to J(f_0) \) from part (a). It remains to show that \( f_n \) are geometrically finite for large \( n \), that is, every critical point is either in the Fatou set or preperiodic (the two cases are not mutually exclusive). Let \( c_n \) be a critical point of \( f_n \) with \( c_n \xrightarrow{n \to \infty} c \). Then \( c \) is a critical point of \( f_0 \). If \( c \in \mathbb{P}^1 \setminus J(f_0) \) then \( c_n \in \mathbb{P}^1 \setminus J(f_n) \) for large \( n \). If \( c \in J(f_0) \), then it is preperiodic, as well as \( c_n \), by assumption of the persistence of critical relations. Hence \( f_n \) is geometrically finite for large \( n \).

6.4. Convergence of Poincaré Series and proof of Proposition 2.12. We now come to the main estimates in the article: controlling the convergence of tails of Poincaré series. We will work under the same assumptions as in §6.1. We will show that if \( \delta_n > p/(p+1) \) and \( \varepsilon > 0 \), there exist \( m_0 \) and \( n_0 \) such that for all \( z \in K \), all \( \delta \in [\delta_0, 2] \), all \( m \geq m_0 \) and all \( n \geq n_0 \), we have \( S_\delta(f_n, z, m) := \sum_{j=m}^{+\infty} |(f_n^j)'(z)|^\delta < \varepsilon \).

Choose \( \alpha, s, \varepsilon \) such that

\[
\alpha' > \alpha > 0, \ s > 0, \ \varepsilon > 0, \ \eta := \delta_0 e^{-s} > \frac{p}{p+1} \quad \text{and} \quad \frac{\eta p + 1}{p + \varepsilon} > 1 .
\]

Recall that \( P_n \) is the monic polynomial which vanish at the \( p+1 \) fixed points of \( f_n \) close to 0 and \( P_0(z) = z^{p+1} \). By assumption all the \( P_n \) are \((\alpha', \gamma_n)\)-stable. Set again \( \Psi_n = \Psi_{P_n, \gamma_n} : S(\alpha') \to \mathbb{C} \). Without loss of generality, conjugating with translations if necessary, we may assume that for all \( n \), the separatrix \( \Psi_n(\mathbb{R}^+) \) lands at 0: \( \Psi_n(t) \to 0 \) as \( t \to +\infty \). In particular \( P_n(0) = 0 \).
By Lemma 5.3, we may fix \( N \) and \( R \) large such that the following two conditions hold.

**Condition 1.** For all \( n \geq N \), \( \Psi_n(S(\alpha)_R) \subset D(r) \) and the lifted dynamics \( F_n : S(\alpha)_R \to S(\alpha)_R \) satisfies:

\[
\forall n \geq N \quad \forall w \in S(\alpha)_R \quad \forall j \in \mathbb{N} \quad d_{S(\alpha)_R}(F_n^{\circ j}(w), w + j) \leq s/2.
\]

**Condition 2.** For all \( n \geq N \), \( \Psi_n'(S(\alpha)_R) \subset \mathbb{D}^* \).

By Lemma 6.2, there are an open set \( L \) relatively compact in \( S(\alpha)_R \), a compact \( K' \subset \Psi_0(L) \), \( n_0 > 0 \) and \( j_0 > 0 \) such that for all \( n \geq n_0 \), we have \( f_n^{\circ j_0}(K) \subset K' \subset \Psi_0(L) \). We can finally fix \( m_0 \geq R \) large enough so that the following condition holds.

**Condition 3.** \( \forall w \in L \) \( \forall j \geq m_0 \) \( \forall t \in [0, 1] \), \( d_{S(\alpha)_R}(w + j, j + t) \leq \frac{s}{2} \).

Note that if \( z \in K \) and \( n \) is large enough, then, \( z_n = f_n^{\circ j_0}(z) \in K' \) and for \( m \geq j_0 \), we have

\[
\sum_{j=m}^{+\infty} |(f_n^{\circ j})'(z)|^\delta = |(f_n^{\circ j_0})'(z)|^\delta S_\delta(f_n, z_n, m - j_0).
\]

Since \( |(f_n^{\circ j})'(z)|^\delta \) is uniformly bounded for \( (z, \delta) \in K \times [\delta_0, 2] \) as \( n \) tends to \( \infty \), in order to prove Proposition 2.12, it is enough to prove that as \( n \) and \( m \) tend to \( \infty \), \( S_\delta(f_n, z, m) \) tends to 0 uniformly with respect to \( (z, \delta) \in K' \times [\delta_0, 2] \). Therefore, Proposition 2.12 follows from Corollary 2.10 and Lemma 6.5 below.

**q.e.d.**

**Lemma 6.5.** There exists a constant \( C \) such that for all \( n > N \) and \( m \geq m_0 \), \( z, \delta \in \mathbb{N}_0, \delta \in [\delta_0, 2] \), we have \( S_\delta(f_n, z, m) \leq C\ell_\eta(P_n, \gamma_n, m) \).

**Proof.** Fix \( n > N \), \( z \in K' \) and \( \delta \in [\delta_0, 2] \). Let \( w_n \in L \) be such that \( \Psi_n(w_n) = z \). Then, it follows from \( \Psi_n \circ F_n^{\circ j} = f_n^{\circ j} \circ \Psi_n \) and the chain rule that

\[
|(f_n^{\circ j})'(z)|^\delta \leq \frac{1}{|\Psi_n'(w_n)|^\delta} |(f_n^{\circ j})'(w_n)|^\delta \cdot |\Psi_n'(F_n^{\circ j}(w_n))|^\delta
\]

\[
\text{Koebe}
\]

where the inequality is due to the bounded Koebe distortion theorem applied to the univalent maps \( F_n^{\circ j} \). By Condition 1, the hyperbolic distance in \( S(\alpha)_R \) between \( F_n^{\circ j}(w_n) \) and \( w_n + j \) is smaller than \( s/2 \). By Condition 3, the hyperbolic distance between \( w + j \) and \( j + t \) is also less than \( s/2 \), for all \( w \in L \) and all \( j \geq m_0 \). Therefore, for all \( j \geq m_0 \) and all \( t \in [0, 1] \),

\[
d_{S(\alpha)_R}(F_n^{\circ j}(w_n), j + t) < s.
\]
Since $\Psi'_n$ maps $S(\alpha)_R$ into $\mathbb{D}^*$ by Condition 2, it follows
\[ d_{\mathbb{D}^*}(\Psi'_n(F^{(j)}_n(w_n)), \Psi'_n(j+t)) \leq s \]
\[ (6.3) \]
by Schwarz lemma. So, \[ |\Psi'_n(F^{(j)}_n(w_n))|^\delta \leq |\Psi'_n(j+t)|^\delta e^{-s} \leq |\Psi'_n(j)|^\delta = |\Psi'_n(j+t)|^\delta. \]

Thus \[ S_\delta(f_n, z, m) = \sum_{j=m}^{\infty} |(f^{(j)}_n)'(z)|^\delta \leq C \sum_{j=m}^{\infty} |\Psi'_n(F^{(j)}_n(w_n))|^\delta \]
\[ (6.2) \]
\[ \leq C \sum_{j=m}^{\infty} \int_0^1 |\Psi'_n(j+t)|^\delta dt = C \int_m^\infty |\Psi'_n(t)|^\delta dt = C\ell_n(P_n, \gamma_n, m). \]

Lemma 6.5 together with Corollary 2.10 proves Proposition 2.12.

**Corollary 6.6.** Assume $(f_n : D(r) \to \mathbb{C})_{n \geq 1}$ is a sequence of univalent maps converging locally uniformly to $f_0 : D(r) \to \mathbb{C}$ with $f_0(z) = z + z^{p+1} + O(z^{p+2})$. If the convergence $f_n \to f_0$ is stable, then, for any compact set $K \subset D(r)$ whose orbit converges to 0 under backward iteration of $f_0$, and for any $\delta_0 > p/(p+1)$, we have \[ \sum_{j=m}^{+\infty} |(f^{-j}_n)'(z)|^\delta_{m,n} \to 0 \]
uniformly with respect to $(z, \delta) \in K \times [\delta_0, 2]$.

**Proof.** By assumption the convergence $f_n \to f_0$ is $(\alpha', k)$-stable for some $\alpha' > 0$ and for every $k$, in particular for every even $k$. Now the compact set $K$ can be written as the disjoint union of finitely many compact sets, each converging to 0 under backward iteration of $f_0$ along some repelling axis. Applying Proposition 2.12 to $f_n^{-1} \to f_0^{-1}$ for each even $k$ gives the corollary.

**Lemma 6.7.** For $a, b \in \mathbb{D}^*$ with $d_{\mathbb{D}^*}(a, b) \leq s$ we have $|a| \leq |b|e^{-s}$.

**Proof.** $s \geq d_{\mathbb{D}^*}(a, b) \geq \int_{|b| \leq |a|} \log \left| \frac{dz}{|z|} \right| \int_{|b| \leq |z|} \frac{dz}{|z|} = \log \frac{1}{|b|}. \quad \text{q.e.d.}$

**7. Proof of Theorem A**

Now we need to recall existing theory about conformal measures and their relation with Hausdorff dimension of Julia sets.

For $f$ a rational map, denote by $\mathcal{E}(f)$ the set of preparable and precritical points in the Julia set. More precisely,
\[ \mathcal{E}(f) = \{ z \in J(f) \mid (\exists n \geq 0), f^{\circ n}(z) \text{ is a parabolic or critical point of } f \}. \]
The map $f$ is geometrically finite if and only if every point in $\mathcal{E}(f)$ is prerepelling or preparable.
An $f$-invariant conformal measure of dimension $\delta > 0$ is a probability measure $\mu$ on $\mathbb{P}^1$ such that $\mu(f(E)) = \int_E \left| f'(x) \right|^\delta \; d\mu$ whenever $f|E$ is injective.

The following result can be found in [McM, DMNU, PU].

**Theorem 7.1.** Assume $f$ is geometrically finite. Then, there exists a unique $f$-invariant conformal measure $\mu_f$ with support in $J(f) \setminus \mathcal{E}(f)$. Furthermore, the dimension $\delta_f$ of $\mu_f$ is equal to the Hausdorff dimension of $J(f)$.

**Sketch of proof.**

**Uniqueness.** Given a rational map $f$, one can define the radial Julia set $J_{\text{rad}}(f)$ as the set of points $z \in J(f)$ such that arbitrarily small neighborhoods of $z$ can be blown up by the dynamics to disks of definite size centered at $f^\circ n(z)$. The radial Julia set $J_{\text{rad}}(f)$ supports at most one conformal measure (see [DMNU] Theorem 1.2 and [McM] Theorem 5.1). Moreover, the dimension of this conformal measure is always equal to the Hausdorff dimension of $J_{\text{rad}}(f)$ (see for example [McM] Theorem 2.1 and Theorem 5.1).

The fact that $f$ is geometrically finite implies that $J(f) \setminus \mathcal{E}(f)$ is contained in the radial Julia set $J_{\text{rad}}(f)$ (see [McM] Theorem 6.5 and [U] Theorem 4.2). Since $\mathcal{E}(f)$ is countable, we have $\text{H.dim } J(f) = \text{H.dim } J_{\text{rad}}(f)$.

**Existence.** In [McM] §4, a conformal measure is constructed with support contained in $J(f) \setminus \{\text{parabolic and prerepelling points}\}$ which is contained in $J(f) \setminus \mathcal{E}(f)$.

The following result is implicit in McMullen [McM], Theorem 11.2.

**Theorem 7.2.** We have $\text{H.dim } J(f_n) \to \text{H.dim } J(f_0)$ under the following assumptions:

- $(A1)$ $f_0$ is geometrically finite;
- $(A2)$ $f_n \to f_0$ algebraically and preserving critical relations;
- $(B1)$ $f_n$ are geometrically finite;
- $(B2)$ $J(f_n) \to J(f_0)$;
- $(B3)$ (see below) the tails of the Poincaré series are uniformly small in neighborhoods of each parapbolic point in $\mathcal{E}(f_0)$.

**Sketch of proof.**

Conditions $(A1)$ and $(B1)$ imply that

$$(\mu_0, \delta_0) := (\mu_{f_0}, \delta_{f_0}) \quad \text{and} \quad (\mu_n, \delta_n) := (\mu_{f_n}, \delta_{f_n})$$

exist, with $\delta_0 = \text{H.dim } J(f_0)$ and $\delta_n = \text{H.dim } J(f_n)$. Assume $\mu_n \to \nu$ weakly and $\delta_n \to \delta'$ (by taking twice subsequences if necessary). Then, $\nu$ is a $f_0$-invariant conformal measure, with $\text{supp}(\nu) \subset \lim_{n \to +\infty} J(f_n) \overset{(B2)}{=} J(f_0)$.
Furthermore, condition (B3) implies that $\nu(\{c\}) = 0$ for any prerepelling $c \in \mathcal{E}(f_0)$ (see below). Thus, the support of $\nu$ is contained in $J(f_0) \setminus \mathcal{E}(f_0)$. It follows from Theorem 7.1 that $\nu = \mu_{f_0}$ and thus,

$$\text{H.dim } J(f_n) = \delta_n \rightarrow \delta' = \delta_0 = \text{H.dim } J(f_0).$$

Let us now explain why a weak limit $\nu$ of the sequence $\mu_n$ cannot charge the points of $\mathcal{E}(f_0)$.

**Technique of change of coordinates** For every parabolic point $c$ of $f_0$, we replace $f_0$ with an iterate, call it $\hat{f}_0$, so that $f_0(c) = c$ and $\hat{f}_0'(c) = 1$. Define $\hat{f}_n$ by the same number of iterates of $f_n$. For a non periodic preparabolic point $b$, we replace again $f_0$ with an iterate to achieve $f_0(b) = c$ and $c$ is a fixed point as above. We then lift the dynamics of $f_0$ near $c$ to a map $\hat{f}_0 = f_0^{-1} \circ f_0 \circ f_0$. For $f_n$, and any $c_n \rightarrow c$ with $f_n(c_n) = c_n$, there is $b_n \rightarrow b$ such that $f_n(b_n) = c_n$. Due to our assumption that critical orbit relations in $J(f_0)$ are preserved, if $b$ is a critical or a postcritical point, then there are unique choices of $c_n$ and $b_n$ such that $b_n$ is again a critical or a postcritical point of $f_n$, with the same local degree. Thus we can form $\hat{f}_n = f_n^{-1} \circ f_n \circ f_n$ and obtain a sequence $(\hat{f}_n, b_n) \rightarrow (\hat{f}_0, b)$.

In any case we consider $\hat{f}_0, \hat{f}_n$ as the induced local dynamics. Note that $\mu_n$ is locally $\hat{f}_n$-conformal.

We can now give a more precise definition of (B3) as follows. Denote by $p$ the maximal petal number of points in $\mathcal{E}(f_0)$. Given a point $b \in \mathcal{E}(f_0)$, change coordinates as above so that this point is 0 with induced local dynamics $\hat{f}_0, \hat{f}_n$. We say that the tails of the Poincaré series are uniformly small in neighborhoods of 0 if there exists $r > 0$, such that for all $\delta_0 > \frac{r}{p+1}$ and all compact set $K \subset \overline{D(r)}$ whose backward orbit under $\hat{f}_0$ remains in $\overline{D(r)}$ and tends to 0 and such that the forward orbit under $\hat{f}_0$ of every point in $J(f_0) \cap D(r)$ eventually falls into $K$ before leaving $D(r)$,

$$\text{(7.1) } (\forall \varepsilon > 0 ) \ (\exists m, N) \ (\forall n \geq N) \ (\forall z \in K) \ \sum_{j=m}^{+\infty} |(\hat{f}_n^{-j})'(z)|^{\delta_n} \leq \varepsilon .$$

Now (B3) together with (A2) guarantee that for any preparabolic $c \in \mathcal{E}(f_0)$, for any $\varepsilon, m > 0$, there are $N > 0$ and a neighborhood $V$ of $c$ such that for $n \geq N$,

$$V \cap J(f_n) \subset \{c_n\} \cup \bigcup_{j=m}^{+\infty} \hat{f}_n^{-j}(K \cap J(f_n))$$

and $\mu_n(V) = \mu_n(V \cap J(f_n)) < \varepsilon$ (see the proof of Theorem 11.2, [McM]). Consequently $\nu(\{c\}) = 0$. q.e.d.
Remark. In [McM], conditions (B1), (B2) and (B3) are replaced by a stronger condition (B'): namely $f_n \to f_0$ dominantly and for the corresponding multipliers $\lambda_n$, either $\lambda_n \to 1$ radially or, $\lambda_n \to 1$ horocyclically with $\lim \inf H.\dim J(f_n) > 2p(f_0)/p(f_0) + 1$ (where $p(f_0)$ is the maximal possible petal number at a point of $E(f_0)$).

Proof of Theorem A. Conditions (A1) and (A2) are kept as such in the hypothesis of our Theorem A. We have assumed that the convergence is stable. This implies conditions (B1) and (B2), due to Theorem 6.4 part (b).

Condition (B3) for any preparabolic point $b \in E(f_0)$, will follow from Corollary 6.6 with the induced local dynamics, together with the following argument.

Denote by $p(b)$ the petal number of $\hat{f}_0$ at $b$ and by $p$ the maximal petal number of points in $E(f_0)$. Assume again $f_0(b) = c$ is a fixed point with multiplier 1. By assumption the convergence $(f_n, c_n) \to (f_0, c)$ is stable. It follows from Lemmas 4.2 and 3.13 that the convergence $(\hat{f}_n, b_n) \to (\hat{f}_0, b)$ is stable at $b$. By Corollary 6.6, we see that for all $\delta_0 > p(b)/p(b)+1$,

$$(7.2) \quad (\forall \varepsilon > 0) (\exists m, N) (\forall n \geq N) (\forall z \in K) (\forall \delta \in [\delta_0, 2]) \sum_{j=m}^{+\infty} |(\hat{f}_{n-j})'(z)|^\delta \leq \varepsilon.$$

In order to get from (7.2) to (7.1), we need two more results in [McM] (see Theorems 3.1, 2.1, 6.1 and Proposition 11.3). At first

$$\delta_{f_0} > \frac{p}{p+1},$$

and at second,

$$\lim \inf \delta_{f_n} \geq \delta_{f_0}$$

(semi-continuity of the dimension). So there is

$$\delta_0 > \frac{p}{p+1} > \frac{p(b)}{p(b)+1}$$

so that $2 \geq \delta_{f_n} \geq \delta_0$ for all large $n$. We may thus apply (7.2) to this $\delta_0$.

The uniform control on $\delta$ in (7.2) leads to (7.1), where $\delta$ is replaced by $\delta_{f_n}$.

Proof of Corollary 2.17. We just need to combine Theorem A with Proposition 3.11.

q.e.d.

References


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E-mail address: xavier.buff@math.ups-tlse.fr

Université Paul Sabatier
Laboratoire Emile Picard
118, route de Narbonne
31062 Toulouse Cedex, France

E-mail address: tanlei@math.u-cergy.fr

CNRS-UMR 8088,
Département de Mathématiques,
Université de Cergy-Pontoise,
2 Avenue Adolphe Chauvin,
92302 Cergy-Pontoise Cedex, France