How regular can the boundary of a quadratic Siegel disk be?

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Abstract. In the family of quadratic polynomials with an irrationally indifferent fixed point, we show the existence of Siegel disks with a fine control on the degree of regularity of the linearizing map on their boundary. A general theorem is stated and proved. As a particular case, we show that in the quadratic family, there are Siegel disks whose boundaries are C^n but not C^{n+1} Jordan curves.

1. Introduction

In 1995, R. Pérez-Marco [**PM1**] constructed the first known examples of Siegel disks with C^∞ boundaries. The authors [**BC**] discovered later an independent proof of the existence of Siegel disks with C^∞ boundaries in the family of quadratic polynomials

$$P_\theta(z) = \exp(i2\pi\theta)z + z^2, \qquad \theta \in \mathbb{R}/\mathbb{Z}.$$

A. Avila formalized, simplified and extended these results in [**A**] (these two preprints have now been published as [**ABC**]). L. Geyer further simplified it in [**G**].

These smooth Siegel disks were provided by an elementary construction based on a perturbation lemma (recalled as lemma 2 below), which was unknown at the time. Here, we describe a procedure which introduces irregularities on the boundaries of Siegel disks. For this, we use an old result of Herman (see lemma 1 below). Combining it with our previous work, it enables us to prove much more: there exist quadratic Siegel disks for which the regularity of the boundary is anywhere in the full spectrum between analytic and continuous. Since the possibilities are numerous, we will formulate our main theorem in a general and abstract form. As a particular case, we will obtain that for all $n \in \mathbb{N}$, there exist quadratic Siegel disks whose boundaries are Jordan curves which are C^n but not C^{n+1} embedded curves.

To state the main theorem, let us introduce a few notations. We will denote by $\mathbb{U}$ the set of complex numbers of norm 1. If P_θ is linearizable in a neighborhood of its indifferent fixed point $z = 0$, we denote by Δ_θ its Siegel disk (maximal domain of linearization) and by $\phi_\theta : r_\theta\mathbb{D} \to \Delta_\theta$ the unique conformal map with $\phi_\theta(0) = 0$ and $\phi'_\theta(0) = 1$. This ϕ_θ is also the linearizing map, and thus satisfies

$$\phi_\theta \circ R_\theta = P_\theta \circ \phi_\theta,$$

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where $R_\theta(z) = \exp(i2\pi\theta)z$. If P_θ is not linearizable, we set $\Delta_\theta = \varnothing$ and $r_\theta = 0$. Let C^0 be the space of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ having a continuous extension to $\overline{\mathbb{D}}$. This is a Banach space for the supremum norm. Let C^ω be the space of functions from $\mathbb{D}$ to $\mathbb{C}$ having a holomorphic extension to a neighborhood of $\overline{\mathbb{D}}$.

Definition 1. *Let X and Y be topological spaces and $X \subset Y$. We write $X \subset_0 Y$ if the canonical injection $X \hookrightarrow Y$ is continuous. If moreover X is a normed vector space and Y a Fréchet space, we write $X \subset_{\rm c} Y$ if every bounded set in X has compact closure in Y.*

Theorem 1. *Let F be any Fréchet space such that $C^\omega \subset F \subset_0 C^0$, and let*

$$B \subset_{\rm c} F$$

be a Banach space. Then, there exists a Bruno number θ such that

- *the map $z \mapsto \phi_\theta(r_\theta z)$ belongs to F but not to B*
- *the boundary of Δ_θ is a Jordan curve that does not contain the critical point of P_θ.*

Equivalently, one can replace the Banach space $B \subset_{\rm c} F$ by a countable union of Banach spaces $B_n \subset_{\rm c} F$ or by a countable union of compact sets $K_n \subset F$.

There are many Fréchet and Banach spaces or compact subsets that can be considered here. To give a sample of what can be obtained from theorem 1, we will restrict ourselves to the following C^α conditions. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. If $\alpha = n$ is an integer, we say that $f \in C^n$ if $f^{(k)} \in C^0$ for $0 \le k \le n$. We say that $f \in C^\infty$ if $f \in C^n$ for every integer n. If $\alpha = n + \varepsilon$ for some integer n and some $\varepsilon \in]0,1[$, we say that $f \in C^{n+\varepsilon}$ if $f \in C^n$ and $f^{(n)}$ is ε-Hölder on $\overline{\mathbb{D}}$. For $\alpha \in [0, +\infty[$, C^α is a Banach space[1] and for $\alpha < \beta$, the injection of C^β in C^α is compact (the proof is similar to the proof of Ascoli's theorem).

Geometry vs Analysis: we control the regularity of the boundaries of Siegel disks via the regularity of the linearizing maps. We will need a few results on the boundary behavior of conformal maps for which we refer the reader to [**Po**]. For $\alpha \ge 1$, there is a notion of C^α Jordan curve (see [**Po**], section 3.3): these are curves that have a parameterization that is C^α with non-vanishing first derivative.

Theorem 2 (Kellog-Warschawski). *Assume $\alpha \in]1, +\infty]$ is not an integer and $\Gamma \subset \mathbb{C}$ is a Jordan curve bounding a simply connected domain U containing 0. Then, Γ is a C^α Jordan curve if and only if the conformal map $\phi : (\mathbb{D}, 0) \to (U, 0)$ belongs to C^α with non-vanishing derivative on $\overline{\mathbb{D}}$.*

The situation is less simple when $\alpha = n$ is a positive integer. If the conformal map of a Jordan curve J is in C^n and if the first derivative does not vanish, then obviously J is a C^n Jordan curve. However, there are C^n Jordan curves whose conformal maps are not in the space C^n.

[1] for $\phi \in C^{n+\varepsilon}$ and $\varepsilon \in]0,1[$, we set

$$\|\phi\|_{C^{n+\varepsilon}} = \sum_{k=0}^{n} \|\phi^{(k)}\|_\infty + \min\left\{\lambda > 0 \mid \lambda x^\varepsilon \text{ is a modulus of continuity for } \phi^{(n)}\right\}.$$

In the quadratic family, if the conformal map of a Siegel disk is at least C^1, then its derivative cannot vanish on $\overline{\mathbb{D}}$ (this is an easy exercise). Theorem 2 has the following consequence: for a non-integer $\alpha \in]1, +\infty]$,

$$z \mapsto \phi_\theta(r_\theta z) \in C^\alpha \iff \partial\Delta_\theta \text{ is a } C^\alpha \text{ Jordan curve.}$$

Let us now state some corollaries of theorem 1.

Corollary 1. *There exist quadratic Siegel disks with C^∞ boundaries.*

Proof. Take $F = C^\infty$ and $K_n = \varnothing$. □

In [**PZ**], C. Petersen and S. Zakeri proved that for almost every $\theta \in \mathbb{R}/\mathbb{Z}$, the quadratic polynomial P_θ has a Siegel disk whose boundary contains the critical point, is a Jordan curve but is not a quasicircle. Theorem 1 implies:

Corollary 2. *There exist quadratic Siegel disks whose boundaries are Jordan curves that do not contain the critical point, and are not quasicircles.*

Proof. The conformal map of a quasidisk is always Hölder (see [**Po**]). So, choose $F = C^0$ and $B_n = C^{1/n}$ $(n \geq 2)$. □

Corollary 3. *For every $n \in \mathbb{N}$, there exist quadratic Siegel disks whose boundaries are Jordan curves that do not contain the critical point, and are C^n but not C^{n+1} Jordan curves.*

Proof. Take $F = C^n$, $B = C^{n+1/2}$. Then, theorem 1 gives Siegel disks whose boundaries are C^n but not $C^{n+1/2}$ Jordan curves. (Note that taking $B = C^{n+1}$ does not work; see the discussion Geometry vs Analysis above). □

If P_θ has a Siegel disk, set

$$I_\theta = \left\{\alpha \in [0, +\infty[\;\middle|\; \phi_\theta(r_\theta z) \in C^\alpha\right\}.$$

This is an interval, either empty or with left bound $= 0$. It contains 0 if and only if the boundary of Δ_θ is a Jordan curve (see [**M**] or [**Ro**]). However, it is yet unknown if $\partial\Delta_\theta$ is always a Jordan curve. So, it is unknown if I_θ can be empty, but the following proves that all other cases happen.

Corollary 4. *For every $\alpha \geq 0$, there exists a Bruno number θ such that $I_\theta = [0, \alpha]$ and for every $\beta > 0$, there exists a Bruno number θ such that $I_\theta = [0, \beta[$.*

Proof. For the first case, choose $F = C^\alpha$ and $B_n = C^{\alpha+1/n}$. For the second case, choose $F = \displaystyle\bigcap_{\alpha \in [0,\beta[} C^\alpha$, $B = C^\beta$. □

Remark. For $n \in \mathbb{N}$, let $C^{n+\mathrm{Lip}}$ be the set of holomorphic functions on $\mathbb{D}$ whose n-th derivative is Lipschitz. The inclusion of $B = C^{n+1}$ in $F = C^{n+\mathrm{Lip}}$ is not compact (it is an isometry). Thus we cannot apply theorem 1. We may wonder whether there exist (fixed quadratic) Siegel disks whose conformal maps belong to $C^{n+\mathrm{Lip}}$ but not C^{n+1}. In fact, for $n > 0$, this is impossible by a theorem of Gottschalk and Hedlund (see [**H1**] proposition 4.2 in part IV).

2. Proof of theorem 1.

It is equivalent to prove theorem 1 for a Banach space $B \subset_{\mathrm{c}} F$, for a countable union of Banach spaces $B_n \subset_{\mathrm{c}} F$ or for a countable union of compact sets $K_n \subset F$, as shown in the following

Proposition 1. *If F is a Fréchet space, then*

1. *the image of any Banach space B under a compact injection is contained in a countable union of compact subsets of F;*
2. *any countable union of compact subsets K_n of F is contained in the image of a Banach space by a compact injection.*

Proof. For (1), let K_n be the closure in F of the ball of radius n in the B-norm. Obviously, $B \subset \bigcup_{n\in\mathbb{N}} K_n$. Moreover, by the very definition of a compact injection, K_n is compact.

For (2), let us reduce the problem to the case of a single compact set. Let $\lambda_n > 0$ be a sequence so that $\lambda_n K_n \longrightarrow 0$. Let $K = \{0\} \cup \bigcup_{n\in\mathbb{N}} \lambda_n K_n$. Then K is compact, and any vector subspace of F containing K will contain $\bigcup_{n\in\mathbb{N}} K_n$. Next, we replace K by $\overline{\mathbb{D}}K = \big\{\lambda f \bigm| |\lambda| \le 1,\ f \in K\big\}$. Then we can replace K by the closure of the convex hull of K, which is compact by a well known property of compact sets in Fréchet spaces (see [**Ru**], Theorem 3.20). Now K is both convex and invariant under multiplication by elements of $\overline{\mathbb{D}}$. Now $B = \bigcup_{t\in\mathbb{R}_+} tK$ equipped with the norm $\|f\|_B = \inf\{t \in \mathbb{R}_+ \mid f \in tK\}$ is a Banach space that is compactly injected in F. $\square$

We will prove theorem 1 for a countable union of compact sets $K_n \subset F$. In fact, we will prove the following stronger version of theorem 1.

Theorem 3. *Let F be any Fréchet space such that*

$$C^\omega \subset F \subset_0 C^0,$$

and let $(K_n)_{n\in\mathbb{N}}$ be any sequence of compact subsets of F. Then, for every Bruno number θ, every $r < r_\theta$ and every $\varepsilon > 0$, there exists a Bruno number θ' such that

1. $|\theta' - \theta| < \varepsilon$
2. $r < r_{\theta'} < r + \varepsilon$
3. *the map $z \mapsto \phi_{\theta'}(r_{\theta'} z)$ belongs to F but to no K_n*
4. *the distance in F between $z \mapsto \phi_{\theta'}(r_{\theta'} z)$ and $z \mapsto \phi_\theta(rz)$ is $< \varepsilon$.*

Moreover, the continuous extension ψ of $\phi_{\theta'}$ to $r_{\theta'}\overline{\mathbb{D}}$, which exists by (3), satisfies

5. $\big\|\psi(r_{\theta'} z) - \phi_\theta(rz)\big\|_\infty < \varepsilon$ *on* $\overline{\mathbb{D}}$
6. $\psi(r_{\theta'}\mathbb{U}) = \partial\Delta_{\theta'}$ *is a Jordan curve that does not contain the critical point.*

It is enough to prove (1)–(4) only. Indeed, (3) implies the existence of the extension since $F \subset C^0$. This extension must be injective (see [**M**], lemma 18.7), which shows that the boundary of the Siegel disk is a Jordan curve. Then, (5) follows from (4) (with a smaller ε), by continuity of the inclusion $F \subset C^0$. And (5) implies that for ε small enough, the boundary of the Siegel disk does not contain the critical point.

2.1. Tools.

The proof is based on a perturbation lemma 9, that will be stated in section 2.4. There are two results that make the perturbation lemma possible: lemma 1 provides irregularity and lemma 2 provides regularity.

Let $\mathcal{D}_2$ denote the set of bounded type irrational numbers. For a proof of the following lemma, see for example [**H2**].

Lemma 1 (Herman). *Let $\theta \in \mathcal{D}_2$, U be a connected open set containing 0 and $f : U \to \mathbb{C}$ be a holomorphic function which fixes 0 with derivative $e^{2\pi i\theta}$. Let Δ be the Siegel disk of f at 0 (which exists by a theorem of Siegel). If U is simply connected and f is univalent, then Δ cannot have a compact closure in U.*

Remark. In fact, Herman's theorem is stronger: U needs not be simply connected, and the condition on θ can be weakened to the so-called Herman-Yoccoz condition. See [**H2**] for a reference. In the case of quadratic polynomials, lemma 1 is a consequence of the following fact: a bounded type quadratic Siegel disk contains the critical point on its boundary.

For a proof of the following lemma, see [**BC**] or [**ABC**].

Lemma 2. *For every Bruno number θ and every $r < r_\theta$, there exists a sequence $\theta_n \in \mathcal{D}_2$ such that $\theta_n \longrightarrow \theta$ and $r_{\theta_n} \longrightarrow r$.*

The references [**A**] and [**G**] contain similar statements without requiring $\theta_n \in \mathcal{D}_2$. It is in fact easy to adapt the proof in [**A**] and get $\mathcal{D}_2$ numbers. We do not know if [**G**] can produce $\mathcal{D}_2$ numbers.

2.2. Elementary properties of the linearizations. Let us state well known properties concerning the dependence of ϕ_θ on θ.

Lemma 3. *Let $\theta_n \longrightarrow \theta$ be real numbers and assume that $r = \liminf r_{\theta_n} > 0$. Then $r_\theta \geq r$ and ϕ_{θ_n} tends to ϕ_θ uniformly on compact subsets of $r\mathbb{D}$.*

Proof. The univalent maps ϕ_{θ_n} all have derivative 1 at 0, hence form a normal family. It is therefore enough to prove that ϕ_θ is the unique adherence value of the sequence ϕ_{θ_n} on $B(0, r)$. Let ψ be the limit of a convergent subsequence (for uniform convergence on compact subsets of $r\mathbb{D}$). Then $\psi'(0) = 1$, so in particular ψ is not constant. Moreover, by passing to the limit in the equation $P_{\theta_n} \circ \phi_{\theta_n} = \phi_{\theta_n} \circ R_{\theta_n}$, we have $P_\theta \circ \psi = \psi \circ R_\theta$. Therefore, uniqueness of the linearization implies $r_\theta \geq r$ and $\psi = \phi_\theta$ on $r\mathbb{D}$. □

Lemma 4. *Assume $r_n > 0$, $r_n \longrightarrow r > 0$, $\theta \in \mathbb{R}$ and $\theta_n \longrightarrow \theta$ is a sequence of real numbers such that $r_{\theta_n} > r_n$, and the maps $z \mapsto \phi_{\theta_n}(r_n z)$ converge uniformly on $\mathbb{U}$ to some ψ. Then $r_\theta \geq r$, and $z \mapsto \psi(z)$ is a continuous extension of $z \mapsto \phi_\theta(rz)$ to the closed unit disk (if $r_\theta > r$, this just means $\phi_\theta(rz) = \psi(z)$). Moreover, ψ is injective. In the case $r_\theta = r$, we have $\partial\Delta_\theta = \psi(\mathbb{U})$.*

2.3. Preparatory work. Note that C^ω is not a Fréchet space. It is the union over $\varepsilon > 0$ of spaces C^ω_ε of holomorphic functions on $(1+\varepsilon)\mathbb{D}$. Each C^ω_ε is endowed with the topology of uniform convergence on compact sets (in fact it is a Fréchet space; if you prefer Banach spaces, take the space of bounded holomorphic functions on $(1+\varepsilon)\mathbb{D}$ and the discussion below will be the same). We do not put a topology on C^ω.

Lemma 5. *Let $F \subset_0 C^0$ be a Fréchet space and K be a compact subset of F. Assume $f_n \in K$ tend to some map $f : \mathbb{D} \to \mathbb{C}$ uniformly on compact subsets of $\mathbb{D}$. Then $f \in K$ and*

a) $d_F(f_n, f) \longrightarrow 0$ *where* d_F *is the distance function of the Fréchet space* F*,*
b) $\|f_n - f\|_\infty \longrightarrow 0$.

Proof. Part b) is a corollary of part a) and the continuity of the injection $F \subset C^0$. Since f_n lies in the compact set K, it has a subsequence f_{n_k} such that $d_F(f_{n_k}, h) \longrightarrow 0$ for some $h \in K$. The injection $F \subset C^0$ being continuous, $\|f_{n_k} - h\|_\infty \longrightarrow 0$. Since f_n tends to f uniformly on compact subsets of $\mathbb{D}$, we must have $h = f$. This proves $f \in K$, and that f is the only limit of convergent subsequences of f_n in the d_F-metric. Since f_n lies in a compact set, this gives part a). $\square$

Lemma 6. *Assume* F *is a Fréchet space with* $C^\omega \subset F \subset_0 C^0$*. Then, for all* ε*, the injection* $C^\omega_\varepsilon \subset F$ *is continuous.*

Proof. This is a corollary of the closed graph theorem, since the injection of C^ω_ε in C^0 is continuous. $\square$

Lemma 7. *There exist subsets* D_n *of* C^ω *with* $C^\omega = \bigcup D_n$ *such that for every Fréchet space* F *with* $C^\omega \subset F \subset_0 C^0$*,* D_n *is compact in* F*.*

Proof. Let D_n be the set of holomorphic functions in $(1 + \frac{1}{n+1})\mathbb{D}$ whose absolute value is bounded by n. Each D_n is compact in $C^\omega_{\frac{1}{n+1}}$ (Montel's theorem). According to lemma 6, D_n is also compact in F. $\square$

We will consider the following property of a subset A of C^0:

$(\mathcal{H})$ For every $\varepsilon > 0$, A contains a neighborhood of 0 in C^ω_ε.

Lemma 8. *For every Fréchet space* F *with* $C^\omega \subset F \subset_0 C^0$*, there exists a compact subset* L *of* F *with property* $\mathcal{H}$*.*

Proof. Consider the D_n from lemma 7. Since D_n is compact in F, λD_n tends to 0 when $\lambda \longrightarrow 0$. Choose $\lambda_n > 0$ such that $\lambda_n D_n$ is included in the ball of F of center 0 and radius $1/(n+1)$. Then the set $L = \bigcup \lambda_n D_n$ is compact (note that $0 \in L$) and has the property $\mathcal{H}$ since D_n contains a neighborhood of 0 in $C^\omega_{\frac{1}{n}}$. $\square$

2.4. Main lemma.

Lemma 9 (perturbation). *Assume that* F *is a Fréchet space with* $C^\omega \subset F \subset_0 C^0$*, and* K *is a compact subset of* F*. For every Bruno number* θ*, every* $\rho_1 < \rho_2 < r_\theta$*, and every* $\varepsilon > 0$ *there exists a Bruno (in fact bounded type) number* θ' *and* $\rho' > 0$ *such that*

(1) $|\theta' - \theta| < \varepsilon$
(2) $\rho_1 < \rho' < \rho_2$
(3) $r_{\theta'} > \rho'$
(4) $d_F(\phi_\theta(\rho' z), \phi_{\theta'}(\rho' z)) < \varepsilon$
(5) $\phi_{\theta'}(\rho' z)$ *does not belong to* K*.*

Proof. Let $\rho_3 = \frac{\rho_1+\rho_2}{2}$. If, for some $\rho' \in]\rho_1, \rho_3[$, $\phi_\theta(\rho' z)$ satisfies (5), then we are done with $\theta' = \theta$.
Otherwise, use lemma 2 to find $\theta_n \longrightarrow \theta$ such that $r_{\theta_n} \longrightarrow \rho_3$. By lemma 3, ϕ_{θ_n} tends to ϕ_θ uniformly on compact subsets of $\rho_3\mathbb{D}$. According to lemma 6, the injection $C^\omega_\varepsilon \subset F$ is continuous for all $\varepsilon > 0$. On the other hand, there is some

$\varepsilon > 0$ such that the function $[\rho_1, \rho_3] \to C^\omega_\varepsilon$ defined by $\rho \mapsto \phi_\theta(\rho z)$ is continuous. Therefore, its image K_0 is compact in C^ω_ε, hence compact in F.

Let L be a compact subset of F with property $\mathcal{H}$ as given by lemma 8. Set $K' = (K+L) \cup (K_0+L)$. We claim that for large enough n, there exists a $\rho \in [\rho_1, r_{\theta_n}[$ such that $\phi_{\theta_n}(\rho z)$ does not belong to K'. Otherwise, by lemma 5 $\phi_{\theta_n}(r_{\theta_n} z)$ would belong to K'. Lemma 5 b) would imply that $\|\phi_{\theta_n}(r_{\theta_n} z) - \phi_\theta(\rho_3 z)\|_\infty \longrightarrow 0$. Thus eventually, $\partial\Delta_{\theta_n}$ would be contained in Δ_θ, contradicting lemma 1 since $\theta_n \in \mathcal{D}_2$.

Let then ρ'_n be the infimum of the set of $\rho \in [\rho_1, r_{\theta_n}[$ such that $\phi_{\theta_n}(\rho z)$ does not belong to K'. Note that $\phi_{\theta_n}(\rho'_n z)$ does not belong to K. Otherwise, since L has property $\mathcal{H}$, $\phi_{\theta_n}(\rho z)$ would belong to $K + L \subset K'$ for all $\rho > \rho'_n$ sufficiently close to ρ'_n, which contradicts the definition of ρ'_n. This gives (5) with $\rho' = \rho'_n$ for n large enough.

Let us prove that $\rho'_n \longrightarrow \rho_3$. Otherwise, for a subsequence, we would have $\rho'_n \longrightarrow r \in [\rho_1, \rho_3[$. With the definition of ρ'_n, this would yield a sequence $\rho''_n \longrightarrow r$ such that $\phi_{\theta_n}(\rho''_n z)$ does not belong to K'. The sequence of holomorphic functions $z \mapsto \phi_{\theta_n}(\rho''_n z)$ would converge uniformly on compact subsets of $\frac{\rho_3}{r}\mathbb{D}$ to $\phi_\theta(rz)$. Because of property $\mathcal{H}$, the function $(\phi_{\theta_n} - \phi_\theta)(\rho''_n z)$ would eventually belong to L, so $\phi_{\theta_n}(\rho''_n z)$ would belong to $L + K_0 \subset K'$, which is a contradiction.

As soon as $\rho'_n > \rho_1$, $\phi_{\theta_n}(\rho'_n z)$ is in K': indeed, for all $\rho \in [\rho_1, \rho'_n[$, $\phi_{\theta_n}(\rho z)$ is in K', and the claim follows from lemma 5.

Lemma 5 a) now implies that $d_F(\phi_{\theta_n}(\rho'_n z), \phi_\theta(\rho_3 z)) \longrightarrow 0$ when $n \longrightarrow +\infty$. And $\rho'_n \longrightarrow \rho_3$ implies $d_F(\phi_\theta(\rho_3 z), \phi_\theta(\rho'_n z)) \longrightarrow 0$. This gives (4). □

2.5. Proof of theorem 3.

Let D_n be given by lemma 7, and L by lemma 8.

We are going to define by induction on $n \geq -1$ a sequence θ_n of parameters, an increasing sequence $\rho'_n \geq r$, and real numbers $\varepsilon_n > 0$.

The induction hypothesis will be H_n:

- $r_{\theta_n} > \rho'_n$
- for all k with $0 \leq k \leq n$, the F-distance between $\phi_{\theta_n}(\rho'_n z)$ and the set $K_k \cup D_k \cup L$ is $> \varepsilon_k$

Let $\theta_{-1} = \theta$, $\rho'_{-1} = r$, $\varepsilon_{-1} = 1$ (ε_{-1} will not be used).

Given $n \geq 0$, assume that θ_k, ρ'_k, ε_k are defined for $-1 \leq k < n$, and that H_{n-1} holds. There exists an $\eta > 0$ such that for all $f \in F$ and all k with $0 \leq k < n$, the condition $d_F(f, \phi_{\theta_{n-1}}(\rho'_{n-1} z)) < \eta$ implies the F-distance between f and $K_k \cup D_k \cup L$ remains $> \varepsilon_k$. Let $\rho_1 = \rho'_{n-1}$ and ρ_2 such that $\rho_1 < \rho_2 < r_{\theta_{n-1}}$, close enough to ρ_1 so that for every $\rho' \in [\rho_1, \rho_2]$,

$$(*) \qquad d_F(\phi_{\theta_{n-1}}(\rho' z), \phi_{\theta_{n-1}}(\rho_1 z)) < \frac{1}{2} \min(\eta, \varepsilon/2^{n+1})$$

(this is possible since the injection of $C^\omega_\varepsilon \subset F$ is continuous for all $\varepsilon > 0$). Let θ_n and ρ'_n be provided by lemma 9 such that

- $|\theta_n - \theta_{n-1}| < \varepsilon/2^{n+1}$
- $\rho_1 < \rho'_n < \rho_2$
- $r_{\theta_n} > \rho'_n$

- $d_F(\phi_{\theta_n}(\rho'_n z), \phi_{\theta_{n-1}}(\rho'_n z)) < \frac{1}{2}\min(\eta, \varepsilon/2^{n+1})$
- $\phi_{\theta_n}(\rho'_n z)$ does not belong to $K_n \cup D_n \cup L$

We then define $\varepsilon_n = \dfrac{1}{2} d_F(K_n \cup D_n \cup L, \phi_{\theta_n}(\rho'_n z))$. It is easy to check that H_n holds.

Now that the sequences θ_n, ρ'_n, ε_n have been defined, let θ' be the limit of the Cauchy sequence θ_n, and ρ' the limit of the increasing sequence ρ'_n (which is bounded from above by 4). Observe that for all n, $r_{\theta_n} > \rho'_n$, and that by equation $(*)$ the sequence of maps $\phi_{\theta_n}(\rho'_n z)$ (restricted to $\mathbb{D}$) is a Cauchy sequence for d_F, thus converges in F to some ψ (we use here the completeness of Fréchet spaces). Convergence in F implies convergence in C^0, thus we can apply lemma 4 to conclude that $\psi(z) = \phi_{\theta'}(\rho' z)$ on $\overline{\mathbb{D}}$. Also, $d_F(K_n \cup D_n \cup L, \psi) \geq \varepsilon_n$, so ψ does not belong to any K_n nor to any D_n. Since $\bigcup D_n = C^\omega$, this implies ψ does not extend holomorphically to a neighborhood of $\mathbb{D}$, thus $r_{\theta'} = \rho'$. $\square$

3. Concluding remark

All of the above discussion extends to any analytic family for which an analogue of lemma 2 can be proved.

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