ON THE BIEBERBACH CONJECTURE AND HOLOMORPHIC DYNAMICS.

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Abstract. In this note we prove that when P is a polynomial of degree d with connected Julia set and when z_0 belongs to the filled-in Julia set $K(P)$, then $|P'(z_0)| \leq d^2$. We also show that equality is achieved if and only if $K(P)$ is a segment of which one extremity is z_0. in that case, P is conjugate to a Tchebycheff polynomial or its opposite. The main tool in our proof is the Bieberbach conjecture proved by de Branges in 1984.

1. Introduction.

Let us first recall two well-known dynamical results which are in the same vein as ours.

Theorem 1. *Let P be a monic centered polynomial with connected Julia set. Then, for any $z_0 \in K(P)$, we have $|z_0| \leq 2$ with equality if and only if $K(P)$ is a segment of which one extremity is z_0.*

Proof. Assume K is a compact connected subset of $\mathbb{C}$ and $\mathbb{C} \setminus K$ is conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$. Let $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ be a conformal isomorphism, with Laurent series expansion

$$\phi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots$$

Then, the Gronwall Area Formula asserts that the area of K is equal to $\pi(|b_1|^2 - \sum_{n\geq 1} n|b_{-n}|^2)$. It follows that $|b_1| \leq |b_{-1}|$, with equality if and only if K is a straight line segment. Moreover, when $b_0 = 0$ and $z_0 \in K$, by considering the map $\psi(w) = \sqrt{\phi(w^2) - z_0}$, we get $|z_0| \leq 2|b_1|$ with equality if and only if K is a straight line segment of which one extremity is z_0.

Then, observe that when P is a monic centered polynomial, the Böttcher coordinate $\phi :\ \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(P)$ has Laurent series expansion of the form

$$\phi(z) = z + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots .$$

Indeed, $b_1 = 1$ because P is monic and $b_0 = 0$ because P is centered. Theorem 1 follows immediately. ■

Theorem 2. *Let P be a polynomial of degree d with connected Julia set. If α is a fixed point of P, then $|P'(\alpha)| \leq d^2$.*

This is a weak version of an inequality due to Pommerenke [Po], Levine [L] and Yoccoz [Y] (see [H] or [Pe]). The idea of the proof goes back to Bers's Inequality in the context of quasi-fuchsian groups. There, Bers proves that the length of a hyperbolic geodesic in $Q(X,Y)$ is bounded by the hyperbolic length of the corresponding geodesic on X or Y (see [B] Theorem 3 and [McM] Prop. 6.4). In [O]

Sect. 5.1, Otal gives a proof of Bers's Inequality based on Koebe's One-Quarter Theorem. His proof is inspired by Ahlfors (see [A] Lemma 1).

In the present article, we present a generalization of those two theorems. We will use the Bieberbach conjecture proved by de Branges in 1984.

De Branges's Theorem. *Let $\phi : \mathbb{D} \to \mathbb{C}$ be a univalent mapping. If $\phi(z) = \sum_{n\geq 1} a_n z^n$, then for any $n \geq 1$, we have $|a_n/a_1| \leq n$. Besides, if $|a_k/a_1| = k$ for some integer $k > 1$, then ϕ is a rotation of the Koebe function, i.e., there exists a real θ such that*

$$\phi(z) = \frac{z}{(1-e^{i\theta}z)^2}.$$

We obtain a result which does not only control the derivative of P at its fixed points, but controls the derivative of P at all the points in the Julia set. Our main observation is the following.

Lemma 1. *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a germ such that 0 is a superattracting fixed point with local degree $k \geq 2$. Let $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a Böttcher coordinate, i.e., a germ which is univalent in a neighborhood of 0 and which satisfies $\phi(z^k) = f(\phi(z))$ for z sufficiently close to 0. If $\phi(z) = \sum_{n\geq 1} a_n z^n$, then*

$$\operatorname{res}\left(\frac{1}{f}, 0\right) = k\frac{a_k}{a_1}.$$

Remark. The result still holds if instead of germs, one considers formal power series but we are not aware of a formal proof.

We say that a polynomial P is a Tchebycheff polynomial if $P(\cos z) = \cos(dz)$, where d is the degree of P. As a corollary of Lemma 1, we will show the following two theorems.

Theorem 3. *Assume P is a polynomial of degree d with connected Julia set. Then, for any $z_0 \in K(P)$, we have $|P'(z_0)| \leq d^2$ with equality if and only if $K(P)$ is a segment, one extremity of which is z_0. In that case, P is conjugate to a Tchebycheff polynomial or to its opposite.*

Theorem 4. *Assume P is a polynomial of degree d with disconnected Julia set. Let $g_P : \mathbb{C} \to \mathbb{R}^+$ be the Green's function of $K(P)$ and set*

$$G(P) = \max_{\{\omega \mid P'(\omega)=0\}} g_P(\omega).$$

Then, for any $z_0 \in \mathbb{C}$ with $g_P(z_0) \leq G(P)$, we have $|P'(z_0)| < d^2 e^{(d-1)G(P)}$.

Remark. This inequality always holds for points in $K(P)$.

2. Proofs of the results.

Proof of Lemma 1. Let γ_1 be a small circle around 0 and let γ_2 be its image by ϕ. Then,

$$\operatorname{res}\left(\frac{1}{f}, 0\right) = \int_{\gamma_2} \frac{dw}{f(w)} \underset{w=\phi(z)}{=} \int_{\gamma_1} \frac{\phi'(z)}{f(\phi(z))} dz = \int_{\gamma_1} \frac{\phi'(z)}{\phi(z^k)} dz = \operatorname{res}\left(\frac{\phi'(z)}{\phi(z^k)}, 0\right).$$

Since $\phi(z) = \sum_{n\geq 1} a_n z^n$, we have

$$\frac{\phi'(z)}{\phi(z^k)} = \frac{a_1 + 2a_2 z + \ldots + k a_k z^{k-1} + \mathcal{O}(|z|^k)}{a_1 z^k (1 + \mathcal{O}(|z|^k))} = \frac{1}{z^k} + \frac{2a_2}{a_1}\frac{1}{z^{k-1}} + \ldots + \frac{k a_k}{a_1}\frac{1}{z} + \mathcal{O}(1).$$

Therefore

$$\mathrm{res}\left(\frac{1}{f},0\right) = \mathrm{res}\left(\frac{\phi'(z)}{\phi(z^k)},0\right) = \frac{ka_k}{a_1}.$$

∎

Proof of Theorem 3. First, observe that when P is conjugate to a Tchebycheff polynomial of degree d (or its opposite) $K(P)$ is a segment and the derivative at an extremity is $\pm d^2$. The proof is not difficult and left to the reader.

Next, assume P is a polynomial of degree d with connected Julia set and z_0 belongs to the filled-in Julia set $K(P)$. Let Ω be the simply connected sub-domain of $\mathbb{P}^1$ defined by

$$\Omega = \left\{ w \in \mathbb{P}^1 \;\middle|\; z_0 + \frac{1}{w} \in \mathbb{P}^1 \setminus K(P) \right\}.$$

Since $z_0 \in K(P)$, we see that $\Omega \subset \mathbb{C}$, and since P has a superattracting fixed point with local degree d at infinity, the rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$f(w) = \frac{1}{P(z_0 + 1/w) - z_0}$$

has a superattracting fixed point at 0 with local degree d. Any Böttcher coordinate of f extends to a univalent mapping $\phi : \mathbb{D} \to \Omega$ and Lemma 1 asserts that writing $\phi(z) = \sum_{n\geq 1} a_n z^n$, we get

$$d\frac{a_d}{a_1} = \mathrm{res}\left(\frac{1}{f},0\right).$$

Since $P(z) = b_0 + b_1(z - z_0) + \ldots + b_d(z - z_0)^d$, we see that

$$\frac{1}{f(w)} = P(z_0 + 1/w) - z_0 = b_0 - z_0 + \frac{b_1}{w} + \ldots + \frac{b_d}{w^d}.$$

Therefore, $\mathrm{res}\,(1/f, 0) = b_1 = P'(z_0)$. It now follows from de Branges's Theorem that

$$|P'(z_0)| = \left| d\frac{a_d}{a_1} \right| \leq d^2,$$

with equality if and only if ϕ is a rotation of the Koebe function. In that case, Ω is a slit plane, and thus, $K(P)$ is a segment of which one extremity is z_0.

We must now show that P is conjugate to a Tchebycheff polynomial or to its opposite. Knowing that $K(P)$ is a segment, this is classical. Conjugating P with an affine map, we may assume that $K(P) = [-1, 1]$. We define $\psi : \mathbb{P}^1 \setminus \overline{\mathbb{D}} \to \mathbb{P}^1 \setminus [-1, 1]$ to be the conformal representation

$$\psi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

The conformal representation $\psi^{-1} : \mathbb{P}^1 \setminus [-1, 1] \to \mathbb{P}^1 \setminus \overline{\mathbb{D}}$ conjugates the proper mapping $P : \mathbb{P}^1 \setminus [-1, 1] \to \mathbb{P}^1 \setminus [-1, 1]$ to a proper mapping from $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$ to itself, having a superattracting fixed point of degree d at infinity. This mapping is necessarily of the form $z \mapsto \lambda z^d$, with $|\lambda| = 1$.

Since $K(P)$ is totally invariant, the polynomial P necessarily maps the set $\{-1, 1\}$ into itself. Besides, $\psi^{-1}(z)$ tends to ± 1 as z tends to ± 1. Therefore, the map $z \mapsto \lambda z^d$ maps the set $\{-1, 1\}$ into itself. This shows that $\lambda = \pm 1$. Hence,

$$(\forall z \in \mathbb{P}^1 \setminus \overline{\mathbb{D}}) \qquad P(\psi(z)) = \psi(\pm z^d).$$

As $z \to e^{i\theta} \in S^1$, we get

$$P(\cos\theta) = P\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \pm\frac{e^{id\theta}+e^{-id\theta}}{2} = \pm\cos(d\theta).$$

■

Proof of Theorem 4. We will mimick the previous proof. We assume that $g_P(z_0) \leq G(P)$ and we set

$$\Omega = \left\{ w \in \mathbb{P}^1 \;\middle|\; g_P\left(z_0 + \frac{1}{w}\right) > G(P) \right\}.$$

We define $f : \mathbb{P}^1 \to \mathbb{P}^1$ by $f(w) = 1/(P(z_0+1/w) - z_0)$. Then the Böttcher coordinate of f at 0 extends to a univalent mapping ϕ between the disk centered at 0 with radius $e^{-G(P)}$ and the domain $\Omega \subset \mathbb{C}$. Since the mapping

$$z \mapsto \phi(e^{-G(P)}z) = \sum_{n\geq 1} a_n e^{-nG(P)} z^n,$$

is univalent in the unit disk, de Branges's Theorem only allows us to conclude that

$$|P'(z_0)| = d\left|\frac{a_d}{a_1}\right| = de^{(d-1)G(P)}\left|\frac{a_d e^{-dG(P)}}{a_1 e^{-G(P)}}\right| < d^2 e^{(d-1)G(P)}.$$

The inequality is strict because the complement of Ω has non-empty interior, and therefore, Ω cannot be a slit plane. ■

3. Application.

A possible application of Theorem 3 is the following.

Corollary 1. *Let $d \geq 3$ be an integer, and $a = (a_2, \ldots, a_{d-1})$ be a point in $\mathbb{C}^{d-2}$. Then, the Julia set of the polynomial $P_a(z) = d^2 z + a_2 z^2 + \ldots + a_{d-1}z^{d-1} + z^d$ is connected, if and only if P_a is conjugate to a Tchebycheff polynomial.*

Proof. On the one hand, if P_a is conjugate to a Tchebycheff polynomial, its Julia set is a segment and therefore, it is connected. On the other hand, observe that 0 is a fixed point with multiplier d^2. Therefore, Theorem 3 shows that if $J(P_a)$ is connected, then P_a is conjugate to a Tchebycheff polynomial or its opposite and 0 is an extremity of $K(P_a)$. Since 0 is fixed, P_a may always be conjugate to a Tchebycheff polynomial. ■

Every polynomial of degree d having a fixed point with multiplier d^2 is conjugate to a polynomial P_a. The family $(P_a)_{a\in\mathbb{C}^{d-2}}$ is a co-dimension 1 algebraic sub-variety of the space of polynomials up to conjugacy. The set of polynomials P_a which are conjugate to a Tchebycheff polynomial is finite but not empty. Therefore, for each degree $d \geq 3$, we produce an example of co-dimension 1 algebraic family of polynomials for which the connectivity locus is non-empty and discrete.

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