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HERMAN RINGS AND ARNOLD DISKS

XAVIER BUFF, NÚRIA FAGELLA, LUKAS GEYER
AND CHRISTIAN HENRIKSEN

Abstract

For $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$, let $f_{\lambda,a}$ be the rational map defined by $f_{\lambda,a}(z) = \lambda z^2(az+1)/(z+a)$. If $\alpha \in \mathbb{R}/\mathbb{Z}$ is a Brjuno number, we let $\mathcal{D}_\alpha$ be the set of parameters (λ, a) such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α (we consider that $(e^{2i\pi\alpha}, 0) \in \mathcal{D}_\alpha$). Results obtained by McMullen and Sullivan imply that, for any $g \in \mathcal{D}_\alpha$, the connected component of $\mathcal{D}_\alpha \cap (\mathbb{C}^* \times (\mathbb{C} \setminus \{0,1\}))$ that contains g is isomorphic to a punctured disk.

We show that there is a holomorphic injection $\mathcal{F}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}_\alpha$ such that $\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0)$ and $\mathcal{F}'_\alpha(0) = (0, r_\alpha)$, where r_α is the conformal radius at 0 of the Siegel disk of the quadratic polynomial $z \longmapsto e^{2i\pi\alpha} z(1+z)$.

As a consequence, we show that for $a \in (0, 1/3)$, if $f_{\lambda,a}$ has a fixed Herman ring with rotation number α and if m_a is the modulus of the Herman ring, then, as $a \to 0$, we have $e^{\pi m_a} = (r_\alpha/a) + \mathcal{O}(a)$.

We finally explain how to adapt the results to the complex standard family $z \longmapsto \lambda z e^{(a/2)(z-1/z)}$.

1. *Introduction*

In this paper, we are mainly concerned with the dynamics of rational maps of the form

$$f_{\lambda,a}(z) = \lambda z^2 \frac{az+1}{z+a}, \qquad \lambda \in \mathbb{C}^*,\ a \in \mathbb{C}.$$

Note that $f_{\lambda,a}$ is conjugate to $f_{\lambda,-a}$ via the conjugacy $z \longmapsto -z$. If $\lambda \in S^1$ and $a = 0$, then the map $f_{\lambda,a}$ is the rotation $z \longmapsto \lambda z$. Observe that when a is real and $|\lambda| = 1$, the map $f_{\lambda,a}$ is a Blaschke fraction $z \longmapsto \lambda z^2(z+b)/(1+\bar{b}z)$ with $b = 1/a$. However, as opposed to families of Blaschke fractions which only depend $\mathbb{R}$-analytically on parameters, our family depends $\mathbb{C}$-analytically on the parameters λ and a and is, in some sense, the simplest one that exhibits families of Herman rings.

For all $\alpha \in \mathbb{R}/\mathbb{Z}$, we denote by $\mathcal{R}_\alpha$ the rigid rotation of the complex plane $\mathcal{R}_\alpha(z) = e^{2i\pi\alpha} z$. When $\lambda \in S^1$ and $a \in (-1/3, 1/3)$, the map $f_{\lambda,a}$ restricts to a diffeomorphism of S^1 which has a rotation number $\rho(\lambda, a) \in \mathbb{R}/\mathbb{Z}$. Given $a \in (-1/3, 1/3)$, the function $t \longmapsto \rho(e^{2i\pi t}, a)$ is continuous and weakly increasing [**15**]. Moreover, for each fixed $a \in (-1/3, 1/3)$ and for each irrational number α, there is a unique angle $t \in \mathbb{R}/\mathbb{Z}$ such that $\rho(e^{2i\pi t}, a) = \alpha$ (see for example [**6**]). By a theorem of Denjoy [**7**], when $\alpha = \rho(\lambda, a)$ is irrational, $f_{\lambda,a} : S^1 \longrightarrow S^1$ is topologically conjugate to the rotation $\mathcal{R}_\alpha : S^1 \longrightarrow S^1$.

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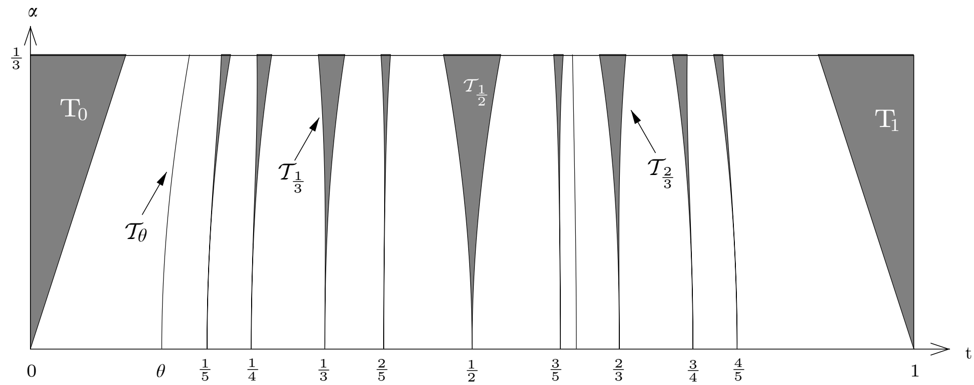

FIGURE 1. *Rational Arnold tongues in the parameter space of the family $f_{\lambda,a}$ for $\lambda = e^{2\pi i t}$, $t \in \mathbb{R}/\mathbb{Z}$, up to denominator* 5, *and irrational tongues for $\gamma = (\sqrt{5}-1)/2$ and $\theta = \sqrt[5]{2}-1$.*

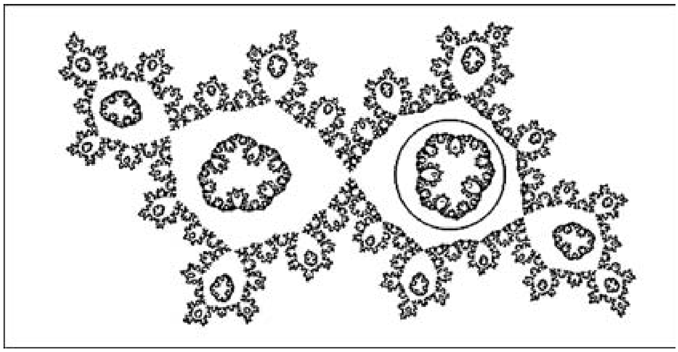

FIGURE 2. *For $t = 0.615\,173\,215\,88\ldots$, the rational map $f_{e^{2i\pi t},1/4}$ leaves the circle $\mathbb{S}^1$ invariant and has a Herman ring.*

Figure 1 shows some of the level sets

$$\mathcal{T}_\alpha = \{(t,a) \in \mathbb{R}/\mathbb{Z} \times [0,1/3] \mid \rho(e^{2i\pi t}, a) = \alpha\}.$$

These sets are called *Arnold tongues*, and they intersect the line $\{a = 0\}$ at the point $t = \alpha$. If α is a rational number, the Arnold tongue $\mathcal{T}_\alpha$ has interior, and if α is irrational, the Arnold tongue $\mathcal{T}_\alpha$ is a Lipschitz curve [**1**].

If $\alpha \in \mathbb{R}/\mathbb{Z}$ is a Brjuno number and if (t,a) belongs to $\mathcal{T}_\alpha$ with a sufficiently close to 0, then the restriction of $f_{e^{2i\pi t},a}$ to S^1 is $\mathbb{R}$-analytically conjugate to the rotation of angle α [**3**]. This conjugacy extends to a conjugacy in a neighborhood of S^1, and so $f_{e^{2i\pi t},a}$ has a fixed Herman ring. Figure 2 shows an example of such a Herman ring for $a = 1/4$ and $t = 0.615\,173\,215\,88\ldots$.

Even in the case when $\lambda \notin S^1$ and $a \notin (0,1/3)$, the rational map $f_{\lambda,a}$ may have a fixed Herman ring. However, the unit circle is no longer invariant, and it is more difficult to locate parameters for which one can find a Herman ring.

It is known that in this particular family $f_{\lambda,a}$, there is a Herman ring if and only if the rotation number is a Brjuno number (see for example [**18**] or [**12**]). This result is proved using a surgery construction due to Shishikura, and the optimality of the Brjuno condition for the existence of Siegel disks in the family of quadratic

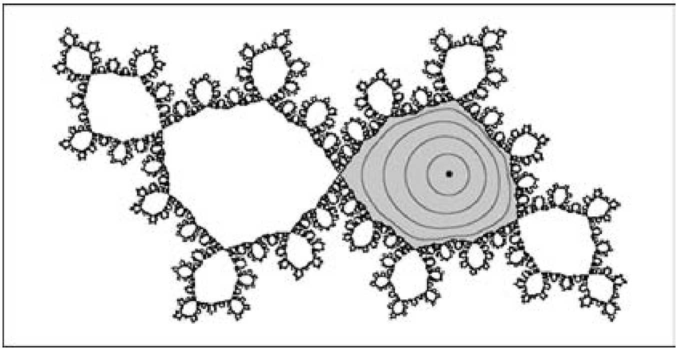

FIGURE 3. *The quadratic polynomial P_α with $\alpha = (\sqrt{5}-1)/2$ has a Siegel disk. The orbits of some points in the Siegel disk are shown. Each orbit accumulates on an $\mathbb{R}$-analytic circle.*

polynomials. More precisely, it uses the fact that the quadratic polynomial $P_\alpha : z \longmapsto e^{2i\pi\alpha} z(1+z)$ is linearizable at 0 if and only if α is a Brjuno number. The sufficiency of this condition was proven by Brjuno [**3**] and Rüssmann [**17**], whereas the optimality was proven by Yoccoz [**20**]. Figure 3 shows the Siegel disk of the quadratic polynomial P_α for $\alpha = (\sqrt{5}-1)/2$.

This paper studies the set of complex parameters (λ, a) for which $f_{\lambda,a}$ has a fixed Herman ring with a given rotation number α. (If $\lambda = e^{2i\pi\alpha}$ and $a = 0$, we consider that there is a Herman ring which is equal to $\mathbb{C}^*$.)

DEFINITION 1.1. Given a Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, we let Δ_α be the Siegel disk of the quadratic polynomial P_α and we let $\mathcal{D}_\alpha$ be the set of parameters $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$ such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α. We call $\mathcal{D}_\alpha$ the *Arnold disk of rotation number α*.

In some sense, this set is the complexification of the Arnold tongue $\mathcal{T}_\alpha$, and it was studied for general families from a local point of view in [**16**]. The term Arnold *disk* is justified by the following theorem, which is proved in Section 4. We first recall the definitions of conformal radius and modulus.

DEFINITION 1.2. If $U \subsetneq \mathbb{C}$ is a simply connected open subset containing 0, then the *conformal radius* of U is $\mathrm{rad}(U) = |\phi'(0)|$, where $\phi : (\mathbb{D}, 0) \longrightarrow (U, 0)$ is any analytic isomorphism. If α is a Brjuno number, then we define $r_\alpha = \mathrm{rad}(\Delta_\alpha)$, where Δ_α is the Siegel disk of P_α.

If A is a round annulus $A = \{z \in \mathbb{C} \mid r < |z| < R\}$, then the *modulus* of A is $\mathrm{mod}(A) = (1/2\pi)\log(R/r)$. If H is any annulus conformally equivalent to A, then we define its modulus to be equal to $\mathrm{mod}(A)$.

THEOREM A. *For any Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, there exists a holomorphic bijection $\mathcal{F}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}_\alpha$ such that*

$$\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0) \quad \text{and} \quad \mathcal{F}'_\alpha(0) = (0, r_\alpha).$$

Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is equal to $(1/\pi)\log(1/|\delta|)$.

Let us give an intuitive idea of how the parameterization $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$ is chosen. For any $\delta \in \mathbb{D}$, the map $f_{\mathcal{F}_\alpha(\delta)}$ possesses a Herman ring whose modulus is determined by $|\delta|$; the argument of δ corresponds to the *twist parameter*, which roughly indicates how much one boundary of the ring is rotated with respect to the other (see Section 2 and Figure 7). Maps in the Arnold tongue $\mathcal{T}_\alpha$ correspond to $\delta \in (0,1)$. (This parameterization was given in [**12**] for the rational family and [**9**] for the complex standard family.)

Theorem B. *Assume that α is a Brjuno number. Then the following hold.*

(a) *The Arnold disk can be locally parameterized by a in a neighborhood of $(e^{2i\pi\alpha}, 0)$ (that is, it is locally the graph of a holomorphic map $a \longmapsto \lambda(a)$).*

(b) *As $|a| \to 0$, the modulus m_a of the Herman ring of $f_{\lambda(a),a}$ satisfies*

$$e^{\pi m_a} = \frac{r_\alpha}{|a|} + \mathcal{O}(a).$$

Corollary 1.3. *Assume that α is a Brjuno number. For $a \in (0, 1/3)$, let $t_a \in \mathbb{R}/\mathbb{Z}$ be the unique parameter such that $(t_a, a) \in \mathcal{T}_\alpha$, and let m_a be the modulus of the Herman ring of $f_{e^{2i\pi t_a},a}$ ($m_a = 0$ if there is no Herman ring). Then, as $a \to 0$, we have*

$$e^{\pi m_a} = \frac{r_\alpha}{a} + \mathcal{O}(a).$$

This improves the estimate one would get by using the techniques developed by Fagella, Seara and Villanueva [**10**] in the case of the complex standard family. Indeed, we would obtain $\mathcal{O}(|\log a|)$ instead of $\mathcal{O}(a)$.

In the last section, we explain how to adapt the arguments to the case of the complex standard family $f_{\lambda,a}(z) = \lambda z e^{(a/2)(z-1/z)}$.

The following is work in progress. Theorem A describes the topology of any given Arnold disk and hence a natural question to ask is how these disks coexist in $\mathbb{C}^2$ when we move the rotation number. We think that there is a nice lamination: if $(\alpha_n)_{n\geqslant 1}$ is a sequence of Brjuno numbers converging to a Brjuno number α_0 such that $\lim_{n\to\infty} r_{\alpha_n} = r$, then the sequence of maps $\mathcal{F}_{\alpha_n}$ converges uniformly on every compact subset of $\mathbb{D}$ to the map

$$\delta \longmapsto \mathcal{F}_{\alpha_0}\left(\frac{r}{r_{\alpha_0}}\delta\right)$$

(the map $\delta \longmapsto \mathcal{F}_{\alpha_0}(r\delta/r_{\alpha_0})$ is well defined because the conformal radius r_α depends upper semi-continuously on α, and so $r \leqslant r_{\alpha_0}$).

We think we can use this result to prove the existence of Brjuno numbers $\alpha \in \mathbb{R}/\mathbb{Z}$ such that the boundary of $\mathcal{D}_\alpha$ is a C^∞ Jordan curve and such that for all $(\lambda, a) \in \partial\mathcal{D}_\alpha$, the map $f_{\lambda,a}$ restricts to a diffeomorphism of a C^∞ Jordan curve and is C^∞ (but not $\mathbb{R}$-analytically) conjugate to the rotation of angle α on this curve. These results will appear in a forthcoming paper.

2. *Preliminaries*

In the whole of this section, we assume that α is a Brjuno number and that (λ, a) is in $\mathcal{D}_\alpha$. For simplicity, we set $f = f_{\lambda,a}$. By the definition of $\mathcal{D}_\alpha$, the map f has a fixed Herman ring H on which it is conjugate to the rotation $\mathcal{R}_\alpha$. More precisely, let m be the modulus of the Herman ring H and set $r = e^{-2\pi m}$. Moreover, denote by A_r the round annulus

$$A_r = \{z \in \mathbb{C}^* \mid r < |z| < 1\}.$$

Then there exists an isomorphism $\phi : A_r \longrightarrow H$ which conjugates $\mathcal{R}_\alpha : A_r \longrightarrow A_r$ to $f : H \longrightarrow H$ and preserves the orientation in $\mathbb{C}$ of the invariant curves (that is, it sends S^1 to the outer boundary of H). This isomorphism is unique up to precomposition with a rotation centered at 0.

2.1. *Basic properties*

The rational map $f_{\lambda,a}$ has four critical points. One is fixed at 0, and one is fixed at ∞. Therefore there are only two *free* critical points. If $f_{\lambda,a}$ has a Herman ring, then the orbits of those critical points are *trapped*: the closure of those orbits must contain the boundary components of the Herman ring. In particular, the immediate basins of the superattracting fixed points at 0 and ∞ are simply connected. We claim that the Herman ring separates those two basins.

PROPOSITION 2.1. *Suppose that $f = f_{\lambda,a}$ has a fixed Herman ring H. Then H separates* 0, *one of the critical points in $\mathbb{C}^*$ and the pole $-a$ on the one hand, from* ∞, *and the other critical point in $\mathbb{C}^*$ and the zero $-1/a$ on the other hand.*

Proof. Let γ be an invariant curve in H, let V_0 be the bounded connected component of $\hat{\mathbb{C}} \setminus \gamma$, and let V_∞ be the unbounded one. Set $U_\infty = f^{-1}(V_\infty) \cap V_0$ and $U_0 = f^{-1}(V_0) \cap V_0$. Since f is proper and since $f(\gamma) = \gamma$, we see that $f : U_\infty \longrightarrow V_\infty$ and $f : U_0 \longrightarrow V_0$ are proper mappings. Let d_∞ and d_0 be their degrees.

We first claim that $d_\infty = 1$ and $d_0 = 2$. Indeed, the image of γ (which is γ itself) turns exactly once around every point in V_0. Therefore, by the argument principle, $d_0 - d_\infty = 1$. Since ∞ has at most one preimage in V_0, we see that the only possibilities for (d_∞, d_0) are $(0, 1)$ or $(1, 2)$. The case $(0, 1)$ is not possible, since otherwise, $f : V_0 \longrightarrow V_0$ would be an isomorphism, V_0 would be contained in the Fatou set of f, and this would contradict the fact that it contains a boundary component of the Herman ring H.

It follows that U_∞ is a topological disk compactly contained in V_0 and U_0 is a topological annulus. By the Riemannin–Hurwitz formula, U_0 contains two critical points of f. The two remaining critical points of f are contained in an annulus $W_\infty \subset V_\infty$ mapped with degree 2 to V_∞. Finally, there is a topological disk $W_0 \subset V_\infty$ mapped to V_0 with degree 1 (see Figure 4).

We must finally show that $0 \in U_0$, which will complete the proof. This last result is more subtle, since we must use a dynamical characterization of 0: it is a superattracting fixed point. There are several possible arguments. It is possible to show that each component of $\hat{\mathbb{C}} \setminus \gamma$ must contain a critical point whose orbit accumulates a boundary component of H. Alternatively, we can argue that every connected component of $\hat{\mathbb{C}} \setminus \gamma$ must contain two fixed points, one of which is repelling or has multiplier 1 (see for example [**19**] or [**4**]). □

As mentioned previously, it is quite difficult to locate parameters in $\mathcal{D}_\alpha$ when a is not real. However, the following proposition asserts that all the sets $\mathcal{D}_\alpha$ are contained in some common compact subset of $\mathbb{C}^* \times \mathbb{C}$.

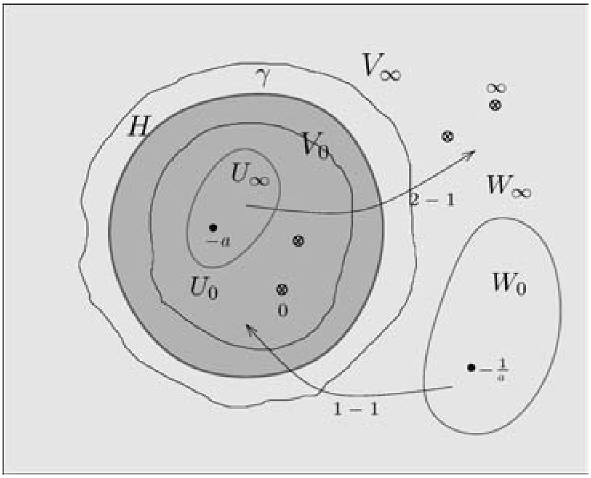

FIGURE 4. *Sketch illustrating the proof of Proposition* 2.1.

PROPOSITION 2.2. *Assume that α is a Brjuno number and $(\lambda, a) \in \mathcal{D}_\alpha$. Then*

$$\tfrac{1}{4} \leqslant |\lambda| \leqslant 4 \quad \textit{and} \quad |a| \leqslant 1.$$

Proof. As mentioned previously, the immediate basins of the superattracting fixed points at 0 and ∞ are simply connected, and the Herman ring separates those two basins. Let W_0 (respectively W_∞) be the immediate basin of 0 (respectively ∞). Let $\phi_0 : W_0 \longrightarrow B(0, r_0)$ (respectively $\phi_\infty : W_\infty \longrightarrow \hat{\mathbb{C}} \setminus \overline{B(0, r_\infty)}$) be the conformal representation which is tangent to the identity at 0 (respectively ∞).

Then ϕ_0 (respectively ϕ_∞) conjugates $f_{\lambda,a}$ to a proper mapping of $B(0, r_0)$ (respectively $\hat{\mathbb{C}} \setminus \overline{B(0, r_\infty)}$) of degree 2 which has a superattracting fixed point at 0 (respectively ∞). This proper mapping is $z \longmapsto z^2/c_0$ (respectively $z \longmapsto z^2/c_\infty$). Since this map must have a fixed point on the boundary of $B(0, r_0)$ (respectively $B(0, r_\infty)$), it follows that $|c_0| = r_0$ (respectively $|c_\infty| = r_\infty$). Now note that

$$f_{\lambda,a}(z) \underset{z\to 0}{\sim} \frac{\lambda}{a} z^2 \quad \text{and} \quad f_{\lambda,a}(z) \underset{z\to\infty}{\sim} \lambda a z^2.$$

Therefore we deduce that

$$r_0 = \frac{|a|}{|\lambda|} \quad \text{and} \quad r_\infty = \frac{1}{|\lambda a|}.$$

Now W_0 does not contain $-a$, which is a preimage of ∞, and W_∞ does not contain $-1/a$, which is a preimage of 0. Therefore, by the Koebe one-quarter theorem, we have

$$r_0 \leqslant 4|a| \quad \text{and} \quad r_\infty \geqslant \frac{1}{4|a|}.$$

This, in turn, yields

$$\tfrac{1}{4} \leqslant |\lambda| \leqslant 4.$$

Finally, the basins W_0 and W_∞ are disjoint. It follows that $r_0 \leqslant r_\infty$ (see for example [**5**, Section 3]). As a consequence, we have

$$\frac{|a|}{|\lambda|} \leqslant \frac{1}{|\lambda a|},$$

which gives $|a| \leqslant 1$. □

2.2. *Shishikura's surgery*

We explain here a surgery construction originally due to Shishikura [**18**] that will be used in several instances in this paper. The general idea of the construction is, starting from a rational map with a Herman ring, to obtain a polynomial with a Siegel disk by means of gluing a rigid rotation to 'fill in the hole' of the Herman ring. In the particular case of our family of rational maps $f_{\lambda,a}$, the polynomial that we obtain is precisely the quadratic polynomial P_α. The result is summarized in the following proposition.

PROPOSITION 2.3. *Suppose that $f = f_{\lambda,a}$ has a fixed Herman ring H with rotation number α, and let $\gamma \subset H$ denote an invariant curve. Let U_0 be the bounded part of $\hat{\mathbb{C}} \setminus \gamma$ and let U_∞ be the unbounded one. Then the polynomial P_α has a Siegel disk Δ_α and there exists a quasiconformal homeomorphism $\psi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ and a P_α-invariant curve Γ in Δ_α such that the following hold.*

(1) *ψ maps γ to Γ and U_∞ to the unbounded component, say V, of $\hat{\mathbb{C}} \setminus \Gamma$.*

(2) *ψ conjugates $f : \overline{U}_\infty \longrightarrow \hat{\mathbb{C}}$ to $P_\alpha : \overline{V} \longrightarrow \hat{\mathbb{C}}$.*

(3) *$\partial\psi/\partial\bar{z} = 0$ almost everywhere on $\hat{\mathbb{C}} \setminus \bigcup_{n \geqslant 0} f^{-n}(U_0)$ (in particular, ψ is conformal in the interior of this set).*

Proof. Shishikura proved this using a surgery construction that has now become classic. We will give an outline of the procedure. Let $\phi : A_r \longrightarrow H$ be a conformal map that conjugates the rigid rotation $\mathcal{R}_\alpha : A_r \longrightarrow A_r$ to $f : H \longrightarrow H$.

Notice that $\phi^{-1}(\gamma)$ is a circle centered at 0 with radius ρ, $r < \rho < 1$. Denote by $\hat{\phi} : \mathbb{D} \longrightarrow H \cup U_0$ a quasiconformal mapping that agrees with ϕ on A_ρ, maps $\mathbb{D}_\rho$ onto U_0, and fixes 0. Define a map $\hat{f} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ by

$$\hat{f} = \begin{cases} f & \text{on } \overline{U}_\infty \\ \hat{\phi} \circ \mathcal{R}_\alpha \circ \hat{\phi}^{-1} & \text{on } \overline{U}_0. \end{cases}$$

The map $\hat{f} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is a proper and quasiregular mapping which maps infinity to itself with local degree 2. By Proposition 2.1, f has a pole in U_0, so by counting the preimages of infinity, we see that $\hat{f}$ has degree 2, it fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0. Finally, since it is of degree 2, it has a unique critical point ω in $\mathbb{C}$.

The map $\hat{f}$ is not holomorphic on U_0, but there it preserves the complex structure defined by the Beltrami from

$$\mu = \frac{\bar{\partial}\hat{\phi}^{-1}}{\partial\hat{\phi}^{-1}},$$

where ∂ denotes $\partial/\partial z$ and $\bar{\partial}$ denotes $\partial/\partial\bar{z}$. Pulling back this Beltrami form via $\hat{f}$, we see that there exists a Beltrami form $\hat{\mu}$ which coincides with μ on U_0, which vanishes on $\hat{\mathbb{C}} \setminus \bigcup_{n\geqslant 0} \hat{f}^{-n}(U_0)$, and which is invariant by $\hat{f}$:

$$\hat{f}^*\hat{\mu} = \hat{\mu}.$$

By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\psi: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ which fixes 0 and ∞, sends ω to $-1/2$, and satisfies

$$\hat{\mu} = \frac{\bar{\partial}\psi}{\partial\psi}.$$

Then the map $\psi \circ \hat{f} \circ \psi^{-1}: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is proper and holomorphic of degree 2, it fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0, and it has a superattracting fixed point at infinity and a critical point at $-1/2$. The only such map is the quadratic polynomial P_α, and thus

$$P_\alpha = \psi \circ \hat{f} \circ \psi^{-1}.$$

The map ψ is the required conjugacy. □

Remark 2.4. The modulus of the annulus bounded by γ and the outer boundary of H is equal to the modulus of the annulus bounded by Γ and the boundary of the Siegel disk Δ_α (since ψ gives a conformal isomorphism between these two annuli).

2.3. *Case when a is real*

Proposition 2.5. *Suppose that $(\lambda, a) \in \mathcal{D}_\alpha$ for some Brjuno number α. If $a \in (0, 1/3)$, then $|\lambda| = 1$. If also $(\lambda', a) \in \mathcal{D}_\alpha$, then $\lambda' = \lambda$.*

Proof. Suppose that $a \in (0, 1/3)$ and $|\lambda| < 1$. Then f maps the unit circle onto the circle of radius $|\lambda| < 1$ as an analytic diffeomorphism. We distinguish between the following two cases.

(a) $\mathbb{D} \subset \hat{\mathbb{C}} \setminus H$.

(b) $\mathbb{D}$ intersects H.

(a) We can cut out the dynamics in the unit disk and replace it by the attracting dynamics $z \longmapsto \lambda z$. Indeed, define $\phi: S^1 \longrightarrow S^1$ by $\phi(z) = f(z)/\lambda$. Extend ϕ to $\overline{\mathbb{D}}$ quasiconformally such that ϕ is the identity on $\lambda\overline{\mathbb{D}}$. Define a new map $\hat{f}: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ by

$$\hat{f} = \begin{cases} f & \text{on } \hat{\mathbb{C}} \setminus \mathbb{D} \\ \phi^{-1} \circ \Lambda \circ \phi & \text{on } \overline{\mathbb{D}}, \end{cases}$$

where Λ denotes multiplication by $\lambda : \Lambda(z) = \lambda z$. Note that we have constructed ϕ so that the two definitions agree on S^1. Now $\hat{f}$ is a proper quasiregular mapping of degree 2 (we removed the pole in $\mathbb{D}$). It is not holomorphic, but there exists an invariant Beltrami form μ that vanishes on $\hat{\mathbb{C}} \setminus \bigcup_{n\geqslant 0} f^{-n}(\mathbb{D})$ and satisfies

$$\mu = \frac{\bar{\partial}\phi}{\partial\phi}$$

on $\mathbb{D}$. It follows that $\hat{f}$ is holomorphic with respect to the complex structure induced by μ, and one can verify that $\hat{f}$ is quasiconformally conjugate to a quadratic polynomial. However, H is a Herman ring for $\hat{f}$, which is impossible.

(b) Since $\mathbb{D}$ intersects H, we can choose an invariant curve $\gamma \subset H$ so that $\mathbb{D}$ intersects the unbounded component of $\hat{\mathbb{C}}\backslash\gamma$. Let ψ and U_0 be defined by Proposition 2.3. Set $W = \psi(\mathbb{D} \cup U_0)$ and $W' = \psi(\lambda\mathbb{D} \cup U_0)$. Then $P_\alpha(W) \subset W'$, and W' is strictly contained in W. Therefore $P_\alpha : W \longrightarrow W'$ is a strong contraction with respect to the hyperbolic metric of W which contradicts zero is an indifferent fixed point.

We have shown that we cannot have $|\lambda| < 1$. The proof that we cannot have $|\lambda| > 1$ is analogous.

Now suppose that $(\lambda', a) \in \mathcal{D}_\alpha$. By what we have just shown, $|\lambda'| = 1$, so $f_{\lambda',\alpha}$ is equal to the composition $\mathcal{R}_\beta \circ f_{\lambda,\alpha}$ for some rigid rotation $\mathcal{R}_\beta$. For an arbitrary circle homeomorphism h, the rotation number of $\mathcal{R}_\beta \circ h$ is a weakly increasing function of β, and there is only one value of β (modulo 1) for which the rotation number is equal to some given irrational number (see for example [**6**]). Since $\mathcal{R}_\beta \circ f_{\lambda,\alpha}|_{S^1}$ and $f_{\lambda,\alpha}|_{S^1}$ have the same irrational rotation number, we have $\beta = 0$ (modulo 1) and $\lambda = \lambda'$. □

3. *The twist coordinate*

As above, let f have a fixed Herman ring H of rotation number α and let $\phi : A_r \longrightarrow H$ be a conformal isomorphism that conjugates $\mathcal{R}_\alpha : A_r \longrightarrow A_r$ to $f : H \longrightarrow H$.

Our goal in this section is to define a *twist coordinate* for the map $f_{\lambda,a}$. To fix ideas, we will first define it for the easiest case, that is, the case when the boundaries of the Herman ring H are Jordan curves and contain the two critical points of the map, say ω_1 on the outer boundary and ω_2 on the inner boundary.

In such a situation, the linearizing map ϕ extends as a homeomorphism $\phi : \partial A_r \longrightarrow \partial H$. Denote by $C(c, r)$ the circle centered at $c \in \mathbb{C}$ of radius $r \in [0, \infty)$. Let $\tilde{\omega}_1 \in C(0, 1)$ and $\tilde{\omega}_2 \in C(0, r)$ be the preimages of ω_1 and ω_2 under ϕ, and let $\Theta_1 = (1/2\pi)\arg(\tilde{\omega}_1)$ and $\Theta_2 = (1/2\pi)\arg(\tilde{\omega}_2)$, taken in $\mathbb{R}/\mathbb{Z}$.

We define the *twist coordinate* of $f_{\lambda,a}$ as the difference between these two arguments (see Figure 5), that is,

$$\Theta = \Theta_2 - \Theta_1.$$

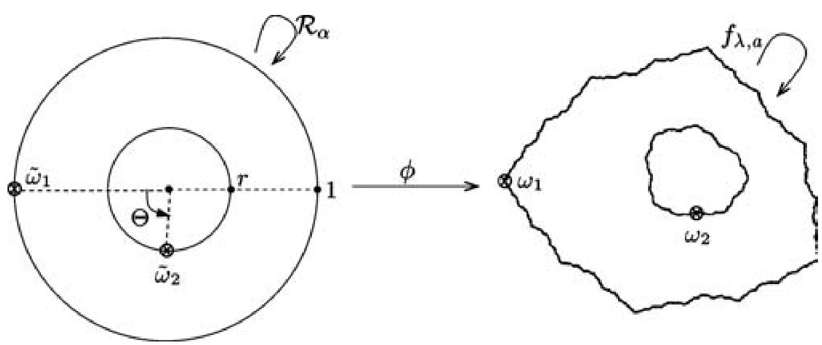

FIGURE 5. *The definition of the twist coordinate in the simplest case, when both critical points lie on the boundary of H.*

Note that even though ϕ is uniquely defined only up to rotation (and hence so are $\tilde{\omega}_1$ and $\tilde{\omega}_2$), the twist coordinate is independent of this choice.

Intuitively, the twist parameter measures the rotation of the boundaries with respect to each other.

We now turn to the general case, when the critical points are not on the boundary of H or ϕ cannot be extended as a homeomorphism $\partial A_r \longrightarrow \partial H$. For this reason, the definition of Θ in this situation will be different. We then show that the two definitions coincide.

We first recall the definition of the *equator* of an annulus.

Definition 3.1. Let H be an annulus conformally equivalent to A_r by a conformal map $\phi : A_r \longrightarrow H$. We define the *equator* of H (or *core geodesic*) as the simple closed curve

$$\gamma = \phi(C(0, \sqrt{r})) \subset H.$$

Remark 3.2. The curve γ subdivides the Herman ring H into two annuli of equal moduli (half the original modulus of H).

Let γ be the equator of H let and $\psi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, U_0, U_∞, Γ and V be as in Proposition 2.3. Recall that ψ conjugates $f_{\lambda,a}$ on U_∞ (the unbounded component of $\hat{\mathbb{C}} \setminus \gamma$) to the polynomial P_α on V (the unbounded component of $\hat{\mathbb{C}} \setminus \Gamma$ with $\Gamma = \psi(\gamma)$). Moreover, ψ fixes 0 and ∞ and sends the critical point in U_∞ to $-1/2$.

We now proceed to create a parallel construction with the map $\hat{f}(u) = 1/\bar{f}(1/\bar{u})$, that is, the conjugate of f under the change of coordinates $u = \tau(z) = 1/\bar{z}$. Observe that $\hat{f}$ has a Herman ring $\widehat{H} = \tau(H)$ of rotation number α. More precisely, let $\hat{\gamma} = \tau(\gamma)$ be the equator of $\widehat{H}$ and let $\hat{\psi}, \widehat{U}_0, \widehat{U}_\infty, \widehat{\Gamma}$ and $\widehat{V}$ be given by Proposition 2.3: $\hat{\psi}$ conjugates $\hat{f}$ in $\widehat{U}_\infty$ to P_α on $\widehat{V}$.

We claim that $\widehat{\Gamma} = \Gamma$ and $\widehat{V} = V$. Indeed, the modulus of H is the same as the modulus of $\widehat{H}$. Since γ and $\hat{\gamma}$ are equators of those annuli, the moduli of the annuli bounded by γ and $\hat{\gamma}$ on the one hand and by the outer boundaries of H and $\widehat{H}$ on the other hand are equal. Therefore, the moduli of the annuli bounded by the curves Γ and $\widehat{\Gamma}$ on the one hand, and by the boundary of the Siegel disk on the other hand, are equal. Therefore $\Gamma = \widehat{\Gamma}$ and $V = \widehat{V}$.

We now denote by $\varphi : \mathbb{D} \longrightarrow \Delta_\alpha$ a linearizing map of the Siegel disk. The map φ conjugates the rigid rotation $\mathcal{R}_\alpha : \mathbb{D} \longrightarrow \mathbb{D}$ to $P_\alpha : \Delta_\alpha \longrightarrow \Delta_\alpha$ and is unique up to precomposition with other rigid rotations. See Figure 6.

We define a map $\mathcal{C} : \varphi^{-1}(\Gamma) \longrightarrow \varphi^{-1}(\Gamma)$ by

$$\mathcal{C} = \varphi^{-1} \circ \psi \circ \tau \circ \hat{\psi}^{-1} \circ \varphi.$$

Observe that, by construction, $\mathcal{C}$ is a conjugacy between the rotation $\mathcal{R}_\alpha$ and itself on the Euclidean circle $\varphi^{-1}(\Gamma)$. However, any such map can only be a rigid rotation itself, that is, $\mathcal{C}(z) = e^{2\pi i \Theta} z$ for some $\Theta \in \mathbb{R}/\mathbb{Z}$. We then define Θ to be the twist coordinate of $f_{\lambda,a}$.

It only remains to prove the following.

Proposition 3.3. *When the two boundary components of the Herman ring H are Jordan curves each containing a critical point, the two preceding definitions coincide.*

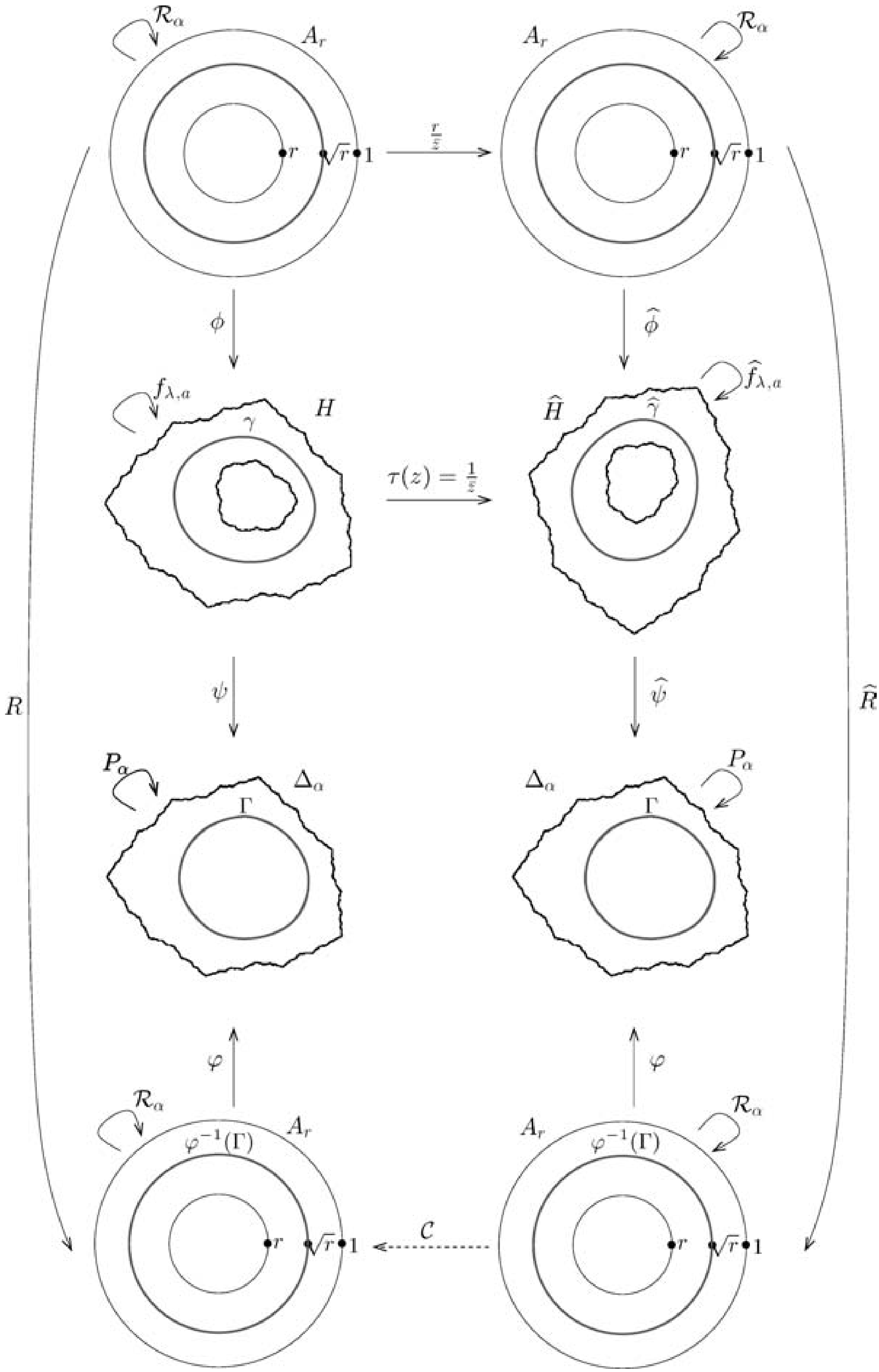

FIGURE 6. *Commutative diagram illustrating the general definition of the twist coordinate of $f_{\lambda,a}$. The conclusion is that $\mathcal{C}(z) = e^{2\pi i\Theta}z$ for some $\Theta \in \mathbb{R}/\mathbb{Z}$ which we define as the twist coordinate.*

Proof. We first observe that if $\phi : A_r \longrightarrow H$ conjugates $\mathcal{R}_\alpha$ to f, then the map $\hat{\phi} : A_r \longrightarrow \widehat{H}$ defined by $\hat{\phi}(u) := \tau \circ \phi(r/\bar{u})$ conjugates $\mathcal{R}_\alpha$ to $\hat{f}$. Hence, observing the commutative diagram in Figure 6, we see that the maps

$$R = \varphi^{-1} \circ \psi \circ \phi$$

and

$$\widehat{R} = \varphi^{-1} \circ \hat{\psi} \circ \hat{\phi}$$

are conformal maps from the annulus $A_{\sqrt{r}}$ to the Euclidean annulus bounded by $\varphi^{-1}(\Gamma)$ and S^1, and they conjugate the rigid rotation $\mathcal{R}_\alpha$ to itself (this in fact implies that $\varphi^{-1}(\Gamma)$ is the Euclidean circle of radius $\sqrt{r}$). Such maps can only be rigid rotations themselves, and we determine their angles now.

Let $\tilde{\omega}_1 \in C(0,1)$ and $\tilde{\omega}_2 \in C(0,r)$ be, as before, the preimages of the two critical points ω_1 and ω_2 under ϕ. If we follow the outer one, $\tilde{\omega}_1$, along the diagram, we see that it is mapped first to ω_1 by ϕ, then to $\omega = -1/2$ by ψ, and finally to some unit vector by φ^{-1}, say $e^{2\pi i\beta}$. Hence

$$R = \mathcal{R}_{\beta - \Theta_1},$$

where we recall that $\Theta_1 = (1/2\pi)\arg(\tilde{\omega}_1)$.

We now do the same for $\widehat{R}$. First observe that the map $r\tau$ sends all points of modulus r to points of modulus 1 and vice versa, but preserves their arguments. Therefore the critical points are interchanged, that is, the point $\tilde{\omega}_2$ is sent to a point of modulus 1 and argument Θ_2. This point is in turn mapped to the critical point on the outer boundary of $\widehat{H}$ which is later sent to $\omega = -1/2$ by $\hat{\psi}$. Finally, its image is, as before, $e^{2\pi i\beta}$. We deduce then that

$$\widehat{R} = \mathcal{R}_{\beta - \Theta_2}.$$

Now, the outmost diagram says that

$$\mathcal{C} = R \circ r\tau \circ \widehat{R}^{-1},$$

or equivalently that

$$\Theta = (\beta - \Theta_1) - (\beta - \Theta_2) = \Theta_2 - \Theta_1. \qquad \square$$

We now want to illustrate the twist coordinate with a picture. If $\lambda \in S^1$ and $a \in (0, 1/3)$, the twist coordinate is always equal to 0, because of the symmetry $z \longmapsto 1/\bar{z}$ with respect to S^1. Now, if $\lambda \in S^1$ and $a \in i\mathbb{R}$, then there is a symmetry $z \longmapsto -1/\bar{z}$, and if $f_{\lambda,a}$ has a Herman ring, the twist parameter is equal to 1/2. Even if S^1 is not invariant, it is still possible to locate parameters (λ, a) such that $f_{\lambda,a}$ has a Herman ring. For example, for $a = i/4$ and $t = 0.624\,098\,187\ldots$, the rational map $f_{e^{2i\pi t},a}$ has a fixed Herman ring with rotation number $(\sqrt{5}-1)/2$. Now the twist coordinate is better seen by working in the universal covering of $\mathbb{C}^*$ given by $Z \longmapsto z = e^{2i\pi Z}$. Figure 7 shows the Julia set of two rational maps having Herman rings with rotation numbers $(\sqrt{5}-1)/2$ and with twist coordinates 0 on the top and 1/2 on the bottom. In both cases, we have marked the critical points with arrows. The real part of Z ranges from -1 to 1.

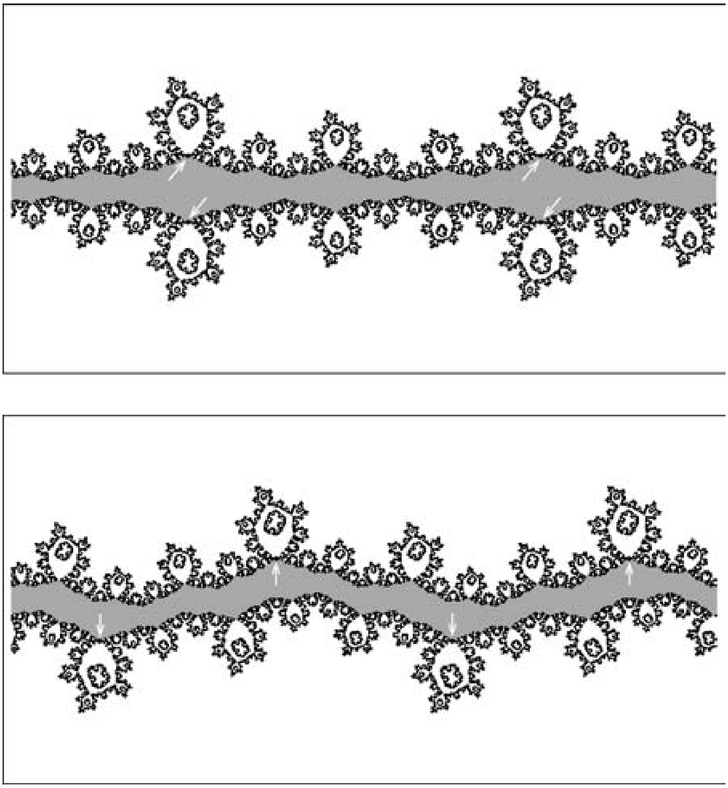

FIGURE 7. *The universal covering of two Herman rings with rotation number* $(\sqrt{5}-1)/2$ *and with twist coordinate* 0 *on the top and* 1/2 *on the bottom.*

4. *Parameterization of Arnold disks*: *proof of Theorem* A

We will work with a slightly different family of rational maps. The purpose of introducing a new family of rational maps is to get rid of the conjugacy between $f_{\lambda,a}$ and $f_{\lambda,-a}$.

DEFINITION 4.1. For $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$, we define

$$g_{\lambda,b} : w \longmapsto \lambda w^2 \frac{1+w}{b+w}.$$

The map $f_{\lambda,a}$ is conjugate to the map g_{λ,a^2} via $z \longmapsto w = az$, that is,

$$f_{\lambda,a}(z) = \frac{1}{a} g_{\lambda,a^2}(az).$$

Now, if g_{λ_1,b_1} is conjugate to g_{λ_2,b_2} by a scaling map, then $(\lambda_1, b_1) = (\lambda_2, b_2)$. Moreover, the maps $g_{\lambda,b}$ are exactly the cubic rational maps which have superattracting fixed points at 0 and ∞ and which map -1 to 0. Observe that $g_{\lambda,0}(z) = \lambda z(1+z)$, a quadratic polynomial, and finally note that if $\lambda \in S^1$ and $b > 0$, then the invariant circle now has radius $\sqrt{b}$.

DEFINITION 4.2. Given a Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\mathcal{D}'_\alpha$ be the set of parameters (λ, b) such that $g_{\lambda,b}$ has a fixed Herman ring with rotation number α.

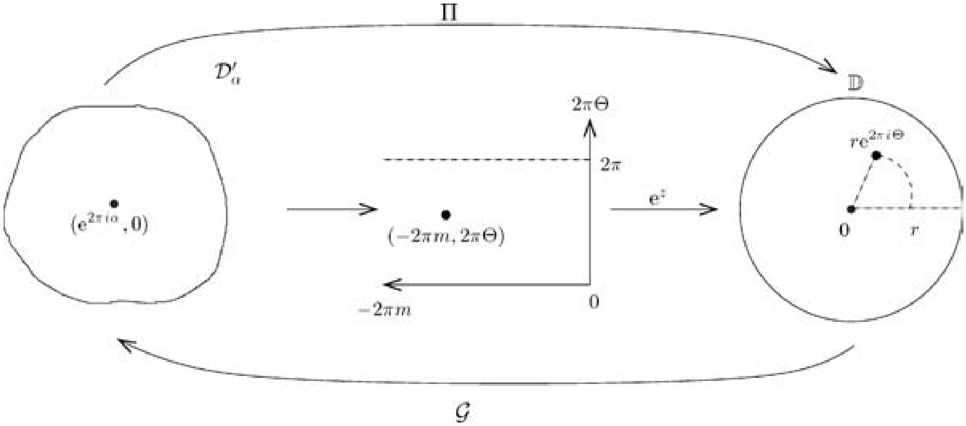

FIGURE 8. *The map* Π *and its inverse* $\mathcal{G}$.

By convention, we consider that for $\lambda = e^{2i\pi\alpha}$ and $b=0$, the quadratic polynomial $g_{\lambda,b} = P_\alpha$ has a Herman ring $\Delta_\alpha \setminus \{0\}$ of infinite modulus. We now state a weak version of Theorem A for the family $g_{\lambda,b}$.

PROPOSITION 4.3. *For any Brjuno number* $\alpha \in \mathbb{R}/\mathbb{Z}$, *there is a holomorphic bijection* $\mathcal{G}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}'_\alpha$ *with* $\mathcal{G}_\alpha(0) = (e^{2i\pi\alpha}, 0)$. *Moreover, for any* $\delta \in \mathbb{D}$, *the modulus of the Herman ring of* $g_{\mathcal{G}_\alpha(\delta)}$ *is equal to* $(1/2\pi)\log(1/|\delta|)$.

4.1. *Proof of Proposition* 4.3

We will prove this result in several steps and later see how it implies Theorem A.

In the previous section, we defined a twist coordinate from $\mathcal{D}_\alpha$ to $\mathbb{R}/\mathbb{Z}$. Similarly, we can define a twist coordinate $\Theta : \mathcal{D}'_\alpha \longrightarrow \mathbb{R}/\mathbb{Z}$. Let us also consider the modulus coordinate $m : \mathcal{D}'_\alpha \longrightarrow (0, +\infty]$ which maps $(\lambda, b) \in \mathcal{D}'_\alpha$ to the modulus of the Herman ring of $g_{\lambda,b}$. We can define a map $\Pi : \mathcal{D}'_\alpha \longrightarrow \mathbb{D}$ by

$$\Pi(\lambda, b) = \exp(-2\pi m(\lambda, b) + 2i\pi\Theta(\lambda, b)).$$

We will show that this map provides an isomorphism between $\mathcal{D}'(\alpha)$ and $\mathbb{D}$. See Figure 8.

For this purpose, given $(\lambda_1, b_1) \in \mathcal{D}'_\alpha$, we will construct an analytic map $\mathcal{G} : \mathbb{D} \longrightarrow \mathbb{C}^* \times \mathbb{C}$ whose image is contained in $\mathcal{D}'_\alpha$ and such that

$$\mathcal{G} \circ \Pi(\lambda_1, b_1) = (\lambda_1, b_1), \quad \mathcal{G}(0) = (e^{2i\pi\alpha}, 0) \quad \text{and} \quad \Pi \circ \mathcal{G} = \mathrm{Id},$$

which shows that Π is injective. We will then show that the map $\mathcal{G}$ does not depend on the choice of (λ_1, b_1), which proves that $\mathcal{G}$ is an isomorphism between $\mathbb{D}$ and $\mathcal{D}'_\alpha$, and Π is its inverse.

Step 1 (*model homeomorphisms*): Given a complex number η in the right half-plane $\mathbb{H}_+ = \{Z \in \mathbb{C} \mid \mathrm{Re}(Z) > 0\}$, let us first consider the $\mathbb{R}$-linear map $L_\eta : \mathbb{C} \longrightarrow \mathbb{C}$, which is the identity on $i\mathbb{R}$ and maps 1 to η:

$$L_\eta(Z) = \tfrac{1}{2}((\eta + 1)Z + (\eta - 1)\bar{Z}),$$

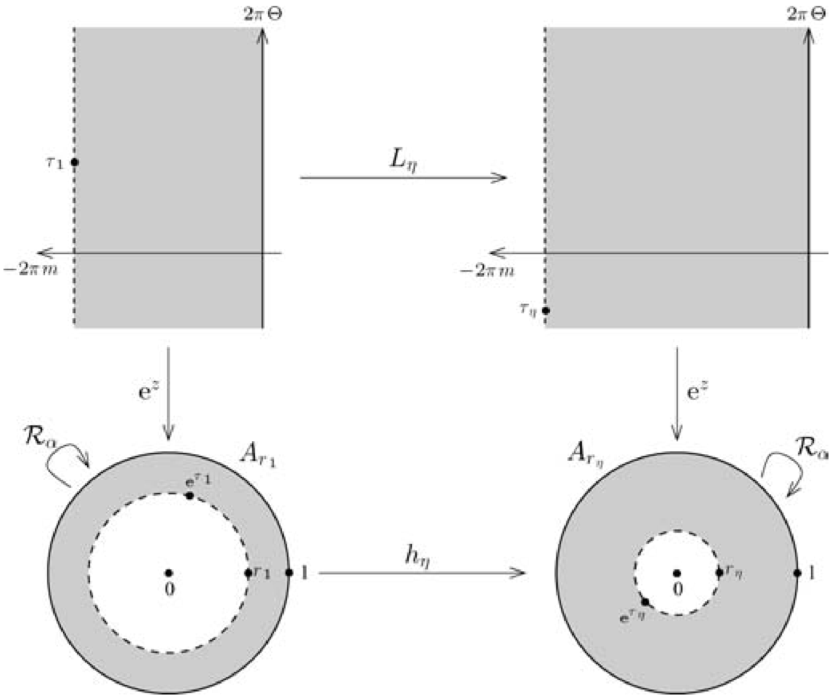

FIGURE 9. *The quasiconformal homeomorphism* L_η.

or, equivalently,

$$L_\eta(x+iy) = x\eta + iy = x(\mathrm{Re}(\eta)) + i(y + x\mathrm{Im}(\eta)).$$

This map is a K-quasiconformal homeomorphism of $\mathbb{C}$ with

$$K = \left|\frac{\eta-1}{\eta+1}\right|.$$

Now, let τ_1 be a complex number in the left half-plane $\mathbb{H}_- = \{Z \in \mathbb{C} \mid \mathrm{Re}(Z) < 0\}$ and set

$$r_1 = |e^{\tau_1}|, \quad \tau_\eta = L_\eta(\tau_1) \quad \text{and} \quad r_\eta = |e^{\tau_\eta}|.$$

Note that L_η is a homeomorphism between the vertical strips

$$\{Z \in \mathbb{C} \mid \mathrm{Re}(\tau_1) < \mathrm{Re}(Z) < 0\} \quad \text{and} \quad \{Z \in \mathbb{C} \mid \mathrm{Re}(\tau_\eta) < \mathrm{Re}(Z) < 0\}$$

which commutes with the vertical translation by $2\pi i$. See Figure 9. Thus it projects to a K-quasiconformal homeomorphism $h_\eta : A_{r_1} \longrightarrow A_{r_\eta}$. Moreover,

$$h_\eta(e^{\tau_1}) = e^{\tau_\eta} \quad \text{and} \quad h_\eta = \mathrm{Id} \text{ on } S^1.$$

Finally, for any angle $\alpha \in \mathbb{R}/\mathbb{Z}$, the homeomorphism $h_\eta : A_{r_1} \longrightarrow A_{r_\eta}$ conjugates the rotation $\mathcal{R}_\alpha : A_{r_1} \longrightarrow A_{r_1}$ to the rotation $\mathcal{R}_\alpha : A_{r_\eta} \longrightarrow A_{r_\eta}$. As a consequence, the rotation $\mathcal{R}_\alpha : A_{r_1} \longrightarrow A_{r_1}$ preserves the complex structure defined by the Beltrami form $\bar\partial h_\eta / \partial h_\eta$.

Step 2 (*a map* $S_{g_1} : \mathbb{H}_+ \longrightarrow \mathcal{D}'_\alpha$): For simplicity, we identify $(\lambda, b) \in \mathcal{D}'_\alpha$ with the rational map $g_{\lambda,b}$. Let us consider a rational map $g_1 := g_{\lambda_1,b_1} \in \mathcal{D}'_\alpha$. Let H_1 be the

$$\begin{array}{ccc}
(A_{r_\eta}, \mu_0) & \xrightarrow{\mathcal{R}_\alpha} & (A_{r_\eta}, \mu_0) \\
{\scriptstyle h_\eta}\big\uparrow & & \big\uparrow{\scriptstyle h_\eta} \\
(A_{r_1}, \frac{\bar\partial h_\eta}{\partial h_\eta}) & \xrightarrow{\mathcal{R}_\alpha} & (A_{r_1}, \frac{\bar\partial h_\eta}{\partial h_\eta}) \\
{\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_1} \\
(H_1, \mu_\eta) & \xrightarrow{g_1} & (H_1, \mu_\eta) \\
{\scriptstyle \chi_\eta}\big\downarrow & & \big\downarrow{\scriptstyle \chi_\eta} \\
(H_\eta, \mu_0) & \xrightarrow{g_\eta} & (H_\eta, \mu_0)
\end{array}$$

FIGURE 10.

Herman ring of g_1 and let (m_1, Θ_1) be the modulus and twist coordinate of H_1. Set $\tau_1 = 2\pi(-m_1 + i\Theta_1)$ and $r_1 = e^{-2\pi m_1}$, and let $\phi_1 : A_{r_1} \longrightarrow H_1$ be an isomorphism which conjugates $\mathcal{R}_\alpha : A_{r_1} \longrightarrow A_{r_1}$ to $g_1 : H_1 \longrightarrow H_1$.

Now, let $\eta \in \mathbb{H}_+$ be an arbitrary complex number and let r_η and $h_\eta : A_{r_1} \longrightarrow A_{r_\eta}$ be as in step 1. Since $\mathcal{R}_\alpha : A_{r_1} \longrightarrow A_{r_1}$ preserves the complex structure defined by the Beltrami form $\bar\partial h_\eta/\partial h_\eta$, and since $\phi_1 : A_{r_1} \longrightarrow H_1$ conjugates $\mathcal{R}_\alpha : A_{r_1} \longrightarrow A_{r_1}$ to $g_1 : H_1 \longrightarrow H_1$, we see that $g_1 : H_1 \longrightarrow H_1$ preserves the complex structure defined by the Beltrami form

$$\mu_\eta = \frac{\bar\partial\big(h_\eta \circ \phi_1^{-1}\big)}{\partial\big(h_\eta \circ \phi_1^{-1}\big)}.$$

REMARK 4.4. Note that μ_η does not depend on the choice of isomorphism $\phi_1 : A_{r_1} \longrightarrow H_1$.

There exists a unique extension of μ_η to $\mathbb{C}$ which is g_1-invariant and is 0 outside the preimages of H_1. This Beltrami form μ_η depends holomorphically on $\eta \in \mathbb{H}_+$.

Next, let $\chi_\eta : \mathbb{C} \longrightarrow \mathbb{C}$ be the unique quasiconformal homeomorphism which fixes 0 and -1 and integrates μ_η:

$$\mu_\eta = \frac{\bar\partial \chi_\eta}{\partial \chi_\eta}.$$

Since g_1 preserves the complex structure defined by μ_η, the map

$$g_\eta = \chi_\eta \circ g_1 \circ \chi_\eta^{-1}$$

is holomorphic. Observe that g_η must be a rational map of degree 3 which has superattracting fixed points at 0 and ∞ and maps -1 to 0. Moreover, by construction, $H_\eta := \chi_\eta(H_1)$ is a Herman ring for g_η with rotation number α. Thus $g_\eta \in \mathcal{D}'_\alpha$ and the above construction defines a map $S_{g_1} : \mathbb{H}_+ \longrightarrow \mathcal{D}'_\alpha$ which maps η to g_η.

The construction above is summarized in the commutative diagram shown in Figure 10.

Finally, μ_η depends holomorphically on η: we have

$$\left.\frac{\partial g_\eta}{\partial \bar\eta}\right|_{\chi_\eta(z)} + \left.\frac{\partial g_\eta}{\partial z}\right|_{\chi_\eta(z)} \cdot \left.\frac{\partial \chi_\eta}{\partial \bar\eta}\right|_z + \left.\frac{\partial g_\eta}{\partial \bar z}\right|_{\chi_\eta(z)} \cdot \left.\frac{\partial \bar\chi_\eta}{\partial \bar\eta}\right|_z = \left.\frac{\partial \chi_\eta}{\partial \bar\eta}\right|_{g_1(z)},$$

and since $\partial\chi_\eta/\partial\bar\eta \equiv 0$, $\partial g_\eta/\partial\bar z \equiv 0$, we get

$$\frac{\partial g_\eta}{\partial\bar\eta}\bigg|_{\chi_\eta(z)} \equiv 0.$$

As a consequence, the map $S_{g_1} : \mathbb{H}_+ \longrightarrow \mathcal{D}'_\alpha$ is analytic.

Step 3 (*modulus and twist of* H_η): Our goal is to compute the modulus and the twist coordinate of the Herman ring H_η of the new map $g_\eta = S_{g_1}(\eta)$, in terms of η, m_1 and Θ_1. We will prove the following.

PROPOSITION 4.5. *Let H_1 be the Herman ring of g_1 and let m_1 and Θ_1 be its modulus and twist coordinate. Given $\eta \in \mathbb{H}_+$, let H_η be the fixed Herman ring of $g_\eta = S_{g_1}(\eta)$. Denote its modulus by m_η and its twist coordinate by Θ_η. Then the following hold.*

(1) $m_\eta = m_1\mathrm{Re}(\eta)$.

(2) $\Theta_\eta = \Theta_1 - m_1\mathrm{Im}(\eta)$.

Proof. For the proof we shall use the notation in step 2.

We start with the modulus. Observe that the map $\phi_\eta := \chi_\eta \circ \phi_1 \circ h_\eta^{-1} : A_{r_\eta} \longrightarrow H_\eta$ is holomorphic by construction and it conjugates $\mathcal{R}_\alpha : A_{r_\eta} \longrightarrow A_{r_\eta}$ to $g_\eta : H_\eta \longrightarrow H_\eta$. Hence it is a linearizing map for the Herman ring, and the modulus of H_η is that of A_{r_η}. That is,

$$m_\eta = \frac{1}{2\pi}\log\frac{1}{r_\eta}.$$

Now, recall that

$$\tau_\eta = L_\eta(-2\pi m_1 + i\Theta_1) = -2\pi m_1\mathrm{Re}(\eta) + 2\pi i(\Theta_1 - m_1\mathrm{Im}(\eta)).$$

Hence $r_\eta = e^{-2\pi m_1 \mathrm{Re}(\eta)}$, and we conclude that $m_\eta = m_1\mathrm{Re}(\eta)$.

We proceed now to compute Θ_η. To this end, we need to create the general construction of Section 3 with the map g_η and its Herman ring H_η. Using the notation in that section, observe that we have the diagram shown in Figure 11 (compare with Figure 6). In Figure 11 we introduce three new composition maps, ψ, $\mathcal{C}$ and $\widehat{\mathcal{C}}$, which are defined so as to make the diagram commute.

We first comment on the bottom row, which is the most important for our argument. Observe that $\mathcal{C}_1$ and $\mathcal{C}_\eta$ are the maps that define the twist coordinates for g_1 and g_η. Indeed, let $\gamma_1, \gamma_\eta, \Gamma_1, \Gamma_\eta$ be as in Section 3. Then $\mathcal{C}_1$ (respectively $\mathcal{C}_\eta$) restricted to the circle $\varphi^{-1}(\Gamma_1)$ (respectively $\varphi^{-1}(\Gamma_\eta)$) is a rigid rotation of angle Θ_1 (respectively Θ_η).

Next observe that both maps $\mathcal{C}, \widehat{\mathcal{C}} : \varphi^{-1}(\Gamma_1) \longrightarrow \varphi^{-1}(\Gamma_\eta)$ are also analytic maps that conjugate the rigid rotation $\mathcal{R}_\alpha$ to itself. Hence they must be rigid rotations composed with scaling maps. That is,

$$\mathcal{C}|_{\varphi^{-1}(\Gamma_1)}(z) = se^{2\pi i\Theta}z$$

and

$$\widehat{\mathcal{C}}|_{\varphi^{-1}(\Gamma_1)}(z) = se^{2\pi i\widehat{\Theta}}z,$$

$$
\begin{tikzcd}
A_{r_\eta} \arrow[d, "\hat{\phi}_\eta"'] & A_{r_\eta} \arrow[l, "r_\eta/\bar{z}"'] \arrow[d, "\phi_\eta"'] & A_1 \arrow[l, "h_\eta"'] \arrow[r, "r_1/\bar{z}"] \arrow[d, "\phi_1"] & A_1 \arrow[d, "\hat{\phi}_1"] \arrow[lll, bend right, "\hat{h}_\eta"'] \\
\widehat{H}_\eta \arrow[d, "\hat{\psi}_\eta"'] & H_\eta \arrow[l, "1/\bar{z}"'] \arrow[d, "\psi_\eta"'] & H_1 \arrow[l, "\chi_\eta"'] \arrow[r, "1/\bar{z}"] \arrow[d, "\psi_1"] & \widehat{H}_1 \arrow[d, "\hat{\psi}_1"] \\
\Delta_\alpha & \Delta_\alpha & \Delta_\alpha \arrow[l, "\chi"] & \Delta_\alpha \\
\mathbb{D} \arrow[u, "\varphi"'] \arrow[r, "\mathcal{C}_\eta"] & \mathbb{D} \arrow[u, "\varphi"'] & \mathbb{D} \arrow[u, "\varphi"'] \arrow[l, "\mathcal{C}"'] & \mathbb{D} \arrow[u, "\varphi"'] \arrow[l, "\mathcal{C}_1"'] \arrow[lll, bend left, "\widehat{\mathcal{C}}"]
\end{tikzcd}
$$

FIGURE 11.

where $s \in \mathbb{R}$ and $\Theta, \widehat{\Theta} \in \mathbb{R}/\mathbb{Z}$. Our goal is then to compute the angles Θ and $\widehat{\Theta}$, since we have

$$\Theta_\eta = -\widehat{\Theta} + \Theta_1 + \Theta. \tag{1}$$

A key observation that we shall use later is the following.

CLAIM 4.6. *The map $\mathcal{C}$ and $\widehat{\mathcal{C}}$ extend as the identity to the unit circle.*

Proof. Observe that the map $\chi := \psi_\eta \circ \chi_\eta \circ \psi_1^{-1}$ is a globally defined quasiconformal map which conjugates the polynomial P_α to itself. It follows that χ can be extended as the identity to the boundary of the Siegel disk (for example, one can argue that χ must fix the orbit of the critical point which accumulates on the boundary of the Siegel disk).

Moreover, the restriction of χ to the annulus bounded by Γ and the boundary of the Siegel disk is quasiconformal. It follows that χ can be extended as the identity to the ideal boundary of the Siegel disk, and hence $\mathcal{C}$ extends as the identity to the unit circle. The same argument applied to $\hat{\chi} := \hat{\psi}_\eta \circ \tau \circ \chi_\eta \circ \tau \circ \hat{\psi}_1^{-1}$ shows the same for $\widehat{\mathcal{C}}$. $\square$

We proceed by computing some rotation angles of the maps in the topmost row. It is easy to check from the expressions of L_η and h_η that

$$h_\eta(re^{2\pi i\Theta}) = \begin{cases} e^{2\pi i\Theta} & r = 1 \\ \sqrt{r_\eta}\, e^{2\pi i\Theta - \pi i m_1 \mathrm{Im}(\eta)} & r = \sqrt{r_1} \\ r_\eta\, e^{2\pi i\Theta - 2\pi i m_1 \mathrm{Im}(\eta)} & r = r_1 \end{cases}$$

and

$$\hat{h}_\eta(re^{2\pi i\Theta}) = \begin{cases} e^{2\pi i\Theta - 2\pi i m_1 \mathrm{Im}(\eta)} & r = 1 \\ \sqrt{r_\eta}\, e^{2\pi i\Theta - \pi i m_1 \mathrm{Im}(\eta)} & r = \sqrt{r_1} \\ r_\eta\, e^{2\pi i\Theta} & r = r_1. \end{cases}$$

$$\begin{tikzcd}
A_{\sqrt{r_\eta}} \arrow[d, "\widehat{R}_\eta"'] & A_{\sqrt{r_\eta}} \arrow[l, "r_\eta/\bar{z}"'] \arrow[d, "R_\eta"'] & A_{\sqrt{r_1}} \arrow[l, "h_\eta"'] \arrow[r, "r_1/\bar{z}"] \arrow[d, "R_1"] & A_{\sqrt{r_1}} \arrow[d, "\widehat{R}_1"] \arrow[lll, bend right, "\hat{h}_\eta"'] \\
A_{\sqrt{r_\eta}} \arrow[r, "\mathcal{C}_\eta"] & A_{\sqrt{r_\eta}} & A_{\sqrt{r_1}} \arrow[l, "\mathcal{C}"'] & A_{\sqrt{r_1}} \arrow[l, "\mathcal{C}_1"'] \arrow[lll, bend left, "\widehat{\mathcal{C}}"]
\end{tikzcd}$$

FIGURE 12.

To transfer this information to the bottom maps, we need to consider the vertical compositions. Let us then consider all the maps of the diagram restricted to the annuli bounded by the equator curves and the outer boundaries. Then all the 'vertical' compositions are conformal. It follows that the circle $\varphi^{-1}(\Gamma_1)$ (respectively $\varphi^{-1}(\Gamma_\eta)$) has radius $\sqrt{r_1}$ (respectively $\sqrt{r_\eta}$), and hence we have the diagram shown in Figure 12, where R_1, R_η, $\widehat{R}_1$ and $\widehat{R}_\eta$ are rigid rotations.

We first analyze the innermost diagram to observe that, given that h_η commutes with rigid rotations, the map $\mathcal{C}$ is equal to h_η up to rotation. However, we observed in Claim 4.6 that $\mathcal{C}$ extends as the identity to the unit circle as h_η does. It follows that $\mathcal{C} = h_\eta$ on $A_{\sqrt{r_1}}$, and hence

$$\Theta = -\tfrac{1}{2} m_1 \mathrm{Im}(\eta).$$

Studying the outermost diagram and arguing as above, we have $\widehat{\mathcal{C}}$ is equal to $\hat{h}_\eta$ up to composition with a rigid rotation. However, again $\widehat{\mathcal{C}}$ is the identity on S^1. It then follows that on $A_{\sqrt{r_1}}$,

$$\widehat{\mathcal{C}} = e^{2\pi i m_1 \mathrm{Im}(\eta)} \cdot \hat{h}_\eta,$$

and hence, by knowing $\hat{h}_\eta$ on the equator, we conclude that

$$\widehat{\Theta} = \tfrac{1}{2} m_1 \mathrm{Im}(\eta).$$

Finally, from equation (1), we obtain

$$\Theta_\eta = -\tfrac{1}{2} m_1 \mathrm{Im}(\eta) + \Theta_1 - \tfrac{1}{2} m_1 \mathrm{Im}(\eta) = \Theta_1 - m_1 \mathrm{Im}(\eta). \qquad \square$$

Step 4 (*a group property*): Let $S : \mathcal{D}'_\alpha \times \mathbb{H}_+ \longrightarrow \mathcal{D}'_\alpha$ be the map $S : (g, \eta) \longmapsto S_g(\eta)$. Assume that η_1 and η_2 are two complex numbers in $\mathbb{H}_+$, and $g_1 \in \mathcal{D}'_\alpha$. Our goal is to prove the following.

LEMMA 4.7.

$$S(S(g_1, \eta_1), \eta_2) = S(g_1, L_{\eta_2}(\eta_1)).$$

Proof. Set

$$g_2 = S(g_1, \eta_1) \quad \text{and} \quad g_3 = S(g_2, \eta_2).$$

Then we want to show that

$$g_3 = S(g_1, \eta_3) \quad \text{with} \quad \eta_3 = L_{\eta_2} \circ L_{\eta_1}(1).$$

Indeed, for $i=1,\ 2$ or 3, let H_i be the Herman ring of g_i and let m_i be the modulus of H_i, and set $r_i = e^{-2\pi m_i}$. Observe that $L_{\eta_3} = L_{\eta_2} \circ L_{\eta_1}$. These homeomorphisms project to homeomorphisms $h_1 : A_{r_1} \longrightarrow A_{r_2}$, $h_2 : A_{r_2} \longrightarrow A_{r_3}$ and $h_3 : A_{r_1} \longrightarrow A_{r_3}$ such that $h_3 = h_2 \circ h_1$.

Now, let μ_1 be the Beltrami form which coincides with 0 outside $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$ with

$$\frac{\bar\partial\big(h_1 \circ \phi_1^{-1}\big)}{\partial\big(h_1 \circ \phi_1^{-1}\big)}$$

on H_1, and which satisfies $g_1^*(\mu_1) = \mu_1$ everywhere. Let $\chi_1 : \mathbb{C} \longrightarrow \mathbb{C}$ be the integrating map that fixes 0 and -1. Observe that the map

$$\phi_2 = \chi_1 \circ \phi_1 \circ h_1^{-1} : A_{r_2} \longrightarrow H_2$$

is a homeomorphism that preserves the standard complex structure, and is thus an isomorphism (and a linearizing map). Therefore we can define μ_2 on H_2 and, $\chi_2 : \mathbb{C} \longrightarrow \mathbb{C}$ and, as above,

$$\phi_3 = \chi_2 \circ \phi_2 \circ h_2^{-1} : A_{r_3} \longrightarrow H_3$$

is a linearizing map. Now we have the following commutative diagram.

$$\begin{array}{ccccc} A_{r_1} & \xrightarrow{h_1} & A_{r_2} & \xrightarrow{h_2} & A_{r_3} \\ {\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle \phi_3} \\ H_1 & \xrightarrow{\chi_1} & H_2 & \xrightarrow{\chi_2} & H_3 \end{array}$$

Let us define $\chi_3 = \chi_2 \circ \chi_1$. On the other hand, let $\tilde\chi_3$, $\tilde\mu_3$ be the map and complex structure used in the construction of $\tilde g_3 := S(g_1, \eta_3)$. We will show that the complex structure μ_3 induced by ϕ_3 equals $\tilde\mu_3$. Since χ_3 and $\tilde\chi_3$ are normalized in the same way, these two integrating maps must be the same. Hence we can conclude that $g_3 = \tilde g_3$.

To this end, observe that the maps ϕ_i are holomorphic and $h_3 = h_2 \circ h_1$. Therefore, on H_1, we have

$$\mu_3 = \frac{\bar\partial \chi_3}{\partial \chi_3} = \frac{\bar\partial(h_3 \circ \phi_1^{-1})}{\partial(h_3 \circ \phi_1^{-1})} = \tilde\mu_3.$$

Moreover, χ_3 conjugates g_1 to the holomorphic map g_3, and so the Beltrami form μ_3 is g_1-invariant. Therefore it follows that μ_3 and $\tilde\mu_3$ coincide on the set $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$. Finally, since χ_1 is holomorphic outside $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$ and χ_2 is holomorphic outside $\bigcup_{n\geqslant 0} g_2^{-n}(H_2) = \chi_1\big(\bigcup_{n\geqslant 0} g_1^{-n}(H_1)\big)$, we see that χ_3 is holomorphic outside $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$. Therefore $\mu_3 = \tilde\mu_3 = 0$ outside $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$, and we conclude that $\mu_3 = \tilde\mu_3$ everywhere. □

Step 5 ($S_{g_1}(\eta + i/m) = S_{g_1}(\eta)$): Again, assume that $g_1 \in \mathcal{D}'_\alpha$, let H_1 be the Herman ring, let m_1 be the modulus of H_1, and set $r_1 = e^{-2\pi m_1}$. Let $\phi_1 : A_{r_1} \longrightarrow H_1$ be an isomorphism, and let the map $S_{g_1} : \mathbb{H}_+ \longrightarrow \mathcal{D}'_\alpha$ be as in step 2. We will show that

$$(\forall\ \eta \in \mathbb{H}_+) \quad S_{g_1}(\eta + i/m) = S_{g_1}(\eta).$$

Thanks to the previous step, we can assume, without loss of generality, that $\eta = 1$. Indeed, if we know that $S_h(1 + i/m) = S_h(1)$ for all $h \in \mathcal{D}'_\alpha$, then, taking $h = S_{g_1}(\eta)$, the general equality follows.

We will construct a quasiconformal homeomorphism $\tilde{\chi}$ that satisfies

$$g_1 = \tilde{\chi} \circ g_1 \circ \tilde{\chi}^{-1}.$$

We will then show that $\tilde{\chi}$ integrates $\mu_{1+i/m}$ and fixes 0, ∞ and -1. Therefore $\chi = \tilde{\chi}$ and

$$S_{g_1}(1) = g_1 = \tilde{\chi} \circ g_1 \circ \tilde{\chi}^{-1} = \chi \circ g_1 \circ \chi^{-1} = S_{g_1}(1+i/m).$$

The quasiconformal homeomorphism $L_{1+i/m}$ preserves vertical strips and, in particular, it preserves

$$B = \{Z \in \mathbb{C} \mid \log r_1 < \mathrm{Re}(Z) < 0\}.$$

It coincides with the identity on $i\mathbb{R}$ and with the translation by $-2i\pi$ on $\log r_1 + i\mathbb{R}$. Therefore the quasiconformal homeomorphism $h_{1+i/m} : A_{r_1} \longrightarrow A_{r_1}$ extends by the identity to the two boundary components of A_{r_1} (it is a Dehn twist).

Let us define $\tilde{\chi} = \phi_1 \circ h_{1+i/m} \circ \phi_1^{-1}$ on the Herman ring. Note that $\bar{\partial}\tilde{\chi}/\partial\tilde{\chi} = \mu_{1+i/m}$ on H_1 (since it preserves the standard complex structure), and $\tilde{\chi}$ is equal to the identity on the two ideal boundary components of H_1. In particular, $\tilde{\chi}$ extends by the identity to the two boundary components of H_1 in $\mathbb{C}$ (see [**8**]). Next, given a connected component U of $\bigcup_{n\geqslant 0} g_1^{-n}(H_1)$, let $n \geqslant 0$ be the smallest integer such that $g_1^n(U) = H_1$. The map $g_1^n : U \longrightarrow H_1$ is an isomorphism, and we set

$$\tilde{\chi}|_U = \left(g_1^{\circ n}|_U\right)^{-1} \circ \tilde{\chi} \circ g_1^n.$$

Note that $\tilde{\chi}|_U$ extends by the identity to ∂U. We can extend $\tilde{\chi}$ to a homeomorphism $\tilde{\chi} : \mathbb{C} \longrightarrow \mathbb{C}$ such that the restriction of $\tilde{\chi}$ to $\mathbb{C} \setminus \bigcup_{n\geqslant 0} g_1^{-n}(H_1)$ is the identity. By a lemma of Rickman, $\tilde{\chi} : \mathbb{C} \longrightarrow \mathbb{C}$ is quasiconformal. By construction,

$$\tilde{\chi} \circ g_1 = g_1 \circ \tilde{\chi} \quad \text{and} \quad \frac{\bar{\partial}\tilde{\chi}}{\partial\tilde{\chi}} = \mu_{1+i/m}.$$

Finally, observe that -1, 0 and ∞ are not eventually mapped to H_1. Hence $\tilde{\chi}$ fixes these three points.

Step 6 (*definition of the map* $G_{g_1} : \mathbb{D} \longrightarrow \mathcal{D}'_\alpha$): Let us still denote by m_1 the modulus of the Herman ring H_1 and by Θ_1 the twist coordinate of g_1. As above, set $r_1 = e^{-2\pi m_1}$. We can define $p : \mathbb{H}_+ \longrightarrow \mathbb{D}^*$ as the universal covering

$$p : \eta \longmapsto \delta = \exp(-2\pi m_1 \eta + 2i\pi\Theta_1) = \exp(L_\eta(\tau_1)).$$

Since $S_{g_1}(\eta + i/m) = S_{g_1}(\eta)$, there exists a holomorphic map $G_{g_1} : \mathbb{D}^* \longrightarrow \mathcal{D}'_\alpha$ such that $S_{g_1} = G_{g_1} \circ p$. That is, the following diagram commutes.

$$\begin{tikzcd} \mathbb{H}_+ \arrow[r, "S_{g_1}"] \arrow[d, "p"'] & \mathcal{D}'_\alpha \\ \mathbb{D}^* \arrow[ur, "G_{g_1}"'] & \end{tikzcd}$$

Claim 4.8. *$G_{g_1}(\Pi(g_1)) = g_1$ and $\Pi \circ G_{g_1} = \mathrm{Id}$.*

Proof. Just from the diagram we see that

$$G_{g_1}(r_1 e^{2i\pi\Theta_1}) = G_{g_1} \circ p(1) = S_{g_1}(1) = g_1.$$

Recall that $\Pi(g_1) = r_1 e^{2\pi i \Theta_1}$, and therefore we have shown that $G_{g_1}(\Pi(g_1)) = g_1$. We now want to show that $\Pi \circ G_{g_1} = \mathrm{Id}$.

In order to do this, let us first determine the 'inverse' of p. It is easy to check that, for $e^{-2\pi i m} = r < 1$ and $\Theta \in \mathbb{R}/\mathbb{Z}$, we have

$$p\left(\frac{m}{m_1} + i\frac{\Theta_1 - \Theta}{m_1}\right) = re^{2\pi i\Theta}.$$

Then

$$\Pi \circ G_{g_1}(re^{2\pi i\Theta}) = \Pi \circ S_{g_1}\left(\frac{m}{m_1} + i\frac{\Theta_1 - \Theta}{m_1}\right) = \Pi(g),$$

where $g \in \mathcal{D}'_\alpha$ has a Herman ring H of modulus $m_1(m/m_1) = m$ and twist $\Theta_1 - m_1(\Theta_1 - \Theta)/m_1 = \Theta$ (see Proposition 4.5). It then follows that $\Pi(g) = re^{2\pi i\Theta}$ and hence $\Pi \circ G_{g_1} = \mathrm{Id}$. □

Claim 4.9. *G_{g_1} extends to 0 by $G_{g_1}(0) = (e^{2i\pi\alpha}, 0)$.*

Proof. By Proposition 2.2, we know that

$$\mathcal{D}'_\alpha \subset \{(\lambda, b) \in \mathbb{C}^* \times \mathbb{C} \mid 1/4 \leqslant |\lambda| \leqslant 4 \text{ and } |b| \leqslant 1\}.$$

Thus, by the removable singularity theorem, $G_{g_1} : \mathbb{D}^* \longrightarrow \mathcal{D}'_\alpha$ extends holomorphically at 0.

We know that the modulus of the Herman ring H_δ of the map $G_{g_1}(\delta)$ is $(1/2\pi)\log(1/|\delta|)$ for $\delta \in \mathbb{D}^*$. Thus the modulus of the Herman ring tends to $+\infty$, when δ tends to the origin. The Herman ring H_δ separates -1, which is mapped to 0, and $-b$, which is mapped to ∞. Therefore we have $b \to 0$ as $\delta \to 0$.

The map $\delta \longmapsto b(\delta)$ is holomorphic, is non-constant, and vanishes at $\delta = 0$. Therefore, there exists $\varepsilon > 0$ such that $b_1(\mathbb{D})$ contains the interval $[0, \varepsilon]$. When $b \in (0, 1/9)$, there is at most one parameter λ such that $(\lambda, b) \in \mathcal{D}'_\alpha$ (see Proposition 2.5). In addition, $\lambda = e^{2i\pi t} \in S^1$ and $(t, \sqrt{b})$ belongs to the Arnold tongue $\mathcal{T}_\alpha$. When $b \in \mathcal{T}_\alpha$ tends to 0, t tends to α and λ tends to $e^{2i\pi\alpha}$, and so $G_{g_1}(0) = (e^{2i\pi\alpha}, 0)$. □

Step 7 (the map G_{g_1} is surjective): In order to complete the proof of Proposition 4.3, it is now enough to show that G_{g_1} is surjective. Thus we must prove that for all $g_1, g_2 \in \mathcal{D}'_\alpha$, there exists $\delta \in \mathbb{D}^*$ such that $g_2 = G_{g_1}(\delta)$. Thanks to step 3, it is enough to prove that there exist δ_1 and δ_2 such that $G_{g_1}(\delta_1) = G_{g_2}(\delta_2)$.

Again, if $G_{g_1}(\delta) = (\lambda_1(\delta), b_1(\delta))$, the map $\delta \longrightarrow b_1(\delta)$ is holomorphic (see step 2) and vanishes at $\delta = 0$. Therefore, there exists $\varepsilon \in (0, 1/9)$ such that $b_1(\mathbb{D})$ contains the interval $[0, \varepsilon]$. Similarly, if $\varepsilon \in (0, 1/9)$ is small enough and $G_{g_2}(\delta) = (\lambda_2(\delta), b_2(\delta))$, then $b_2(\mathbb{D})$ contains the interval $[0, \varepsilon]$. Therefore, if we pick $b \in (0, \varepsilon]$, there exist $\delta_1, \delta_2 \in \mathbb{D}^*$ such that $b_1(\delta_1) = b_2(\delta_2) = b$. Now, since $b \in (0, 1/9)$, there is at most one parameter λ such that $(\lambda, b) \in \mathcal{D}'_\alpha$ (see Proposition 2.5). Moreover $\lambda = e^{2i\pi t} \in S^1$ and $(t, \sqrt{b})$ belongs to the Arnold tongue $\mathcal{T}_\alpha$. Therefore $\lambda_1(\delta_1) = \lambda_2(\delta_2)$. Thus $G_{g_1}(\delta_1) = G_{g_2}(\delta_2)$, and the proof is completed. □

We can then take $\mathcal{G}_\alpha = G_{g_1}$, and this concludes the proof of Proposition 4.3.

Corollary 4.10. *The map $\mathcal{G}_\alpha$ induces a continuous injection from $[0, 1)$ to $\mathcal{T}_\alpha$. In particular, the only maps in $\mathcal{D}'_\alpha$ with twist parameter 0 are those corresponding to the Arnold tongue $\mathcal{T}_\alpha$.*

4.2. *Conclusion of the proof of Theorem* A

Observe that the map $(\lambda, a) \longmapsto (\lambda, a^2)$ provides a ramified covering of degree 2 from $\mathcal{D}_\alpha$ to $\mathcal{D}'_\alpha$, ramified at $(e^{2i\pi\alpha}, 0)$ and above $(e^{2i\pi\alpha}, 0)$. Thus the map $\mathcal{G}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}'_\alpha$ lifts to a map $\mathcal{F}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\mathcal{F}_\alpha} & \mathcal{D}_\alpha \\ {\scriptstyle \delta \longmapsto \delta^2}\Big\downarrow & & \Big\downarrow{\scriptstyle (\lambda,a) \longmapsto (\lambda,a^2)} \\ \mathbb{D} & \xrightarrow[\mathcal{G}_\alpha]{} & \mathcal{D}'_\alpha \end{array} \tag{2}$$

The map $\mathcal{F}_\alpha$ is a holomorphic bijection of $\mathbb{D}$ onto $\mathcal{D}_\alpha$, it maps 0 to $(e^{2i\pi\alpha}, 0)$, and the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is

$$\frac{1}{2\pi} \log \frac{1}{|\delta|^2} = \frac{1}{\pi} \log \frac{1}{|\delta|}.$$

Let us set $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$. Since $\lambda(\delta) = \lambda(-\delta)$, we have

$$\frac{\partial \lambda}{\partial \delta}(0) = 0.$$

Moreover we claim that

$$\frac{\partial a}{\partial \delta}(0) = r_\alpha.$$

This is a consequence of the following proposition. The analogue for the complex standard family has been proved by Fagella, Seara and Villanueva in [**10**].

Proposition 4.11. *Let us fix a Brjuno number α, and for $a \in (0, 1/3)$ small enough, let $t_a \in \mathbb{R}/\mathbb{Z}$ be the unique angle such that (t_a, a) belongs to the Arnold tongue $\mathcal{T}_\alpha$. Then, as $a \to 0$, the modulus m_a of the Herman ring of $f_{e^{2i\pi t_a}, a}$ satisfies*

$$m_a = \frac{1}{\pi} \log \frac{r_\alpha}{a} + o(1).$$

Proof. Let us again work with the family $g_{\lambda,b}$ with $b = a^2$. In the whole proof, we consider that (λ, b) varies in the Arnold disk $\mathcal{D}'_\alpha$. More precisely, we assume that

$$(\lambda, b) = (\lambda(\delta), b(\delta)) = \mathcal{G}_\alpha(\delta) \quad \text{with } \delta \in \mathbb{D}.$$

Observe that when $\delta \to 0$, the maps $g_{\lambda,b}$ converge to P_α uniformly on every compact subset of $\mathbb{C}^*$. In particular, one of the critical points of $g_{\lambda,b}$, say $\omega(\delta)$, converges to the critical point $\omega(0) = -1/2$ of P_α.

The maps

$$\delta \longmapsto g^n_{\lambda(\delta), b(\delta)}(\omega(\delta))$$

are all well defined and holomorphic for $\delta \in \mathbb{D}$. Moreover they define a holomorphic motion of the orbit of the critical point $\omega(\delta)$, parameterized by $\delta \in \mathbb{D}$. By the Mañe–Sad–Sullivan λ-lemma [**13**], this holomorphic motion extends to a holomorphic motion of the closure of this critical orbit.

Let $B(\delta)$ be the outer boundary component of the Herman ring of $g_{\lambda,b}$ for $\delta \neq 0$, and the boundary of the Siegel disk for $\delta = 0$. Moreover, let $\Delta(\delta)$ be the bounded component of $\mathbb{C} \setminus B(\delta)$. Since $B(\delta)$ is contained in the closure of the orbit of $\omega(\delta)$, we see that $B(\delta)$ moves holomorphically with respect to $\delta \in \mathbb{D}$. As a consequence,

$(\Delta(\delta), 0)$ depends continuously on $\delta \in \mathbb{D}$ for the Carathéodory topology. Thus its conformal radius at 0 varies continuously with respect to $\delta \in \mathbb{D}$.

Let us now assume that $\delta \in (0,1)$, and thus that $b(\delta) > 0$ (see Corollary 4.10). Then the Herman ring is symmetric with respect to the circle centered at 0 with radius $\sqrt{b}$. It follows that the modulus $m(\delta)$ of the Herman ring is equal to twice the modulus of the annulus bounded by this circle and $B(\delta)$. As $\delta \to 0$, we have $b \to 0$, and the conformal radius of $\Delta(\delta)$ tends to r_α. It follows from Lemma 4.12 below that

$$m(\delta) = 2 \cdot \frac{1}{2\pi} \log \frac{r_\alpha}{\sqrt{b}} + o(1).$$

Using the relation $b = a^2$ and $m_a = m(\delta)$, we get

$$m_a = \frac{1}{\pi} \log \frac{r_\alpha}{a} + o(1). \qquad \square$$

LEMMA 4.12. *There exists a function* $h : [0, 1/4) \longrightarrow \mathbb{R}$ *with* $h(x) = \mathcal{O}(x)$ *such that, for any topological disk* $D \subset \mathbb{C}$ *containing* 0 *and any* $\varepsilon \in (0, \mathrm{rad}(D)/4)$, *the set* $U_\varepsilon = D \setminus \overline{\mathbb{D}}_\varepsilon$ *is an annulus of modulus* m_ε *with*

$$h\left(\frac{\varepsilon}{\mathrm{rad}(D)}\right) \leqslant m_\varepsilon - \frac{1}{2\pi} \log\left(\frac{\mathrm{rad}(D)}{\varepsilon}\right) \leqslant 0.$$

Proof. Rescaling if necessary, we may suppose that $\mathrm{rad}(D) = 1$. Note that when $\varepsilon < 1/4$, it follows from Koebe's quarter-theorem that U_ε is an annulus. Let $\phi : (\mathbb{D}, 0) \longrightarrow (D, 0)$ be a conformal representation. By Koebe's distortion theorem, for all $z \in \mathbb{D}$,

$$\frac{|z|}{(1+|z|)^2} \leqslant |\phi(z)| \leqslant \frac{|z|}{(1-|z|)^2}.$$

Therefore, $\gamma_\varepsilon = \phi^{-1}(C(0, \varepsilon))$ is contained in the annulus $\{r_\varepsilon < |z| < R_\varepsilon\}$, with $r_\varepsilon, R_\varepsilon \in (0,1)$ defined by

$$\varepsilon = \frac{r_\varepsilon}{(1-r_\varepsilon)^2} = \frac{R_\varepsilon}{(1+R_\varepsilon)^2}.$$

The annulus bounded by γ_ε and S^1 is isomorphic to U_ε, and so

$$\frac{1}{2\pi} \log \frac{1}{R_\varepsilon} \leqslant m \leqslant \frac{1}{2\pi} \log \frac{1}{r_\varepsilon}.$$

Therefore we can take

$$h(\varepsilon) = \frac{1}{2\pi} \log \frac{\varepsilon}{R_\varepsilon} = \mathcal{O}(\varepsilon).$$

We also see that

$$m \leqslant \frac{1}{2\pi} \log \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon).$$

This last inequality can be improved as follows. Choose $\varepsilon' < \varepsilon$ small. The circle of radius ε subdivides the annulus $U_{\varepsilon'}$ into the two annuli U_ε and $\{z \mid \varepsilon' < |z| < \varepsilon\}$. By the Grötzsch inequality, we have

$$\frac{1}{2\pi} \log \frac{\varepsilon}{\varepsilon'} + m_\varepsilon \leqslant m_{\varepsilon'} = \frac{1}{2\pi} \log \frac{1}{\varepsilon'} + \mathcal{O}(\varepsilon').$$

We get the required inequality by letting ε' tend towards 0. $\square$

We now want to conclude from Proposition 4.11 the fact that $\partial a/\partial\delta(0) = r_\alpha$. Remember that for $\delta \in \mathbb{D}$ and $(\lambda, a) = \mathcal{F}_\alpha(\delta)$, we have

$$m_a = \frac{1}{\pi}\log\frac{1}{|\delta|}.$$

Thus when $\delta \in (0,1)$, we have

$$\frac{1}{\pi}\log\frac{1}{\delta} = \frac{1}{\pi}\log\frac{r_\alpha}{a} + o(1) = \frac{1}{\pi}\log\left(\frac{r_\alpha}{a}(1+o(1))\right).$$

Therefore, as $\delta \to 0$, we get

$$\frac{1}{\delta} = \frac{r_\alpha}{a}(1+o(1)).$$

In particular, we see that $a = \delta r_\alpha + \delta o(1) = \delta r_\alpha + o(\delta)$, which proves that

$$\frac{\partial a}{\partial \delta}(0) = r_\alpha$$

and completes the proof of Theorem A.

5. *Proof of Theorem* B

Part (a) follows from $\partial a/\partial\delta(0) = r_\alpha \neq 0$.

Part (b) is obtained as follows. Since $\delta \longmapsto a(\delta)$ is holomorphic at 0, we know that

$$a(\delta) = r_\alpha\delta + \mathcal{O}(\delta^2).$$

Moreover, by following the commutative diagram (2), we have $\mathcal{F}_\alpha(-\delta) = (\lambda(\delta), -a(\delta))$ and therefore $a(\delta)$ is an odd function and

$$a(\delta) = r_\alpha\delta + \mathcal{O}(\delta^3).$$

By the inverse function theorem, it follows that

$$\delta(a) = \frac{a}{r_\alpha} + \mathcal{O}(a^3).$$

In particular, as $a \to 0$, we have

$$e^{\pi m_a} = \frac{1}{|\delta(a)|} = \frac{r_\alpha}{|a|} + \mathcal{O}(a).$$

6. *The complex standard family*

We now explain how to adapt the results to the complex standard family

$$f_{\lambda,a}(z) = \lambda z e^{(a/2)(z-1/z)}.$$

In this section, we will no longer consider cubic rational maps and quadratic polynomials. We will therefore feel free to use the same notations as in the rest of the paper to denote similar but different maps and objects.

The complex standard family has been extensively studied in the case $\lambda = e^{2i\pi t} \in S^1$ and $a \in \mathbb{R}$: it lifts via $Z \longmapsto z = e^{2i\pi Z}$ to the Arnold family

$$F_{\alpha,a}(Z) = Z + t + \frac{a}{2\pi}\sin(2\pi Z).$$

For $t \in \mathbb{R}$ and $a \in (-1,1)$, these maps restrict to analytic diffeomorphisms of $\mathbb{R}/\mathbb{Z}$.

Let us now assume that λ and a are complex parameters, and that $f_{\lambda,a}$ has a fixed Herman ring $H \subset \mathbb{C}^*$ with rotation number α. By the maximum modulus principle, H separates the essential singularities 0 and ∞. Following [**11**], we can perform Shishikura's surgery in order to obtain an entire mapping fixing 0 and having a Siegel disk with rotation number α around 0. By construction, this map does not vanish, except at 0, and it may be normalized to have a critical point at -1. In fact, Geyer has shown that the resulting map is given by

$$E_\alpha(z) = e^{2i\pi\alpha} z e^z,$$

and, moreover, that E_α has a Siegel disk around 0 if and only if α is a Brjuno number. Therefore a map $f_{\lambda,a}$ may have a fixed Herman ring only when α is a Brjuno number.

Definition 6.1. Given a Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, we let Δ_α be the Siegel disk of the entire mapping E_α, and we let $\mathcal{D}_\alpha$ be the set of parameters $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$ such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α. We call $\mathcal{D}_\alpha$ the *Arnold disk of rotation number* α.

As in the case of cubic rational maps, to any map $f \in \mathcal{D}_\alpha$ we can associate two coordinates: the modulus of the Herman ring and a twist coordinate defined via the surgery construction (see Section 3).

Also, the change of coordinates $z \longmapsto -z$ conjugates $f_{\lambda,a}$ to $f_{\lambda,-a}$, and it is thus useful to introduce a new family

$$g_{\lambda,b} : w \longmapsto \lambda w e^{w-b/(4w)}.$$

The map $f_{\lambda,a}$ is conjugate to the map g_{λ,a^2} via $z \longmapsto w = az/2$, that is,

$$f_{\lambda,a}(z) = \frac{2}{a} g_{\lambda,a^2}\left(\frac{a}{2}z\right).$$

Note that $g_{\lambda,0}$ is the entire mapping $w \longmapsto \lambda w e^w$.

Definition 6.2. Given a Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\mathcal{D}'_\alpha$ be the set of parameters (λ, b) such that $g_{\lambda,b}$ has a fixed Herman ring with rotation number α.

By convention, we consider that for $\lambda = e^{2i\pi\alpha}$ and $b = 0$, the entire mapping $g_{\lambda,b} = E_\alpha$ has a Herman ring $\Delta_\alpha \setminus \{0\}$ of infinite modulus. As in the case of the cubic rational maps studied in this paper, one can show the following proposition.

Proposition 6.3. *For any Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, there is a holomorphic bijection $\mathcal{G}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}'_\alpha$ with $\mathcal{G}_\alpha(0) = (e^{2i\pi\alpha}, 0)$. Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $g_{\mathcal{G}_\alpha(\delta)}$ is equal to $(1/2\pi)\log(1/|\delta|)$.*

The only difference in the proof appears when one shows that a certain map $G : \mathbb{D}^* \longrightarrow \mathcal{D}'_\alpha$ has a removable singularity at 0 (this corresponds to Claim 4.9). Indeed, we do not know whether the Arnold disks $\mathcal{D}'_\alpha$ are compactly contained in $\mathbb{C}^* \times \mathbb{C}$.

Remark 6.4. This raises the following questions. Are the Siegel disks Δ_α bounded in $\mathbb{C}$? Are the Arnold disks $\mathcal{D}_\alpha$ and $\mathcal{D}'_\alpha$ bounded in $\mathbb{C}^* \times \mathbb{C}$?

In order to overcome this difficulty, we set $G(\delta) = (\lambda(\delta), b(\delta))$ for $\delta \in \mathbb{D}^*$ and argue as follows. First, as in the case of the cubic rational maps, the modulus of the Herman ring tends to ∞ as $\delta \to 0$. The Herman ring separates the critical points which are the roots of the equation $4z^2 + 4z + b(\delta) = 0$. It follows that as $\delta \to 0$, we have $b(\delta) \to 0$ (one of the critical points tends to 0 and the other tends to -1). To show that G has a removable singularity at $\delta = 0$, it is sufficient to show that when $\delta \to 0$, $\lambda(\delta)$ remains uniformly bounded away from 0 and ∞. This is a consequence of the following lemma and the fact that $b(\delta) \underset{\delta \to 0}{\longrightarrow} 0$.

LEMMA 6.5. *Suppose that $|a| < 1$ and $(\lambda, a) \in \mathcal{D}_\alpha$. Then*

$$e^{-|a|} \leqslant |\lambda| \leqslant e^{|a|}.$$

Proof. Suppose that $(\lambda, a) \in \mathcal{D}_\alpha$, $|a| < 1$ and $|\lambda| < e^{-|a|}$. Then the image $\Gamma = f_{\lambda,a}(S^1)$ of the unit circle is an $\mathbb{R}$-analytic Jordan curve in $\mathbb{D}$, and $f_{\lambda,a} : S^1 \longrightarrow \Gamma$ is a diffeomorphism (this is easily seen by working in the lifted $F_{\omega,a}(Z) = Z + \omega + (a/2\pi)\sin(2\pi Z)$ with $\mathrm{Im}(\omega) > 0$ and $\lambda = e^{2i\pi\omega}$). We then get a contradiction as in the proof of Proposition 2.5. Similarly, we cannot have $|\lambda| > e^{|a|}$. $\square$

With the help of Proposition 6.3, we can now proceed exactly as in the case of cubic rational maps, and we obtain the following results.

THEOREM A′. *For any Brjuno number $\alpha \in \mathbb{R}/\mathbb{Z}$, there exists a holomorphic bijection $\mathcal{F}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}_\alpha$ of $\mathbb{D}$ onto $\mathcal{D}_\alpha$ that satisfies*

$$\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0) \quad \textit{and} \quad \mathcal{F}'_\alpha(0) = (0, 2r_\alpha),$$

where r_α is the conformal radius of the entire mapping $E_\alpha(z) = e^{2i\pi\alpha} z e^z$. Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is equal to $(1/\pi)\log(1/|\delta|)$.

THEOREM B′. *Assume that α is a Brjuno number. Then the following hold.*

(a) *The Arnold disk can be locally parameterized by a in a neighborhood of $(e^{2i\pi\alpha}, 0)$ (that is, it is locally the graph of a holomorphic map $a \longmapsto \lambda(a)$).*

(b) *As $|a| \to 0$, the modulus m_a of the Herman ring of $f_{\lambda(a),a}$ satisfies*

$$e^{\pi m_a} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

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Xavier Buff
Laboratoire Emile Picard
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex
France

buff@picard.ups-tlse.fr

Lukas Geyer
Department of Mathematics
Montana State University
PO Box 172400
Bozeman
MT 59717-2400
USA

geyer@math.montana.edu

Núria Fagella
Departimento de Matematica Aplicada i Analisi
Universitat de Barcelona
Gran via 585
08007 Barcelona
Spain

fagella@maia.ub.es

Christian Henriksen
Department of Mathematics
Technical University of Denmark
Matematiktorvet
Building 303
DK-2800 Kgs. Lyngby
Denmark

christian.henriksen@mat.dtu.dk