Subshift on an infinite alphabet

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Abstract

We study transfer operators over general subshifts of sequences of an infinite alphabet. We introduce a family of Banach spaces of functions satisfying a regularity condition and a deacreasing condition. Under some assumptions on the transfer operator, we prove its continuity and quasi-compactness on these spaces. Under additional assumptions - existence of a conformal measure and topological mixing - we prove that its peripheral spectrum is reduced to 1 and that this eigenvalue is simple. We describe the consequences of these results in terms of existence and properties of invariant measures absolutely continuous with respect to the conformal measure. We also give some examples of contexts in which this setting can be used - expansive maps of the interval, statistical mechanics.

1 Introduction

Given a measured space (X, m) and an application T mapping X into itself, it is a classical question to ask if it admits an invariant measure absolutely continuous with respect to m ([15],[7],[18]). It is then natural to study the ergodic and statistical properties of the system with respect to this invariant measure.

When T is a non invertible, piecewise differentiable and expansive map from $\mathbf{R}^n$ into itself, the Ruelle-Perron-Frobenius operator has proved to be a rather efficient tool for these aims:

$$P(f)(x) = \sum_{y|T(y)=x} \frac{1}{T'(y)} f(y)$$

The study of the spectral properties of transfer operators is a slightly more general setting. Given a function ϕ - that will be called a potential - defined on X, the transfer operator associated with ϕ is a linear operator acting on Banach functional spaces ([19],[16],[25],[27]):

$$L_\phi(f)(x) = \sum_{y|T(y)=x} e^{\phi(y)} f(y)$$

Under reasonable assumptions, one proves the existence of a measure for which L_ϕ is the RPF operator - such a measure is said to be conformal. A classical and important result is the quasicompactness of the transfer operators over the Banach space of Hölder functions on the shift with finite alphabet, if ϕ itself is Hölder ([19],[16]). It is sufficient to obtain the existence of a measure absolutely continuous with respect to the conformal measure. Its density is the eigenfunction associated to the real eigenvalue of maximal modulus of the transfer operator. One then obtains exponential decay of correlations for Hölder continuous functions. It is also possible, using the perturbation theory of linear operators to obtain limit theorems for the sums $\sum_n f \circ T^n$, for f Hölder ([17],[10],[24]).

This formulation comes from statistical mechanics ([19]). But these results can be used for the study of interval maps ([4]) or differentiable hyperbolic dynamical systems ([3]).

These methods are well developed and efficient. But two assumptions are usually required and limit their possible applications:

- The considered transfer operators are acting on the shift with a finite alphabet. This ensures the compactness of the original space and hence, the existence of a conformal measure. It also simplifies the obtention of various uniform bounds.
- The corresponding results for subshifts concern subshifts of finite type. In more general cases, the space of Hölder functions might not be stable for the transfer operator.

Different authors have studied different ways to extend these results in various particular cases. For interval maps, see for example [20], [26], [14], [6], [1], [13], for symbolic dynamics, [22], [9], [23].

In this paper we prove a quasi compactness result for some transfer operators defined on a general closed subshift of a shift over an infinite alphabet. The assumptions are formulated on the "potential" ϕ associated with the transfer operator.

Let us precise the content of our result.

One of the main point is the introduction of particular Banach spaces that will allow us to deal with the infinite case. These Banach spaces contain bounded functions satisfying a decreasing condition and a regularity condition . Roughly speaking, they go to zero exponentially fast with the first coordinate. A weighted sum, $V_\theta()$, of the oscillations on each cylinder is asked to be finite.

On one hand, the usual pressure $P(\phi)$ is defined to be the logarithm of the spectral radius of the transfer operator. We introduce another quantity, called *pressure at infinity*, $P_\infty(\phi)$, which plays the same role as the pressure, but taking in account only what happens at infinity. The system has a nice behavior if $P_\infty(\phi) < P(\phi)$.

On the other hand, we have to control the regularity of the $L_\phi(f)$. For this aim, we ask ϕ to be regular enough ($V_\theta(e^\phi) < +\infty$), and small enough to compress the oscillations ($\|\phi\|_\infty < P(\phi)$).

Under these conditions, we prove that the transfer operator acting on the introduced Banach spaces is quasicompact. (Theorem 1)

This first result does not involve any measure. The existence of a conformal measure is possible to prove under the same kind of assumptions (See [5]) but, here, we assume there exists one. In this case and under a topological mixing condition, we prove that the only peripheral eigenvalue is 1 and that it is simple (Theorem 2).

We deduce that there exists a unique invariant measure absolutely continous with respect to the conformal measure; it is mixing and has exponential decay of correlations. (Corollary 1). We then apply these results to the case of a statistical mechanics setting (Corollary 2) and to the case of an interval map with an infinite coding partition (Corollary 3 and 4).

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2 Statement of results

2.1 Shift and subshift

Let A be a countable set (say $A = \mathbf{N}^*$) and $\mathcal{S} = A^{\mathbf{N}^*}$ the set of all the sequences of elements of A. The elements of $\mathcal{S}$ are denoted by $x = (x_n)_{n \in \mathbf{N}^*} = (x_1, \ldots, x_n, \ldots)$. For any integer n and any elements $i_1, \ldots, i_n$ of A, the subset $\{x \in \mathcal{S} | x_1 = i_1, \ldots, x_n = i_n\}$ of $\mathcal{S}$ is called a $n-$cylinder and is simply denoted by $(i_1, \ldots, i_n)$. $\mathcal{S}$ is endowed with the $\sigma-$algebra $\mathcal{B}$, generated by the finite cylinders.

Let then $T : \mathcal{S} \to \mathcal{S}$, $x \to Tx$, where : $\forall n \in \mathbf{N}^*, \quad (Tx)_n = x_{n+1}$. T is not invertible; let ix denote the preimage of x in (i): $ix = (i, x_1, \ldots, x_n, \ldots)$. In the same way, let $i_1 \cdots i_n x = (i_1, \ldots, i_n, x_1, \ldots, x_n, \ldots)$. T is called *one-sided full shift* (or simply *shift*). We shall denote by U the linear operator (acting on various functional spaces) defined by: $Uf = f \circ T$.

Let $\mathcal{F}$ be a T-invariant closed subset of $\mathcal{S}$. The induced $\sigma-$algebra on $\mathcal{F}$ is generated by the traces $(i_1, \ldots, i_n) \cap \mathcal{F}$ of the cylinders. The restriction of the map T to the subset $\mathcal{F}$ is called a one-sided *subshift*. Notice that it is either of finite or infinite type.

Definition 1 (Topological mixing) *We shall say that the subshift $T_{\mathcal{F}}$ is* topologically mixing *if and only if for any cylinder $(i_1, \ldots, i_n)$ intersecting $\mathcal{F}$, there exists an integer N such that*

$$T^N((i_1, \ldots, i_n) \cap \mathcal{F}) = \mathcal{F}$$

2.2 Banach spaces

Let α and θ be two reals, $\alpha > 1$ and $0 < \theta < 1$. We shall denote by $\tilde{\alpha}$ the application $\tilde{\alpha} : \mathcal{S} \to \mathbf{R}$ depending only of the first coordinate, defined for all i and all x in the cylinder (i) by $\tilde{\alpha}(x) = \alpha^{-i}$.

Given a function $f : \mathcal{F} \to \mathbf{C}$, we can define the followings:

$$var_{i_1,\ldots,i_n}(f) := \sup_{x,y \in (i_1,\ldots,i_n) \cap \mathcal{F}} (|f(x) - f(y)|)$$

$$V_\theta(f) := \sum_{n=1}^{+\infty} \theta^n \Big(\sum_{i_1,\ldots,i_n} var_{i_1,\ldots,i_n}(f) \Big) \quad \textit{and} \quad [f]_\alpha := \sup_{j \in \mathbf{N}} (\alpha^j \sup_{x \in (j) \cap \mathcal{F}} (f(x))) = \| \frac{f}{\tilde{\alpha}} \|_\infty$$

Let now consider the family of sets $\mathcal{X}_{\alpha,\theta} := \{f : \mathcal{F} \to \mathbf{C} \quad | \quad V_\theta(f) < +\infty, [f]_\alpha < +\infty\}$ endowed with the norm $\|.\| := V_\theta(.) + [.]_\alpha$. We will prove that these spaces are Banach spaces (proposition 3).

Let then $\mathcal{X}_\theta$ denote the space $\mathcal{X}_{1,\theta}$ and $\mathcal{X}_\alpha$ the space $\mathcal{X}_{\alpha,0}$.

Let us define notations for the corresponding local quantities. Given a cylinder $(j_1, \ldots, j_N)$, let

$$f(i_1, \ldots, i_n) := \sup_{x \in (i_1,\ldots,i_n) \cap \mathcal{F}} |f(x)|$$

and let $V_\theta^{(j_1,\ldots,j_N)}(f)$ denote the variation of f, restricted to the cylinder $(j_1, \ldots, j_N)$. Namely:

$$\begin{aligned} V_\theta^{(j_1,\ldots,j_N)}(f) &:= \sum_{n=1}^{+\infty} \theta^n \Big(\sum_{i_1,\ldots,i_n | (i_1,\ldots,i_n) \subset (j_1,\ldots,j_N)} var_{i_1,\ldots,i_n}(f) \Big) \\ &= \sum_{n > N} \theta^n \Big(\sum_{i_{N+1},\ldots,i_n} var_{j_1,\ldots,j_N,i_{N+1},\ldots,i_n}(f) \Big) \end{aligned}$$

Let us introduce a very similar quantity that will be useful to deal with the case of subshifts of infinite type.

Definition 2 *Let $N \in \mathbf{N}^*$. We shall say that the N-cylinder $(j_1, \ldots, j_N)$ in $\mathcal{S}$ is* non regular *for $\mathcal{F}$ if and only if*

$$\exists x, y \in (j_2, \cdots, j_N) \cap \mathcal{F} \mid j_1 x \in (j_1, \ldots, j_N) \cap \mathcal{F} \text{ and } j_1 y \notin (j_1, \ldots, j_N) \cap \mathcal{F}$$

We shall then distinguish:

- The set $\mathcal{M}^o$ of the non regular cylinders
- The set $\mathcal{M}^+$ of the regular cylinders that intersect $\mathcal{F}$.
- The set $\mathcal{M}^-$ of the regular cylinders that do not intersect $\mathcal{F}$.

Let now set:

$$\overline{var}_C(f) := \begin{cases} var_C(f) & if\ C \in \mathcal{M}^+ \\ \sup_{x \in C} f(x) & if\ C \in \mathcal{M}^o \\ 0 & if\ C \in \mathcal{M}^- \end{cases}$$

We then denote by $\overline{V}_\theta(f)$ the corresponding total variation:

$$\overline{V}_\theta(f) := \sum_{n=1}^{+\infty} \theta^n (\sum_{i_1, \ldots, i_n} \overline{var}_{i_1, \ldots, i_n}(f))$$

Remark 1 *It is equivalent to count the variations of f as a function defined on $T^{-1}\mathcal{F}$ and equal to zero on $T^{-1}\mathcal{F} \setminus \mathcal{F}$.*

Remark 2 *Notice that if all the cylinders are regular (the subshift is then called Markovian), or equivalently if the sets $\{j \in \mathbf{N}^* | jx \in \mathcal{F}\}$ do not depend on $i_1(x)$, then $\overline{V}_\theta(f) = V_\theta(f)$ for all f.*

2.3 Transfer operators

To any function $\phi : \mathcal{F} \longrightarrow \mathbf{R}$, one can associate an operator, called *transfer operator* , $\mathcal{L}_\phi$, by :

$$\mathcal{L}_\phi(f)(x) := \sum_{y | T(y) = x} e^{\phi(y)} f(y)$$

or, using the introduced notations:

$$\mathcal{L}_\phi(f)(x) = \sum_{j \in \mathbf{N}^* | jx \in \mathcal{F}} e^{\phi(jx)} f(jx)$$

Notice that it is defined only if the serie converges. Assumptions on ϕ will make it defined at any x if f remains in a correct functional space. For example, if $\|\mathcal{L}_\phi(1)\|_\infty < +\infty$, it is defined on the space of bounded functions.

Let ϕ_N denote : $\phi_N = \sum_{k=0}^{N-1} \phi \circ T^k$. The iterates of the operator have the following expression:

$$\mathcal{L}_\phi^N(f)(x) = \sum_{i_1, \ldots, i_N | j_1 \cdots j_N x \in \mathcal{F}} e^{\phi_N(j_1 \cdots j_N x)} f(j_1 \cdots j_N x)$$

Remark 3 *We also can think of ϕ as defined over $\mathcal{S}$ and set $\phi(x) = -\infty$ if $x \notin \mathcal{F}$. It then makes sense to write: $\mathcal{L}_\phi^N(f)(x) = \sum_{i_1, \ldots, i_N} e^{\phi_N(j_1 \cdots j_N x)} f(j_1 \cdots j_N x)$ without defining f out of $\mathcal{F}$.*

We shall say that a potential ϕ defined over a subset $\mathcal{F}$ of $\mathcal{S}$ is topologically mixing if the subshift over the subset $\mathcal{F}$ is.

When it is not ambiguous, we shall write simply $\mathcal{L}$ instead of $\mathcal{L}_\phi$.

2.3.1 Pressure

We now introduce a quantity which will play the same role as the usual pressure in statistical mechanics.

Definition 3 *We will denote by $P(\phi)$ and call* Pressure *of ϕ over L^∞ the real number :*

$$P(\phi) := \lim_{n\to+\infty} \frac{1}{n} \log(\|\mathcal{L}_\phi^n(1)\|_\infty).$$

The following proposition gives a criterion for the existence of the pressure.

Proposition 1 *If $\|\mathcal{L}(1)\|_\infty < +\infty$, then, the sequence $(\frac{1}{n}\log(\|\mathcal{L}^n(1)\|_\infty))_{n\in\mathbf{N}^*}$ converges to a limit $-\infty \leq P(\phi) < +\infty$.*

The sequence is convergent because $(\|\mathcal{L}^n(1)\|_\infty)_{n\in\mathbf{N}}$ is sub-multiplicative. □

In practice, we will use a condition stronger than $\|\mathcal{L}(1)\|_\infty < +\infty$. We will ask ϕ to satisfy:

$$\sup_{j\in\mathbf{N}^*} \sum_{i=1}^{+\infty} e^{\phi(ij)} < +\infty$$

Notice that:

$$\|\mathcal{L}(1)\|_\infty = \sup_{j\in\mathbf{N}^*} \sup_{x\in(j)} \sum_{i=1}^{+\infty} e^{\phi(ix)} \leq \sup_{j\in\mathbf{N}^*} \sum_{i=1}^{+\infty} e^{\phi(ij)}$$

To deal with this kind of quantities that take in account only the supremum of e^ϕ on cylinders, let introduce the following notations:

$$\Delta_p(\phi) := \sup_{j>p}(\sum_{i=1}^{+\infty} e^{\phi(ij)}) \quad \textit{and} \quad \Delta_{N,p}(\phi) := \sup_{j>p}(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)})$$

The next proposition gives a relationship between pressures of different potentials.

Proposition 2 *If $P(\phi) < +\infty$ then, for any c real and h function satisfying $\|\log h\|_\infty < +\infty$, one has:*

- $P(\phi + c) = P(\phi) + c$
- $P(\phi + \log h \circ T - \log h) = P(\phi)$

The proof is immediate. □

The first equality implies $P(\phi - P(\phi)) = 0$. Hence, it always is possible to reduce a problem to the case of potentials with pressure zero. Notice that the second equality does not hold if h is only non negative or if h is unbounded.

2.3.2 Pressure at infinity

The set $\mathcal{F}$ is not compact. Usual results existing for the shift over a finite alphabet use the compactness of the set on which acts the shift. For some potentials, some of these results do not stand.

To overcome this difficulty, and be able to state results for some classes of potentials, we introduce a quantity that measures the importance of what 'happens' outside of compact subsets of $\mathcal{F}$. In a general setting, these quantities would be of the form: $\inf_{K\ compact}(\sup_{x\in K^c} \mathcal{L}_\phi^n 1(x))$.

To state our result, we shall use simpler criterions. Let then introduce:

$$\Delta_\infty(\phi) := \inf_{p\in\mathbf{N}}(\Delta_p(\phi)) = \inf_{p\in\mathbf{N}}(\sup_{j>p}(\sum_{i=1}^{+\infty} e^{\phi(ij)})) = \limsup_{j\to+\infty} \sum_{i=1}^{+\infty} e^{\phi(ij)}$$

We can define the same kind of quantity for iterated operators:

$$\Delta_{N,\infty}(\phi) := \inf_{p\in\mathbf{N}}(\Delta_{N,p}(\phi)) = \inf_{p\in\mathbf{N}}(\sup_{j>p}(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)})) = \limsup_{j\to+\infty} \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}$$

We define the pressure at infinity as follows:

Definition 4 *If $\Delta_0(\phi) < +\infty$, we define the* Pressure at infinity *of ϕ by:*

$$P_\infty(\phi) := \limsup_{N\to\infty} \frac{1}{N} \log \Delta_{N,\infty}(\phi)$$

It is not necessary to estimate this quantity. We only need upper bounds. If there exists δ such that $P_\infty(\phi) < \log\delta$, then, there exists two integers N and p_N such that $\Delta_{N,p_N} < \delta^N$.

2.4 Properties of the invariant elements

Assume that the spectral radius of $\mathcal{L}_\phi$ is 1.

Definition 5 *We shall say that a measure m is* conformal *if it satisfies*

$$\forall f \in \mathcal{X}_{\alpha,\theta}, \quad \int_{\mathcal{F}} \mathcal{L}_\phi(f)dm = \int_{\mathcal{F}} f dm$$

It is quite easy to check that if m is a conformal measure for the potential ϕ and h is in $\mathcal{X}_\theta$, then $h.m$ is a conformal measure for the potential $\phi - \log(h\circ T) + \log(h)$.

If m is a conformal measure and ρ is an invariant function ($\mathcal{L}_\phi(\rho) = \rho$), then the measure $d\mu = \rho dm$ absolutely continuous with respect to m is invariant (we shall say it is an a.c.i.m.).

We shall denote by $\Sigma(\mathcal{P})$ the spectrum of a bounded operator $\mathcal{P}$ and by $\varrho(\mathcal{P})$ its spectral radius. The essential spectral radius of $\mathcal{P}$ is the infimum over all the compact operators $\mathcal{Q}$ of the spectral radius of the operators $\mathcal{P} - \mathcal{Q}$. An operator is said to be quasicompact, or equivalently to have a spectral gap, if its essential spectral radius is strictly smaller than its spectral radius. Then, its dominating behaviour is compact-like.

The existence of a spectral gap for $\mathcal{L}$ induces ergodic properties for the corresponding a.c.i.m. because of the relation:

$$\int_{\mathcal{F}} f.g\circ T^n d\mu = \int_{\mathcal{F}} \mathcal{L}^n_\phi(f.\rho).gdm$$

2.5 Suitable classes of potential

Let $\alpha > 1$ and $0 < \theta < 1$ be two reals.

Definition 6 *We will say that ϕ is in the class $\mathcal{C}_\theta$ if it satisfies the following conditions :*

($C_0$0) $P(\phi) = 0$

($C_0$1) $\lim_{N\to\infty} \|e^{\phi_N}\|_\infty^{\frac{1}{N}} < \theta^2$

($C_0$2) $P_\infty(\phi) < P(\phi)$.

($C_0$3) $\forall N \in \mathbf{N}^*, V_\theta(e^{\phi_N}) < +\infty$

We want to deal with some potentials for which the condition $(C_0 3)$ is not satisfied, but only the weaker: $\sup_i(V_\theta^{(i)}(e^\phi)) < +\infty$. This means the variation of e^ϕ on each cylinder is uniformly bounded but that the sum of these variations can be infinite. (This is for example the case in [2]).

We have to enforce the other assumptions, to compensate this effect. Mainly, we will ask $(C_0 0)$ and $(C_0 2)$ to remain satisfied for a potential which is 'less expansive at infinity'.

Let us introduce $\varphi := \phi - \log(\tilde{\alpha} \circ T) + \log \tilde{\alpha}$. Roughly speaking, $e^{\varphi(ij)}$ is much greater than $e^{\phi(ij)}$ when i is fixed and j goes to infinity. If x is in the cylinder (ij), we have $e^{\varphi(x)} = \alpha^{j-i} e^{\phi(x)}$.

Remark 4 *We set $\tilde{\alpha}(x) = \alpha^{-i}$ for $x \in (i)$ for simplicity. But the results and the proofs remain the same if we replace α^{-i} by any sequence α_i with $\sum_i \alpha_i < +\infty$.*

We shall denote by $P^\alpha(\phi) := P(\varphi)$ and by $P^\alpha_\infty(\phi) := P_\infty(\varphi)$. In general, the exponent α will mean that the quantity is calculated for φ instead of ϕ.

Definition 7 *We will say that ϕ is in the class $\mathcal{C}_{\alpha,\theta}$ if it satisfies the following conditions :*

$$\begin{array}{ll} (C0) & P^\alpha(\phi) = 0 \\ (C1) & \lim_{N\to\infty} \|e^{\phi_N}\|_\infty^{\frac{1}{N}} < \theta^2 \\ (C2) & P^\alpha_\infty(\phi) < P^\alpha(\phi) \\ (C3) & \sup_i(\overline{V}_\theta^{(i)}(e^\phi)) < +\infty \end{array}$$

As $(C2)$ is difficult to check, we will also give some criterions that make easy a majoration of $P^\alpha_\infty(\phi)$. Namely, one has the following results:

Proposition 6 *If (a) $\Delta_0(\phi) < +\infty$, (b) $\Delta_\infty(\phi) < \delta$ and (c) $\lim_{j\to\infty} e^{\phi(ij)} = 0$, then $P_\infty(\phi) < \log \delta$*

The proof is in section 3.6.4, page 20.

We will also introduce other classes of potentials more directly adapted to the proofs.

2.6 Statement of the main results

We now are able to state the main result we will prove:

Theorem 1 *Let ϕ be a potential of $\mathcal{C}_{\alpha,\theta}$. The operator $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ is quasicompact. More precisely, there exists $\tau' < 1$ such that it has the following spectral decompostion :*

$$\mathcal{L}_\phi = \mathcal{Q} + \mathcal{R}, \ \ \mathcal{Q} \ \textit{compact}, \ \ \{1\} \subset \Sigma(\mathcal{Q}) \subset \{0\} \cup \{z||z| = 1\}, \ \ \varrho(\mathcal{R}) \leq \tau' < 1, \ \ \mathcal{QR} = \mathcal{RQ} = 0$$

It is enforced by the following:

Theorem 2 *Let ϕ be a potential of $\mathcal{C}_{\alpha,\theta}$. Assume that there exists a conformal measure and that the system is mixing. Then, the operator has only one eigenvalue of modulus 1. This eigenvalue is simple. It is 1. Let ρ denote the corresponding eigenfunction. It is strictly positive. The conformal measure is unique. It charges all the cylinders in $\mathcal{F}$. The spectral projector on the eigenfunction ρ is:*

$$\mathcal{Q}(f) = (\int_{\mathcal{F}} f dm)\rho$$

Let us make a few remarks about these results.

Remark 5 *$\tau_\infty = max(\theta^{-1}\lim_{N\to\infty}\|e^{\phi_N}\|_\infty^{\frac{1}{N}}, exp(P^\alpha_\infty(\phi)))$ is an upper bound of the essential spectral radius of $\mathcal{L}_\phi$ over $\mathcal{X}_{\alpha,\theta}$. Notice that we do not give any bound for the modulus τ' of the second eigenvalue of the operator. We only know it is strictly less than 1.*

Remark 6 *In many cases, such results also imply the existence of a lower bound (away from 0) for the density ρ. Under our assumptions, the density is strictly positive but can tend to zero.*

Remark 7 *The fact that the conformal measure charges all the cylinders comes from the topological mixing condition and is important. If we already knew that it is true, then, we could use a weaker topological mixing property as, for example:*

$$\forall n, i_1, \ldots, i_n,\ \forall x \in \mathcal{F},\ \ \exists N, j_1, \ldots, j_N,\ \mid i_1 \cdots i_n j_1 \cdots j_N x \in \mathcal{F}$$

.

2.7 Consequences

2.7.1 Ergodic results

The following corollary is a direct consequence of the spectral decomposition of the transfer operator.

Corollary 1 *Let T be a mixing subshift over a part $\mathcal{F}$ of $\mathcal{S}$. Let $\alpha > 1$, $0 < \theta < 1$ be two reals and ϕ a potential in ($\mathcal{C}_{\alpha,\theta}$). Denote by m a conformal measure of the system.*

T has a unique invariant probability μ, absolutely continuous with respect to m. Its density ρ with respect to m is in $\mathcal{X}_{\alpha,\theta}$. The measure $d\mu = \rho dm$ is ergodic, mixing and the functions of $\mathcal{X}_\theta$ have exponential decay of correlations with respect to μ.

Under the same assumptions, and using quite classical methods, it is possible to prove limit theorems for the system. The central limit theorem follows directly of the summability of the iterates $\mathcal{L}^n 1$ (see [12]). Large deviations results can be obtained using pertubations of quasicompact operators theory (see [6], [5]). Convergence of the laws of entrance times in families of rare events can be obtained following [8].

2.7.2 Statistical mechanics

One can notice that ϕ gives raise to a compatible system of conditional probabilities:

$$\Pi_\phi(X_1 = i_1, \ldots, X_n = i_n | \sigma\left((X_p)_{p>n}\right))(x) = \frac{e^{\phi_n(i_1\cdots i_n T^n x)}}{\sum_{j_1,\cdots,j_n} e^{\phi_n(j_1\cdots j_n T^n x)}}$$

A Gibbs state for this system of conditional probabilities is a probability measure whose conditional probabilities are precisely these values. Namely, m is a Gibbs state for the system of conditional probabilities associated to ϕ if and only if it satisfies:

$$m(f|T^n\mathcal{B})(x) = \frac{\mathcal{L}^n_\phi f(T^n x)}{\mathcal{L}^n_\phi 1(T^n x)}, \quad for \quad m - a.e.\ x$$

Adopting this point of view, it is not very difficult to deduce from theorem 1 and theorem 2 the following:

Corollary 2 *Let $(\mathcal{F}, T)$ be a one-sided mixing subshift. Let ϕ be a potential of the class $\mathcal{C}_{\alpha,\theta}$. Let $m = m_\phi$ be a conformal measure for this potential. It is also a Gibbs state for this potential.*

There exists a unique function ρ and a unique invariant measure $\mu = \mu_\phi$ satisfying,

$$\mu(f|T^n\mathcal{B})(x) = \mathcal{L}^n_{\phi-\log(\rho\circ T)+\log(\rho)} f(T^n x), \quad for \quad \mu - a.e.\ x$$

This measure is a Gibbs state for the new potential, absolutely continuous with respect to m. It is translation invariant, ergodic and mixing. The decay of correlations for functions of $\mathcal{X}_\theta$ is exponential.

2.7.3 Interval maps

Let (I, F) be a topologically mixing, piecewise C^1 interval map with a countable number of subintervals of monotonicity. If it is expanding, it is possible to encode it using its monotonicity partition. We shall identify it with a subshift $\mathcal{F}$ on an infinite alphabet.

Let m be the image of the Lebesgue measure on $\mathcal{F}$. It is conformal for the potential $\phi = -\log F'$. The corresponding transfer operator is then called Ruelle-Perron-Frobenius operator.

We can deduce from theorem 1 and theorem 2 the following:

Corollary 3 *Let (I, F) be a topologically mixing interval map, with an infinite coding partition. If $-\log F'$ is in $\mathcal{C}_{\alpha,\theta}$ then there exists a unique probability measure μ, invariant and absolutely continuous with respect to the Lebesgue measure. Moreover, μ is ergodic, mixing and functions with bounded variations have exponential decay of correlations.*

We now consider as an example a set of assumptions - close to the assumptions of the Folklore Theorem - under which it is possible to use our result.

Corollary 4 *Let $I = [0, 1]$ be the unit interval, and suppose $\{I_1, I_2, ...\}$ is a countable collection of disjoint open subintervals of I such that $\cup_i I_i$ has the full Lebesgue (dl) measure in I. Suppose there are constants $K_0 > 1$ and $K_1 > 0$ and mappings $f_i : I_i \to I$ satisfying the following conditions.*

1. *f_i extends to a C^2 diffeomorphism from $\bar{I}_i$ onto I, and $\inf_{x\in I_i} |f_i'(x)| > K_0$ for all i.*

2. $\sup_{x\in I_i} \frac{f_i''(x)}{f_i'(x)} |I_i| < K_1$ *for all i.*

3. $M = -\sum_i |I_i| \log |I_i| < +\infty$

Then, the mapping F defined by $F(x) = f_i(x)$ for $x \in I_i$, has unique invariant ergodic probability measure μ absolutely continuous with respect to Lebesgue measure on I. This measure is mixing and functions with bounded variations have exponential decays of correlations.

In this context, the result cannot be applied directly because the corresponding potential does not satisfy $P^\alpha_\infty < 0$ for any suitable sequence α. We introduce a conjugate potential which fits the assumptions of our result and leads to the conclusion for the original system.

3 Proof of the theorems

3.1 Main steps of the proof

The section 3.2 contains preliminary results. The introduced functional spaces are proved to be complete. Then, we prove an inequality bounding the variation of the image of a function of $\mathcal{X}_{\alpha,\theta}$. In particular, we precise how it can be proved in the case of the subshift. This inequality plays the same kind of role as the "basic inequality" in [16].

Let us introduce the following notations:

- $\gamma_N = \theta^{-N} \|e^{\phi_N}\|_\infty$
- $v_N = \theta^{-2N} \sup_{j_1,\dots,j_N \in \mathbf{N}^N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N})$
- $a_N^\alpha = \theta^{-1} \sum_{i=1}^{+\infty} \alpha^{-i} \overline{V}_\theta^{(i)}(e^{\phi_N}) = \sum_{n=1}^{+\infty} \sum_{i_1,\dots,i_n} \theta^n \alpha^{-i_1} \overline{var}_{i_1,\dots,i_n}(e^{\phi_N})$
- $\delta_N > \Delta_{N,\infty}^\alpha(\phi)$
- $\tau_N = \sqrt[N]{4max(\gamma_N; v_N; \delta_N)}$

In the section 3.3, we exhibit a criterion for the continuity of a transfer operator over both $\mathcal{X}_\alpha$ and $\mathcal{X}_{\alpha,\theta}$.

> **Lemma 2** *If γ_N, a_N^α and $\Delta_{N,0}^\alpha$ are finite, then, the operators $\mathcal{L}_\phi : \mathcal{X}_\alpha \to \mathcal{X}_\alpha$ and $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ are continuous.*

In section 3.4, under the same assumptions we obtain an upper bound for the spectral radius of $\mathcal{L}_\phi$ as an operator over $\mathcal{X}_{\alpha,\theta}$.

> **Proposition 4** *If γ_N, a_N^α and $\Delta_{N,0}^\alpha$ are finite, then, the essential spectral radius of $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ is strictly less than τ_N*

We complete the proof of this result in three steps:

- We explicit a linear decomposition of the operator in three parts:
  - A part deals with the action of $\mathcal{L}_\phi$ on the oscillations of f on the N-cylinders.
  - A part deals with what happens to f on each cylinder except a finite number of them.
  - The last part concerns what happens on the remaining finite number of cylinders. The corresponding operator is compact. It will not contribute to the essantial spectral radius.
- We introduce a balanced norm $\|\|_b$ wich is equivalent to the usual norm on $\mathcal{X}_{\alpha,\theta}$. It depends on the properties of the potential ϕ:
  - $[f] = \alpha^{-p_N}[f]_\alpha$, where p_N is such that $\Delta_{N,p_N}^\alpha < \delta_N$.
  - $\|f\|_b = V_\theta(f) + [f]$
- We then need inequalities to control the norm of the two first parts of the decomposition. As the last part is compact, this provides an upper bound for the essential radius.

These two results make natural the introduction of the following classes of potentials depending on an integer N:

Definition 8 *We shall say that ϕ is in $\mathcal{C}_{\alpha,\theta}^{(N)}$ if:*

$$\begin{array}{ll} (C_N 0) & P^\alpha(\phi) = 0 \\ (C_N 1) & \|e^{\phi_N}\|_\infty < \frac{\theta^N}{4} \\ (C_N 2) & \Delta_{N,0}^\alpha < +\infty \ \text{ and } \ \Delta_{N,\infty}^\alpha < \frac{1}{4} \\ (C_N 3.1) & \sup_{i_1,\dots,i_N}(\overline{V}_\theta^{(i_1,\dots,i_N)}(e^{\phi_N})) < \frac{\theta^{2N}}{4} \\ (C_N 3.2) & a_N^\alpha < +\infty \end{array}$$

The potentials of these classes will give raise to continous transfer operators with essantial spectral radius strictly smaller than 1, whereas their spectral radius is exactly 1. This will ensure the existence of a spectral gap.

More precisely, in section 3.5, we shall prove the following lemma,

> **Lemma 4** *Let a positive operator $\mathcal{L}$ be continuous on $\mathcal{X}_\alpha$, with a spectral radius ϱ. Assume it is also continuous on $\mathcal{X}_{\alpha,\theta}$, with an essential spectral radius $\varrho_e < \varrho$.*

The spectral radius of $\mathcal{L}_\phi$ on $\mathcal{X}_{\alpha,\theta}$ is exactly ϱ. Moreover, the real ϱ is an eigenvalue of $\mathcal{L}$.

Using this lemma, we will be able to prove a result slightly more general than theorem 1:

Theorem 1-(N) *If ϕ is in $\mathcal{C}_{\alpha,\theta}^{(N)}$, then, there exists $\tau' < 1$ such that the operator $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ has the following spectral decomposition.*

$$\mathcal{L}_\phi = \mathcal{Q} + \mathcal{R}, \ \ \mathcal{Q} \ \textit{compact}, \ \ \Sigma(\mathcal{Q}) \subset \{0\} \cup \{z||z| = 1\}, \ \ \varrho(\mathcal{R}) \leq \tau' < 1, \ \ \mathcal{Q}\mathcal{R} = \mathcal{R}\mathcal{Q} = 0$$

To conclude, we have to explicit the links between the different introduced classes of potentials. In section 3.6, we shall see that for any $\phi \in \mathcal{C}_{\alpha,\theta}$, there exists an integer N such that $\phi \in \mathcal{C}_{\alpha,\theta}^{(N)}$.

Proposition 5

$$\mathcal{C}_{\alpha,\theta} \subset \bigcup_{M \geq 1} \bigcap_{N \geq M} \mathcal{C}_{\alpha,\theta}^{(N)}$$

This achieves the proof of theorem 1.

In the section 3.7, we prove:

Proposition 7 *If ϕ is in $\mathcal{C}_{\alpha,\theta}$, is topologically mixing, and if there exists a conformal measure for ϕ, then 1 is a simple eigenvalue of $\mathcal{L}$, over $\mathcal{X}_{\alpha,\theta}$, and it is the only eigenvalue of modulus 1.*

The proof is completed in three steps:

- The existence of a conformal measure implies that the positive part of any invariant function is almost invariant. This provides a simple way to exhibit a positive invariant function.
- Using the topological mixing property, we prove that such a positive invariant function is strictly positive. From this argument we deduce that there cannot exist any other invariant direction. We deduce that the conformal measure is the only one.
- We then prove that 1 is the only eigenvalue of modulus 1.

Remark 8 *The property of topological mixing is equivalent to the irreductibility of the operator in the sense of Schaefer ([21], Définition III.8.1, page 186). The previous results could be obtained using the results describing the peripheral spectrum of a positive irreductible operator ([21] V.5, pages 328-333).*

The theorem 2 easily follows from proposition 7.

3.2 Preliminary results

3.2.1 Completness of the spaces

We first prove:

Proposition 3 *The spaces $\mathcal{X}_{\alpha,\theta}$ are Banach spaces.*

Proof We must prove they are complete. Let $(f_n)_{n \in \mathbf{N}}$ be a Cauchy sequence of $\mathcal{X}_{\alpha,\theta}$. The sequence $(g_n)_{n \in \mathbf{N}}$ defined by $g_n = \frac{f_n}{\tilde{\alpha}}$ is a Cauchy sequence in L^∞, which is a Banach space. Hence, it converges in L^∞ toward a limit g. We deduce that f_n converges in $(\mathcal{X}_\alpha)$ toward $f = \tilde{\alpha} g$.

We choose a real $\epsilon > 0$ ($\epsilon < \frac{1}{2(1+\theta)}$). We will exhibit an integer N_1 such that if $n > N_1$, we have:

$$V_\theta(f_n - f) \leq 4\epsilon$$

To do this, let first choose a way of counting the cylinders: to each integer k we associate a cylinder C_k in such a way that each cylinder is counted (the set of all the finite cylinders is countable). Let $l(k)$ denote the lenght of the cylinder C_k.

To the kth cylinder $C_k = (i_1, \dots, i_{l(k)})$, we now associate an integer N_k with the following properties:

- $\forall n \geq N_k, \|f_n - f\|_\infty \leq \epsilon^{i_1+\dots+i_{l(k)}}$ ($f_n \to f$ in $\mathcal{X}_\alpha$ and also in L^∞)
- $\forall n, m \geq N_k, V_\theta(f_n - f_m) \leq \epsilon^k$ ((f_n) is a Cauchy sequence in $\mathcal{X}_{\alpha,\theta}$)

Let $n > N_1$. We have, for all integer k (and hence for all cylinder C_k):

$$var_{C_k}(f_n - f) \leq var_{C_k}(f_n - f_{N_2}) + \sum_{j=3}^{k} var_{C_k}(f_{N_{j-1}} - f_{N_j}) + var_{C_k}(f_{N_k} - f)$$

Hence, summing:

$$\begin{aligned}
V_\theta(f_n - f) &= \sum_{k=1}^{+\infty} \theta^{l(k)} var_{C_k}(f_n - f) \\
&\leq V_\theta(f_n - f_{N_2}) + \sum_{j=3}^{k} V_\theta(f_{N_{j-1}} - f_{N_j}) + \sum_{k=1}^{+\infty} \theta^{l(k)} \|f_{N_k} - f\|_\infty \\
&\leq \sum_{j=2}^{k} \epsilon^{j-1} + \sum_{l=1}^{+\infty} \theta^l \sum_{i_1,\dots,i_l} \epsilon^{i_1+\dots+i_l} \\
&\leq \frac{\epsilon}{1-\epsilon} + \frac{\theta\epsilon}{1-\epsilon-\theta\epsilon} \\
&\leq 4\epsilon
\end{aligned}$$

The proof is complete. □

3.2.2 Regularity of the images

We wish to prove the following inequality:

Lemma 1 *If γ_N and v_N are finite, the following inequality holds:*

$$V_\theta(\mathcal{L}^N f) \leq \gamma_N \sum_{j_1,\dots,j_N} V_\theta^{(j_1,\dots,j_N)}(f) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) \|f.1_{(j_1,\dots,j_N)}\|_\infty \tag{1}$$

Proof Let complete the proof only for $N = 1$. It works exactly in the same way when N is larger. We first assume that all the cylinders are regular. For all x and y in $\mathcal{F}$, one has:

$$\begin{aligned}
|\mathcal{L}f(x) - \mathcal{L}f(y)| &= |\sum_{j=1}^{+\infty} (e^{\phi(jx)} f(jx) - e^{\phi(jy)} f(jy))| \\
&\leq \sum_{j=1}^{+\infty} e^{\phi(jx)} |f(jx) - f(jy)| + \sum_{j=1}^{+\infty} |f(jy)(e^{\phi(jx)} - e^{\phi(jy)})|
\end{aligned}$$

Hence, if $x, y \in (i_1, \dots, i_n) \cap \mathcal{F}$,

$$|\mathcal{L}f(x) - \mathcal{L}f(y)| \leq \|e^\phi\|_\infty \sum_{j=1}^{+\infty} |f(jx) - f(jy)| + \sum_{j=1}^{+\infty} \|f.1_{(ji_1\cdots i_n)}\|_\infty |e^{\phi(jx)} - e^{\phi(jy)}|$$

Taking the supremum, we obtain:

$$
\begin{aligned}
var_{i_1,\ldots,i_n}(\mathcal{L}f) &= \sup_{x,y\in(i_1,\ldots,i_n)\cap\mathcal{F}} |\mathcal{L}f(x)-\mathcal{L}f(y)| \\
&\leq \|e^{\phi}\|_\infty \sum_{j=1}^{+\infty} var_{j,i_1,\ldots,i_n}(f) + \sum_{j=1}^{+\infty} \|f.1_{(ji_1\cdots i_n)}\|_\infty var_{j,i_1,\ldots,i_n}(e^{\phi})
\end{aligned}
$$

A sum over all the cylinders gives:

$$
\begin{aligned}
V_\theta(\mathcal{L}f) &= \sum_{n=1}^{+\infty}\theta^n \sum_{i_1,\ldots,i_n} var_{i_1,\ldots,i_n}(\mathcal{L}f) \\
&\leq \|e^{\phi}\|_\infty \theta^{-1} V_\theta(f) + \sum_{n=1}^{+\infty}\theta^n \sum_{j,i_1,\ldots,i_n} var_{j,i_1,\ldots,i_n}(e^{\phi})\|f.1_{(j)}\|_\infty
\end{aligned}
$$

If some cylinders are not regular, then the sum $\mathcal{L}_\phi(f)(x) = \sum_{j\in\mathbf{N}|jx\in\mathcal{F}} e^{\phi(jx)} f(jx)$ might depend on x. Then, for two x et y in the same 1-cylinder, the sum is made over two different subsets of the alphabet $\mathbf{N}^*$. More precisely, $|\mathcal{L}f(x)-\mathcal{L}f(y)|$ can not any longer be written as $|\sum(e^{\phi(jx)}f(jx) - e^{\phi(jy)}f(jy))|$. It is to overcome this problem that we introduced the $\overline{V}_\theta()$. Let verify that this fits the problem. For $x, y \in (i_1,\ldots,i_n)$,

$$
|\mathcal{L}f(x)-\mathcal{L}f(y)| \leq |\sum_{j|(j,i_1,\ldots,i_n)\in\mathcal{M}^+} (e^{\phi(jx)}f(jx)-e^{\phi(jy)}f(jy))| + |\sum_{j|(j,i_1,\ldots,i_n)\in\mathcal{M}^o} e^{\phi(ji_1\cdots i_n)} f(ji_1\cdots i_n)|
$$

We get:

$$
\begin{aligned}
V_\theta(\mathcal{L}f) &= \sum_{n=1}^{+\infty}\theta^n \sum_{i_1,\ldots,i_n} var_{(i_1,\ldots,i_n)}(\mathcal{L}f) \\
&\leq \|e^{\phi}\|_\infty \theta^{-1} V_\theta(f) \\
&\quad + \sum_{n=1}^{+\infty}\theta^n \sum_{j,i_1,\ldots,i_n\in\mathcal{M}^+} var_{(j,i_1,\ldots,i_n)}(e^{\phi})\|f.1_{(j)}\|_\infty \\
&\quad + \sum_{n=1}^{+\infty}\theta^n \sum_{j,i_1,\ldots,i_n\in\mathcal{M}^o} e^{\phi(ji_1\cdots i_n)}|f|(ji_1\cdots i_n)
\end{aligned}
$$

And finally,

$$
V_\theta(\mathcal{L}f) \leq \|e^{\phi}\|_\infty \theta^{-1} V_\theta(f) + \sum_{n=1}^{+\infty}\theta^n \sum_{j,i_1,\ldots,i_n} \overline{var}_{(j,i_1,\ldots,i_n)\cap\mathcal{F}}(e^{\phi})\|f.1_{(j)}\|_\infty
$$

or,

$$
V_\theta(\mathcal{L}f) \quad \leq \quad \gamma_1 V_\theta(f) + \theta^{-1}\sum_{j=1}^{+\infty} \overline{V}_\theta^{(j)}(e^{\phi})\|f.1_{(j)}\|_\infty
$$

The proof of the iterated inequality is the same. The only point to notice is that the non regular cylinders for $\mathcal{L}^N$ (Let denote by $\mathcal{M}_N^o$ their set) also are non regular for $\mathcal{L}$ (Meaningly: $\mathcal{M}_N^o \subset \mathcal{M}^o$), because, if $i_1,\ldots,i_n$ is such that there exist $x,y \in (i_{N+1},\ldots,i_n)$ with $i_1\cdots i_N x \in \mathcal{F}$ and $i_1\cdots i_N y \notin \mathcal{F}$, then, $x' = i_2\cdots i_N x$ and $y' = i_2\cdots i_N y$ are such that $i_1 x' \in \mathcal{F}$ and $i_1 y' \notin \mathcal{F}$. □

3.3 Continuity of $\mathcal{L}$

Let us recall the introduced notations:

- $\gamma_N = \theta^{-N} \|e^{\phi_N}\|_\infty$
- $\delta_N > \Delta^\alpha_{N,\infty}(\phi)$
- $v_N = \theta^{-2N} \sup_{j_1,\dots,j_N \in \mathbf{N}^N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N})$
- $a^\alpha_N = \theta^{-1} \sum_{i=1}^{+\infty} \alpha^{-i} \overline{V}_\theta^{(i)}(e^{\phi_N}) = \sum_{n=1}^{+\infty} \sum_{i_1,\dots,i_n} \theta^n \alpha^{-i_1} \overline{var}_{i_1,\dots,i_n}(e^{\phi_N})$
- $\tau_N = \sqrt[N]{4max(\gamma_N; v_N; \delta_N)}$

Lemma 2 *If γ_N, a^α_N and $\Delta^\alpha_{N,0}$ are finite, then, the operators $\mathcal{L}_\phi : \mathcal{X}_\alpha \to \mathcal{X}_\alpha$ and $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ are continuous.*

Proof

- On one hand, according to (1), we have:

$$V_\theta(\mathcal{L}^N f) \leq \gamma_N \sum_{j_1,\dots,j_N} V_\theta^{(j_1,\dots,j_N)}(f) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) \|f.1_{(j_1,\dots,j_N)}\|_\infty$$

Using the finiteness of a^α_N, we deduce:

$$\begin{aligned} V_\theta(\mathcal{L}^N f) &\leq \gamma_N V_\theta(f) + \sum_{n=1}^{+\infty} \theta^n \sum_{j_1,\dots,j_N,i_1,\dots,i_n} \overline{var}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N}) \|f.1_{(j_1,\dots,j_N)}\|_\infty \\ &\leq \gamma_N V_\theta(f) + \sum_{j_1,\dots,j_N} \alpha^{-j_1} \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \overline{var}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N}) \alpha^{j_1} \|f.1_{(j_1)}\|_\infty \\ &\leq \gamma_N V_\theta(f) + a^\alpha_N \, [f]_\alpha \end{aligned}$$

- On the other hand, we have:

$$\begin{aligned} |\alpha^i \mathcal{L}^N f(ix)| &\leq (\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1 \cdots i_N ix)} \alpha^{i_1} \|f 1_{(i_1)}\|_\infty) \\ &\leq \Delta^\alpha_{N,0} [f]_\alpha \end{aligned}$$

We can conclude that $\mathcal{L}^N$ is bounded on $\mathcal{X}_\alpha$ and $\mathcal{X}_{\alpha,\theta}$. □

3.4 Essential spectral radius

We shall now prove the following proposition:

Proposition 4 *If γ_N, a^α_N and $\Delta^\alpha_{N,0}$ are finite, then, the essential spectral radius of $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ is strictly less than τ_N*

3.4.1 Explicit decomposition of $\mathcal{L}$

If $\mathcal{L}^N = Q + R$, with Q a compact operator, the spectral radius of R maximizes the essential spectral radius of $\mathcal{L}^N$. The following decomposition of $\mathcal{L}^N$ will enable us to derive the anounced estimation.

The compact part will concern the action of the operator over finite approximations of functions measurable with respect to the N-cylinders. Let m be a measure and consider:

$$E^{(N)}(f) = \sum_{i_1,\dots,i_n} 1_{(i_1,\dots,i_n)} \frac{1}{m(i_1,\dots,i_n)} \int_{(i_1,\dots,i_n)} f dm$$

For any finite family of N-cylinder $\mathcal{K}$ (this letter shall denote the family as well as the union of the corresponding cylinders), we define:

$$E^{(N)}_{\mathcal{K}}(f) = E^{(N)}(f).1_{\mathcal{K}}$$

The operator $Q^{(N)}_{\mathcal{K}} = \mathcal{L}^N E^{(N)}_{\mathcal{K}}$ clearly is compact. Let then write:

$$\bar{f} = f - E^{(N)}(f) \quad and \quad f^* = E^{(N)}(f) - E^{(N)}_{\mathcal{K}}(f)$$

The next relations clearly stand for any function f in $\mathcal{X}_{\alpha,\theta}$ and any $(i_1,\dots,i_N)$, N-cylinder,

- $V_\theta^{(i_1,\dots,i_N)}(E^{(N)}(f)) = 0$.
- $\|\bar{f}1_{(i_1,\dots,i_N)}\|_\infty \le var_{i_1,\dots,i_N}(\bar{f}) = var_{i_1,\dots,i_N}(f)$
- $[\bar{f}]_\alpha \le 2[f]_\alpha$

3.4.2 Inequalities

Let p_N be an integer such that $\Delta^\alpha_{N,p_N} < \delta_N$ and set $[f] = \alpha^{-p_N}[f]_\alpha$. We define the balanced norm $\|\|_b$ as follows: $\|f\|_b = V_\theta(f) + [f]$. The next lemma will enable us to control the norm $\|\|_b$ of the operator $\mathcal{L}^N - Q^{(N)}_{\mathcal{K}}$:

Lemma 3 *There exists $\mathcal{K}$ such that the following inequalities stand for all f in $\mathcal{X}_{\alpha,\theta}$.*

$$\begin{aligned}
V_\theta(\mathcal{L}^N f^*) &\le \delta_N\,[f] && (2)\\
V_\theta(\mathcal{L}^N \bar{f}) &\le (\gamma_N + v_N)V_\theta(f) && (3)\\
[\mathcal{L}^N f^*] &\le \delta_N[f] && (4)\\
[\mathcal{L}^N \bar{f}] &\le \gamma_N V_\theta(f) + 2\delta_N[f] && (5)
\end{aligned}$$

Proof

- Using (1) and $V_\theta^{(i_1,\dots,i_N)}(f^*) = 0$, we obtain:

$$\begin{aligned}
V_\theta(\mathcal{L}^N f^*) &\le \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N})\|f^*.1_{(j_1,\dots,j_N)}\|_\infty \\
&\le \sum_{j_1,\dots,j_N \in \mathcal{K}^c} \alpha^{-j_1} \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \overline{var}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N})\alpha^{j_1}\|f.1_{(j_1)}\|_\infty \\
&\le a^\alpha_N(\mathcal{K})\ \alpha^{p_N}[f]
\end{aligned}$$

  Where $a^\alpha_N(\mathcal{K})$ denotes the sum a^α_N restricted to the cylinders which are not in $\mathcal{K}$. As a^α_N is finite, one can choose $\mathcal{K}$ enough big for $a^\alpha_N(\mathcal{K})$ to be small. In particular, it can be chosen such that $a^\alpha_N(\mathcal{K}) \le \delta_N \alpha^{-p_N}$.

- Using (1) and $\|\bar{f}1_C\|_\infty \le var_C(f)$, we obtain:

$$\begin{aligned}
V_\theta(\mathcal{L}^N \bar{f}) &\le \gamma_N \sum_{j_1,\dots,j_N} (V_\theta^{(j_1,\dots,j_N)}(\bar{f})) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N})\|\bar{f}.1_{(j_1,\dots,j_N)}\|_\infty \\
&\le \gamma_N \sum_{j_1,\dots,j_N} (V_\theta^{(j_1,\dots,j_N)}(f)) + \theta^{-N} \sup_{j_1,\dots,j_N} (\overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N})) \sum_{j_1,\dots,j_N} var_{j_1,\dots,j_N}(f) \\
&\le (\gamma_N + v_N)V_\theta(f)
\end{aligned}$$

- The hypothesis says that the sum $\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}$ is finite for all i and, hence, converges uniformly for $i \le p_N$. $\mathcal{K}$ can then be chosen such that

$$\max_{i\le p_N}\Big(\sum_{i_1,\dots,i_N\in\mathcal{K}^c} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}\Big) < \delta_N$$

Then, if $i \le p_N$:

$$\begin{aligned}|\alpha^i \mathcal{L}^N f^*(ix)| &\le \max_{i\le p_N}\Big(\sum_{i_1,\dots,i_N\in\mathcal{K}^c} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}\Big)[f]_\alpha \\ &\le \delta_N [f]_\alpha\end{aligned}$$

If $i > p_N$, let use the definition of δ_N:

$$\begin{aligned}|\alpha^i \mathcal{L}^N f^*(ix)| &\le \sup_{i>p_N}\Big(\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}\Big)[f]_\alpha \\ &\le \delta_N [f]_\alpha\end{aligned}$$

Finally,

$$[\mathcal{L}^N f^*] \le \delta_N [f]$$

- If $i \le p_N$:

$$\begin{aligned}|\alpha^i \mathcal{L}^N \bar{f}(ix)| &\le \alpha^i \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N i)} var_{i_1,\dots,i_N}(f) \\ &\le \alpha^{p_N} \|e^{\phi_N}\|_\infty \sum_{i_1,\dots,i_N} var_{i_1,\dots,i_N}(f) \\ &\le \alpha^{p_N}\gamma_N V_\theta(f)\end{aligned}$$

Notice that the balanced norm will be useful here to erase the term α^{p_N}.

And, if $i > p_N$:

$$\begin{aligned}|\alpha^i \mathcal{L}^N \bar{f}(ix)| &\le \sup_{i>p_N}\Big(\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}\Big)[\bar{f}]_\alpha \\ &\le \delta_N 2[f]_\alpha\end{aligned}$$

Hence,

$$[\mathcal{L}\bar{f}] \le \gamma_N V_\theta(f) + 2\delta_N[f]$$

The lemma 3 is proved. □

3.4.3 Conclusion

Let us choose $\mathcal{K}$ such that the inequalities of the lemma 3 are satisfied. We obtain, by combination of these inequalities:

$$\|\mathcal{L}^N(\bar{f}+f^*)\|_b \le (2\gamma_N + v_N)V_\theta(f) + 4\delta_N[f]$$

We have established, for all N, the existence of a compact operator $Q_{\mathcal{K}}^{(N)}$ such that:

$$\|\mathcal{L}^N - Q_{\mathcal{K}}^{(N)}\|_b \le max(2\gamma_N + v_N; 4\delta_N) \le \tau_N^N$$

We can conclude that the essential spectral radius of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$ is less than τ_N. As the spectrum of an operator does not change when a norm is replaced by an equivalent norm, this achieves the proof of the proposition 4. □

3.5 Spectral decomposition of $\mathcal{L}$

We first prove:

Lemma 4 *Let a positive operator $\mathcal{L}$ be continuous on $\mathcal{X}_\alpha$, with a spectral radius ϱ. Assume it is also continuous on $\mathcal{X}_{\alpha,\theta}$, with an essential spectral radius $\varrho_e < \varrho$.*

The spectral radius of $\mathcal{L}$ on $\mathcal{X}_{\alpha,\theta}$ is exactly ϱ. Moreover, the real ϱ is an eigenvalue of $\mathcal{L}$.

Proof The points of the spectrum of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$ whose modulus is more than the essential spectral radius are eigenvalues. But, an eigenfunction of $\mathcal{L}$ in $\mathcal{X}_{\alpha,\theta}$ also is in $\mathcal{X}_\alpha$: the modulus of the corresponding eigenvalue is less than the spectral radius of $\mathcal{L}$ over $\mathcal{X}_\alpha$. Hence, the spectral radius of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$ is less than ϱ.

As $\tilde{\alpha}$ is in $\mathcal{X}_{\alpha,\theta}$, the following inequalities: $\frac{1}{n}\log\|\mathcal{L}^n(\tilde{\alpha})\| \geq \frac{1}{n}\log\left[\mathcal{L}^n(\tilde{\alpha})\right]_\alpha \geq \log\varrho$ prove that the spectral radius of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$ is greater than ϱ.

Let then λ be one of the eigenvalues of modulus ϱ. There exists a sequence of complex numbers with modulus strictly greater than ϱ, $(\lambda_n)_{n\in\mathbf{N}}$, going to λ as n tend to infinity, and a function h such that the sequence: $(\|R(\lambda_n,\mathcal{L})h\| = \|\sum_{k=0}^{+\infty}\lambda_n^k, \mathcal{L}^k(h)\|)_{n\in\mathbf{N}}$ is not bounded.

We deduce that the sequence $(\|R(|\lambda_n|,\mathcal{L})|h|\|)_{n\in\mathbf{N}}$ is not bounded and that, hence, $\lim_n|\lambda_n| = \varrho$ is in the spectrum of $\mathcal{L}$ (for more details, see proposition V.4.1 in [21]). As the elements of the spectrum with modulus ϱ are isolated, they are eigenvalues: ϱ is an eigenvalue of $\mathcal{L}$. $\square$

We now are able to finish the proof of the theorem 1-(N). Let us recall:

Definition 8 *We shall say that ϕ is in $\mathcal{C}_{\alpha,\theta}^{(N)}$ if:*

$$\begin{array}{ll}
(C_N0) & P^\alpha(\phi) = 0 \\
(C_N1) & \|e^{\phi_N}\|_\infty < \frac{\theta^N}{4} \\
(C_N2) & \Delta_{N,0}^\alpha < +\infty \ \textit{ and } \ \Delta_{N,\infty}^\alpha < \frac{1}{4} \\
(C_N3.1) & \sup_{i_1,\dots,i_N}(\overline{V}_\theta^{(i_1,\dots,i_N)}(e^{\phi_N})) < \frac{\theta^{2N}}{4} \\
(C_N3.2) & a_N^\alpha < +\infty
\end{array}$$

We shall prove that if ϕ is in $\mathcal{C}_{\alpha,\theta}^{(N)}$, then the conclusion of theorem 1 holds (This is what says the theorem 1-(N)) . In this case, the assumptions of the proposition 4 are satisfied and we have $\tau_N < 1$. The lemma 4 implies the existence of a spectral gap.

It becomes possible to precise the decomposition of $\mathcal{L}_\phi$. One can find a curve (for example, a circle) that separates the spectrum in two different subsets:

- One, Σ_0 included in the disk of radius τ_N.
- The other one made of a finite number, k, of isolated points, $(\lambda_i)_{i=1..k}$ out of this circle.

According to Kato ([11], theorem 6.17, page 178) one can define two spectral projectors, P_0 and P_1, each of them associated to one of the subsets, such that, if $\mathcal{R}_0 = \mathcal{L}P_0$ et $\mathcal{Q}_0 = \mathcal{L}P_1$, then,

$$\mathcal{L} = \mathcal{Q}_0 + \mathcal{R}_0$$

with $\mathcal{Q}_0\mathcal{R}_0 = \mathcal{R}_0\mathcal{Q}_0 = 0$, $\Sigma(\mathcal{Q}_0) = (\lambda_i)_{i=1..p}$, $\Sigma(\mathcal{R}_0) = \Sigma_0$.

The operator $\mathcal{Q}_0$ can itself be expressed as the finite sum of the spectral projectors on the (finite dimensional) eigenspaces associated to each non zero eigenvalue $(\lambda_i)_{i=1..k}$. Let τ' be the biggest modulus of the $(\lambda_i)_{i=1..k}$ whose modulus is strictly less than one. It is possible to use the following decomposition: $\mathcal{L} = \mathcal{Q} + \mathcal{R}$ where the eigenvalues of the compact operator $\mathcal{Q}$ all are of modulus one and the spectrum of $\mathcal{R}$ is included in the disk of radius $\tau' < 1$. $\square$

3.6 Classes of potential

We first will prove intermediate results in order to prove the following proposition:

Proposition 5

$$\mathcal{C}_{\alpha,\theta} \subset \bigcup_{M\geq 1} \bigcap_{N\geq M} \mathcal{C}_{\alpha,\theta}^{(N)}$$

Then, we will give independantly a proof of the proposition 6 providing a way to estimate the pressure at infinity.

3.6.1 Oscillations

A first result concerns the local oscillations of the iterated potentials.

Lemma 5 *For any integer p, there exists a constant C_p such that for any N, r integers, with $0 \leq r \leq p$,*

$$V_\theta^{(j_1,\dots,j_{pN+r})}(e^{\phi_{pN+r}}) \leq C_p \|e^{\phi_p}\|_\infty^N V_\theta^{(j_1,\dots,j_p)}(e^{\phi_p})$$

$$\textit{and, more precisely,} \qquad v_{pN+r} \leq C_p \|e^{\phi_p}\|_\infty^N v_p$$

Proof We will only prove the result for $p = 1$ (and $r = 0$). The general proof works in the same way. So, we have to prove the existence of a constant C such that for any integer N,

$$\sup_{i_1,\dots,i_N} (V_\theta^{(i_1,\dots,i_N)}(e^{\phi_N})) \leq C \|e^\phi\|_\infty^{N-1} \sup_i (V_\theta^{(i)}(e^\phi)) \tag{6}$$

We first are going to prove that for any integer N, any integer $n > N$ and any cylinder $(i_1, \dots, i_n)$ of lenght n, one has:

$$var_{i_1,\dots,i_n}(e^{\phi_N}) \leq \|e^\phi\|_\infty^{N-1} \sum_{k=1}^{N} (var_{i_k,\dots,i_n}(e^\phi)) \tag{7}$$

It clearly stands if $N = 1$ (equality). Assume it holds at the step $N-1$ and write $var_{i_1,\dots,i_n}(e^{\phi_N})$:

$$\begin{aligned}
var_{i_1,\dots,i_n}(e^{\phi_N}) &= var_{i_1,\dots,i_n}(e^{\phi_{N-1}} e^\phi \circ T^{N-1}) \\
&\leq \|e^\phi\|_\infty var_{i_1,\dots,i_n}(e^{\phi_{N-1}}) + var_{i_N,\dots,i_n}(e^\phi)\|e^\phi\|_\infty^{N-1} \\
&\leq \|e^\phi\|_\infty \left(\|e^\phi\|_\infty^{N-2} \sum_{k=1}^{N-1} (var_{i_k,\dots,i_n}(e^\phi)) \right) + var_{i_N,\dots,i_n}(e^\phi)\|e^\phi\|_\infty^{N-1} \\
&\leq \|e^\phi\|_\infty^{N-1} \sum_{k=1}^{N} (var_{i_k,\dots,i_n}(e^\phi))
\end{aligned}$$

Hence, (7) holds for any N, $n > N$ and any cylinder $i_1, \dots, i_n$. We deduce, by summing over all the cylinders:

$$\begin{aligned}
V_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) &= \sum_{n>N} \theta^n (\sum_{i_{N+1},\dots,i_n} var_{j_1,\dots,j_N,i_{N+1},\dots,i_n}(e^{\phi_N})) \\
&\leq \sum_{n>N} \theta^n \sum_{i_N,\dots,i_n} \left(\|e^\phi\|_\infty^{N-1} \sum_{k=1}^{N-1} (var_{j_k,\dots,j_N,i_{N+1},\dots,i_n}(e^\phi)) \right) \\
&\leq \|e^\phi\|_\infty^{N-1} \sum_{k=1}^{N-1} \theta^k \left(\sum_{n>N} \theta^{n-k} \sum_{i_N,\dots,i_n} var_{j_k,\dots,j_N,i_{N+1},\dots,i_n}(e^\phi) \right) \\
&\leq \|e^\phi\|_\infty^{N-1} \sum_{k=1}^{N-1} \theta^k \sup_{j_k,\dots,j_N} (V_\theta^{(j_k,\dots,j_N)}(e^\phi)) \\
&\leq C \|e^\phi\|_\infty^{N-1} \sup_i (V_\theta^{(i)}(e^\phi))
\end{aligned}$$

Hence, the inequality (6) is proved.

Using this result, we obtain the corresponding result taking in account the non regular cylinders:

$$\begin{aligned} v_N &\leq \sup_{i_1,\dots,i_N}(V_\theta^{(i_1,\dots,i_N)}(e^{\phi_N})) + \sum_{n=0}^{+\infty}\theta^n \sum_{i_1,\dots,i_n\in\mathcal{M}_N^o} e^{\phi_N(i_1\cdots i_n)} \\ &\leq C\|e^\phi\|_\infty^N \sup_i(V_\theta^{(i)}(e^\phi)) + \|e^{\phi_{N-1}}\|_\infty \sum_{n=0}^{+\infty}\theta^n \sum_{i_1,\dots,i_n\in\mathcal{M}^o} e^{\phi(i_1\cdots i_n)} \\ &\leq C'\|e^\phi\|_\infty^N v_1 \end{aligned}$$

The anounced result can be proved exactly in the same way. □

3.6.2 Total weighted variation

The following lemma refers to the quantities a_N^α, weighted sums of the local oscillations of the potential defined in section 3.3, page 14. It provides a simple criterion for their finitness.

Lemma 6 *If $a_1^\alpha < +\infty$, then, for all $N \geq 1$, $a_N^\alpha < +\infty$.*

Proof We need an inequality a bit more precise than (7). For any $N > 1$, and $n > 1$:

$$\begin{aligned} var_{i_1,\dots,i_n}(e^{\phi_N}) &= var_{i_1,\dots,i_n}(e^\phi e^{\phi_{N-1}\circ T}) \\ &\leq e^{\phi(i_1\cdots i_n)} var_{i_1,\dots,i_n}(e^{\phi_{N-1}}\circ T) + var_{i_1,\dots,i_n}(e^\phi) e^{\phi_{N-1}\circ T(i_1\cdots i_n)} \\ &\leq e^{\phi(i_1 i_2)} var_{i_2,\dots,i_n}(e^{\phi_{N-1}}) + var_{i_1,\dots,i_n}(e^\phi)\|e^{\phi_{N-1}}\|_\infty \end{aligned}$$

If $n = 1$:

$$var_{i_1}(e^{\phi_N}) \leq e^{\phi(i_1)}\|e^{\phi_{N-1}}\|_\infty + var_{i_1}(e^\phi)\|e^{\phi_{N-1}}\|_\infty$$

Assume that all the cylinders are regular. Then, by recurence, :

$$\begin{aligned} a_N^\alpha &= \sum_{n=1}^{+\infty}\theta^n \sum_{i_1,\dots,i_n} \alpha^{-i_1} var_{i_1,\dots,i_n}(e^{\phi_N}) \\ &\leq \theta\sum_{i_1}\alpha^{-i_1}(e^{\phi(i_1)}\|e^{\phi_{N-1}}\|_\infty + var_{i_1}(e^\phi)\|e^{\phi_{N-1}}\|_\infty) \\ &\quad + \sum_{n=2}^{+\infty}\theta^n \sum_{i_1,\dots,i_n}\alpha^{-i_1}\Big((e^{\phi(i_1 i_2)} var_{i_2,\dots,i_n}(e^{\phi_{N-1}}) + var_{i_1,\dots,i_n}(e^\phi)\|e^{\phi_{N-1}}\|_\infty\Big) \\ &\leq \theta\|e^\phi\|_\infty^N(\sum_{i_1}\alpha^{-i_1}) + \|e^{\phi_{N-1}}\|_\infty\theta\sum_{i_1}\alpha^{-i_1} var_{i_1}(e^\phi) \\ &\quad + \sum_{n=2}^{+\infty}\theta^n \sum_{i_1,\dots,i_n}\alpha^{i_2-i_1} e^{\phi(i_1 i_2)}\alpha^{-i_2} var_{i_2,\dots,i_n}(e^{\phi_{N-1}}) \\ &\quad + \|e^{\phi_{N-1}}\|_\infty \sum_{n=2}^{+\infty}\theta^n \sum_{i_1,\dots,i_n}\alpha^{-i_1} var_{i_1,\dots,i_n}(e^\phi) \\ &\leq \theta\|e^\phi\|_\infty^N(\sum_i \alpha^{-i}) \\ &\quad + \sup_{i_2}(\sum_{i_1}\alpha^{i_2-i_1}e^{\phi(i_1 i_2)})\theta\sum_{n=2}^{+\infty}\theta^{n-1}\sum_{i_2,\dots,i_n}\alpha^{-i_2} var_{i_2,\dots,i_n}(e^{\phi_{N-1}}) \end{aligned}$$

$$+\|e^{\phi_{N-1}}\|_\infty \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \alpha^{-i_1} var_{i_1,\dots,i_n}(e^\phi)$$
$$\leq \theta \|e^\phi\|_\infty^N (\sum_i \alpha^{-i}) + \Delta_0^\alpha \theta a_{N-1}^\alpha + \gamma_{N-1} a_1^\alpha$$
$$< +\infty$$

If some cylinders are not regular, the same argument as in the previous lemma ensures the result. □

3.6.3 Conclusion

Let ϕ be a potential in $\mathcal{C}_{\alpha,\theta}$. We are going to exhibit an integer N_3 such that if $N > N_3$, $\phi \in \mathcal{C}_{\alpha,\theta}^{(N)}$.

- $(C0)$ is the same as (C_N0)
- There exists an integer N_0 such that $\|e^{\phi_{N_0}}\|_\infty$ is bounded. For $N > N_0$, $\|e^{\phi_{N_0}}\|_\infty$ decreases exponentially fast, strictly faster than θ^{2N}. Hence there exists some integer N_1 such that if $N > N_1$, $\|e^{\phi_N}\|_\infty \leq \frac{\theta^{2N}}{4}$. Meaning that as soon as $N > N_1$, (C_N1) holds for ϕ.
- According to $(C2)$, there exists an integer N_2 (say $N_2 \geq N_1$) such that as soon as $N > N_2$, (C_N2) holds.
- $(C3)$ and the inequality (6) prove that $v_{N_1} < +\infty$. The lemma 5 proves that for $N > N_1$, v_N decreases exponentially fast with rate $\sqrt[N_1]{\|e^{\phi_{N_1}}\|_\infty} < \theta^2$. Hence, there exists $N_3 > N_2$ such that as soon as $N > N_3$, $v_N < \frac{\theta^{2N}}{4}$. This proves that $(C_N3.1)$ holds.
- $(C_N3.2)$ holds for any N, according to the lemma 6, because $a_1^\alpha \leq \sum_i \alpha^{-i} v_1 < +\infty$.

It appears that $\phi \in \mathcal{C}_{\alpha,\theta}^{(N)}$ for any $N > N_3$. □

Remark 9 *We also get a bound for the essential spectral radius of the operator. As it is smaller than τ_N for any $N > N_3$, we can say that it is smaller to the corresponding limits when N tends to infinity. Hence, using again lemma 5, we obtain:*

$$\log \tau \leq \max\left(\lim_{N\to\infty} \frac{1}{N} \log \gamma_N, P_\infty^\alpha(\phi)\right)$$

3.6.4 Pressure at infinity

The following proposition shall be useful to obtain an upper bound of the pressure at infinity of a potential.

Proposition 6 *If (a) $\Delta_0(\phi) < +\infty$, (b) $\Delta_\infty(\phi) < \delta$ and (c) $\lim_{j\to\infty} e^{\phi(ij)} = 0$, then $P_\infty(\phi) < \log \delta$*

Remark 10 *We also could state it in the following way. Any potential ϕ satisfying $(C0)$, $(C1)$, $(C3)$ and (a) $\Delta_0(\varphi) < +\infty$, (b) $\Delta_\infty(\varphi) < \delta$ and (c) $\lim_{j\to\infty} e^{\varphi(ij)} = 0$ for $\varphi = \phi - \log\tilde{\alpha} \circ T + \log\tilde{\alpha}$ is in $\mathcal{C}_{\alpha,\theta}$.*

Proof Let first notice that, as the sequence $(\Delta_{N,0}^\alpha)_{N\in\mathbf{N}}$ is subadditive, $\Delta_0^\alpha < +\infty$ implies for all N, $\Delta_{N,0}^\alpha < +\infty$.

For any $N \in \mathbf{N}^*$, consider the statement H_N:

$$\exists p_N \in \mathbf{N}, \quad \sup_{j>p_N} \left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)} \right) < \delta^N$$

The statement H_N holds for $N=1$, as this is one of the assumptions. Assume it holds for an integer N. Let p_N be the corresponding integer (it can be chosen greater than p). Then:

$$\begin{aligned}
\sum_{i_1,\ldots,i_{N+1}} e^{\phi_{N+1}(i_1\cdots i_{N+1}j)} &= \sum_{i_{N+1}} e^{\phi(i_{N+1}j)} \sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} \\
&= \sum_{i_{N+1}\le p_N} e^{\phi(i_{N+1}j)} \sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} + \sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)} \sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} \\
&< (\sum_{i\le p_N} e^{\phi(ij)}) \sup_{i\le p_N} (\sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_N i)}) \\
&\quad + (\sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)}) \sup_{i>p_N} (\sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_{N+1} i)}) \\
&< (\sum_{i\le p_N} e^{\phi(ij)})\Delta_{N,0} + (\sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)})\delta^N
\end{aligned}$$

For all $\epsilon>0$, one can choose p_ϵ enough big to ensure that as soon as $j>p_\epsilon$, $\sum_i e^{\phi(ij)}<\delta$ (Assumption (b)) and $\sum_{i\le p_N} e^{\phi(ij)}\le\epsilon$ (Assumption (c)). One then has:

$$\sup_{j>p_\epsilon}(\sum_{i_1,\ldots,i_N} e^{\phi_N(i_1\cdots i_N j)}) \le \epsilon\Delta_{N,0}+\delta^{N+1}$$

This is sufficient to see that H_{N+1} is satisfied. Hence, H_N is satisfied for all integers. We deduce that $P_\infty(\phi)<\log\delta$. □

3.7 Mixing

We now assume that the system has a conformal measure, say m. We also assume that ϕ is topologically mixing.

Proposition 7 (Mixing) *If $\phi\in\mathcal{C}_{\alpha,\theta}$, is topologically mixing and if there exists a conformal measure m for ϕ, then 1 is a simple eigenvalue of $\mathcal{L}$, over $\mathcal{X}_{\alpha,\theta}$, and it is the only eigenvalue of modulus 1.*

3.7.1 Existence of a positive invariant function

Lemma 7 *There exists a positive invariant function*

Proof We first prove that the conformal measure charges all the cylinders of $\mathcal{F}$. Let $(i_1,\ldots,i_n)$ be a cylinder intersecting $\mathcal{F}$. The topological mixing property says that an integer N can be chosen large enough to ensure that for all x in $\mathcal{F}$, there is a $(j_1,\ldots,j_N)$ with $i_1\cdots i_n j_1\cdots j_N x\in\mathcal{F}$ and hence $e^{\phi_{n+N}(i_1\cdots i_n j_1\cdots j_N x)}>0$. As m is conformal,

$$m(i_1,\ldots,i_n)=m(\mathcal{L}^{n+N}1_{i_1,\ldots,i_n})=\sum_{j_1,\ldots,j_N} m\left(e^{\phi_{n+N}(i_1\cdots i_n j_1\cdots j_N x)}\right)>0$$

Hence, we can use the conformal measure m in the definition of the part $E^{(N)}$ of the explicit decomposition of $\mathcal{L}^N$ (section 3.4). If w is in $\mathcal{X}_{\alpha,\theta}$ and $\int_{\mathcal{F}}|w|dm=0$, then $E^{(N)}w=0$. We deduce from the lemma 3 that for such a w, $\|\mathcal{L}^N w\|_b\le\tau_N^N\|w\|_b$.

Let now ρ be an invariant function. $\rho^+ := \max(\rho,0)$ is invariant m-almost everywhere in the following sense. We have $\mathcal{L}(\rho^+)\ge\rho^+$ and $m(\mathcal{L}(\rho^+))=m(\rho^+)$.

Set $w=\mathcal{L}(\rho^+)-\rho^+$. As $w\ge 0$ and $m(w)=0$, we know that $\|\mathcal{L}^N w\|_b\le\tau_N^N\|w\|_b$. We deduce that $\mathcal{L}^n(\rho^+)$ converges in $\mathcal{X}_{\alpha,\theta}$ as n goes to infinity. The limit is a positive invariant function. □

3.7.2 Simplicity of the eigenvalue 1

Lemma 8 *Let $\rho \in \mathcal{X}_{\alpha,\theta}$ be a non zero positive invariant function. ρ is strictly positive. All the invariant functions of $\mathcal{L}_\phi$ are proportional to ρ. There are no other elements in the characteristic space associated with the eigenvalue 1. The conformal measure is unique.*

Proof Assume that for any cylinder C, $\inf_{C\cap\mathcal{F}} \rho = 0$. Then, for all $x \in \mathcal{F}$ and all integer n:

$$\rho(x) = \mathcal{L}^n \rho(x) = \sum_{i_1,\dots,i_n} e^{\phi_n(i_1\cdots i_n x)} \rho(i_1 \cdots i_n x) \leq \theta^n \sum_{i_1,\dots,i_n} var_{i_1,\dots,i_n}(\rho)$$

and hence, as $V_\theta(\rho) < +\infty$, $\forall x \in \mathcal{F}, \rho(x) = 0$. As ρ is non zero, we deduce that there exists a cylinder C over which ρ is greater than a strictly positive real.

The topological mixing condition ensures the existence of an integer n such that $T^n(C\cap\mathcal{F}) = \mathcal{F}$. Hence, for all $x \in \mathcal{F}$, exists $i_1, \dots, i_n$ such that $i_1 \cdots i_n x \in C \cap \mathcal{F}$. Hence, $\rho(x) = \mathcal{L}(\rho)(x) \geq e^{\phi(i_1\cdots i_n x)} \rho(i_1 \cdots i_n x) \geq e^{\phi(i_1\cdots i_n x)} \inf_C \rho$. Finally:

$$\forall x \in \mathcal{F}, \quad \rho(x) > 0$$

Let ρ' be another invariant function, λ a real, and set $\rho_\lambda := \rho' - \lambda\rho$. The function ρ_λ is invariant. For the same reasons as in the lemma 7, there exists a set $X \subset \mathcal{F}$, with $m(X) = 1$, such that for all $x \in X$, $\mathcal{L}(\rho_\lambda^+)(x) = \rho_\lambda^+(x)$. On this set, the same argument as for ρ proves that either ρ_λ^+ is zero either it is strictly positive. Hence, the set $\{\rho' \leq \lambda\rho\} = \{\rho_\lambda^+ = 0\}$ is either of measure 0 or of measure 1.

We now choose λ as the infimum of the reals such that $m\{\rho' \leq \lambda\rho\} = 0$. We then have $\rho' = \lambda\rho$, m-almost everywhere. The same argument as in the proof of the lemma 7 ensures that $\|\rho_\lambda\|_b = \|\mathcal{L}^N(\rho_\lambda)\|_b \leq \tau_N^N \|\rho_\lambda\|_b$ and, hence, $\|\rho_\lambda\|_b = 0$. We conclude that $\rho' = \lambda\rho$ in $\mathcal{X}_{\alpha,\theta}$. ρ is the only eigendirection associated to the eigenvalue 1.

Let now assume that ρ' is such that $\mathcal{L}_\phi(\rho') = \rho + \rho'$. This implies $m(\rho+\rho') = m(\mathcal{L}_\phi(\rho')) = m(\rho')$ and, hence, $m(\rho) = 0$. But ρ is strictly positive. We deduce that such a ρ' can not exist. Hence, the eigenvalue 1 is simple.

The unicity of the conformal measure follows from the theorem 6.22, page 184 in [11]. It ensures that the eigenvalue 1 of the adjoint operator of $\mathcal{L}$ also has multiplicity 1. $\square$

3.7.3 Renormalization

Definition 9 *The only strictly positive invariant function, ρ, being given, one can define the* renormalised operator*, $\mathcal{P}$, by: $\mathcal{P} = \mathcal{L}_{\phi+\log\rho-\log\rho\circ T}$, or equivalently:*

$$\forall f \in \mathcal{X}_{\alpha,\theta}, \quad \mathcal{P}(f) = \frac{1}{\rho}\mathcal{L}_\phi(f.\rho)$$

The real 1 is eigenvalue of $\mathcal{P}$ with eigenfunction 1. Notice also that $\mathcal{P}$ is the adjoint of U in L^2_μ because:

$$\int_\mathcal{F} U(f).g d\mu = \int_\mathcal{F} U(f).g.\rho dm = \int_\mathcal{F} f.\mathcal{L}_\phi(g\rho) dm = \int_\mathcal{F} f.\mathcal{P}(g) d\mu$$

3.7.4 Other eigenvalues

Lemma 9 *The only eigenvalue of modulus 1 is 1.*

Proof We first prove that that if λ is an eigenfunction of modulus 1 of $\mathcal{L}$, then λ is a root of the unity.

The set of the eigenvalues of U is a subgroup of the complex unit circle. $\mathcal{P}$ is its adjoint: the eigenvalues of modulus 1 of $\mathcal{P}$ also form a subgroup of the circle.

Let $h \in \mathcal{X}_{\alpha,\theta}$ be an eigenfunction of $\mathcal{L}$, associated to the eigenvalue λ of modulus 1. Then, $|h| = \rho$ (because it is an invariant function μ-almost everywhere) and, if we set $h = \rho e^{i\omega}$, $\mathcal{P}(e^{i\omega}) = \lambda e^{i\omega}$. We see that λ^p also is eigenvalue of $\mathcal{P}$ (with eigenfunction $e^{ip\omega}$). Hence λ^p also is an eigenvalue of $\mathcal{L}$.

The total number of eigenvalues with modulus greater than τ is finite. The subgroup of eigenvalues of modulus 1 must then be finite. Hence, it is made of a finite number of roots of the unity.

The eigenfunctions of $\mathcal{L}$ corresponding to eigenvalues of modulus 1 are invariant with respect to an iterate of $\mathcal{L}$. We can prove that they are strictly positive and hence, proportional to ρ, using the same argument as in lemma 8.

We conclude that 1 is the only eigenvalue of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$. The lemma 9 is proved. $\square$

The proposition 7 is proved. $\square$

3.7.5 Decomposition

It now is possible to precise the spectral decomposition of $\mathcal{L}$.

- On one hand, $\mathcal{Q}(f) = \rho . \int_{\mathcal{F}} f dm$
- On the other hand, $\mathcal{R}$ with spectral radius lower than some $\tau' < 1$.

4 Consequences

We shall now briefly present the links between the main results and the settings in which they might be used. The only aim is to give an idea of the possible use of these results.

4.1 Ergodic results

The existence of a real eigenfunction associated with the eigenvalue 1 of $\mathcal{L}$ in $\mathcal{X}_{\alpha,\theta}$ implies the existence of an a.c.i.m.. Choose ρ such that $m(\rho) = 1$ and set $d\mu = \rho dm$. The measure μ is invariant and absolutely continuous with respect to m.

The following proposition proves that the functions of $\mathcal{X}_\theta$ have exponential decay of correlations.

Proposition 8 (Exponential decay of correlations) *There exists $\tau' \in]0,1[$ such that for all functions f, g in $\mathcal{X}_\theta$, there exists a constant $C = C(f,g)$ with:*

$$Cor_n(f,g) := |\int_{\mathcal{F}} f.g \circ T^n d\mu - \int_{\mathcal{F}} f d\mu . \int_{\mathcal{F}} g d\mu| \leq C.\tau'^n$$

Proof In that case, $f\rho \in \mathcal{X}_{\alpha,\theta}$ et $g \in L^\infty$. We get:

$$\begin{aligned} Cor_n(f,g) &= |\int_{\mathcal{F}} f.g \circ T^n d\mu - \int_{\mathcal{F}} f d\mu . \int_{\mathcal{F}} g d\mu| \\ &\leq \int_{\mathcal{F}} \left| \mathcal{L}_\phi^n(f\rho) - \int_{\mathcal{F}} f\rho dm.\rho \right| |g| dm \\ &\leq \int_{\mathcal{F}} |\mathcal{R}^n(f\rho)||g| dm \leq [\mathcal{R}^n(f\rho)]_\alpha \int_{\mathcal{F}} |g| \tilde{\alpha} dm \leq \tau'^n \|f\rho\| \|g\|_\infty \\ &\leq C.\tau'^n \quad \square \end{aligned}$$

The set $\{f|\ f\rho \in \mathcal{X}_{\alpha,\theta}\}$ which contains $\mathcal{X}_\theta$ is dense in L^2_μ. This proves the strong mixing of the system because we have the convergence of $\int_\mathcal{F} f.f\circ T^n d\mu$ toward $\left(\int_\mathcal{F} f d\mu\right)^2$ for all f in a dense subset of L^2_μ. The corollary 1 is proved. $\Box$

4.2 Statistical mechanics

The only point we have to notice is the following:

Lemma 10 *The conformal measure m_ϕ of the operator $\mathcal{L}_\phi$ is a Gibbs state of the potential ϕ.*

Proof We have to verify that, over m, a version of the conditional expectation of f knowing that $T^n x = T^n y$ can be written:

$$m(f|T^n\mathcal{B})(y) = \frac{(U\mathcal{L})^n(f)(y)}{(U\mathcal{L})^n(1)(y)}$$

First notice that this last quantity is measurable with respect to $T^n\mathcal{B}$. Then, let g be a function measurable with respect to $T^n\mathcal{B}$ (depending only of the coordinates whose index is more than n) ; let write $g = h\circ T^n$,

$$\begin{aligned}\int_\mathcal{F} g.\frac{(U\mathcal{L})^n(f)}{(U\mathcal{L})^n(1)}dm &= \int_\mathcal{F}\left(h.\frac{\mathcal{L}^n(f)}{\mathcal{L}^n(1)}\right)\circ T^n dm\\ &= \int_\mathcal{F} h.\mathcal{L}^n(f)dm\\ &= m(g.f) \quad \Box\end{aligned}$$

The same computation can prove that if ρ is the unique eigenfunction of $\mathcal{L}$, the measure $d\mu = \rho dm$ is a Gibbs state on $\mathbf{N}^*$, translation invariant, associated to the potential $\phi - \log\rho\circ T + \log\rho$.

4.3 Interval maps

4.3.1 General result

A simple change of variable proves that the Lebesgue measure is conformal for the RPF operator.

The functions of the interval with bounded variations are in $\mathcal{X}_\theta$. The partition of $\mathcal{F}$ in cylinders $(i_1,\ldots,i_n)$ corresponds to a partition of the original interval in subintervals $I_{i_1,\ldots,i_n}$. Hence, if $Var_J(f)$ denotes the (usual) variation of f on the interval J,

$$\sum_{i_1,\ldots,i_n} var_{i_1,\ldots,i_n}(f) \le \sum_{I_{i_1,\ldots,i_n}} Var_{I_{i_1,\ldots,i_n}}(f) \le Var_I(f)$$

and, hence,

$$V_\theta(f) \le (\sum_n \theta^n) Var_I(f)$$

4.3.2 Example

We now give the proof of the corollary 4.

Proof We denote by dl the Lebesgue measure on I, as well as its image on the shift. We shall denote by $a_i := |I_i| = l(I_i) = l(i)$. Without loss of generality, we can assume that this sequence is decreasing.

Let notice that the condition 2. implies:

$$\forall x,y\in(i),\quad \exp(-K_1\frac{|x-y|}{a_i}) \le \frac{F'(y)}{F'(x)} \le \exp(K_1\frac{|x-y|}{a_i}) \quad \textit{and} \quad a_i e^{-K_1} \le \frac{1}{F'(x)} \le a_i e^{K_1}$$

Let $\psi(x) := -\log F'(x)$. This potential does not satisfy the required properties for the application of our theorem. We shall use another one. Let i_0 be the first integer i such that $a_i \leq \frac{e^{-K_1}}{K_0}$. Let $(\underline{a}_n)$ be the sequence defined as follows:

$$\underline{a}_i := \begin{cases} a_{i_0} & if\ i < i_0 \\ a_i & if\ i \geq i_0 \end{cases}$$

Let then set:

$$\forall x \in (ij), \quad \phi(x) = \psi(x) + \log \underline{a}_j - \log \underline{a}_i$$

Let us set $\frac{1}{\sqrt{K_0}} < \theta < 1$ and for all integer i, $\alpha_i = -\underline{a}_i \log \underline{a}_i$. According to the condition 3., the sequence (α_i) is summable. We shall use our result enlarged by the remark 4 ; the results also apply when the sequence is only summable and not exponentially decreasing. Let then set $\varphi(ijx) = \phi(ijx) - \log \alpha_j + \log \alpha_i$ and notice that there is a constant $\underline{M} > 0$ such that,

$$\sum_{i=1}^{+\infty} e^{\varphi(ij)} \leq \sum_{i=1}^{+\infty} \frac{\alpha_i}{\alpha_j} a_i e^{K_1} \frac{\underline{a}_j}{\underline{a}_i} \leq \frac{\underline{a}_j}{\alpha_j} e^{K_1} (\sum_{i=1}^{+\infty} \frac{a_i \alpha_i}{\underline{a}_i}) \leq \frac{e^{K_1}}{\log \underline{a}_j} (\sum_{i=1}^{+\infty} a_i \log \underline{a}_i) \leq \frac{-\underline{M}}{\log \underline{a}_j} \tag{8}$$

Let now verify that the potential ϕ is in $\mathcal{C}_{\alpha,\theta}$.

- $(C0)$ The pressure $P^\alpha(\phi) = P(\varphi)$ is finite because $\sum_{i=1}^{+\infty} e^{\varphi(ij)}$ is uniformly bounded (proposition 1). The measure $dm_\varphi = \frac{\alpha_i}{\underline{a}_i} dl$ is finite and satisfies $m_\varphi(1) = m_\varphi(\mathcal{L}_\varphi^n 1) \leq m_\varphi(1) \|\mathcal{L}_\varphi^n 1\|_\infty$. We deduce that, $P^\alpha(\phi) \geq 0$. It is enough here because in what follows, we can replace ϕ by $\phi - P^\alpha(\phi)$. But in facts it is possible to prove $P^\alpha(\phi) = 0$.
- $(C1)$ Let us estimate $e^\phi(ijx) = e^\psi(ijx) \frac{\underline{a}_j}{\underline{a}_i}$. Let distiguish three cases:
  - If $i \leq j$, $e^\phi(ijx) \leq e^\psi(ijx) \leq \frac{1}{K_0}$
  - If $j < i < i_0$, $e^\phi(ijx) = e^\psi(ijx) \leq \frac{1}{K_0}$
  - If $j < i$ and $i > i_0$, $e^\phi(ijx) \leq a_i e^{K_1} \frac{\underline{a}_j}{a_i} \leq \underline{a}_j e^{K_1} \leq a_{i_0} e^{K_1} \leq \frac{1}{K_0}$
- $(C2)$ From (8), we deduce that

  (a) $\Delta_0(\varphi) = \sup_j (\sum_{i=1}^{+\infty} e^{\varphi(ij)}) \leq \sup_j (\frac{-1}{\log \underline{a}_j}) \underline{M} < +\infty$

  (b) $\Delta_p(\varphi) \leq \sup_{j>p} (\frac{-1}{\log \underline{a}_j}) \underline{M}$ is smaller than 1 if p is large enough.

  (c) $\lim_{j\to\infty} e^{\varphi(ij)} \leq \underline{M} \lim_{j\to\infty} \frac{-1}{\log \underline{a}_j} = 0$

  We now apply the proposition 6 to conclude that $P_\infty^\alpha(\phi) < 0$.
- $(C3)$ If $x, y \in (ij)$,

  $$|e^{\phi(x)} - e^{\phi(y)}| = e^{\phi(y)} |e^{\phi(x)-\phi(y)} - 1| \leq a_i e^{K_1} \frac{\underline{a}_j}{\underline{a}_i} |e^{K_1 \frac{|x-y|}{a_i}} - 1| \leq a_i e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} K_1 \frac{|x-y|}{a_i}$$

  We deduce,

  $$\sum_{i_3,\cdots,i_n} var_{i,j,i_3,\cdots,i_n}(e^\phi) \leq \sum_{i_3,\cdots,i_n} a_i K_1 e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} \frac{l(i_3,\cdots,i_n)}{a_i} \leq K_1 e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} l(i,j)$$

  Hence,

  $$V_\theta^{(i)}(e^\phi) = \sum_{n>1} \theta^n \sum_{i_2,\ldots,i_n} var_{i,i_2,\ldots,i_n}(e^\phi) \leq (\sum_{n>1} \theta^n) K_1 e^{2K_1} \frac{a_i}{\underline{a}_i} \leq K < +\infty$$

We have proved that the potential ϕ is in $\mathcal{C}_{\alpha,\theta}$. The measure $\underline{a}_i^{-1} dl$ is conformal. The system is topologically mixing. Hence, applying the corollary 1, we obtain the existence of an invariant function ρ for $\mathcal{L}_\phi$ in $\mathcal{X}_{\alpha,\theta}$. The measure $d\mu = \rho \underline{a}_i^{-1} dl$ is invariant for the original system and absolutely continuous with respect to the Lebesgue measure on I. The ergodic properties of this invariant measure also follow from corollary 1. The proof is complete. □

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